3D COORDINATE TRANSFORMATIONS

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ABSTRACT

A three-dimensional (3D) conformal coordinate transformation, combining axes rotations, scale change and origin shifts is a practical mathematical model of the relationships between different 3D coordinate systems. Applications in geodesy and photogrammetry often use simplified transformation models under the assumption of small or negligible rotations, but in other areas of interest rotations may be large. In such other cases, approximate values of rotations are required to perform initial transformations before the simplified models are employed. This paper uses an example applicable to the construction industry to demonstrate methods of calculating approximate rotations and performing initial transformations prior to computing transformation parameters. A rigorous development and proof of the 3D conformal transformation is given as well as the necessary assumptions for the simplified model. In addition, this paper also explains how least squares may be used in determining transformation parameters.

INTRODUCTION

Coordinate transformations, conformal and otherwise, are widely used in surveying and related professions. For instance, in geodesy, 3D transformations are used to convert coordinates related to the Australian Geodetic Datum to the new Geocentric Datum of Australia (Featherstone 1996), in engineering they form part of monitoring and control systems used in large manufacturing projects such as the construction of the ANZAC frigates for the Australian and New Zealand Navies (Bellman & Anderson 1995) and in photogrammetry they are used in the orientation (interior and exterior) of aerial photographs. In two-dimensional (2D) form, transformations are used in cadastral survey re-establishments (Bebb 1981, Sprott 1983 and Bird 1984), matching digitized cadastral maps (Shmutter and Doytsher 1991) and "sewing together" the edges of strips of digital images (Bellman, Deakin and Rollings 1992).

In general, the effect of a transformation on a 2D or 3D object will vary from a simple change of location and orientation (with no change in shape or size) to a uniform change in scale (no change in shape) and finally to changes of shape and size of different degrees of nonlinearity (Mikhail 1976). The most common transformations in surveying applications, and the only type dealt with in this paper, are conformal, i.e. transformations that preserve angles and thus the shape of objects. Theory and applications of other coordinate transformations, such as affine, polynomial, projective etc. can be found in Mikhail (1976) and Moffitt and Mikhail (1980).

3D conformal coordinate transformations – also known as similarity transformations (Blais 1972 and Bervoets 1992) and in the 2D case, sometimes called Helmert transformations after the German geodesist F.R. Helmert (1843-1917) – are often given in the form

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \lambda \mathbf{R}_{\kappa\phi\omega} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \\ T_U \end{bmatrix} \qquad (1)$$

E, N and U (East, North, Up) and X, Y and Z are 3D "design" and "survey" coordinates respectively, λ is a scale factor, $\mathbf{R}_{\kappa\phi\omega}$ is a rotation matrix (the product of rotations ω, ϕ and κ about the X, Y and Z axes in turn) and T_E, T_N and T_U are translations between the origins of the two coordinate systems. In the context of this paper, (1) – often called a seven-parameter transformation, three rotations three translations and one scale factor – represents the mathematical relationship between a constructed object in the XYZ survey system and its transformed position in the ENU design system. Its practical use in the construction industry, where components of structures are manufactured "off-site" and brought together "on-site" can be explained in the following way.

At the off-site location the object is measured in situ and coordinated in an arbitrary XYZ survey system. To confirm that it will fit in its designed on-site location its XYZ coordinates are scaled and rotated until the coordinate axes are parallel with the ENU design axes. The origins of the two coordinate systems are then brought together by adding the translations. This has the effect of superimposing the object over its design location and its "fit" can be checked by comparing coordinate differences.

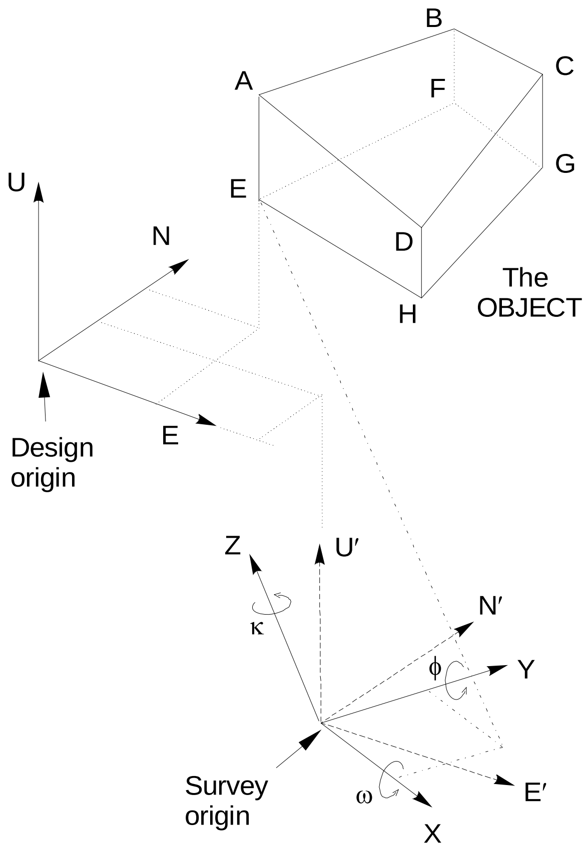

Figure 1. An object with coordinates in two 3D rectangular systems

The seven parameters in (1) can be determined by solving a system of equations derived from "common" points whose coordinate values are known in both the survey and design systems. Each common point (or control point) generates three equations, thus a minimum of three such points is required to solve for the parameters, but it is usual (and also good practice) to include extra control points in surveys of this type; the additional points leading to a redundant system of equations (more equations than unknowns). Least squares can then be employed to determine the best estimates of the parameters which minimize the sum of the squares of the residuals (small corrections to the transformed coordinates) at the control points. The residuals, three for each point in the direction of the coordinate axes, can then be used as a measure of how well the constructed object fits its design values.

Translation parameters may be eliminated from the solution by using "centroidal" coordinates $\overline{E}, \overline{N}$ and $\overline{U}$ and $\overline{X}, \overline{Y}$ and $\overline{Z}$ having a common origin at the centroid of the control points. The transformation then combines scale and rotation only

$$\begin{bmatrix} \overline{E} \\ \overline{N} \\ \overline{U} \end{bmatrix} = \lambda \mathbf{R}_{\kappa\phi\omega} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} \tag{2a}$$

[Translations can also be eliminated by moving the origins to a common control point, both techniques having the desirable effect of reducing the size of the numbers involved in the computations.]

Due to the nature of the matrix $\mathbf{R}_{\kappa\phi\omega}$ it is not possible to solve directly for the individual rotations ω, ϕ and κ. But if the coordinate axes are approximately parallel, a matrix $\mathbf{R}_S$ of independent small rotations $\delta\omega, \delta\phi$ and $\delta\kappa$ can be developed, which together with an approximate value of the scale factor λ' with its small unknown correction $\delta\lambda$ $(\lambda = \lambda' + \delta\lambda)$ enables a least squares solution of $\delta\omega, \delta\phi$, $\delta\kappa$ and $\delta\lambda$ using

$$\begin{bmatrix} \overline{E} \\ \overline{N} \\ \overline{U} \end{bmatrix} = (\lambda' + \delta\lambda) \mathbf{R}_S \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} \tag{2b}$$

This solution technique requires iteration, each iteration preceded by a transformation using (2b) with the process being terminated when $\delta\omega, \delta\phi$, $\delta\kappa$ and $\delta\lambda$ reach some predetermined negligible values.

This paper shows how the general principles of conformal transformation, originally developed by C.F. Gauss (1777-1855), are used to derive a transformation between two plane rectangular coordinate systems which is equivalent to 2D rotation, scaling and translation. This concept is extended to 3D systems and the derivation of (1) and (2) is set out together with a proof showing that angles are preserved in the transformation.

The application of the least squares principle to the solution of the transformation parameters is explained and the necessary system of equations is developed in matrix form. Since the solution is iterative, requiring an initial transformation to approximately align the two coordinate axes, a simple method of determining the parameters of the initial transformation is given and a worked example of a problem is provided.

2D CONFORMAL TRANSFORMATIONS

C.F. Gauss showed that the necessary and sufficient condition for a conformal transformation from the ellipsoid to the map plane is given by the complex expression (Lauf 1983)

$$y + \mathrm{i}x = f(\chi + \mathrm{i}\omega) \tag{3}$$

where the function $f(\chi + \mathrm{i}\omega)$ is analytic, containing isometric parameters χ (isometric latitude) and ω (longitude). i is the imaginary number $(\mathrm{i}^2 = -1)$ and the xy rectangular coordinates have the y-axis "up the page" and the x-axis "across the page". [It should be noted here that isometric means: *of equal measure*, and on the surface of the ellipsoid (or sphere) latitude and longitude are not equal measures of length. This is obvious if we consider a point near the pole where similar distances along a meridian and a parallel of latitude will correspond to vastly different angular values of latitude and longitude. Hence in conformal map projections, isometric latitude is determined to ensure that angular changes correspond to linear changes.]

A necessary condition for an analytic function is that it must satisfy the Cauchy-Riemann equations

$$\frac{\partial y}{\partial \chi} = \frac{\partial x}{\partial \omega} \quad \text{and} \quad \frac{\partial y}{\partial \omega} = -\frac{\partial x}{\partial \chi} \tag{4}$$

Using this theorem, a conformal transformation from the XY rectangular coordinate system (isometric parameters) to the EN (East, North) rectangular system (also isometric parameters) is given by the complex expression

$$N + \mathrm{i}E = f(Y + \mathrm{i}X) \tag{5}$$

A function $f(Y + \mathrm{i}X)$ which satisfies the Cauchy-Riemann equations is a complex polynomial, hence (5) can be given as

$$N + iE = \sum_{k=0}^{n} \left(a_k + i\,b_k\right)\left(Y + i\,X\right)^k \qquad (6)$$

Expanding (6) to the first power (k = 1) and equating real and imaginary parts gives

$$\begin{aligned} N &= a_0 + a_1 Y - b_1 X \\ E &= b_0 + a_1 X + b_1 Y \end{aligned} \qquad (7)$$

which are essentially the same equations as in Jordan/Eggert/Kneissal (1963, pp. 70-73) in the section headed "Das Helmertsche Verfahren (Helmertsche Transformation)" although as noted by Bervoets (1992) in his bibliography, there is no reference to the original source. It is probable that Helmert developed this conformal transformation in his masterpiece on geodesy, *Die mathematischen und physikalischen Theorem der höheren Geodäsie,* (The mathematics and physical theorems of higher geodesy) on which he worked from 1877 and published in two parts: vol. 1, *Die mathematischen Theorem* (1880) and vol. 2, *Die physikalischen Theorem* (1884) [DSB 1972].

In (7), a_0 and b_0 are translations between the coordinate axes and the coefficients a_1 and b_1 can be considered as functions of scale λ and rotation α between the coordinate axes

$$\begin{aligned} a_1 &= \lambda \cos\alpha \\ b_1 &= \lambda \sin\alpha \end{aligned} \qquad (8)$$

Substituting (8) into (7), re-arranging and using matrix notation gives the familiar 2D conformal transformation

$$\begin{bmatrix} E \\ N \end{bmatrix} = \lambda \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix} \qquad (9)$$

where the coefficient matrix on the right-hand-side is the rotation matrix $\mathbf{R}_\alpha$

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad (10)$$

The 2D conformal transformation can be represented by the familiar diagram

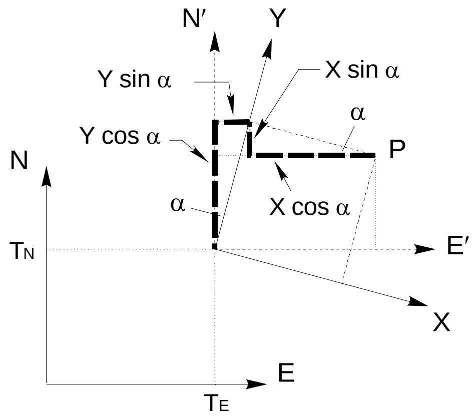

Figure 2. 2D Conformal Transformation (Rotation and Translation)

Thus it is seen that the 2D conformal transformation is equivalent to a translation and rotation of rectangular axes with a scale factor between the EN and XY coordinates.

THE 3D ROTATION MATRIX

The 2D transformation can be extended to three dimensions by firstly considering a sequence of rotations of ω, ϕ and κ about the X, Y and Z-axes in turn.

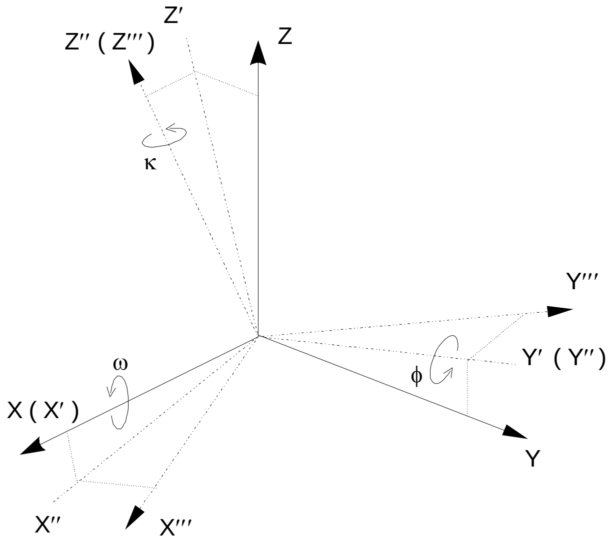

Figure 3. 3D rotations ω, ϕ and κ

Rotations are considered as positive anti-clockwise when looking along the axis towards the origin; the positive sense of rotations being determined by the *right-hand-grip rule* where an imaginary right hand grips the axis with the thumb pointing in the positive direction of the axis and the natural curl of the fingers indicating positive direction of rotation.

The three rotations in order are:

(i) Rotation of ω about the X-axis. This rotates the Y and Z axis to Y' and Z' with the X and X' axes coincident.

Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \underset{(\mathbf{R}_\omega)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{11}$$

(ii) Rotation of ϕ about the new Y' axis. This rotates the X' and Z' to X'' and Z'' with the Y' and Y'' axes coincident.

Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \underset{(\mathbf{R}_\phi)}{\begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{12}$$

(iii) Rotation of κ about the new Z'' axis. This rotates the X'' and Y'' to X''' and Y''' with the Z'' and Z''' axes coincident.

Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X''' \\ Y''' \\ Z''' \end{bmatrix} = \underset{(\mathbf{R}_\kappa)}{\begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} \tag{13}$$

The coefficient matrices $\mathbf{R}_\kappa$, $\mathbf{R}_\phi$, $\mathbf{R}_\omega$ above are 3D rotation matrices which can be multiplied together (in that order) to give another rotation matrix $\mathbf{R}_{\kappa\phi\omega}$ (Mikhail and Moffitt 1980).

$$\begin{bmatrix} X''' \\ Y''' \\ Z''' \end{bmatrix} = \mathbf{R}_\kappa \mathbf{R}_\phi \mathbf{R}_\omega \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{R}_{\kappa\phi\omega} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{14}$$

with

$$\mathbf{R}_{\kappa\phi\omega} = \begin{bmatrix} c_\phi c_\kappa & c_\omega s_\kappa + s_\omega s_\phi c_\kappa & s_\omega s_\kappa - c_\omega s_\phi c_\kappa \\ -c_\phi s_\kappa & c_\omega c_\kappa - s_\omega s_\phi s_\kappa & s_\omega c_\kappa + c_\omega s_\phi s_\kappa \\ s_\phi & -s_\omega c_\phi & c_\omega c_\phi \end{bmatrix} \tag{15}$$

where, for instance, $c_\kappa s_\phi s_\omega = \cos\kappa \sin\phi \sin\omega$.

Rotation matrices, e.g. $\mathbf{R}_\alpha$, $\mathbf{R}_\kappa$, $\mathbf{R}_\phi$, $\mathbf{R}_\omega$ and $\mathbf{R}_{\kappa\phi\omega}$ are *orthogonal,* i.e. the sum of squares of the elements of any row or column is equal to unity. They have the unique property that their inverse is equal to their transpose, i.e. $\mathbf{R}^{-1} = \mathbf{R}^{T}$ which will be used in later developments.

THE 3D CONFORMAL TRANSFORMATION

The 3D conformal coordinate transformation is an extension of the 2D case; (14) and (15) combined with a scale factor and translations to give (as previously stated)

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \lambda \mathbf{R}_{\kappa\phi\omega} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_E \\ T_N \\ T_U \end{bmatrix} \tag{1}$$

To prove that this transformation is indeed conformal, i.e. angles between points in the XYZ survey system are preserved when transformed into the ENU design system, consider the following.

(i) Let three points a, b and c in the survey system be transformed into A, B and C in the design system; points in both systems located by vectors **a**, **b**, **c** and **A**, **B**, **C**.

Using (1) with **R** as the rotation matrix and **t** as the vector of translations, we may write the transformation of **a** to **A** as

$$\mathbf{A} = \lambda \mathbf{R} \mathbf{a} + \mathbf{t} \tag{16}$$

and similarly for the other vectors.

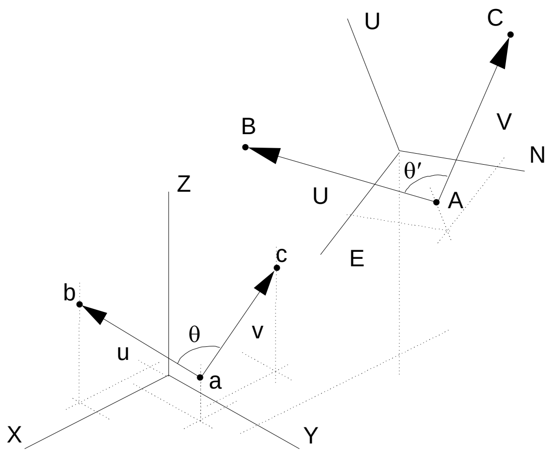

Figure 4. Conformal transformation of two vectors.

(ii) Let θ be the angle at a in the survey system between the vectors $\mathbf{u} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v} = \mathbf{c} - \mathbf{a}$; and θ' the complementary angle at A in the design system between vectors $\mathbf{U} = \mathbf{B} - \mathbf{A}$ and $\mathbf{V} = \mathbf{C} - \mathbf{A}$.

(iii) In the survey system, the angle between the vectors is found from the vector dot product of unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$

$$\cos\theta = \hat{\mathbf{u}} \bullet \hat{\mathbf{v}} = \hat{u}_1\hat{v}_1 + \hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3$$

where $\hat{u}_1, \hat{u}_2, \hat{u}_3$ and $\hat{v}_1, \hat{v}_2, \hat{v}_3$ are the components of the two unit vectors, noting that a unit vector is defined as $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$ where $|\mathbf{u}|$ is the magnitude.
The dot product is equivalent to a matrix multiplication, hence

$$\cos\theta = \hat{\mathbf{u}} \bullet \hat{\mathbf{v}} = \hat{\mathbf{u}}^T\hat{\mathbf{v}} \tag{17}$$

(iv) In the design system we may use (16) to write

$$\mathbf{U} = \mathbf{B} - \mathbf{A} = \lambda\,\mathbf{R}\,\mathbf{b} + \mathbf{t} - \left(\lambda\,\mathbf{R}\,\mathbf{a} + \mathbf{t}\right) = \lambda\,\mathbf{R}\,\mathbf{u}$$

and the unit vector $\hat{\mathbf{U}}$

$$\hat{\mathbf{U}} = \frac{\lambda\,\mathbf{R}\,\mathbf{u}}{|\lambda\,\mathbf{R}\,\mathbf{u}|} = \frac{\mathbf{R}\,\mathbf{u}}{|\mathbf{R}\,\mathbf{u}|} \tag{18}$$

Now using matrix algebra we may write two results (a) and (b)

(a) $\quad |\mathbf{u}|^2 = \mathbf{u}^T\mathbf{u}$

and remembering that for orthogonal rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^T$, hence $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix

(b) $\quad |\mathbf{Ru}|^2 = (\mathbf{Ru})^T\mathbf{Ru} = \mathbf{u}^T\mathbf{R}^T\mathbf{Ru} = \mathbf{u}^T\mathbf{u}$

Equating (a) and (b) gives $|\mathbf{Ru}| = |\mathbf{u}|$ which can be substituted into (18) to give

$$\hat{\mathbf{U}} = \frac{\mathbf{R}\mathbf{u}}{|\mathbf{u}|} = \mathbf{R}\frac{\mathbf{u}}{|\mathbf{u}|} = \mathbf{R}\hat{\mathbf{u}} \tag{19}$$

and using similar reasoning

$$\hat{\mathbf{V}} = \mathbf{R}\hat{\mathbf{v}} \tag{20}$$

(v) In the design system the angle θ' between $\mathbf{U}$ and $\mathbf{V}$ is given by the dot product $\cos\theta' = \hat{\mathbf{U}} \bullet \hat{\mathbf{V}} = \hat{\mathbf{U}}^T\hat{\mathbf{V}}$ and using (19) and (20) we may write

$$\cos\theta' = (\mathbf{R}\hat{\mathbf{u}})^T\mathbf{R}\hat{\mathbf{v}} = \hat{\mathbf{u}}^T\mathbf{R}^T\mathbf{R}\hat{\mathbf{v}} = \hat{\mathbf{u}}^T\hat{\mathbf{v}} \tag{21}$$

Comparing (17) and (21) shows that $\theta = \theta'$ and constitutes a proof that angles are preserved by the transformation (1). [Baetslé (1966) has a similar proof but in a slightly different form.]

Conformal mapping has a long mathematical history. Vlcek (1966) in a discussion paper on the topic notes that the famous French mathematician Liouville (1847) determined all the conformal transformations in an analytical way and in a theorem bearing his name showed that the only ways of conformal representation of space on itself are:

1. by translation and rotation, accompanied by constant magnification,
2. by inversion with respect to the sphere,

which, at the very least, demonstrates that conformal 3D transformations have been with us for a long time.

THE 3D ROTATION MATRIX FOR SMALL ANGLES

For small angles $\delta\omega$, $\delta\phi$, $\delta\kappa$ the rotation matrix $\mathbf{R}_{\kappa\phi\omega}$ may be simplified by the approximations

$$\cos\delta\omega \approx 1$$
$$\sin\delta\phi \approx \delta\phi(\text{radians})$$
$$\sin\delta\kappa\ \sin\delta\omega \approx 0$$

and (15) becomes the anti-symmetric (or skew-symmetric) matrix (Harvey 1986).

$$\mathbf{R}_S = \begin{bmatrix} 1 & \delta\kappa & -\delta\phi \\ -\delta\kappa & 1 & \delta\omega \\ \delta\phi & -\delta\omega & 1 \end{bmatrix} \tag{22}$$

It should be noted that $\mathbf{R}_S$ is no longer orthogonal but its inverse will, nevertheless, be given by its transpose ($\mathbf{R}_S^{-1} = \mathbf{R}_S^T$), since it is the approximate form of the orthogonal matrix $\mathbf{R}_{\kappa\phi\omega}^T$ (Hotine 1969, p. 263).

In the least squares development to follow it is useful to split $\mathbf{R}_S$ (the rotation matrix for small angles) into two parts

$$\mathbf{R}_S = \mathbf{I} + \delta\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \delta\kappa & -\delta\phi \\ -\delta\kappa & 0 & \delta\omega \\ \delta\phi & -\delta\omega & 0 \end{bmatrix} \tag{23}$$

Assuming small angles is a convenient and practical technique of simplifying $\mathbf{R}_{\kappa\phi\omega}$ (whose non-independent elements are functions of the rotation angles ω, ϕ and κ) to $\mathbf{R}_S$ (whose independent elements are $\delta\omega$, $\delta\phi$ and $\delta\kappa$), thus enabling the solution of $\delta\omega$, $\delta\phi$ and $\delta\kappa$ via a system of linear equations (see the following section on least squares solution). Solutions, based on this assumption, will not be theoretically correct but will be "practically correct" if an iterative process is adopted where each n^{th} iteration is preceded by a transformation using previously derived values, and the process terminated when $\delta\omega_n$, $\delta\phi_n$ and $\delta\kappa_n$ converge to negligible values. A practical test on the assumption of small angles in the initial transformation will be revealed by noting whether the solution converges.

CENTROIDAL COORDINATES

In a solution for the transformation parameters, the three translations T_E, T_N and T_U can be eliminated by adopting a system of "centroidal" coordinates, i.e. a system of coordinates whose origin is at the centroid of the n control points.

Denoting the coordinates of the centroid (in both systems) with a subscript g

$$\begin{aligned} X_g &= \frac{X_1 + X_2 + \cdots + X_n}{n} \\ Y_g &= \frac{Y_1 + Y_2 + \cdots + Y_n}{n} \\ Z_g &= \frac{Z_1 + Z_2 + \cdots + Z_n}{n} \end{aligned} \tag{24}$$

and similarly for E_g, N_g and U_g. Centroidal coordinates (denoted with an over-bar) are

$$\begin{aligned} \overline{X}_i &= X_i - X_g \\ \overline{Y}_i &= Y_i - Y_g \\ \overline{Z}_i &= Z_i - Z_g \end{aligned} \tag{25}$$

with similar expressions for $\overline{E}, \overline{N}$ and $\overline{U}$.

Thus, using centroidal coordinates [see (2), where the origins of both systems are common] the transformation is reduced to a combination of scale and rotation only and the size of the numbers involved in the computations is reduced. After solving for the scale factor λ and the elements of the rotation matrix $\mathbf{R}_{\kappa\phi\omega}$, the translations can be found by substituting the coordinates of the centroid into (1) and re-arranging

$$\begin{bmatrix} T_E \\ T_N \\ T_U \end{bmatrix} = \begin{bmatrix} E_g \\ N_g \\ U_g \end{bmatrix} - \lambda\mathbf{R}_{\kappa\phi\omega}\begin{bmatrix} X_g \\ Y_g \\ Z_g \end{bmatrix} \tag{26}$$

LEAST SQUARES SOLUTION OF TRANSFORMATION PARAMETERS

To develop the least squares solution of the parameters, consider (2b) which pre-supposes that the rotation angles are <u>small</u> which may be due to (i) an initial transformation which approximately aligns the coordinate axes, or (ii) a knowledge that the rotations are small. Bearing this in mind (noting that a method of determining the parameters of an initial transformation is discussed later) we may use (23) to write for each of the n control points an equation of the form

$$\begin{bmatrix} \bar{E} \\ \bar{N} \\ \bar{U} \end{bmatrix} = (\lambda' + \delta\lambda)(\mathbf{I} + \delta\mathbf{R})\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} + \begin{bmatrix} v_E \\ v_N \\ v_U \end{bmatrix} \qquad (27)$$

where v_E, v_N and v_U are residuals at the control points. Now, since the rotation angles and $\delta\lambda$ are small, their products will be negligible ($\delta\lambda\,\delta\mathbf{R} \approx \mathbf{0}$), thus (27) can be expanded and re-arranged as

$$\begin{bmatrix} v_E \\ v_N \\ v_U \end{bmatrix} + \lambda' \begin{bmatrix} 0 & -\bar{Z} & \bar{Y} & \bar{X} \\ \bar{Z} & 0 & -\bar{X} & \bar{Y} \\ -\bar{Y} & \bar{X} & 0 & \bar{Z} \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta\phi \\ \delta\kappa \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} \bar{E} \\ \bar{N} \\ \bar{U} \end{bmatrix} - \lambda' \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} \qquad (28)$$

An equation of the form of (28) can be written for each control point and represented symbolically in partitioned matrix form as

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} \qquad (29a)$$

where each component $\mathbf{v}_i$ $(i = 1, 2, \ldots, n)$ of the vector of residuals $\mathbf{v}$ and $\mathbf{B}_i$ of the matrix of coefficients $\mathbf{B}$ is

$$\mathbf{v}_i = \begin{bmatrix} v_{Ei} \\ v_{Ni} \\ v_{Ui} \end{bmatrix}, \quad \mathbf{B}_i = \lambda' \begin{bmatrix} 0 & -\bar{Z}_i & \bar{Y}_i & \bar{X}_i \\ \bar{Z}_i & 0 & -\bar{X}_i & \bar{Y}_i \\ -\bar{Y}_i & \bar{X}_i & 0 & \bar{Z}_i \end{bmatrix}$$

and the vector of parameters $\mathbf{x}$ and the components $\mathbf{f}_i$ of the vector of numeric terms $\mathbf{f}$ are

$$\mathbf{x} = \begin{bmatrix} \delta\omega \\ \delta\phi \\ \delta\kappa \\ \delta\lambda \end{bmatrix}, \quad \mathbf{f}_i = \begin{bmatrix} \bar{E}_i - \lambda'\bar{X}_i \\ \bar{N}_i - \lambda'\bar{Y}_i \\ \bar{U}_i - \lambda'\bar{Z}_i \end{bmatrix}$$

Equation (29a) is a "standard" least squares form (Mikhail 1976)

$$\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f} \qquad (29b)$$

with the solution for the parameters as

$$\mathbf{x} = \left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{W}\mathbf{f} \qquad (30a)$$

where $\mathbf{W}$ is the weight matrix associated with the observations, which in this case are the triplets of centroidal coordinates $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}_i$ and $\bar{E}_i$, $\bar{N}_i$, $\bar{U}_i$. Mikhail (1976 pp. 64-66) defines weight matrices $\mathbf{W}$, cofactor matrices $\mathbf{Q}$ and variance-covariance matrices $\boldsymbol{\Sigma}$ as follows

$$\mathbf{W} = \mathbf{Q}^{-1} \qquad (30b)$$

$$\boldsymbol{\Sigma} = \sigma_0^2\mathbf{Q} \qquad (30c)$$

where the elements of $\mathbf{Q}$ are estimates of the variances and covariances of the observations and σ_0^2 is the reference variance whose posteriori estimate is (Mikhail 1976, p. 288)

$$\sigma_0^2 = \frac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{n-u} = \frac{\mathbf{f}^T\mathbf{W}\mathbf{f} - \mathbf{x}^T\left(\mathbf{B}^T\mathbf{W}\mathbf{f}\right)}{n-u} \qquad (30d)$$

n and u are the number of observations and unknown parameters respectively.

The partitioned form of $\mathbf{W}$ in (30a) is

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_n \end{bmatrix} \qquad (30e)$$

where each diagonal element $\mathbf{W}_i$ contains weights associated with the coordinate triplets of each i^{th} control point, and the elements of $\mathbf{W}_i$ must be obtained from a pre-analysis of the survey techniques used to determine their values. The off-diagonal elements of $\mathbf{W}$ are null matrices, indicating that the observations (coordinate triplets) are considered to be independent of each other.

In this paper, and the example following, it is assumed that the coordinates (derived from survey measurements) are independent of each other and all having equal precision. This allows **W** to be replaced by the identity matrix **I** in (30a) to give

$$\mathbf{x} = \left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{f} \tag{31}$$

CALCULATION OF APPROXIMATE ROTATIONS

The solution above requires that rotations be small, i.e. $\mathbf{R}_S$ approximates $\mathbf{R}_{\kappa\phi\omega}$. To ensure this a preliminary (or initial) transformation is made using centroidal coordinates and approximations for the rotation matrix $\mathbf{R}_A$ and scale factor λ_A

$$\begin{bmatrix}\overline{X}\\\overline{Y}\\\overline{Z}\end{bmatrix}_{\text{INITIAL}} = \lambda_A\,\mathbf{R}_A\begin{bmatrix}\overline{X}\\\overline{Y}\\\overline{Z}\end{bmatrix}_{\text{ORIGINAL}} \tag{32}$$

An approximate vale of λ_A can be obtained by ratios of distances in the survey and design systems, but for most practical applications will be very close to unity. The approximate rotation matrix $\mathbf{R}_A$ can be derived in the following manner.

(i) Shifting both the survey and design systems to the centroid and choosing two control points A and B gives the vector pairs **a** and **b** in the survey system, and **A** and **B** in the design system whose components are the centroidal coordinates of the respective points. Each vector can be reduced to a unit vector using the following: if $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ is a vector whose components are v_1, v_2, v_3 ($\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors in the directions of the X, Y and Z-axes respectively) the unit vector $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \hat{v}_1\,\mathbf{i} + \hat{v}_2\,\mathbf{j} + \hat{v}_3\,\mathbf{k}$ has components $\hat{v}_1 = \dfrac{v_1}{r}$, $\hat{v}_2 = \dfrac{v_2}{r}$, $\hat{v}_3 = \dfrac{v_3}{r}$ where $|\mathbf{v}| = r = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the magnitude.

(ii) In the survey system, the angle θ between the vectors is found from the dot product

$$\cos\theta = \hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = \hat{a}_1\hat{b}_1 + \hat{a}_2\hat{b}_2 + \hat{a}_3\hat{b}_3 \tag{33}$$

where $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and $\hat{b}_1, \hat{b}_2, \hat{b}_3$ are the components of the two unit vectors.

A unit vector $\hat{\mathbf{p}}$, perpendicular to the plane containing **a** and **b**, can be obtained from the vector cross product

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{a}}}{\sin\theta} = \hat{p}_1\,\mathbf{i} + \hat{p}_2\,\mathbf{j} + \hat{p}_3\,\mathbf{k} \tag{34}$$

where

$$\hat{p}_1 = \frac{\hat{b}_2\hat{a}_3 - \hat{b}_3\hat{a}_2}{\sin\theta}$$

$$\hat{p}_2 = \frac{\hat{b}_3\hat{a}_1 - \hat{b}_1\hat{a}_3}{\sin\theta}$$

$$\hat{p}_3 = \frac{\hat{b}_1\hat{a}_2 - \hat{b}_2\hat{a}_1}{\sin\theta}$$

A second cross product $(\hat{\mathbf{a}} \times \hat{\mathbf{p}})$ gives a third unit vector $\hat{\mathbf{q}}$ perpendicular to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{p}}$

$$\hat{\mathbf{q}} = \hat{\mathbf{a}} \times \hat{\mathbf{p}} = \hat{q}_1\,\mathbf{i} + \hat{q}_2\,\mathbf{j} + \hat{q}_3\,\mathbf{k} \tag{35}$$

where

$$\hat{q}_1 = \hat{a}_2\hat{p}_3 - \hat{a}_3\hat{p}_2$$
$$\hat{q}_2 = \hat{a}_3\hat{p}_1 - \hat{a}_1\hat{p}_3$$
$$\hat{q}_3 = \hat{a}_1\hat{p}_2 - \hat{a}_2\hat{p}_1$$

Thus $\hat{\mathbf{q}}$, $\hat{\mathbf{a}}$ and $\hat{\mathbf{p}}$ are the unit vectors of another orthogonal centroidal coordinate system ξ(Xi), η(Eta) and ζ(Zeta).

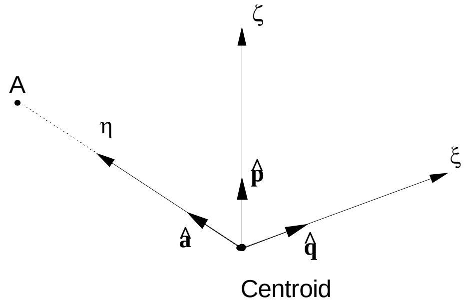

Figure 5. $\xi\eta\zeta$ coordinate system

(iii) The $\xi\eta\zeta$ and $\overline{X}\,\overline{Y}\,\overline{Z}$ centroidal systems, are rotated with respect to each other, the ξ-axis making angles $\alpha_1, \beta_1, \gamma_1$ with the $\overline{X}, \overline{Y}$ and $\overline{Z}$-axes respectively. Similarly, the η and ζ-axes make angles $\alpha_2, \beta_2, \gamma_2$ and $\alpha_3, \beta_3, \gamma_3$. The elements of $\hat{\mathbf{q}}$ are the *direction cosines* $\cos\alpha_1$, $\cos\beta_1$ and $\cos\gamma_1$ (Wolf 1974) and

$$\xi = \overline{X}\cos\alpha_1 + \overline{Y}\cos\beta_1 + \overline{Z}\cos\lambda_1 \qquad (36a)$$

or

$$\xi = \overline{X}\,\hat{\mathbf{q}}_1 + \overline{Y}\,\hat{\mathbf{q}}_2 + \overline{Z}\,\hat{\mathbf{q}}_3 \qquad (36b)$$

Similarly for the elements of $\hat{\mathbf{a}}$ and $\hat{\mathbf{p}}$ we may write

$$\begin{aligned} \eta &= \overline{X}\,\hat{\mathbf{a}}_1 + \overline{Y}\,\hat{\mathbf{a}}_2 + \overline{Z}\,\hat{\mathbf{a}}_3 \\ \zeta &= \overline{X}\,\hat{\mathbf{p}}_1 + \overline{Y}\,\hat{\mathbf{p}}_2 + \overline{Z}\,\hat{\mathbf{p}}_3 \end{aligned} \qquad (36c)$$

Combining (36b) and (36c) in the form of a rotation matrix $\mathbf{R}_1$ gives

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}_3 \\ \hat{\mathbf{p}}_1 & \hat{\mathbf{p}}_2 & \hat{\mathbf{p}}_3 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} = \mathbf{R}_1 \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} \qquad (37)$$

(iv) Replacing survey system vectors **a** and **b** with design system vectors **A** and **B** in steps (ii) and (iii) yields unit vectors $\hat{\mathbf{Q}}, \hat{\mathbf{A}}$ and $\hat{\mathbf{P}}$, and a second rotation matrix $\mathbf{R}_2$

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Q}}_1 & \hat{\mathbf{Q}}_2 & \hat{\mathbf{Q}}_3 \\ \hat{\mathbf{A}}_1 & \hat{\mathbf{A}}_2 & \hat{\mathbf{A}}_3 \\ \hat{\mathbf{P}}_1 & \hat{\mathbf{P}}_2 & \hat{\mathbf{P}}_3 \end{bmatrix} \begin{bmatrix} \overline{E} \\ \overline{N} \\ \overline{U} \end{bmatrix} = \mathbf{R}_2 \begin{bmatrix} \overline{E} \\ \overline{N} \\ \overline{U} \end{bmatrix} \qquad (38)$$

(v) Re-arranging (38) and substituting into (37), using the orthogonal property of rotation matrices $\left(\mathbf{R}_2^{-1} = \mathbf{R}_2^{T}\right)$, gives

$$\begin{bmatrix} \overline{E} \\ \overline{N} \\ \overline{U} \end{bmatrix} = \mathbf{R}_2^{T}\mathbf{R}_1 \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} \qquad (39)$$

where the left-hand-side of (39) can be considered as an initial transformation of the survey coordinates. Thus with an approximate rotation matrix $\mathbf{R}_A = \mathbf{R}_2^{T}\mathbf{R}_1$ and the approximate scale factor λ_A we have (32) as given above where

$$\mathbf{R}_A = \mathbf{R}_2^{T}\mathbf{R}_1 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \qquad (40)$$

$\mathbf{R}_A$ has the same form as $\mathbf{R}_{\kappa\phi\omega}$ (15) and values of the approximate rotations ω_A, ϕ_A and κ_A can be calculated from

$$\tan\kappa_A = \frac{-R_{21}}{R_{11}} \qquad (41)$$

$$\cos\phi_A = \frac{R_{11}}{\cos\kappa'} \qquad (42)$$

$$\cos\omega_A = \frac{R_{33}\cos\kappa'}{R_{11}} \qquad (43)$$

An example of the computation of approximate rotations is contained in the Appendix.

CONCLUSION

This paper has presented the necessary equations and computation technique for the practical use of 3D conformal transformations in survey measurement. These transformations, combined with least squares, can be usefully employed on large construction projects where components, often manufactured "off-site", must be brought together "on-site" to fit within a design coordinate system. The least squares transformation method can be used to compare the off-site component, measured in situ in XYZ survey coordinates, with its proposed location in the on-site ENU design coordinates; comparison being via the coordinate residuals at common points. This pre-analysis "tool" can be used to prevent costly (and often embarrassing) misalignment of components in a large engineering structure.

An example computation of the transformation parameters and residuals of a simple plane figure ABC with measured XYZ coordinates and ENU design coordinates is provided in the Appendix.

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APPENDIX

Calculation of transformation parameters between survey and design locations of the figure ABC.

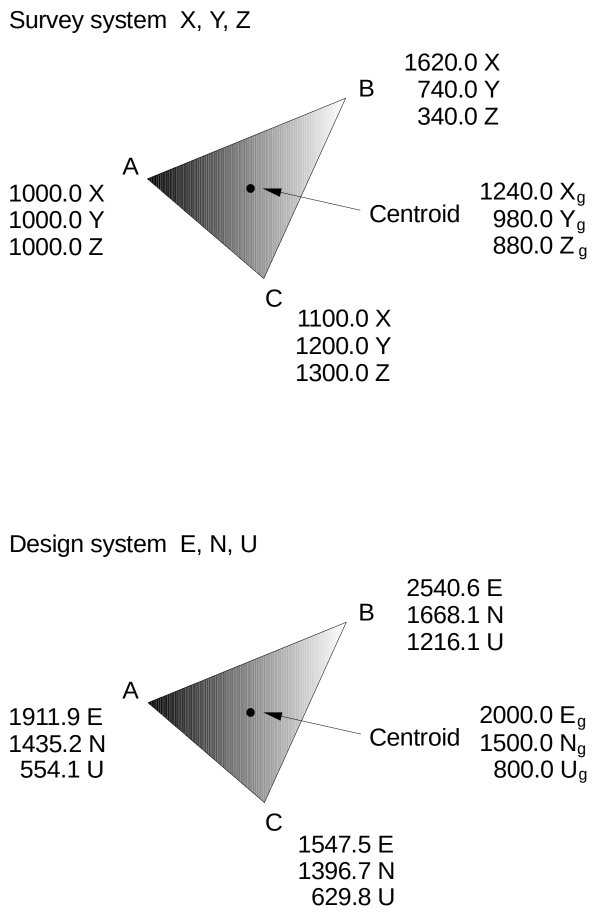

Figure A

Note: In Figure A the ENU design system coordinates have been generated from the XYZ survey system coordinates by (i) shifting to the centroid of ABC, (ii) transforming to $\overline{E}, \overline{N}$ and $\overline{U}$ using (2) with a scale factor $\lambda = 1$ and rotation matrix $\mathbf{R}_{\kappa\phi\omega}$ with values $\kappa = 10^\circ$, $\phi = 94^\circ$, $\omega = 310^\circ$ then (iii) rounding the transformed centroidal coordinates to the nearest 0.1m and "adding back" E_g, N_g and U_g.

The solution for the transformation parameters is accomplished in the following steps.

Step 1: Shift the origin of the survey system to the centroid by subtracting X_g, Y_g and Z_g from the coordinates of A B and C then calculate the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

	A	B	C
$\overline{X}$	-240.000	380.000	-140.000
$\overline{Y}$	20.000	-240.000	220.000
$\overline{Z}$	120.000	-540.000	420.000

Table 1. Survey system centroidal coordinates

$$\mathbf{a} = -240\,\mathbf{i} + 20\,\mathbf{j} + 120\,\mathbf{k}$$
$$\hat{\mathbf{a}} = -0.891953\,\mathbf{i} + 0.074329\,\mathbf{j} + 0.445976\,\mathbf{k}$$
$$\mathbf{b} = 380\,\mathbf{i} - 240\,\mathbf{j} - 540\,\mathbf{k}$$
$$\hat{\mathbf{b}} = 0.540874\,\mathbf{i} - 0.341605\,\mathbf{j} - 0.768610\,\mathbf{k}$$

Step 2: Use the vector dot product (33) to calculate angle θ between $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned} \hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = \cos\theta &= \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \hat{a}_3 \hat{b}_3 \\ &= -0.850607457 \\ \theta &= 148.277801360 \end{aligned}$$

Step 3: Use the vector cross product (34) to calculate the unit vector $\hat{\mathbf{p}}$ perpendicular to the plane containing A, B and the centroid

$$\begin{aligned} \hat{\mathbf{p}} &= \frac{\hat{\mathbf{b}} \times \hat{\mathbf{a}}}{\sin\theta} = \hat{p}_1\,\mathbf{i} + \hat{p}_2\,\mathbf{j} + \hat{p}_3\,\mathbf{k} \\ &= -0.181090\,\mathbf{i} + 0.845086\,\mathbf{j} - 0.503027\,\mathbf{k} \end{aligned}$$

Step 4: Use the vector cross product (35) to calculate the unit vector $\hat{\mathbf{q}}$ perpendicular to $\hat{\mathbf{a}}$ and $\hat{\mathbf{p}}$

$$\hat{\mathbf{q}} = \hat{\mathbf{a}} \times \hat{\mathbf{p}} = \hat{q}_1 \mathbf{i} + \hat{q}_2 \mathbf{j} + \hat{q}_3 \mathbf{k}$$
$$= -0.414278\,\mathbf{i} - 0.529438\,\mathbf{j} - 0.740317\,\mathbf{k}$$

Step 5: Form the rotation matrix $\mathbf{R}_1$ in (37) from the elements of the unit vectors $\hat{\mathbf{q}}, \hat{\mathbf{a}}$ and $\hat{\mathbf{p}}$

$$\mathbf{R}_1 = \begin{bmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}_3 \\ \hat{\mathbf{p}}_1 & \hat{\mathbf{p}}_2 & \hat{\mathbf{p}}_3 \end{bmatrix} = \begin{bmatrix} -0.414278 & -0.529438 & -0.740317 \\ -0.891953 & 0.074329 & 0.445976 \\ -0.181090 & 0.845086 & -0.503027 \end{bmatrix}$$

Step 6: Shift the origin of the design system to the centroid by subtracting E_g, N_g and U_g from the coordinates of A B and C then calculate the unit vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$.

	A	B	C
$\overline{E}$	-88.100	540.600	-452.500
$\overline{N}$	-64.800	168.100	-103.300
$\overline{U}$	-245.900	416.100	-170.200

Table 2. Design system centroidal coordinates

$$\mathbf{A} = -88.1\,\mathbf{i} - 64.8\,\mathbf{j} - 245.9\,\mathbf{k}$$
$$\hat{\mathbf{A}} = -0.327359\,\mathbf{i} - 0.240782\,\mathbf{j} - 0.913707\,\mathbf{k}$$
$$\mathbf{B} = 540.6\,\mathbf{i} + 168.1\,\mathbf{j} + 416.1\,\mathbf{k}$$
$$\hat{\mathbf{B}} = 0.769429\,\mathbf{i} + 0.239255\,\mathbf{j} + 0.592230\,\mathbf{k}$$

Step 7: Use the unit vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ in steps 2 and 3 to calculate the angle between **A** and **B**, then the unit vector $\hat{\mathbf{P}}$ followed by the unit vector $\hat{\mathbf{Q}}$

$$\hat{\mathbf{P}} = -0.144563\,\mathbf{i} + 0.968366\,\mathbf{j} - 0.203392\,\mathbf{k}$$
$$\hat{\mathbf{Q}} = 0.933776\,\mathbf{i} + 0.065506\,\mathbf{j} - 0.351812\,\mathbf{k}$$

Step 8: Form the rotation matrix $\mathbf{R}_2$ in (38) from the elements of the unit vectors $\hat{\mathbf{Q}}, \hat{\mathbf{A}}$ and $\hat{\mathbf{P}}$

$$\mathbf{R}_2 = \begin{bmatrix} \hat{\mathbf{Q}}_1 & \hat{\mathbf{Q}}_2 & \hat{\mathbf{Q}}_3 \\ \hat{\mathbf{A}}_1 & \hat{\mathbf{A}}_2 & \hat{\mathbf{A}}_3 \\ \hat{\mathbf{P}}_1 & \hat{\mathbf{P}}_2 & \hat{\mathbf{P}}_3 \end{bmatrix} = \begin{bmatrix} 0.933776 & 0.065506 & -0.351812 \\ -0.327359 & -0.240782 & -0.913707 \\ -0.144563 & 0.968366 & -0.203392 \end{bmatrix}$$

Step 9: Compute the numeric values in the approximate rotation matrix $\mathbf{R}_A$ using (40).

$$\mathbf{R}_A = \mathbf{R}_2^T \mathbf{R}_1 = \begin{bmatrix} -0.068675 & -0.640878 & -0.764565 \\ 0.012267 & 0.765774 & -0.642993 \\ 0.997564 & -0.053536 & -0.044728 \end{bmatrix}$$

$\mathbf{R}_A$ is of similar form to $\mathbf{R}_{\kappa\phi\omega}$ and, if required, values of the approximate rotations ω_A, ϕ_A and κ_A can be calculated from it by using (41), (42) and (43).

$$\tan\kappa_A = \frac{-R_{21}}{R_{11}} = \frac{-0.012267}{-0.068675}$$
$$\kappa_A = 190^\circ\,07'\,39''$$

$$\cos\phi_A = \frac{R_{11}}{\cos\kappa'}$$
$$\phi_A = 85^\circ\,59'\,59''$$

$$\cos\omega_A = \frac{R_{33}\cos\kappa'}{R_{11}}$$
$$\omega_A = 129^\circ\,52'\,40''$$

Step 10: Use (32) with $\mathbf{R}_A$ from Step 9 and $\lambda_A = 1$ to transform the original $\overline{X}\,\overline{Y}\,\overline{Z}$ centroidal coordinates (Table 1) to an initial set of approximately transformed centroidal coordinates to be used in the least squares solution.

	A	B	C
$\overline{X}$	-88.0833	540.5791	-452.4958
$\overline{Y}$	-64.7877	168.0919	-103.3042
$\overline{Z}$	-245.8534	416.0761	-170.2227

Table 3. Initial Survey system centroidal coordinates (Iteration 1)

The least squares solution of $\delta\lambda$ (correction to the approximate scale factor) $\delta\omega$, $\delta\phi$ and $\delta\kappa$ (small rotation angles) follows; noting that the approximate scale factor is assumed to be unity $(\lambda' = 1)$.

Step 11: With a re-arrangement of the order of residuals in (30) the coefficient matrix **B** becomes

$$\begin{bmatrix} 0 & 245.8534 & -64.7877 & -88.0833 \\ 0 & -416.0761 & 168.0919 & 540.5791 \\ 0 & 170.2227 & -103.3042 & -452.4958 \\ \hdashline -245.8534 & 0 & 88.0833 & -64.7877 \\ 416.0761 & 0 & -540.5791 & 168.0919 \\ -170.2227 & 0 & 452.4958 & -103.3042 \\ \hdashline 64.7877 & -88.0833 & 0 & -245.8534 \\ -168.0919 & 540.5791 & 0 & 416.0761 \\ 103.3042 & -452.4958 & 0 & -170.2227 \end{bmatrix}$$

and the vector of numeric terms is

$$\mathbf{f} = \begin{bmatrix} -0.0167 \\ 0.0209 \\ -0.0042 \\ \hdashline -0.0123 \\ 0.0081 \\ 0.0042 \\ \hdashline -0.0466 \\ 0.0239 \\ 0.0227 \end{bmatrix}$$

Step 12: Forming the matrix products $\mathbf{B}^T\mathbf{B}$ (normal coefficient matrix) and $\mathbf{B}^T\mathbf{f}$ (vector of numeric terms) and solving (31) gives the solution as

$$\mathbf{x} = \begin{bmatrix} \delta\omega \\ \delta\phi \\ \delta\kappa \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} 1.38\times10^{-6} \\ -8.30\times10^{-6} \\ 1.93\times10^{-6} \\ 4.1870\times10^{-5} \end{bmatrix} \begin{matrix} \text{radians} \\ \text{radians} \\ \text{radians} \\ \end{matrix} = \begin{bmatrix} 0.28 \\ -1.71 \\ 0.40 \\ 41.870 \end{bmatrix} \begin{matrix} \text{sec} \\ \text{sec} \\ \text{sec} \\ \text{ppm} \end{matrix}$$

Step 13: Substituting the vector **x** into (30) gives the residuals at A, B and C

$$\begin{bmatrix} v_{E_A} \\ v_{E_B} \\ v_{E_C} \\ \hdashline v_{N_A} \\ v_{N_B} \\ v_{N_C} \\ \hdashline v_{U_A} \\ v_{U_B} \\ v_{U_C} \end{bmatrix} = \begin{bmatrix} 0.011 \\ 0.006 \\ -0.016 \\ \hdashline 0.009 \\ -0.002 \\ -0.008 \\ \hdashline 0.037 \\ -0.011 \\ -0.026 \end{bmatrix} \text{metres}$$

and the transformed centroidal coordinates of A, B and C can be found from (2b) with the elements of $\mathbf{R}_S$ given above ($\delta\omega = 0.28''$, $\delta\phi = -1.71''$, $\delta\kappa = 0.40''$) and $\lambda' + \delta\lambda = 1.000\ 041\ 870$.

	A	B	C
$\overline{X}$	-88.0892	540.6055	-452.5164
$\overline{Y}$	-64.7906	168.0985	-103.3079
$\overline{Z}$	-245.8635	416.0888	-170.2259

Table 4. Survey system centroidal coordinates (Iteration 2)

Step 14: A second iteration using the centroidal coordinates of Table 4 gives solutions as

$$\mathbf{x} = \begin{bmatrix} \delta\omega \\ \delta\phi \\ \delta\kappa \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} 3.27\times10^{-8} \\ 3.22\times10^{-9} \\ 1.76\times10^{-8} \\ -3.13\times10^{-8} \end{bmatrix} \begin{matrix} \text{radians} \\ \text{radians} \\ \text{radians} \\ \end{matrix} = \begin{bmatrix} 0.01 \\ 0.01 \\ 0.00 \\ 0.031 \end{bmatrix} \begin{matrix} \text{sec} \\ \text{sec} \\ \text{sec} \\ \text{ppm} \end{matrix}$$

These solutions, which are at least an order of magnitude less than the corrections of the first iteration (demonstrating that the solution is converging) are probably due to rounding errors in the calculations and the first iteration results can be assumed as "exact".

To transform any additional points in the XYZ survey system to the ENU design system the following process should be adopted:

(a) convert to centroidal coordinates then perform an initial transformation as per Step 10 using (32) with the numerical values of $\mathbf{R}_A$ as determined in Step 9,

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_{\text{INITIAL}} = \lambda_A \mathbf{R}_A \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_{\text{ORIGINAL}} \tag{32}$$

(b) then transform to the ENU system using (2b) modified by setting the approximate scale factor to unity $(\lambda' = 1)$ and adding back the coordinates of the centroid in the design system.

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = (1 + \delta\lambda) \mathbf{R}_S \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} + \begin{bmatrix} E_g \\ N_g \\ U_g \end{bmatrix}$$

with the values for $\delta\omega$, $\delta\phi$, $\delta\kappa$ and $\delta\lambda$ from the least squares solution as per Step 12