3. ANALYTICAL KINEMATICS

In planar mechanisms, kinematic analysis can be performed either analytically or graphically. In this course we first discuss analytical kinematic analysis.

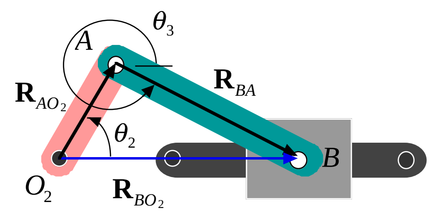
Analytical kinematics is based on projecting the vector loop equation(s) of a mechanism onto the axes of a non-moving Cartesian frame. This projection transforms a vector equation into two algebraic equations. Then, for a given value of the position (or orientation) of the input link, the algebraic equations are solved for the position/orientation of the remaining links. The first and second time derivative of the algebraic position equations provide the velocity and acceleration equations for the mechanism. For given values of the velocity and acceleration of the input link, these equations are solved to find the velocity and acceleration of the other links in the system.

Analytical kinematics is a systematic process that is most suitable for developing into a computer program. However, for very simple systems, analytical kinemtics can be performed by hand calculation. As it will be seen in the upcoming examples, even simple mechanisms can become a challenge for analysis without the use of a computer program.

As a reminder, by definition, a mechanism is a collection of links that are interconnected by kinematic joints forming a single degree-of-freedom system. Therefore, in a kinematic analysis, the position, velocity, and acceleration of the input link must be given or assumed (one coordinate, one velocity and one acceleration). The task is then to compute the other coordinates, velocities, and accelerations.

Slider-crank (inversion 1)

In a slider-crank mechanism, depending on its application, either the crank is the input link and the objective is to determine the kinematics of the connecting rod and the slider, or the slider is the input link and the objective is to determine the kinematics of the connecting rod and the crank. In this example, we assume the first case: For known values of θ_2, ω_2, and α_2 we want to determine the kinematics of the other links.



We start the analysis by defining vectors and constructing the vector loop equation:

$$\mathbf{R}_{AO_2} + \mathbf{R}_{BA} - \mathbf{R}_{BO_2} = \mathbf{0}$$

The constant lengths are: $R_{AO_2} = L_2$, $R_{BA} = L_3$. We place the *x*-*y* frame at a convenient location.

We define an angle (orientation) for each vector according to our convention (CCW with respect to the positive *x*-axis).

<u>Position equations</u>

The vector loop equation is projected onto the *x* and *y* axes to obtain two algebraic equations

$$R_{AO_2}\cos\theta_2 + R_{BA}\cos\theta_3 - R_{BO_2}\cos\theta_1 = 0$$

$$R_{AO_2}\sin\theta_2 + R_{BA}\sin\theta_3 - R_{BO_2}\sin\theta_1 = 0$$

Since $\theta_1 = 0$, we have:

$$\begin{aligned} L_2\cos\theta_2 + L_3\cos\theta_3 - R_{BO_2} &= 0 \\ L_2\sin\theta_2 + L_3\sin\theta_3 &= 0 \end{aligned} \qquad \text{(sc1.p.1)}$$

For known values of L_2 and L_3, and given value for θ_2, these equations can be solved θ_3 and R_{BO_2}:

$$\sin\theta_3 = -(L_2 / L_3)\sin\theta_2 \quad \Rightarrow \quad \theta_3 = \sin^{-1}\theta_3$$

$$R_{BO_2} = L_2\cos\theta_2 + L_3\cos\theta_3$$

Velocity equations

The time derivative of the position equations yields the velocity equations:

$$\begin{aligned} -L_2 \sin\theta_2 \omega_2 - L_3 \sin\theta_3 \omega_3 - \dot{R}_{BO_2} &= 0 \\ L_2 \cos\theta_2 \omega_2 + L_3 \cos\theta_3 \omega_3 &= 0 \end{aligned} \qquad \text{(sc1.v.1)}$$

These equations can also be represented in matrix form, where the terms associated with the known crank velocity are moved to the right-hand-side:

$$\begin{bmatrix} -L_3 \sin\theta_3 & -1 \\ L_3 \cos\theta_3 & 0 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{R}_{BO_2} \end{Bmatrix} = \begin{Bmatrix} L_2 \sin\theta_2 \omega_2 \\ -L_2 \cos\theta_2 \omega_2 \end{Bmatrix} \qquad \text{(sc1.v.2)}$$

Solution of these equations provides values of ω_3 and $\dot{R}_{BO_2}$.

Acceleration equations

The time derivative of the velocity equations yields the acceleration equations:

$$\begin{aligned} -L_2 \sin\theta_2 \alpha_2 - L_2 \cos\theta_2 \omega_2^2 - L_3 \sin\theta_3 \alpha_3 - L_3 \cos\theta_3 \omega_3^2 - \ddot{R}_{BO_2} &= 0 \\ L_2 \cos\theta_2 \alpha_2 - L_2 \sin\theta_2 \omega_2^2 + L_3 \cos\theta_3 \alpha_3 - L_3 \sin\theta_3 \omega_3^2 &= 0 \end{aligned} \qquad \text{(sc1.a.1)}$$

These equations can also be represented in matrix form, where the terms associated with the known crank acceleration and the quadratic velocity terms are moved to the right-hand-side:

$$\begin{bmatrix} -L_3 \sin\theta_3 & -1 \\ L_3 \cos\theta_3 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{R}_{BO_2} \end{Bmatrix} = \begin{Bmatrix} L_2(\sin\theta_2 \alpha_2 + \cos\theta_2 \omega_2^2) + L_3 \cos\theta_3 \omega_3^2 \\ -L_2(\cos\theta_2 \alpha_2 - \sin\theta_2 \omega_2^2) + L_3 \sin\theta_3 \omega_3^2 \end{Bmatrix} \qquad \text{(sc1.a.2)}$$

Solution of these equations provides values of α_3 and $\ddot{R}_{BO_2}$.

Kinematic analysis

For the slider-crank mechanism consider the following constant lengths: $L_2 = 0.12$ and $L_3 = 0.26$ (SI units). For $\theta_2 = 65^o$, $\omega_2 = 1.6$ rad/sec, and $\alpha_2 = 0$, solve the position, velocity and acceleration equations for the unknowns.

Position analysis

For $\theta_2 = 65^o$, we need to solve the position equations for θ_3 and R_{BO_2}. Substituting the known values in equations (sc1.p.1), we have

$$\begin{aligned} 0.12\cos(65) + 0.26\cos\theta_3 - R_{BO_2} &= 0 \\ 0.12\sin(65) + 0.26\sin\theta_3 &= 0 \end{aligned} \qquad \text{(a)}$$

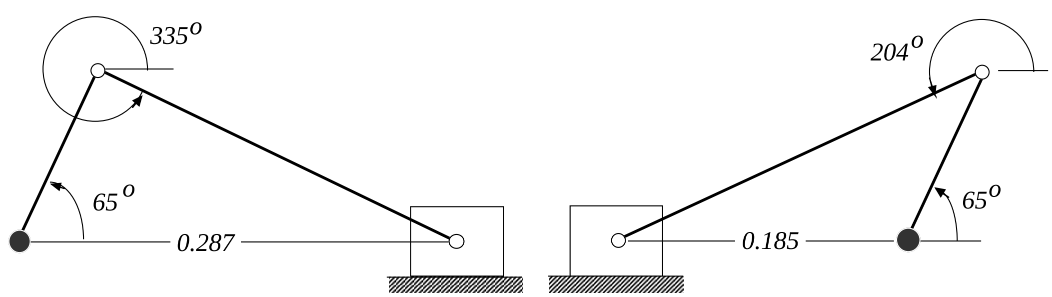
The second row of the equation that simplifies to

$$\sin\theta_3 = -\frac{0.12}{0.26}\sin(65) \quad \Rightarrow \quad \theta_3 = \sin^{-1}(-0.418) \quad \Rightarrow \quad \theta_3 = -24.73^o (335.27^o) \text{ or } 204.73^o$$

There are two solutions for θ_3. Substituting any of these values in the first equation of (a) yields the position of the slider:

$$R_{BO_2} = 0.12\cos(65) + 0.26\cos(335.27) = 0.287 \quad \text{for } \theta_3 = 335.27^o$$

$$R_{BO_2} = 0.12\cos(65) + 0.26\cos(204.73) = -0.185 \quad \text{for } \theta_3 = 204.73^o$$



The two solutions are shown in the diagram. We select the solution that fits our application—here we select the first solution and continue with the rest of the kinematic analysis.

Velocity analysis

For $\theta_2 = 65^o$, $\theta_3 = 335.27^o$, $R_{BO_2} = 0.287$, and $\omega_2 = 1.6$ rad/sec, the velocity equations in (sc1.v.2) become

$$\begin{bmatrix} 0.109 & -1 \\ 0.236 & 0 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{R}_{BO_2} \end{Bmatrix} = \begin{Bmatrix} 0.174 \\ -0.081 \end{Bmatrix}$$

Solving these two equations in two unknowns yields

$$\omega_3 = -0.344 \text{ rad/sec}, \ \dot{R}_{BO_2} = -0.211$$

Acceleration analysis

Substituting all the known values for the coordinates and velocities in (sc1.a.2) provides the acceleration equations as

$$\begin{bmatrix} 0.109 & -1 \\ 0.236 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{R}_{BO_2} \end{Bmatrix} = \begin{Bmatrix} 1.577 \\ 2.656 \end{Bmatrix}$$

Solving these equations yields

$$\alpha_3 = 1.125 \text{ rad/sec}^2, \ \ddot{R}_{BO_2} = -0.035$$

Observations

The analytical process for the kinematics of the slider-crank mechanism reveals the following observations:

- A mechanism with a single kinematic loop yields one vector-loop equation.
- A vector loop equation can be represented as two algebraic position equations.
- Position equations are *non-linear* in the coordinates (angles and distances). Non-linear equations are difficult and time consuming to solve by hand. Numerical methods, such as Newton-Raphson, are recommended for solving non-linear algebraic equations.
- The time derivative of position equations yields velocity equations.
- Velocity equations are *linear* in the velocities.
- The time derivative of velocity equations yields acceleration equations.
- Acceleration equations are *linear* in the accelerations.
- The coefficient matrix of the velocities in the velocity equations and the coefficient matrix of the accelerations in the acceleration equations are identical. This characteristic can be used to simplify the solution process of these equations.

Four-bar

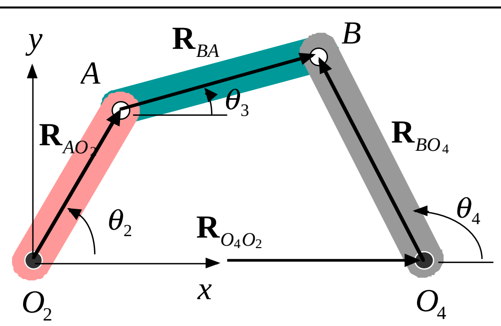
In a four-bar mechanism, generally for a known angle, velocity and acceleration of the input link, we attempt to find the angles, velocities and accelerations of the other two links

The vector loop equation for this four-bar is constructed as

$$\mathbf{R}_{AO_2} + \mathbf{R}_{BA} - \mathbf{R}_{BO_4} - \mathbf{R}_{O_4O_2} = \mathbf{0}$$

The length of the links are

$$R_{O_4O_2} = L_1, R_{AO_2} = L_2, R_{BA} = L_3, R_{BO_4} = L_4$$



We place the x-y frame at a convenient location as shown. We define an angle (orientation) for each link according to our convention (CCW with respect to the positive x-axis).

Position equations

The vector loop equation is projected onto the *x*- and *y*-axes to obtain two algebraic equations:

$$\begin{aligned} R_{AO_2}\cos\theta_2 + R_{BA}\cos\theta_3 - R_{BO_4}\cos\theta_4 - R_{O_4O_2}\cos\theta_1 &= 0 \\ R_{AO_2}\sin\theta_2 + R_{BA}\sin\theta_3 - R_{BO_4}\sin\theta_4 - R_{O_4O_2}\sin\theta_1 &= 0 \end{aligned} \quad \text{(fb-p.1)}$$

Since $\theta_1 = 0$ and the link lengths are known constants, the equations are simplified to:

$$\begin{aligned} L_2\cos\theta_2 + L_3\cos\theta_3 - L_4\cos\theta_4 - L_1 &= 0 \\ L_2\sin\theta_2 + L_3\sin\theta_3 - L_4\sin\theta_4 &= 0 \end{aligned} \quad \text{(fb-p.2)}$$

Velocity equations

The time derivative of the position equations yields:

$$\begin{aligned} -L_2\sin\theta_2\omega_2 - L_3\sin\theta_3\omega_3 + L_4\sin\theta_4\omega_4 &= 0 \\ L_2\cos\theta_2\omega_2 + L_3\cos\theta_3\omega_3 - L_4\cos\theta_4\omega_4 &= 0 \end{aligned} \quad \text{(fb.v.1)}$$

Assuming the angular velocity of the crank, ω_2, is known, we re-arrange and express these equations in matrix form as

$$\begin{bmatrix} -L_3\sin\theta_3 & L_4\sin\theta_4 \\ L_3\cos\theta_3 & -L_4\cos\theta_4 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \end{Bmatrix} \quad \text{(fb.v.2)}$$

Acceleration equations

The time derivative of the velocity equations yields the acceleration equations:

$$\begin{aligned} -L_{BA}\sin\theta_3\alpha_3 - L_{BA}\cos\theta_3\omega_3^2 + L_{BO_4}\sin\theta_4\alpha_4 + L_{BO_4}\cos\theta_4\omega_4^2 &= L_{AO_2}\sin\theta_2\alpha_2 + L_{AO_2}\cos\theta_2\omega_2^2 \\ L_{BA}\cos\theta_3\alpha_3 - L_{BA}\sin\theta_3\omega_3^2 - L_{BO_4}\cos\theta_4\alpha_4 + L_{BO_4}\sin\theta_4\omega_4^2 &= -L_{AO_2}\cos\theta_2\alpha_2 + L_{AO_2}\sin\theta_2\omega_2^2 \end{aligned}$$

(fb.a.1)

Assuming that α_2 is known, we re-arrange the equations as

$$\begin{bmatrix} -L_{BA}\sin\theta_3 & L_{BO_4}\sin\theta_4 \\ L_{BA}\cos\theta_3 & -L_{BO_4}\cos\theta_4 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} L_{AO_2}(\sin\theta_2\alpha_2 + \cos\theta_2\omega_2^2) + L_{BA}\cos\theta_3\omega_3^2 - L_{BO_4}\cos\theta_4\omega_4^2 \\ -L_{AO_2}(\cos\theta_2\alpha_2 - \sin\theta_2\omega_2^2) + L_{BA}\sin\theta_3\omega_3^2 - L_{BO_4}\sin\theta_4\omega_4^2 \end{Bmatrix}$$

(fb.a.2)

Kinematic analysis

Let us consider the following constant lengths: $L_1 = 5$, $L_2 = 2$, $L_3 = 6$, $L_4 = 4$. For $\theta_2 = 120^\circ$, $\omega_2 = 1.0$ rad/sec, CCW, and $\alpha_2 = -1.0$ rad/sec^2, determine the other two angles, angular velocities, and angular accelerations.

Position analysis

For $\theta_2 = 120^\circ$ we solve the position equations for θ_3 and θ_4. Substituting the known lengths and the input angle in (fb.p.2), we get

$$\begin{aligned} 2\cos(120^o) + 6\cos\theta_3 - 4\cos\theta_4 - 5 &= 0 \\ 2\sin(120^o) + 6\sin\theta_3 - 4\sin\theta_4 &= 0 \end{aligned}$$

We have two nonlinear equations in two unknowns. We will consider a numerical method (Newton-Raphson) for solving these equations, as will be seen next. At this point let us consider the solution to be

$$\theta_3 = 0.3834 = 21.98^\circ$$

$$\theta_4 = 1.6799 = 96.24^\circ$$

Velocity analysis

With known values for the angles and the given input velocity, the velocity equation of (fb.v.2) becomes:

$$\begin{bmatrix} -2.2442 & 3.9762 \\ 5.5645 & 0.4355 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} 1.7321 \\ 1.0000 \end{Bmatrix}$$

The solution to these linear equations yields: $\omega_3 = 0.1395,\ \omega_4 = 0.5143$ rad/sec. Both velocities are positive, which means both are CCW.

Acceleration analysis

For known values for the angles and velocities, and the given input acceleration, the acceleration equations become:

$$\begin{bmatrix} -2.2442 & 3.9762 \\ 5.5645 & 0.4355 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} -2.7390 \\ -0.2761 \end{Bmatrix}$$

The solution yields: $\alpha_3 = 0.0041,\ \alpha_4 = -0.6865$ rad/sec^2. One acceleration is positive; i.e., CCW, and one is negative; i.e., CW.

Newton-Raphson Method

Newton-Raphson is a numerical method for solving non-linear algebraic equations. The method is based on linearizing nonlinear equation(s) using Taylor series, then solving the approximated linear equation(s) iteratively.

One Equation in One Unknown

Consider the nonlinear equation $f(x) = 0$ which contains one unknown x. The approximated linearized equation is written as

$$f(x) + \frac{df}{dx}\Delta x \approx 0$$

The Newton-Raphson iterative formula is expressed as

$$\Delta x = -f(x) / \left(\frac{df}{dx}\right) \qquad \text{(N-R.1)}$$

The process requires an initial estimate for the solution. This value is used in (N-R.1) to compute Δx. Then the computed value for Δx is used to update x as

$$x + \Delta x \quad \rightarrow \quad x \qquad \text{(N-R.2)}$$

The process is repeated until a solution is found; i.e., until $f(x) = 0$.

Note: In iterative procedures such as N-R, if $|f(x)| \le \varepsilon$, where ε is a small positive number, we must accept that a solution has been found.

Example

Find the root(s) of $x^3 - 3x^2 - 10x + 24 = 0$ using Newton-Raphson process.

Solution

We re-state the equation as $f = x^3 - 3x^2 - 10x + 24$. The derivative of this function with respect to the unknown is $df / dx = 3x^2 - 6x - 10$. To start the N-R process, we assume the solution is at $x = 10$. The following table shows the results from the iterative N-R process:

Iteration #	x	f	df/dx	Δx	$x + \Delta x$
1	10	624.0000	230	-2.7130	7.2870
2	7.2870	178.7667	105.5775	-1.6932	5.5937
3	5.5937	49.2200	50.3070	-0.9784	4.6153
4	4.6153	12.2555	26.2120	-0.4676	4.1478
5	4.1478	2.2688	16.7257	-0.1356	4.0121
6	4.0121	0.1713	14.2189	-0.0120	4.0001
7	4.0001	0.0013	14.0017	$-9.3\text{x}10^{-5}$	4.0000
8	**4.0000**	0.0000			

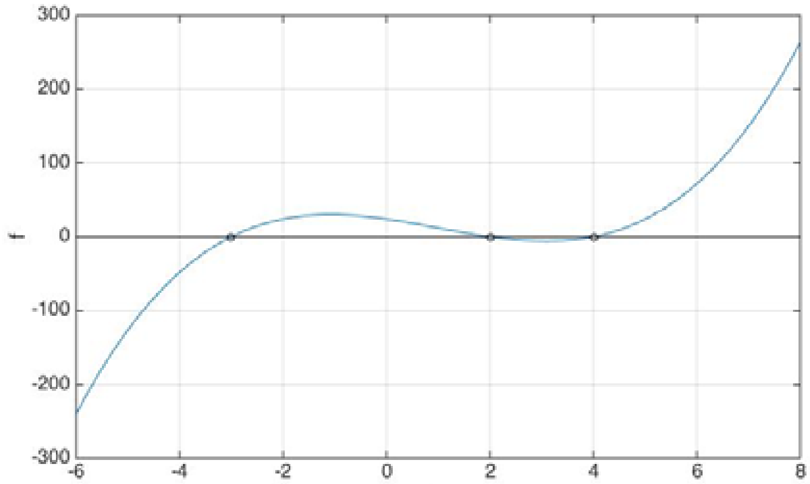
The process converges to $x = 4.0$ as the answer.

We now consider a different initial estimate for the solution. Instead of $x = 10$ we repeat the process from $x = -5$.

Iteration #	x	f	df/dx	Δx	$x + \Delta x$
1	-5	-126	95	1.3263	-3.6737
2	-3.6737	-29.3309	52.5300	0.5584	-3.1153
3	-3.1153	-4.1973	37.8076	0.1110	-3.0043
4	-3.0043	-0.1508	35.1033	0.0043	-3.0000
5	-3.0000	$-2.2\text{x}10^{-4}$	35.0002	$6.3\text{x}10^{-6}$	-3.0000
6	**-3.0000**	$-4.8\text{x}10^{-8}$			

We now know that $x = -3.0$ is another solution to this problem.

Obviously there should be a third solution since we are dealing with a quadratic function. The following figure should clarify what the solutions are.



Two Equations in Two Unknowns

Consider the following two non-linear equations in *x* and *y*:

$$f_1(x,y) = 0$$

$$f_2(x,y) = 0$$

The approximated linearized equations are written as

$$f_1(x,y) + \frac{\partial f_1}{\partial x}\Delta x + \frac{\partial f_1}{\partial y}\Delta y \approx 0$$

$$f_2(x,y) + \frac{\partial f_2}{\partial x}\Delta x + \frac{\partial f_2}{\partial y}\Delta y \approx 0$$

The Newton-Raphson iterative formula is expressed as

$$\begin{Bmatrix} \Delta x \\ \Delta y \end{Bmatrix} = -\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1} \begin{Bmatrix} f_1(x,y) \\ f_2(x,y) \end{Bmatrix} \qquad \text{(N-R.3)}$$

The process requires an initial estimate for the unknowns *x* and *y*. These value are used in (N-R.3) to compute Δx and Δy. Then the computed values are used to update the approximated solution:

$$\begin{aligned} x + \Delta x &\rightarrow x \\ y + \Delta y &\rightarrow y \end{aligned} \qquad \text{(N-R.4)}$$

The process is repeated until a solution is found. Rather than checking whether each function meets the condition $|f| \le \varepsilon$, we consider $\sqrt{f_1^2 + f_2^2} \le \varepsilon$ for terminating the process.

Example (four-bar)

We apply the Newton-Raphson process to solve the position equations for a four-bar mechanism. The position equations from Example 1 are expressed as:

$$\begin{aligned} f_1 &= 2\cos(120^o) + 6\cos\theta_3 - 4\cos\theta_4 - 5 \\ f_2 &= 2\sin(120^o) + 6\sin\theta_3 - 4\sin\theta_4 \end{aligned} \qquad \text{(a)}$$

Then N-R formula for these equations becomes:

$$\begin{Bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{Bmatrix} = -\begin{bmatrix} \dfrac{\partial f_1}{\partial\theta_3} & \dfrac{\partial f_1}{\partial\theta_4} \\ \dfrac{\partial f_2}{\partial\theta_3} & \dfrac{\partial f_2}{\partial\theta_4} \end{bmatrix}^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = -\begin{bmatrix} -6\sin\theta_3 & 4\sin\theta_4 \\ 6\cos\theta_3 & -4\cos\theta_4 \end{bmatrix}^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \qquad \text{(b)}$$

From a rough sketch of the mechanism for the given input angle, we estimate the values for the two unknowns to be:

$$\theta_3 \approx 30^o = 0.5236 \text{ rad}, \; \theta_4 \approx 90^o = 1.5708 \text{ rad}$$

We start the Newton-Raphson process by evaluating the two functions in (a):

$$f_1 = 2\cos(120^o) + 6\cos(30^o) - 4\cos(90^o) - 5 = -0.8038$$

$$f_2 = 2\sin(120^o) + 6\sin(30^o) - 4\sin(90^o) = 0.7321$$

These values show that our estimates are far from zeros. We evaluate (b):

$$\begin{Bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{Bmatrix} = -\begin{bmatrix} -3.0000 & 4.0000 \\ 5.1962 & 0.0000 \end{bmatrix}^{-1} \begin{Bmatrix} -0.8038 \\ 0.7321 \end{Bmatrix} = \begin{Bmatrix} -0.1409 \\ 0.0953 \end{Bmatrix}$$

Note that the corrections for the two angles are in radians not in degrees (this is always true). Therefore the estimated values of the two angles are corrected as

$$\theta_3 \approx 0.5236 - 0.1409 = 0.3827 \text{ and } \theta_4 \approx 1.5708 + 0.0953 = 1.6661$$

The two equations in (a) are re-evaluated:

$$f_1 = -0.0535, \; f_2 = -0.0092$$

Since these values are not zeros, the process is continued. After two more iterations the process yields:

$$\theta_3 = 0.3834 = 21.98^\circ, \; \theta_4 = 1.6799 = 96.24^\circ$$

With these values, f_1 and f_2 are small enough to be considered zeros.

The Newton-Raphson process can be extended to *n* equations in *n* unknowns. The formulas are similar to those for two equations. It should be obvious that the N-R process is not suitable for hand calculation. The method is suitable for implementation in a computer program.

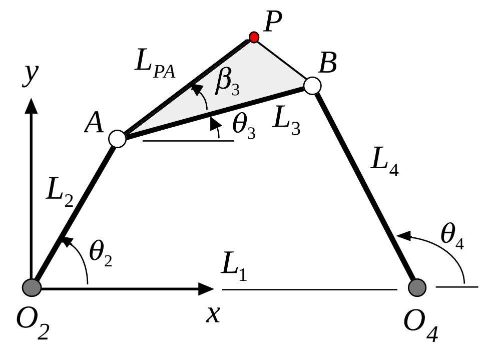
Secondary Computations

In addition to solving the kinematic equations for the coordinates, velocities and accelerations, we may need to determine the kinematics of a point that is defined on one of the links of the mechanism. Determining the kinematics of a point on a link is a secondary process and it does not require solving any set of algebraic equations—we only need to evaluate one or more expressions.

Four-bar coupler point

Assume that the coupler of a four-bar is in the shape of a triangle, and the location of the coupler point P relative to A and B is defined by the angle β_3 and the length L_{PA} (two constants). This coupler point can be positioned with respect to the origin of the x-y frame as

$$\mathbf{R}_{PO_2} = \mathbf{R}_{AO_2} + \mathbf{R}_{PA}$$



Coupler point expressions

Algebraically, the above equation becomes:

$$\begin{aligned} x_P &= L_2 \cos\theta_2 + L_{PA} \cos(\theta_3 + \beta_3) \\ y_P &= L_2 \sin\theta_2 + L_{PA} \sin(\theta_3 + \beta_3) \end{aligned} \qquad \text{(fb.cp.1)}$$

The time derivative of the position expressions provides the velocity of point P:

$$\begin{aligned} \dot{x}_P \equiv V_{P(x)} &= -L_2 \sin\theta_2 \omega_2 - L_{PA} \sin(\theta_3 + \beta_3)\omega_3 \\ \dot{y}_P \equiv V_{P(y)} &= \quad L_2 \cos\theta_2 \omega_2 + L_{PA} \cos(\theta_3 + \beta_3)\omega_3 \end{aligned} \qquad \text{(fb.cv.1)}$$

Similarly, the time derivative of the velocity expressions yields the acceleration of point P:

$$\begin{aligned} \ddot{x}_P \equiv A_{P(x)} &= -L_2(\sin\theta_2\alpha_2 + \cos\theta_2\omega_2^2) - L_{PA}(\sin(\theta_3 + \beta_3)\alpha_3 + \cos(\theta_3 + \beta_3)\omega_3^2) \\ \ddot{y}_P \equiv A_{P(y)} &= \quad L_2(\cos\theta_2\alpha_2 - \sin\theta_2\omega_2^2) + L_{PA}(\cos(\theta_3 + \beta_3)\alpha_3 - \sin(\theta_3 + \beta_3)\omega_3^2) \end{aligned} \qquad \text{(fb.ca.1)}$$

Example (four-bar)

We continue with the data for the four-bar example. Assume the coupler point is positioned at $\beta_3 = 22.5^o$, $L_{PA} = 5.5$. Substituting the known values for the angles, angular velocities, and angular accelerations yields the coordinate, velocity, and acceleration of the coupler point:

$$\mathbf{R}_{PO_2} = \begin{Bmatrix} x_P \\ y_P \end{Bmatrix} = \begin{Bmatrix} 5\cos(120^o) + 5.5\cos(21.98^\circ + 22.5^\circ) = 2.9253 \\ 5\sin(120^o) + 5.5\sin(21.98^\circ + 22.5^\circ) = 5.5846 \end{Bmatrix}$$

$$\mathbf{V}_P = \begin{Bmatrix} \dot{x}_P \\ \dot{y}_P \end{Bmatrix} = \begin{Bmatrix} -2.2693 \\ -0.4526 \end{Bmatrix}, \quad \mathbf{A}_P = \begin{Bmatrix} \ddot{x}_P \\ \ddot{y}_P \end{Bmatrix} = \begin{Bmatrix} 2.6399 \\ -0.7908 \end{Bmatrix}$$

Matlab Programs

Two Matlab programs (**`fourbar.m`** and **`fourbar_anim.m`**) are provided for kinematic analysis of a four-bar mechanism containing a coupler point. The program **`fourbar.m`** performs position, velocity, and acceleration analysis for a given angle of the crank. The program solves for the unknown coordinates, velocities, and accelerations, and reports the results in numerical form. The program **`fourbar_anim.m`** only performs position analysis. However, it repeatedly increments the crank angle and reports the results in the form of an animation. Both programs obtain the data for the four-bar from the file **`fourbar_data`**.

`fourbar_data.m`

The user is required to provide in this file the following data for the four-bar of interest:

- Constant values for the link lengths (L_1, L_2, L_3, L_4) †✻
- Initial angle of the crank (θ_2) †✻
- Estimates for the initial angles of the coupler and the follower (θ_3, θ_4) †✻

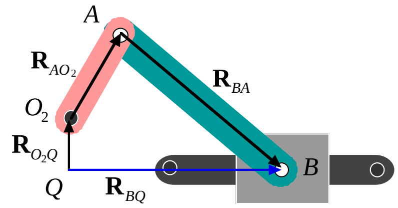
- Constant values for the coupler point position (L_{PA} , β_3) † ✻
- Angular velocity and acceleration of the crank (ω_2 , α_2) †
- Animation increment for the crank angle ($\Delta\theta_2$) ✻
- Limits for the plot axes [`x_min  x_max  y_min  y_max`] ✻
  † Needed by **`fourbar.m`**
  ✻ Needed by **`fourbar_anim.m`**

The program **`fourbar_anim.m`** requires an accompanying M-file named **`fourbar_plot.m`** that must reside in the same directory. The program first checks and reports whether the four-bar is *Grashof* or not. Then it computes the unknown angles, using the N-R process, and the coordinates of the coupler point. If a solution is found the results are depicted graphically. After the four-bar is displayed in its initial state, if any keys is pressed the program will increment the crank angle and solves for the new angles and coordinates. The solution and animation will be continued for two complete cycles of the crank rotation. For a *non-Grashof* four-bar, or if the four-bar is *Grashof* but the input link is not able to rotate completely, the input link is rotated between its limits.

Other Mechanisms

Kinematic analysis of other mechanisms is similar to, and in some cases simpler than, that of a four-bar. The followings are examples of position, velocity, and acceleration equations, and the underlying objectives of kinematic analysis for some commonly used mechanisms.

Slider-crank (inversion 1 - offset)



Vector loop equation

$$\mathbf{R}_{AO_2} + \mathbf{R}_{BA} - \mathbf{R}_{BO_2} + \mathbf{R}_{O_2Q} = \mathbf{0}$$

Constant: $R_{AO_2} = L_2$, $R_{BA} = L_3$, $R_{O_2Q} = a$ (offset)

Position equations

$$R_{AO_2}\cos\theta_2 + R_{BA}\cos\theta_3 - R_{BQ}\cos\theta_1 + R_{O_2Q}\cos(90^o) = 0$$

$$R_{AO_2}\sin\theta_2 + R_{BA}\sin\theta_3 - R_{BQ}\sin\theta_1 + R_{O_2Q}\sin(90^o) = 0$$

Since $\theta_1 = 0$, we have:

$$\begin{aligned} &L_2\cos\theta_2 + L_3\cos\theta_3 - R_{BQ} = 0 \\ &L_2\sin\theta_2 + L_3\sin\theta_3 + a = 0 \end{aligned} \qquad \text{(sc1-o.p.1)}$$

Position analysis

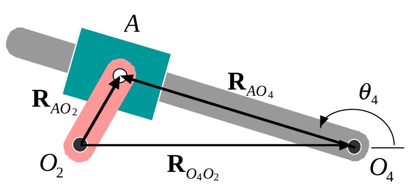
For a given crank angle θ_2, solve the position equations for θ_3 and R_{BQ}.

Velocity and acceleration equations

The equations are identical to those of the non-offset system.

Q: What are the angular velocity and acceleration of the slider-block?

Slider-crank (inversion 2)



Vector loop equation

$$\mathbf{R}_{AO_2} - \mathbf{R}_{AO_4} - \mathbf{R}_{O_4O_2} = \mathbf{0}$$

Constant: $R_{AO_2} = L_2$, $R_{O_4O_2} = L_1$

Position equations

$$\begin{aligned} &L_2\cos\theta_2 - R_{AO_4}\cos\theta_4 - L_1 = 0 \\ &L_2\sin\theta_2 - R_{AO_4}\sin\theta_4 = 0 \end{aligned} \qquad \text{(sc2.p.1)}$$

Position analysis

For a given crank angle θ_2, solve the position equations for θ_4 and R_{AO_4}.

Velocity equations (in expanded and matrix forms)

$$-L_2 \sin\theta_2\omega_2 + R_{AO_4}\sin\theta_4\omega_4 - \dot{R}_{AO_4}\cos\theta_4 = 0$$
$$L_2 \cos\theta_2\omega_2 - R_{AO_4}\cos\theta_4\omega_4 - \dot{R}_{AO_4}\sin\theta_4 = 0 \quad \text{(sc2.v.1)}$$

$$\begin{bmatrix} R_{AO_4}\sin\theta_4 & -1 \\ -R_{AO_4}\cos\theta_4 & 0 \end{bmatrix} \begin{Bmatrix} \omega_4 \\ \dot{R}_{AO_4} \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \end{Bmatrix} \quad \text{(sc2.v.2)}$$

Velocity analysis

For a given crank velocity ω_2, solve the velocity equations for ω_4 and $\dot{R}_{AO_4}$.

Acceleration equations (in expanded and matrix forms)

$$-L_2\sin\theta_2\alpha_2 - L_2\cos\theta_2\omega_2^2 + R_{AO_4}\sin\theta_4\alpha_4 + R_{AO_4}\cos\theta_4\omega_4^2 + 2\dot{R}_{AO_4}\sin\theta_4\omega_4 - \ddot{R}_{AO_4}\cos\theta_4 = 0$$
$$L_2\cos\theta_2\alpha_2 - L_2\sin\theta_2\omega_2^2 - R_{AO_4}\cos\theta_4\alpha_4 + R_{AO_4}\sin\theta_4\omega_4^2 - 2\dot{R}_{AO_4}\cos\theta_4\omega_4 - \ddot{R}_{AO_4}\sin\theta_4 = 0 \quad \text{(sc2.a.1)}$$

$$\begin{bmatrix} R_{AO_4}\sin\theta_4 & -1 \\ -R_{AO_4}\cos\theta_4 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \ddot{R}_{AO_4} \end{Bmatrix} = \begin{Bmatrix} L_2(\sin\theta_2\alpha_2 + \cos\theta_2\omega_2^2) - R_{AO_4}\cos\theta_4\omega_4^2 - 2\dot{R}_{AO_4}\sin\theta_4\omega_4 \\ -L_2(\cos\theta_2\alpha - \sin\theta_2\omega_2^2) - R_{AO_4}\sin\theta_4\omega_4^2 + 2\dot{R}_{AO_4}\cos\theta_4\omega_4 \end{Bmatrix} \quad \text{(sc2.a.2)}$$

Acceleration analysis

For a given crank acceleration α_2, solve the acceleration equations for α_4 and $\ddot{R}_{AO_4}$.

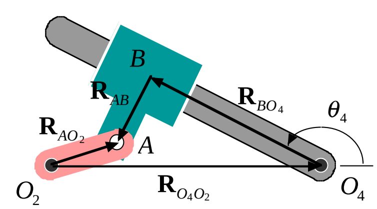
Q: What are the angular velocity and acceleration of the slider-block?

Slider-crank (inversion 2 - offset)

Vector loop equation

$$\mathbf{R}_{AO_2} - \mathbf{R}_{AB} - \mathbf{R}_{BO_4} - \mathbf{R}_{O_4O_2} = \mathbf{0}$$

Constants: $R_{AO_2} = L_2$, $R_{O_4O_2} = L_1$, $R_{AB} = a$ (offset)



Position equations

$$L_2\cos\theta_2 - a\cos(\theta_4 + 90^o) - R_{BO_4}\cos\theta_4 - L_1 = 0$$
$$L_2\sin\theta_2 - a\sin(\theta_4 + 90^o) - R_{BO_4}\sin\theta_4 = 0 \quad \text{(sc2-o.p.1)}$$

Position analysis

For a given crank angle θ_2, solve the position equations for θ_4 and R_{BO_4}.

Velocity equations (in expanded and matrix forms)

$$-L_2\sin\theta_2\omega_2 + a\sin(\theta_4 + 90^o)\omega_4 + R_{BO_4}\sin\theta_4\omega_4 - \dot{R}_{BO_4}\cos\theta_4 = 0$$
$$L_2\cos\theta_2\omega_2 - a\cos(\theta_4 + 90^o)\omega_4 - R_{BO_4}\cos\theta_4\omega_4 - \dot{R}_{BO_4}\sin\theta_4 = 0 \quad \text{(sc2-o.v.1)}$$

$$\begin{bmatrix} \left(a\sin(\theta_4 + 90^o) + R_{BO_4}\sin\theta_4\right) & -\cos\theta_4 \\ -\left(a\cos(\theta_4 + 90^o) - R_{BO_4}\cos\theta_4\right) & -\sin\theta_4 \end{bmatrix} \begin{Bmatrix} \omega_4 \\ \dot{R}_{BO_4} \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \end{Bmatrix} \quad \text{(sc2-o.v.2)}$$

Velocity analysis

For a given crank velocity ω_2, solve the velocity equations for ω_4 and $\dot{R}_{BO_4}$.

Acceleration equations (in expanded and matrix forms)

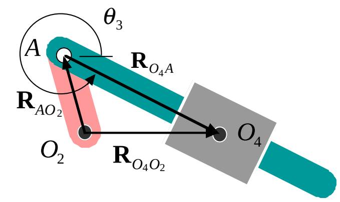
$$-L_2\sin\theta_2\alpha_2 - L_2\cos\theta_2\omega_2^2 + \left(a\sin(\theta_4 + 90^o) + R_{BO_4}\sin\theta_4\right)\alpha_4$$
$$+\left(a\cos(\theta_4 + 90^o) + R_{BO_4}\cos\theta_4\right)\omega_4^2 + 2\dot{R}_{BO_4}\sin\theta_4\omega_4 - \ddot{R}_{BO_4}\cos\theta_4 = 0$$
$$L_2\cos\theta_2\alpha_2 - L_2\sin\theta_2\omega_2^2 - \left(a\cos(\theta_4 + 90^o) + R_{BO_4}\cos\theta_4\right)\alpha_4 \quad \text{(sc2-o.a.1)}$$
$$+\left(a\sin(\theta_4 + 90^o) + R_{BO_4}\sin\theta_4\right)\omega_4^2 - 2\dot{R}_{BO_4}\cos\theta_4\omega_4 - \ddot{R}_{BO_4}\sin\theta_4 = 0$$

$$\begin{bmatrix} \left(a\sin(\theta_4+90^o)+R_{BO_4}\sin\theta_4\right) & -\cos\theta_4 \\ -\left(a\cos(\theta_4+90^o)-R_{BO_4}\cos\theta_4\right) & -\sin\theta_4 \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \ddot{R}_{BO_4} \end{Bmatrix} =$$
$$\begin{Bmatrix} L_2(\sin\theta_2\alpha_2+\cos\theta_2\omega_2^2)-\left(a\cos(\theta_4+90^o)+R_{BO_4}\cos\theta_4\right)\omega_4^2-2\dot{R}_{BO_4}\sin\theta_4\omega_4 \\ -L_2(\cos\theta_2\alpha_2-\sin\theta_2\omega_2^2)-\left(a\sin(\theta_4+90^o)+R_{BO_4}\sin\theta_4\right)\omega_4^2+2\dot{R}_{BO_4}\cos\theta_4\omega_4 \end{Bmatrix} \quad \text{(sc2-o.a.2)}$$

Acceleration analysis

For a given crank acceleration α_2, solve the acceleration equations for α_4 and $\ddot{R}_{BO_4}$.

Slider-crank (inversion 3)



Vector loop equation

$$\mathbf{R}_{AO_2}+\mathbf{R}_{O_4A}-\mathbf{R}_{O_4O_2}=\mathbf{0}$$

Constant: $R_{AO_2}=L_2,\ R_{O_4O_2}=L_1$

Position equations

$$\begin{aligned} &L_2\cos\theta_2+R_{O_4A}\cos\theta_3-L_1=0 \\ &L_2\sin\theta_2+R_{O_4A}\sin\theta_3=0 \end{aligned} \quad \text{(sc3.p.1)}$$

Position analysis

For a given crank angle θ_2, solve the position equations for θ_3 and R_{O_4A}.

Velocity equations (in expanded and matrix forms)

$$\begin{aligned} &-L_2\sin\theta_2\omega_2-R_{O_4A}\sin\theta_3\omega_3+\dot{R}_{O_4A}\cos\theta_3=0 \\ &L_2\cos\theta_2\omega_2+R_{O_4A}\cos\theta_3\omega_3+\dot{R}_{O_4A}\sin\theta_3=0 \end{aligned} \quad \text{(sc3.v.1)}$$

$$\begin{bmatrix} -R_{O_4A}\sin\theta_3 & \cos\theta_3 \\ R_{O_4A}\cos\theta_3 & \sin\theta_3 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{R}_{O_4A} \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \end{Bmatrix} \quad \text{(sc3.v.2)}$$

Velocity analysis

For a given crank velocity ω_2, solve the velocity equations for ω_3 and $\dot{R}_{O_4A}$.

Acceleration equations (in expanded and matrix forms)

$$\begin{aligned} &-L_2\sin\theta_2\alpha_2-L_2\cos\theta_2\omega_2^2-R_{O_4A}\sin\theta_3\alpha_3-R_{O_4A}\cos\theta_3\omega_3^2-2\dot{R}_{O_4A}\sin\theta_3\omega_3+\ddot{R}_{O_4A}\cos\theta_3=0 \\ &L_2\cos\theta_2\alpha_2-L_2\sin\theta_2\omega_2^2+R_{O_4A}\cos\theta_3\alpha_3-R_{O_4A}\sin\theta_3\omega_3^2+2\dot{R}_{O_4A}\cos\theta_3\omega_3+\ddot{R}_{O_4A}\sin\theta_3=0 \end{aligned} \quad \text{(sc3.a.1)}$$

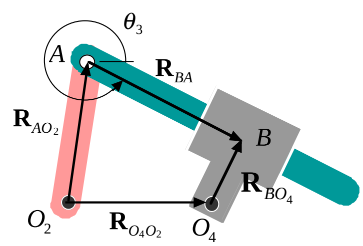
$$\begin{bmatrix} -R_{O_4A}\sin\theta_3 & \cos\theta_3 \\ R_{O_4A}\cos\theta_3 & \sin\theta_3 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{R}_{O_4A} \end{Bmatrix} = \begin{Bmatrix} L_2(\sin\theta_2\alpha_2+\cos\theta_2\omega_2^2)+R_{O_4A}\cos\theta_3\omega_3^2+2\dot{R}_{O_4A}\sin\theta_3\omega_3 \\ -L_2(\cos\theta_2\alpha_2-\sin\theta_2\omega_2^2)+R_{O_4A}\sin\theta_3\omega_3^2-2\dot{R}_{O_4A}\cos\theta_3\omega_3 \end{Bmatrix} \quad \text{(sc3.a.2)}$$

Acceleration analysis

For a given crank acceleration α_2, solve the acceleration equations for α_3 and $\ddot{R}_{O_4A}$.

Q: What are the angular velocity and acceleration of the slider-block?

Slider-crank (inversion 3 - offset)



Vector loop equation

$$\mathbf{R}_{AO_2}+\mathbf{R}_{BA}-\mathbf{R}_{BO_4}-\mathbf{R}_{O_4O_2}=\mathbf{0}$$

Constant: $R_{AO_2}=L_2,\ R_{O_4O_2}=L_1,\ R_{BO_4}=a$ (offset)

Position equations

$$\begin{aligned} &L_2\cos\theta_2+R_{BA}\cos\theta_3-a\cos(\theta_3+90^o)-L_1=0 \\ &L_2\sin\theta_2+R_{BA}\sin\theta_3-a\sin(\theta_3+90^o)=0 \end{aligned} \quad \text{(sc3-o.p.1)}$$

<u>Position analysis</u>

For a given crank angle θ_2, solve the position equations for θ_3 and R_{BA}.

<u>Velocity equations</u> (in expanded and matrix forms)

$$-L_2 \sin\theta_2\omega_2 - R_{BA}\sin\theta_3\omega_3 + \dot{R}_{BA}\cos\theta_3 + a\sin(\theta_3 + 90^o)\omega_3 = 0$$
$$L_2 \cos\theta_2\omega_2 + R_{BA}\cos\theta_3\omega_3 + \dot{R}_{BA}\sin\theta_3 - a\cos(\theta_3 + 90^o)\omega_3 = 0 \qquad \text{(sc3-o.v.1)}$$

$$\begin{bmatrix} \left(-R_{BA}\sin\theta_3 + a\sin(\theta_3+90^o)\right) & \cos\theta_3 \\ \left(R_{BA}\cos\theta_3 - a\cos(\theta_3+90^o)\right) & \sin\theta_3 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{R}_{BA} \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \end{Bmatrix} \qquad \text{(sc3-o.v.2)}$$

<u>Velocity analysis</u>

For a given crank angle ω_2, solve the position equations for ω_3 and $\dot{R}_{BA}$.

<u>Acceleration equations</u> (in expanded and matrix forms)

$$\begin{aligned} &-L_2\sin\theta_2\alpha_2 - L_2\cos\theta_2\omega_2^2 - \left(R_{BA}\sin\theta_3 - a\sin(\theta_3+90^o)\right)\alpha_3 \\ &\quad + \ddot{R}_{BA}\cos\theta_3 - \left(R_{BA}\cos\theta_3 - a\cos(\theta_3+90^o)\right)\omega_3^2 - 2\dot{R}_{BA}\sin\theta_3\omega_3 = 0 \\ &L_2\cos\theta_2\alpha_2 - L_2\sin\theta_2\omega_2^2 + \left(R_{BA}\cos\theta_3 - a\cos(\theta_3+90^o)\right)\alpha_3 \\ &\quad + \ddot{R}_{BA}\sin\theta_3 - \left(R_{BA}\sin\theta_3 - a\sin(\theta_3+90^o)\right)\omega_3^2 + 2\dot{R}_{BA}\cos\theta_3\omega_3 = 0 \end{aligned} \qquad \text{(sc3-o.a.1)}$$

$$\begin{bmatrix} \left(-R_{BA}\sin\theta_3 + a\sin(\theta_3+90^o)\right) & \cos\theta_3 \\ \left(R_{BA}\cos\theta_3 - a\cos(\theta_3+90^o)\right) & \sin\theta_3 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{R}_{BA} \end{Bmatrix} =$$
$$\begin{Bmatrix} L_2(\sin\theta_2\alpha_2 + \cos\theta_2\omega_2^2) + \left(R_{BA}\cos\theta_3 - a\cos(\theta_3+90^o)\right)\omega_3^2 + 2\dot{R}_{BA}\sin\theta_3\omega_3 \\ -L_2(\cos\theta_2\alpha_2 - \sin\theta_2\omega_2^2) + \left(R_{BA}\sin\theta_3 - a\sin(\theta_3+90^o)\right)\omega_3^2 - 2\dot{R}_{BA}\cos\theta_3\omega_3 \end{Bmatrix} \qquad \text{(sc3-o.a.2)}$$

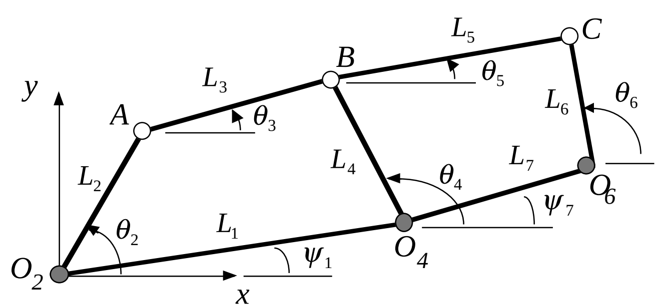
<u>Acceleration analysis</u>

For a given crank acceleration α_2, solve the acceleration equations for α_3 and $\ddot{R}_{BA}$.

In any of the inversions of the slider-crank, the position equations contain one unknown length and one unknown angle. These equations, compared to the position equations of a four-bar, are simpler to solve by hand. However, it is highly recommended that the Matlab program `fourbar.m` to be revised from a four-bar to any of the slider-crank inversions.

Six-bar Mechanism

This six-bar mechanism is constructed from two four-bars in series. There are three ground attachment joints at O_2, O_4 and O_6.



<u>Position loop equations</u>

$$\mathbf{R}_{AO_2} + \mathbf{R}_{BA} - \mathbf{R}_{BO_4} - \mathbf{R}_{O_4O_2} = 0$$
$$\mathbf{R}_{BO_4} + \mathbf{R}_{CB} - \mathbf{R}_{CO_6} - \mathbf{R}_{O_6O_4} = 0$$

Constant angles:

$\theta_1 = \psi_1$ and $\theta_7 = \psi_7$

Constant lengths: $R_{O_4O_2} = L_1$, $R_{AO_2} = L_2$, $R_{BA} = L_3$, $R_{BO_4} = L_4$, $R_{CB} = L_5$, $R_{CO_6} = L_6$, $R_{O_6O_4} = L_7$

<u>Position equations</u>

$$L_2\cos\theta_2 + L_3\cos\theta_3 - L_4\cos\theta_4 - L_1\cos\psi_1 = 0$$
$$L_2\sin\theta_2 + L_3\sin\theta_3 - L_4\sin\theta_4 - L_1\sin\psi_1 = 0$$
$$L_4\cos\theta_4 + L_5\cos\theta_5 - L_6\cos\theta_6 - L_7\cos\psi_7 = 0$$
$$L_4\sin\theta_4 + L_5\sin\theta_5 - L_6\sin\theta_6 - L_7\sin\psi_7 = 0$$

<u>Velocity equations</u>

$$\begin{bmatrix} -L_3\sin\theta_3 & L_4\sin\theta_4 & 0 & 0 \\ L_3\cos\theta_3 & -L_4\cos\theta_4 & 0 & 0 \\ 0 & -L_4\sin\theta_4 & -L_5\sin\theta_5 & L_6\sin\theta_6 \\ 0 & L_4\cos\theta_4 & L_5\cos\theta_5 & -L_6\cos\theta_6 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{Bmatrix} = \begin{Bmatrix} L_2\sin\theta_2\omega_2 \\ -L_2\cos\theta_2\omega_2 \\ 0 \\ 0 \end{Bmatrix}$$

<u>Acceleration equations</u>

$$\begin{bmatrix} -L_3\sin\theta_3 & L_4\sin\theta_4 & 0 & 0 \\ L_3\cos\theta_3 & -L_4\cos\theta_4 & 0 & 0 \\ 0 & -L_4\sin\theta_4 & -L_5\sin\theta_5 & L_6\sin\theta_6 \\ 0 & L_4\cos\theta_4 & L_5\cos\theta_5 & -L_6\cos\theta_6 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} =$$

$$\begin{Bmatrix} L_2(\sin\theta_2\alpha_2 + \cos\theta_2\omega_2^2) + L_3\cos\theta_3\omega_3^2 - L_4\cos\theta_4\omega_4^2 \\ -L_2(\cos\theta_2\alpha_2 + \sin\theta_2\omega_2^2) + L_3\sin\theta_3\omega_3^2 - L_4\sin\theta_4\omega_4^2 \\ L_4\cos\theta_4\omega_4^2 + L_5\cos\theta_5\omega_5^2 - L_6\cos\theta_6\omega_6^2 \\ L_4\sin\theta_4\omega_4^2 + L_5\sin\theta_5\omega_5^2 - L_6\sin\theta_6\omega_6^2 \end{Bmatrix}$$