3 The Mathematics of Sharing

Divider-Chooser for Three Players?

The Divider-Chooser method is simple and appears to work very well. Why do we need to study other methods?

The Divider-Chooser method actually breaks down if there are three or more parties.

As an example, suppose Damian goes to the state fair with Cleo and her twin sister Cloe.

They win a cake that is one third chocolate, one third strawberry, and one third pineapple.

Divider-Chooser for Three Players?

Damian likes chocolate, strawberry, and pineapple equally. Cleo and Chloe both hate chocolate and pineapple.

Damian divides the cake into three pieces. Since Damian has no preference between the flavors, he views any physical third of the cake as a fair share.

Suppose Damian divides the cake so that one piece is entirely chocolate, one piece is entirely strawberry, and the remaining piece is entirely pineapple.

Divider-Chooser for Three Players?

As far as Damian is concerned, each piece is worth 1/3 of the whole.

However, in Cleo and Chloe's value system the chocolate piece and the pineapple piece are worthless, and the strawberry piece contains all of the value of the cake

Now, lets suppose Cleo gets to choose first. Obviously, Cleo would choose the strawberry piece.

Divider-Chooser for Three Players?

Is this fair?

Cleo receives at least 1/3 of the value of the cake (as perceived by Cleo's value system)

Damian receives exactly 1/3 the value of cake (as perceived by Damian's value system)

But Chloe receives either the chocolate piece or the pineapple piece, both of which she values as worthless.

Divider-Chooser for Three Players?

The trouble with this naïve extension of Divider-Chooser is that one of the players was allowed to choose (Cleo) BEFORE one of the other players (Chloe) had a chance impose their value system on the game.

Generally, the analysis of fair division games with three or more players is considerably more technical than their two player counterparts.

Lone-Divider Method

The following extension of the Divider-Chooser method to three players was only discovered in 1943 (whereas the two player Divider-Chooser method is ancient)

In the Lone-Divider Method, one player serves as a divider and the other two players serve as choosers.

Lone-Divider Method for Three Players

Preliminaries

One of the three players will be the divider; the other two players will be choosers.

Since it is better to be a chooser than a divider, the decision of who is what is made by a random draw (rolling dice, drawing cards from a deck, etc.). We'll call the divider D and the choosers C_1 and C_2.

Lone-Divider Method for Three Players

Step 1 (Division)

The divider *D* divides the cake into three shares (s_1, s_2, and s_3).

D will get one of these shares, but does not know which one.

(Not knowing which share will be his is critical – it forces *D* to divide the cake into three shares of equal value.)

Lone-Divider Method for Three Players

Step 2 (Bidding)

C_1 and C_2 each declare which of the three pieces are fair shares to them. These are the *bids*. A chooser's bid *must list every single piece that he or she considers to be a fair share* (i.e.,worth one-third or more of the cake)–bidding only for the very best piececan easily backfire.

To preserve the privacy requirement, the bids should be made independently, without the choosers being privy to each other's bids.

Lone-Divider Method for Three Players

Step 3 (Distribution)

Who gets which piece? The answer depends on which pieces are listed in the bids.

We separate the pieces into two types:

C-pieces (pieces *chosen* by at least one of the choosers)

U-pieces (*unwanted* pieces that did not appear in either of the bids).

Lone-Divider Method for Three Players

Step 3 (Distribution) continued

A U-piece is a piece that *both* choosers value at less than 1/3 of the value of the cake

A C-piece is a piece that *at least* one of the choosers (maybe both) value at 1/3 of the value of the cake or more .

Depending on the number of *C*-pieces, there are two separate cases to consider.

Lone-Divider Method for Three Players

Case 1

There are two or more *C*-pieces.

In this case it is always possible to give each chooser a piece from among the pieces listed in her bid.

Once each chooser gets her piece, the divider gets the last remaining piece.

At this point every player has received a fair share, and a fair division has been accomplished.

Lone-Divider Method for Three Players

Case 1

(We could end up in a situation in which C_1 likes C_2' s piece better than her own and vice versa. It is perfectly reasonable to allow the choosers to swap pieces at this point–this would make each of them happier than they already were, and who could be against that?)

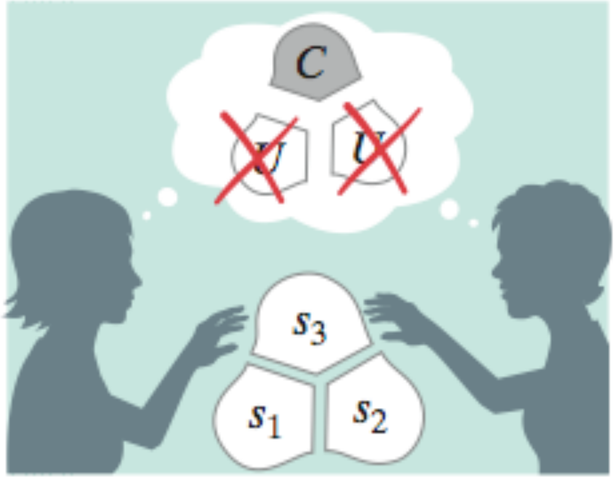
Lone-Divider Method for Three Players

Case 2

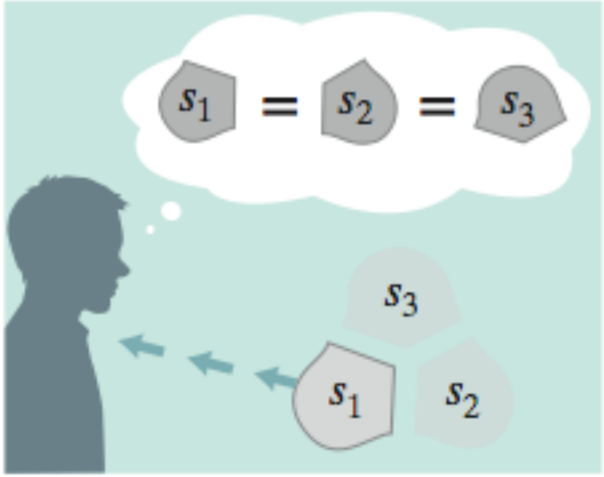
There is only one *C*-piece

In this case, both choosers are bidding for the same piece.

(Like the Cleo-Chloe example)



In this case we first take care of the divider *D* by giving him one of the pieces that neither chooser wants. (All pieces are equal as far as D is concerned)



Lone-Divider Method for Three Players

Case 2

After *D* gets his piece, the remaining pieces are recombined into one piece that we call the *B*-piece.

The two choosers can now use the two player divider-chooser method on the B-piece to finish the fair division.

Lone-Divider Method for Three Players

Case 2

This process guarantees fair shares for all players. WHY?

D ends up with a fair since *D* did the original division, but what about C_1 and C_2?

According to each of C_1 and C_2 the *B*-piece is worth *more than two-thirds of the value of the original cake* (think of the *B*-piece as 100% of the original cake minus a *U*-piece worth less than 33 1/3%), so when we divide it fairly into *two* shares, each party is guaranteed *more than one-third* of the original cake.

3 Player Lone Divider(Case 1, V. 1)

Dale, Cindy, and Cher are dividing a cake using Steinhaus's lone-divider method. They draw cards from a well-shuffled deck of cards, and Dale draws the low card (bad luck!) and has to be the divider.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-1 shows the values of the three pieces according to each of the players.

3 Player Lone Divider (Case 1, V1)

TABLE 3-1

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	35%	10%	55%
Cher	40%	25%	35%

Step 2 (Bidding)

We can assume that Cindy's bid list is $\{s_1, s_3\}$ and Cher's bid list is also $\{s_1, s_3\}$.

3 Player Lone Divider (Case 1, V1)

Step 3 (Distribution)

The *C*-pieces are s_1 and s_3. There are two possible distributions. One distribution would be: Cindy gets s_1, Cher gets s_3, and Dale gets s_2. An even better distribution (the *optimal* distribution) would be: Cindy gets s_3, Cher gets s_1, and Dale gets s_2. In the case of the first distribution, both Cindy and Cher would benefit by swapping pieces, and there is no rational reason why they would not do so.

3 Player Lone Divider (Case 1, V1)

Step 3 (Distribution) continued

Thus, using the rationality assumption, we can conclude that in either case the final result will be the same: Cindy gets s_3, Cher gets s_1, and Dale gets s_2.

3 Player Lone Divider (Case 1, V 2)

We'll use the same setup as in Example 3.2– Dale is the divider, Cindy and Cher are the choosers.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-2 shows the values of the three pieces according to each of the players.

3 Player Lone Divider (Case 1, V 2)

TABLE 3-2

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	30%	40%	30%
Cher	60%	15%	25%

Step 2 (Bidding)

Here Cindy's bid list is $\{s_2\}$ only, and Cher's bid list is $\{s_1\}$ only.

3 Player Lone Divider (Case 1, V 2)

Step 3 (Distribution)

This is the simplest of all situations, as there is only one possible distribution of the pieces: Cindy gets s_2, Cher gets s_1, and Dale gets s_3.

3 Player Lone Divider (Case 2)

The gang of Examples 3.2 and 3.3 are back at it again.

Step 1 (Division)

Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-3 shows the values of the three pieces according to each of the players.

3 Player Lone Divider (Case 2)

TABLE 3-3

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	20%	30%	50%
Cher	10%	20%	70%

Step 2 (Bidding)

Here Cindy's and Cher's bid list consists of just $\{s_3\}$.

3 Player Lone Divider (Case 2)

Step 3 (Distribution)

The only C-piece is s_3.

Of the two *U*-pieces, one of them is selected at random and given to Dale. Suppose s_1 is the piece given to Dale.

The remaining pieces (s_2 and s_3) are then recombined to form the B-piece, to be divided between Cindy and Cher using the divider-chooser method (one of them divides the *B*-piece into two shares, the other one chooses the share she likes better).

3 Player Lone Divider (Case 2)

Step 3 (Distribution) continued
Regardless of how this plays out, both of them will get a very healthy share of the cake:

Cindy will end up with a piece worth at least 40% of the original cake (the *B*-piece is worth 80% of the original cake to Cindy)

Cher will end up with a piece worth at least 45% of the original cake (the *B*-piece is worth 90% of the original cake to Cher).

The Lone-Divider Method for More Than Three Players

Some extra complications can arise when applying the Lone-Divider Method to games with more than three players.

Rather than trying to describe the method in full generality, we will only give an outline of the method and illustrate the details with a couple of examples for $N = 4$ players.

The Lone-Divider Method for More Than Three Players

Preliminaries

One of the players is chosen to be the divider *D*, and the remaining $N - 1$ players are all going to be choosers. As always, it's better to be a chooser than a divider, so the decision should be made by a random draw.

The Lone-Divider Method for More Than Three Players

Step 1 (Division)

The divider *D* divides the set *S* into *N* shares $s_1, s_2, s_3, \ldots, s_N$. *D* is guaranteed of getting one of these shares, but doesn't know which one.

The Lone-Divider Method for More Than Three Players

Step 2 (Bidding)

Each of the N – 1 choosers independently submits a bid list consisting of every share that he or she considers to be a fair share (i.e., worth 1/Nth or more of the goods S).

The Lone-Divider Method for More Than Three Players

Step 3 (Distribution)

The bid lists are opened.

We will have to consider two separate cases, depending on how these bid lists turn out.

The Lone-Divider Method for More Than Three Players

Case 1.

If there is a way to assign a different share to each of the N – 1 choosers, then that should be done. (Needless to say, the share assigned to a chooser should be from his or her bid list.) The divider gets the last unassigned share.

At the end, players may choose to swap pieces if they want.

The Lone-Divider Method for More Than Three Players

Case 2.

There is a *standoff*–in other words, there are two choosers both bidding for just one share, or three choosers bidding for just two shares, or *K* choosers bidding for less than *K* shares. This is a much more complicated case, and what follows is a rough sketch of what to do. To resolve a standoff, we first set aside the shares involved in the standoff from the remaining shares.

The Lone-Divider Method for More Than Three Players

Case 2.

Likewise, the players involved in the standoff are temporarily separated from the rest. Each of the remaining players (including the divider) can be assigned a fair share from among the remaining shares and sent packing. All the shares left are recombined into a new set *S* to be divided among the players involved in the standoff, and the process starts all over again.

4 Player Lone Divider (Case 1)

We have one divider, Demi, and three choosers, Chan, Chloe, and Chris.

Step 1 (Division)

Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-4 shows how each of the players values each of the four shares. Remember that the information on each row of Table 3-4 is private and known only to that player.

4 Player Lone Divider (Case 1)

TABLE 3-4

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	30%	20%	35%	15%
Chloe	20%	20%	40%	20%
Chris	25%	20%	20%	35%

Step 2 (Bidding)

Chan's bid list is $\{s_1, s_3\}$; Chloe's bid list is $\{s_3\}$ only; Chris's bid list is $\{s_1, s_4\}$.

4 Player Lone Divider (Case 1)

Step 3 (Distribution)

The bid lists are opened. It is clear that for starters Chloe must get s_3 – there is no other option. This forces the rest of the distribution: s_1 must then go to Chan, and s_4 goes to Chris. Finally, we give the last remaining piece, s_2, to Demi.

4 Player Lone Divider (Case 1)

This distribution results in a fair division of the cake, although it is not entirely “envy-free”

Chan wishes he had Chloe’s piece (35% is better than 30%) but Chloe is not about to trade pieces with him, so he is stuck with s_1.

(From a strictly rational point of view, Chan has no reason to gripe–he did not get the best piece, but got a piece worth 30% of the total, better than the 25% he is entitled to.)

4 Player Lone Divider (Case 2)

Once again, we will let Demi be the divider and Chan, Chloe, and Chris be the three choosers (same players, different game).

Step 1 (Division)

Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-5 shows how each of the players values each of the four shares.

4 Player Lone Divider (Case 2)

TABLE 3-5

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	20%	20%	20%	40%
Chloe	15%	35%	30%	20%
Chris	22%	23%	20%	35%

Step 2 (Bidding)

Chan's bid list is $\{s_4\}$; Chloe's bid list is $\{s_2, s_3\}$ only; Chris's bid list is $\{s_4\}$.

4 Player Lone Divider (Case 2)

Step 3 (Distribution)

The bid lists are opened, there is a standoff: Chan and Chris are both bidding for s_4

The first step is to set aside and assign Chloe and Demi a fair share from s_1, s_2, and s_4. Chloe could be given either s_2 or s_3. (She would rather have s_2, but it's not for her to decide.)

4 Player Lone Divider (Case 2)

Step 3 (Distribution)

A coin toss is used to determine which one. Let's say Chloe ends up with s_3 (bad luck!).

Demi could be now given either s_1 or s_2. Another coin toss, and Demi ends up with s_1.

Now recombine s_2 and s_4 into a single piece to be divided between Chan and Chris using the divider-chooser method.

4 Player Lone Divider (Case 2)

Step 3 (Distribution)

Since (s_2 + s_4) is worth 60% to Chan and 58% to Chris (you can check it out in Table 3-5), regardless of how this final division plays out they are both guaranteed a final share worth more than 25% of the cake.

Mission accomplished! We have produced a fair division of the cake.

Summarizing Lone-Divider Method

Goods S are to be divided amongst N players. One player serves as a divider, the remaining players serve as choosers.

If there are no stand-offs in the bidding, then each of the players is guaranteed a fair share from among the original N pieces.

If there is a stand-off, the players are divided into those that are involved in the stand-off and those that are not involved in the stand-off.

Summarizing Lone-Divider Method

Those that are not involved in the stand-off are guaranteed a fair share among the original N pieces.

The remaining pieces are re-combined (into the B-piece), and the Lone-Divider method can be applied to the new fair division game in which the B-piece is the goods and the players in this new game are the players that were involved in the stand-off in the original game.

Summarizing Lone-Divider Method

This new game could result in more stand-offs, in which case the new game is decomposed into an even newer game.

However, each time we pass from the Lone-Divider Method on a set to a subset, the number of players reduces by at least one (the divider always gets his fair share), so eventually we will reduce to a Lone-Divider game that does not produce stand-offs.