4.2 Trigonometric Functions: The Unit Circle

What you should learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use the domain and period to evaluate sine and cosine functions.
- Use a calculator to evaluate trigonometric functions.

Why you should learn it

Trigonometric functions are used to model the movement of an oscillating weight. For instance, in Exercise 60 on page 298, the displacement from equilibrium of an oscillating weight suspended by a spring is modeled as a function of time.

Richard Megna/Fundamental Photographs

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$x^2 + y^2 = 1$ Unit circle

as shown in Figure 4.20.

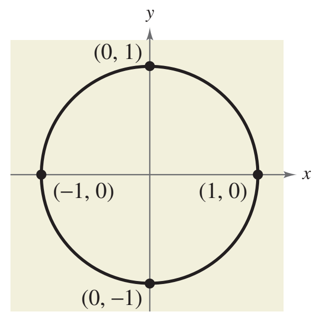


FIGURE 4.20

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown in Figure 4.21.

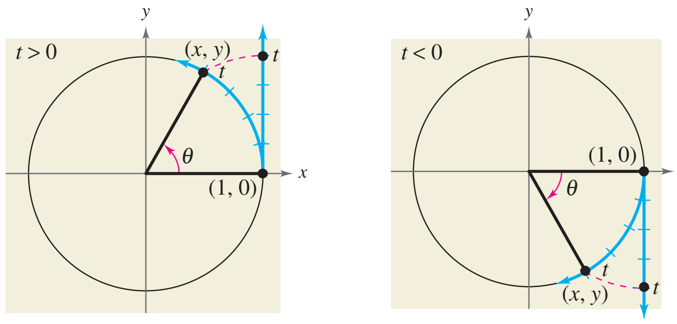


FIGURE 4.21

As the real number line is wrapped around the unit circle, each real number t corresponds to a point (x, y) on the circle. For example, the real number 0 corresponds to the point $(1, 0)$. Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point $(1, 0)$.

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t. With this interpretation of t, the arc length formula $s = r\theta$ (with $r = 1$) indicates that the real number t is the (directional) length of the arc intercepted by the angle θ, given in radians.

The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t. You can use these coordinates to define the six trigonometric functions of t.

sine **cosecant** **cosine** **secant** **tangent** **cotangent**

These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

Study Tip

Note in the definition at the right that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$. For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*. Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, cot 0 and csc 0 are *undefined*.

In Figure 4.22, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

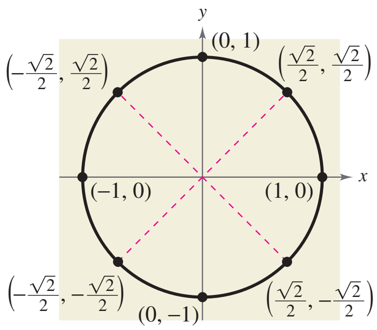


FIGURE 4.22

Similarly, in Figure 4.23, the unit circle has been divided into 12 equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

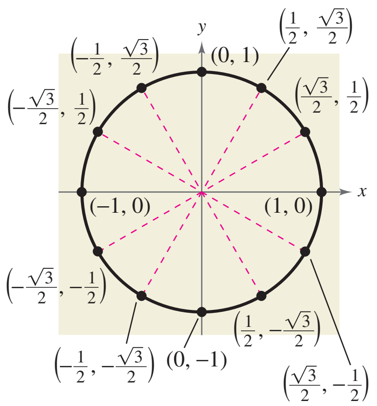


FIGURE 4.23

To verify the points on the unit circle in Figure 4.22, note that $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ also lies on the line $y = x$. So, substituting x for y in the equation of the unit circle produces the following.

$$x^2 + x^2 = 1 \quad \Rightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{\sqrt{2}}{2}$$

Because the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$ and because $y = x$, you also have $y = \frac{\sqrt{2}}{2}$. You can use similar reasoning to verify the rest of the points in Figure 4.22 and the points in Figure 4.23.

Using the (x, y) coordinates in Figures 4.22 and 4.23, you can evaluate the trigonometric functions for common t-values. This procedure is demonstrated in Examples 1, 2, and 3. You should study and learn these exact function values for common t-values because they will help you in later sections to perform calculations.

Algebra Help

You can review dividing fractions and rationalizing denominators in Appendix A.1 and Appendix A.2, respectively.

Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solution

For each t-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 293.

a. $t = \frac{\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$$\sin \frac{\pi}{6} = y = \frac{1}{2} \qquad \csc \frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos \frac{\pi}{6} = x = \frac{\sqrt{3}}{2} \qquad \sec \frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan \frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot \frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = \frac{5\pi}{4}$ corresponds to the point $(x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\sin \frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc \frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos \frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan \frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0 \qquad \csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1 \qquad \sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot 0 = \frac{x}{y} \text{ is undefined.}$$

d. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0 \qquad \csc \pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos \pi = x = -1 \qquad \sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot \pi = \frac{x}{y} \text{ is undefined.}$$

CHECKPoint Now try Exercise 23.

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\frac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \qquad \cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

CHECKPoint Now try Exercise 33.

Domain and Period of Sine and Cosine

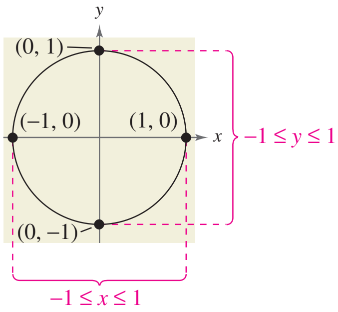


FIGURE 4.24

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.24. By definition, $\sin t = y$ and $\cos t = x$. Because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$-1 \le y \le 1 \qquad \text{and} \qquad -1 \le x \le 1$$
$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.25. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

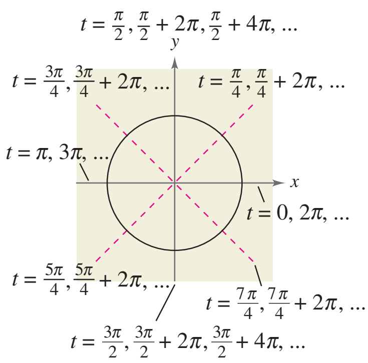


FIGURE 4.25

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

Recall from Section 1.5 that a function f is *even* if $f(-t) = f(t)$, and is *odd* if $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

Example 3 Using the Period to Evaluate the Sine and Cosine

a. Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have $\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$.

b. Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.

CHECKPoint Now try Exercise 37.

Study Tip

From the definition of periodic function, it follows that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic, and will be discussed further in Section 4.6.

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (*degree* or *radian*).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the [x^{-1}] key with their respective reciprocal functions sine, cosine, and tangent. For instance, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc \frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

[(] [SIN] [(] [π] [÷] 8 [)] [)] [x^{-1}] [ENTER]   Display 2.6131259

TECHNOLOGY

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, if you want to evaluate *sin t* for $t = \pi/6$, you should enter

[SIN] [(] [π] [÷] 6 [)] [ENTER].

These keystrokes yield the correct value of 0.5. Note that some calculators automatically place a left parenthesis after trigonometric functions. Check the user's guide for your calculator for specific keystrokes on how to evaluate trigonometric functions.

Example 4 Using a Calculator

Function	*Mode*	*Calculator Keystrokes*	*Display*
a. $\sin \frac{2\pi}{3}$	Radian	[SIN] [(] 2 [π] [÷] 3 [)] [ENTER]	0.8660254
b. $\cot 1.5$	Radian	[(] [TAN] [(] 1.5 [)] [)] [x^{-1}] [ENTER]	0.0709148

CHECKPoint Now try Exercise 55.

4.2 EXERCISES

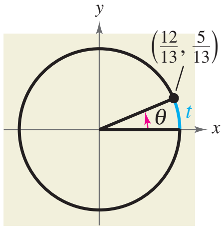
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

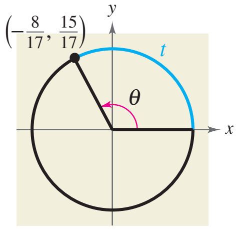
VOCABULARY: Fill in the blanks.

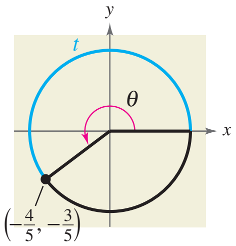
1. Each real number t corresponds to a point (x, y) on the ________ ________.
2. A function f is ________ if there exists a positive real number c such that $f(t + c) = f(t)$ for all t in the domain of f.
3. The smallest number c for which a function f is periodic is called the ________ of f.
4. A function f is ________ if $f(-t) = -f(t)$ and ________ if $f(-t) = f(t)$.

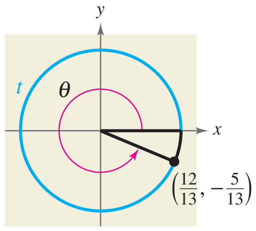
SKILLS AND APPLICATIONS

In Exercises 5–8, determine the exact values of the six trigonometric functions of the real number t.

5.


6.


7.


8.


In Exercises 9–16, find the point (x, y) on the unit circle that corresponds to the real number t.

9. $t = \frac{\pi}{2}$
10. $t = \pi$
11. $t = \frac{\pi}{4}$
12. $t = \frac{\pi}{3}$
13. $t = \frac{5\pi}{6}$
14. $t = \frac{3\pi}{4}$
15. $t = \frac{4\pi}{3}$
16. $t = \frac{5\pi}{3}$

In Exercises 17–26, evaluate (if possible) the sine, cosine, and tangent of the real number.

17. $t = \frac{\pi}{4}$
18. $t = \frac{\pi}{3}$
19. $t = -\frac{\pi}{6}$
20. $t = -\frac{\pi}{4}$
21. $t = -\frac{7\pi}{4}$
22. $t = -\frac{4\pi}{3}$
23. $t = \frac{11\pi}{6}$
24. $t = \frac{5\pi}{3}$
25. $t = -\frac{3\pi}{2}$
26. $t = -2\pi$

In Exercises 27–34, evaluate (if possible) the six trigonometric functions of the real number.

27. $t = \frac{2\pi}{3}$
28. $t = \frac{5\pi}{6}$
29. $t = \frac{4\pi}{3}$
30. $t = \frac{7\pi}{4}$
31. $t = \frac{3\pi}{4}$
32. $t = \frac{3\pi}{2}$
33. $t = -\frac{\pi}{2}$
34. $t = -\pi$

In Exercises 35–42, evaluate the trigonometric function using its period as an aid.

35. $\sin 4\pi$
36. $\cos 3\pi$
37. $\cos \frac{7\pi}{3}$
38. $\sin \frac{9\pi}{4}$
39. $\cos \frac{17\pi}{4}$
40. $\sin \frac{19\pi}{6}$
41. $\sin\left(-\frac{8\pi}{3}\right)$
42. $\cos\left(-\frac{9\pi}{4}\right)$

In Exercises 43–48, use the value of the trigonometric function to evaluate the indicated functions.

43. $\sin t = \frac{1}{2}$
 (a) $\sin(-t)$
 (b) $\csc(-t)$
44. $\sin(-t) = \frac{3}{8}$
 (a) $\sin t$
 (b) $\csc t$
45. $\cos(-t) = -\frac{1}{5}$
 (a) $\cos t$
 (b) $\sec(-t)$
46. $\cos t = -\frac{3}{4}$
 (a) $\cos(-t)$
 (b) $\sec(-t)$
47. $\sin t = \frac{4}{5}$
 (a) $\sin(\pi - t)$
 (b) $\sin(t + \pi)$
48. $\cos t = \frac{4}{5}$
 (a) $\cos(\pi - t)$
 (b) $\cos(t + \pi)$

In Exercises 49–58, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

49. $\sin \frac{\pi}{4}$
50. $\tan \frac{\pi}{3}$
51. $\cot \frac{\pi}{4}$
52. $\csc \frac{2\pi}{3}$
53. $\cos(-1.7)$
54. $\cos(-2.5)$
55. $\csc 0.8$
56. $\sec 1.8$
57. $\sec(-22.8)$
58. $\cot(-0.9)$

59. HARMONIC MOTION The displacement from equilibrium of an oscillating weight suspended by a spring is given by $y(t) = \frac{1}{4}\cos 6t$, where y is the displacement (in feet) and t is the time (in seconds). Find the displacements when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

60. HARMONIC MOTION The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by $y(t) = \frac{1}{4}e^{-t}\cos 6t$, where y is the displacement (in feet) and t is the time (in seconds).

(a) Complete the table.

t	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y					

(b) Use the *table* feature of a graphing utility to approximate the time when the weight reaches equilibrium.

(c) What appears to happen to the displacement as t increases?

EXPLORATION

TRUE OR FALSE? In Exercises 61–64, determine whether the statement is true or false. Justify your answer.

61. Because $\sin(-t) = -\sin t$, it can be said that the sine of a negative angle is a negative number.

62. $\tan a = \tan(a - 6\pi)$

63. The real number 0 corresponds to the point (0, 1) on the unit circle.

64. $\cos\left(-\frac{7\pi}{2}\right) = \cos\left(\pi + \frac{\pi}{2}\right)$

65. Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = \pi - t_1$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(\pi - t_1)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(\pi - t_1)$.

66. Use the unit circle to verify that the cosine and secant functions are even and that the sine, cosecant, tangent, and cotangent functions are odd.

67. Verify that $\cos 2t \neq 2\cos t$ by approximating $\cos 1.5$ and $2\cos 0.75$.

68. Verify that $\sin(t_1 + t_2) \neq \sin t_1 + \sin t_2$ by approximating $\sin 0.25$, $\sin 0.75$, and $\sin 1$.

69. THINK ABOUT IT Because $f(t) = \sin t$ is an odd function and $g(t) = \cos t$ is an even function, what can be said about the function $h(t) = f(t)g(t)$?

70. THINK ABOUT IT Because $f(t) = \sin t$ and $g(t) = \tan t$ are odd functions, what can be said about the function $h(t) = f(t)g(t)$?

71. GRAPHICAL ANALYSIS With your graphing utility in *radian* and *parametric* modes, enter the equations

$X_{1T} = \cos T$ and $Y_{1T} = \sin T$

and use the following settings.

Tmin = 0, Tmax = 6.3, Tstep = 0.1
Xmin = −1.5, Xmax = 1.5, Xscl = 1
Ymin = −1, Ymax = 1, Yscl = 1

(a) Graph the entered equations and describe the graph.

(b) Use the *trace* feature to move the cursor around the graph. What do the t-values represent? What do the x- and y-values represent?

(c) What are the least and greatest values of x and y?

72. CAPSTONE A student you are tutoring has used a unit circle divided into 8 equal parts to complete the table for selected values of t. What is wrong?

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0
$\sin t$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1
$\cos t$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0
$\tan t$	Undef.	1	0	-1	Undef.

4.3 Right Triangle Trigonometry

What you should learn

- Evaluate trigonometric functions of acute angles.
- Use fundamental trigonometric identities.
- Use a calculator to evaluate trigonometric functions.
- Use trigonometric functions to model and solve real-life problems.

Why you should learn it

Trigonometric functions are often used to analyze real-life situations. For instance, in Exercise 76 on page 309, you can use trigonometric functions to find the height of a helium-filled balloon.

Joseph Sohm/Visions of America/Corbis

The Six Trigonometric Functions

Our second look at the trigonometric functions is from a *right triangle* perspective. Consider a right triangle, with one acute angle labeled θ, as shown in Figure 4.26. Relative to the angle θ, the three sides of the triangle are the **hypotenuse,** the **opposite side** (the side opposite the angle θ), and the **adjacent side** (the side adjacent to the angle θ).

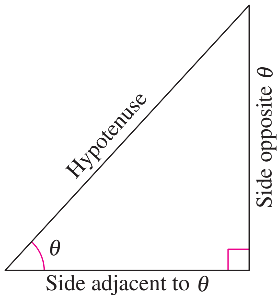


FIGURE 4.26

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle θ.

sine cosecant cosine secant tangent cotangent

In the following definitions, it is important to see that $0° < \theta < 90°$ (θ lies in the first quadrant) and that for such angles the value of each trigonometric function is *positive*.

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows. (Note that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent to* θ

hyp = the length of the *hypotenuse*

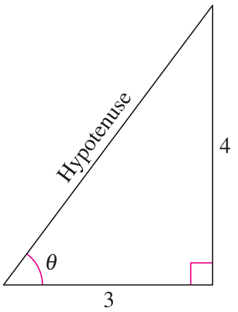


FIGURE 4.27

Algebra Help

You can review the Pythagorean Theorem in Section 1.1.

HISTORICAL NOTE

Georg Joachim Rhaeticus (1514–1574) was the leading Teutonic mathematical astronomer of the 16th century. He was the first to define the trigonometric functions as ratios of the sides of a right triangle.

Example 1 Evaluating Trigonometric Functions

Use the triangle in Figure 4.27 to find the values of the six trigonometric functions of θ.

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\begin{aligned}\text{hyp} &= \sqrt{4^2 + 3^2}\\ &= \sqrt{25}\\ &= 5.\end{aligned}$$

So, the six trigonometric functions of θ are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}.$$

CHECKPoint Now try Exercise 7.

In Example 1, you were given the lengths of two sides of the right triangle, but not the angle θ. Often, you will be asked to find the trigonometric functions of a *given* acute angle θ. To do this, construct a right triangle having θ as one of its angles.

Example 2 Evaluating Trigonometric Functions of 45°

Find the values of sin 45°, cos 45°, and tan 45°.

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 4.28. Choose the length of the adjacent side to be 1. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

CHECKPoint Now try Exercise 23.

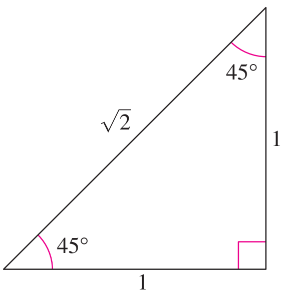


FIGURE 4.28

Study Tip

Because the angles 30°, 45°, and 60° ($\pi/6$, $\pi/4$, and $\pi/3$) occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 4.28 and 4.29.

Example 3 Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle shown in Figure 4.29 to find the values of sin 60°, cos 60°, sin 30°, and cos 30°.

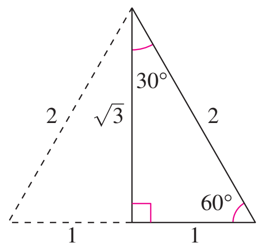


FIGURE 4.29

Solution

Use the Pythagorean Theorem and the equilateral triangle in Figure 4.29 to verify the lengths of the sides shown in the figure. For $\theta = 60°$, you have adj $= 1$, opp $= \sqrt{3}$, and hyp $= 2$. So,

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30°$, adj $= \sqrt{3}$, opp $= 1$, and hyp $= 2$. So,

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

CHECK*Point* Now try Exercise 27.

TECHNOLOGY

You can use a calculator to convert the answers in Example 3 to decimals. However, the radical form is the exact value and in most cases, the exact value is preferred.

Sines, Cosines, and Tangents of Special Angles

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

In the box, note that $\sin 30° = \frac{1}{2} = \cos 60°$. This occurs because 30° and 60° are complementary angles. In general, it can be shown from the right triangle definitions that *cofunctions of complementary angles are equal*. That is, if θ is an acute angle, the following relationships are true.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta$$

Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Note that $\sin^2\theta$ represents $(\sin\theta)^2$, $\cos^2\theta$ represents $(\cos\theta)^2$, and so on.

Example 4 Applying Trigonometric Identities

Let θ be an acute angle such that $\sin\theta = 0.6$. Find the values of (a) $\cos\theta$ and (b) $\tan\theta$ using trigonometric identities.

Solution

a. To find the value of $\cos\theta$, use the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

So, you have

$$(0.6)^2 + \cos^2\theta = 1 \qquad \text{Substitute 0.6 for } \sin\theta.$$

$$\cos^2\theta = 1 - (0.6)^2 = 0.64 \qquad \text{Subtract } (0.6)^2 \text{ from each side.}$$

$$\cos\theta = \sqrt{0.64} = 0.8. \qquad \text{Extract the positive square root.}$$

b. Now, knowing the sine and cosine of θ, you can find the tangent of θ to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75.$$

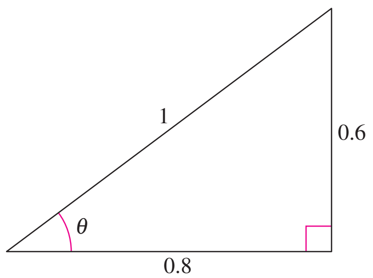


FIGURE 4.30

Use the definitions of $\cos\theta$ and $\tan\theta$, and the triangle shown in Figure 4.30, to check these results.

CHECKPoint Now try Exercise 33.

Example 5 Applying Trigonometric Identities

Let θ be an acute angle such that $\tan\theta = 3$. Find the values of (a) $\cot\theta$ and (b) $\sec\theta$ using trigonometric identities.

Solution

a. $\cot\theta = \dfrac{1}{\tan\theta}$ Reciprocal identity

$\cot\theta = \dfrac{1}{3}$

b. $\sec^2\theta = 1 + \tan^2\theta$ Pythagorean identity

$\sec^2\theta = 1 + 3^2$

$\sec^2\theta = 10$

$\sec\theta = \sqrt{10}$

Use the definitions of $\cot\theta$ and $\sec\theta$, and the triangle shown in Figure 4.31, to check these results.

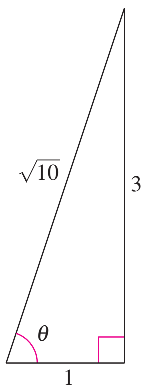


FIGURE 4.31

CHECK*Point* Now try Exercise 35.

Study Tip

You can also use the reciprocal identities for sine, cosine, and tangent to evaluate the cosecant, secant, and cotangent functions with a calculator. For instance, you could use the following keystroke sequence to evaluate sec 28°.

1 [÷] [COS] 28 [ENTER]

The calculator should display 1.1325701.

Evaluating Trigonometric Functions with a Calculator

To use a calculator to evaluate trigonometric functions of angles measured in degrees, first set the calculator to *degree* mode and then proceed as demonstrated in Section 4.2. For instance, you can find values of cos 28° and sec 28° as follows.

Function	*Mode*	*Calculator Keystrokes*	*Display*
a. cos 28°	Degree	[COS] 28 [ENTER]	0.8829476
b. sec 28°	Degree	[(] [COS] [(] 28 [)] [)] [x^{-1}] [ENTER]	1.1325701

Throughout this text, angles are assumed to be measured in radians unless noted otherwise. For example, sin 1 means the sine of 1 radian and sin 1° means the sine of 1 degree.

Example 6 Using a Calculator

Use a calculator to evaluate $\sec(5°\,40'\,12'')$.

Solution

Begin by converting to decimal degree form. [Recall that $1' = \frac{1}{60}(1°)$ and $1'' = \frac{1}{3600}(1°)$].

$$5°\,40'\,12'' = 5° + \left(\frac{40}{60}\right)^{\circ} + \left(\frac{12}{3600}\right)^{\circ} = 5.67°$$

Then, use a calculator to evaluate sec 5.67°.

Function	*Calculator Keystrokes*	*Display*
$\sec(5°\,40'\,12'') = \sec 5.67°$	[(] [COS] [(] 5.67 [)] [)] [x^{-1}] [ENTER]	1.0049166

CHECK*Point* Now try Exercise 51.

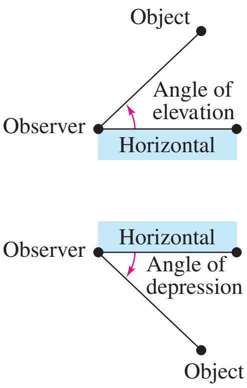


FIGURE 4.32

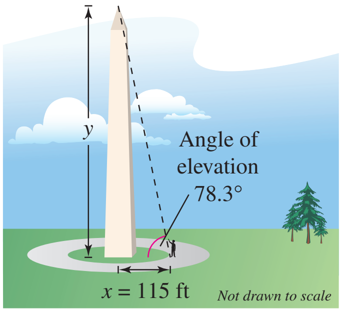


FIGURE 4.33

Applications Involving Right Triangles

Many applications of trigonometry involve a process called **solving right triangles.** In this type of application, you are usually given one side of a right triangle and one of the acute angles and are asked to find one of the other sides, *or* you are given two sides and are asked to find one of the acute angles.

In Example 7, the angle you are given is the **angle of elevation,** which represents the angle from the horizontal upward to an object. For objects that lie below the horizontal, it is common to use the term **angle of depression,** as shown in Figure 4.32.

Example 7 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 115 feet from the base of the Washington Monument, as shown in Figure 4.33. The surveyor measures the angle of elevation to the top of the monument as 78.3°. How tall is the Washington Monument?

Solution

From Figure 4.33, you can see that

$$\tan 78.3° = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

where $x = 115$ and y is the height of the monument. So, the height of the Washington Monument is

$$y = x \tan 78.3° \approx 115(4.82882) \approx 555 \text{ feet.}$$

CHECKPoint Now try Exercise 67.

Example 8 Using Trigonometry to Solve a Right Triangle

A historic lighthouse is 200 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 400 yards long. Find the acute angle θ between the bike path and the walkway, as illustrated in Figure 4.34.

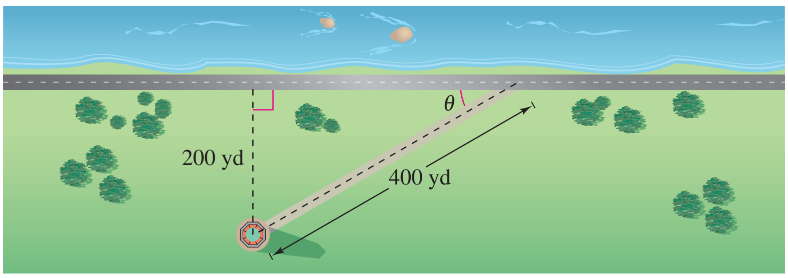


FIGURE 4.34

Solution

From Figure 4.34, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now you should recognize that $\theta = 30°$.

CHECKPoint Now try Exercise 69.

By now you are able to recognize that $\theta = 30°$ is the acute angle that satisfies the equation $\sin \theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin \theta = 0.6$ and were asked to find the acute angle θ. Because

$$\sin 30° = \frac{1}{2}$$
$$= 0.5000$$

and

$$\sin 45° = \frac{1}{\sqrt{2}}$$
$$\approx 0.7071$$

you might guess that θ lies somewhere between 30° and 45°. In a later section, you will study a method by which a more precise value of θ can be determined.

Example 9 Solving a Right Triangle

Find the length c of the skateboard ramp shown in Figure 4.35.

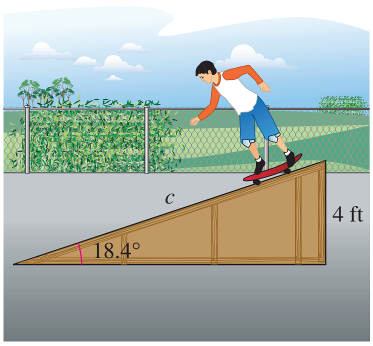


FIGURE 4.35

Solution

From Figure 4.35, you can see that

$$\sin 18.4° = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{4}{c}.$$

So, the length of the skateboard ramp is

$$c = \frac{4}{\sin 18.4°}$$
$$\approx \frac{4}{0.3156}$$
$$\approx 12.7 \text{ feet.}$$

CHECKPoint Now try Exercise 71.

4.3 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY

1. Match the trigonometric function with its right triangle definition.

(a) Sine (b) Cosine (c) Tangent (d) Cosecant (e) Secant (f) Cotangent

(i) $\dfrac{\text{hypotenuse}}{\text{adjacent}}$ (ii) $\dfrac{\text{adjacent}}{\text{opposite}}$ (iii) $\dfrac{\text{hypotenuse}}{\text{opposite}}$ (iv) $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ (v) $\dfrac{\text{opposite}}{\text{hypotenuse}}$ (vi) $\dfrac{\text{opposite}}{\text{adjacent}}$

In Exercises 2–4, fill in the blanks.

2. Relative to the angle θ, the three sides of a right triangle are the ________ side, the ________ side, and the ________.

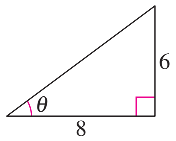
3. Cofunctions of ________ angles are equal.

4. An angle that measures from the horizontal upward to an object is called the angle of ________, whereas an angle that measures from the horizontal downward to an object is called the angle of ________.

SKILLS AND APPLICATIONS

In Exercises 5–8, find the exact values of the six trigonometric functions of the angle θ shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)

5.



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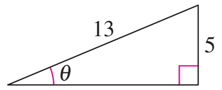

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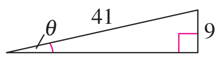

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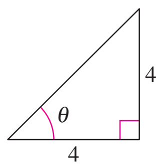
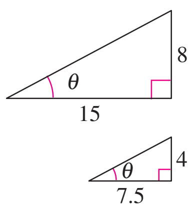
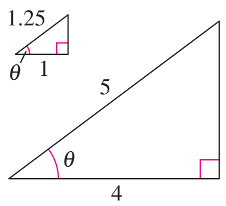


In Exercises 9–12, find the exact values of the six trigonometric functions of the angle θ for each of the two triangles. Explain why the function values are the same.

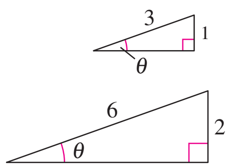
9.



10.



11.



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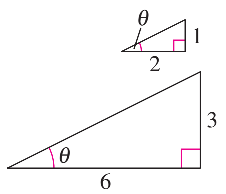


In Exercises 13–20, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of θ.

13. $\tan\theta = \frac{3}{4}$

14. $\cos\theta = \frac{5}{6}$

15. $\sec\theta = \frac{3}{2}$

16. $\tan\theta = \frac{4}{5}$

17. $\sin\theta = \frac{1}{5}$

18. $\sec\theta = \frac{17}{7}$

19. $\cot\theta = 3$

20. $\csc\theta = 9$

In Exercises 21–30, construct an appropriate triangle to complete the table. $(0° \le \theta \le 90°,\ 0 \le \theta \le \pi/2)$

	Function	*θ (deg)*	*θ (rad)*	*Function Value*
21.	sin	30°		
22.	cos	45°		
23.	sec		$\frac{\pi}{4}$	
24.	tan		$\frac{\pi}{3}$	
25.	cot			$\frac{\sqrt{3}}{3}$
26.	csc			$\sqrt{2}$
27.	csc		$\frac{\pi}{6}$	
28.	sin		$\frac{\pi}{4}$	
29.	cot			1
30.	tan			$\frac{\sqrt{3}}{3}$

In Exercises 31–36, use the given function value(s), and trigonometric identities (including the cofunction identities), to find the indicated trigonometric functions.

31. $\sin 60° = \dfrac{\sqrt{3}}{2}, \quad \cos 60° = \dfrac{1}{2}$

(a) $\sin 30°$ (b) $\cos 30°$
(c) $\tan 60°$ (d) $\cot 60°$

32. $\sin 30° = \dfrac{1}{2}, \quad \tan 30° = \dfrac{\sqrt{3}}{3}$

(a) $\csc 30°$ (b) $\cot 60°$
(c) $\cos 30°$ (d) $\cot 30°$

33. $\cos \theta = \frac{1}{3}$

(a) $\sin \theta$ (b) $\tan \theta$
(c) $\sec \theta$ (d) $\csc(90° - \theta)$

34. $\sec \theta = 5$

(a) $\cos \theta$ (b) $\cot \theta$
(c) $\cot(90° - \theta)$ (d) $\sin \theta$

35. $\cot \alpha = 5$

(a) $\tan \alpha$ (b) $\csc \alpha$
(c) $\cot(90° - \alpha)$ (d) $\cos \alpha$

36. $\cos \beta = \dfrac{\sqrt{7}}{4}$

(a) $\sec \beta$ (b) $\sin \beta$
(c) $\cot \beta$ (d) $\sin(90° - \beta)$

In Exercises 37–46, use trigonometric identities to transform the left side of the equation into the right side $(0 < \theta < \pi/2)$.

37. $\tan \theta \cot \theta = 1$

38. $\cos \theta \sec \theta = 1$

39. $\tan \alpha \cos \alpha = \sin \alpha$

40. $\cot \alpha \sin \alpha = \cos \alpha$

41. $(1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta$

42. $(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$

43. $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = 1$

44. $\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta - 1$

45. $\dfrac{\sin \theta}{\cos \theta} + \dfrac{\cos \theta}{\sin \theta} = \csc \theta \sec \theta$

46. $\dfrac{\tan \beta + \cot \beta}{\tan \beta} = \csc^2 \beta$

In Exercises 47–56, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)

47. (a) $\sin 10°$ (b) $\cos 80°$

48. (a) $\tan 23.5°$ (b) $\cot 66.5°$

49. (a) $\sin 16.35°$ (b) $\csc 16.35°$

50. (a) $\cot 79.56°$ (b) $\sec 79.56°$

51. (a) $\cos 4° 50' 15''$ (b) $\sec 4° 50' 15''$

52. (a) $\sec 42° 12'$ (b) $\csc 48° 7'$

53. (a) $\cot 11° 15'$ (b) $\tan 11° 15'$

54. (a) $\sec 56° 8' 10''$ (b) $\cos 56° 8' 10''$

55. (a) $\csc 32° 40' 3''$ (b) $\tan 44° 28' 16''$

56. (a) $\sec\left(\frac{9}{5} \cdot 20 + 32\right)°$ (b) $\cot\left(\frac{9}{5} \cdot 30 + 32\right)°$

In Exercises 57–62, find the values of θ in degrees $(0° < \theta < 90°)$ and radians $(0 < \theta < \pi/2)$ without the aid of a calculator.

57. (a) $\sin \theta = \frac{1}{2}$ (b) $\csc \theta = 2$

58. (a) $\cos \theta = \dfrac{\sqrt{2}}{2}$ (b) $\tan \theta = 1$

59. (a) $\sec \theta = 2$ (b) $\cot \theta = 1$

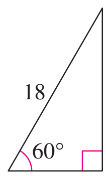
60. (a) $\tan \theta = \sqrt{3}$ (b) $\cos \theta = \frac{1}{2}$

61. (a) $\csc \theta = \dfrac{2\sqrt{3}}{3}$ (b) $\sin \theta = \dfrac{\sqrt{2}}{2}$

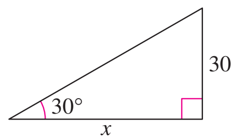
62. (a) $\cot \theta = \dfrac{\sqrt{3}}{3}$ (b) $\sec \theta = \sqrt{2}$

In Exercises 63–66, solve for x, y, or r as indicated.

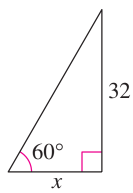
63. Solve for y.



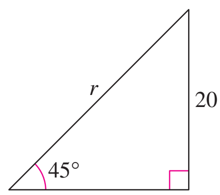
64. Solve for x.



65. Solve for x.



66. Solve for r.



67. EMPIRE STATE BUILDING You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the 86th floor (the observatory) is 82°. If the total height of the building is another 123 meters above the 86th floor, what is the approximate height of the building? One of your friends is on the 86th floor. What is the distance between you and your friend?

68. HEIGHT A six-foot person walks from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person is 132 feet from the tower and 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow.

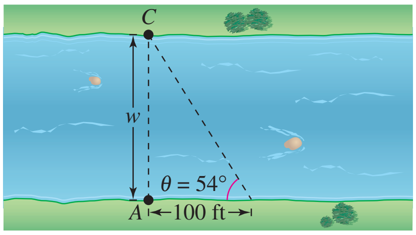
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the tower.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

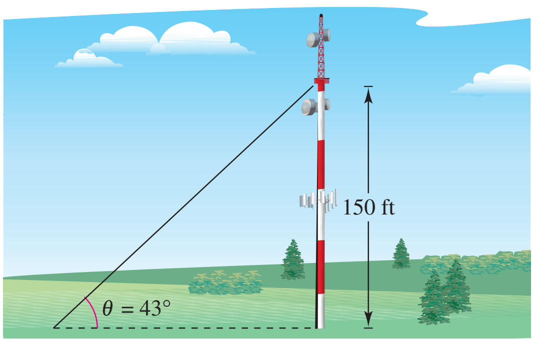
(c) What is the height of the tower?

69. ANGLE OF ELEVATION You are skiing down a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?

70. WIDTH OF A RIVER A biologist wants to know the width w of a river so that instruments for studying the pollutants in the water can be set properly. From point A, the biologist walks downstream 100 feet and sights to point C (see figure). From this sighting, it is determined that $\theta = 54°$. How wide is the river?



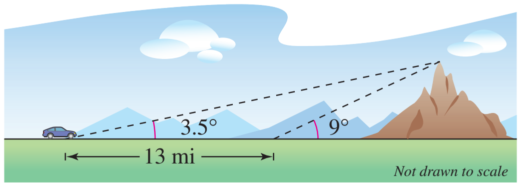
71. LENGTH A guy wire runs from the ground to a cell tower. The wire is attached to the cell tower 150 feet above the ground. The angle formed between the wire and the ground is 43° (see figure).



(a) How long is the guy wire?

(b) How far from the base of the tower is the guy wire anchored to the ground?

72. HEIGHT OF A MOUNTAIN In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.



73. MACHINE SHOP CALCULATIONS A steel plate has the form of one-fourth of a circle with a radius of 60 centimeters. Two two-centimeter holes are to be drilled in the plate positioned as shown in the figure. Find the coordinates of the center of each hole.

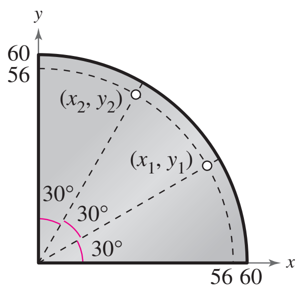


FIGURE FOR 73

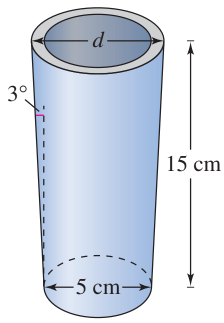
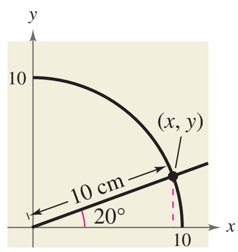


FIGURE FOR 74

74. MACHINE SHOP CALCULATIONS A tapered shaft has a diameter of 5 centimeters at the small end and is 15 centimeters long (see figure). The taper is 3°. Find the diameter d of the large end of the shaft.

75. GEOMETRY Use a compass to sketch a quarter of a circle of radius 10 centimeters. Using a protractor, construct an angle of 20° in standard position (see figure). Drop a perpendicular line from the point of intersection of the terminal side of the angle and the arc of the circle. By actual measurement, calculate the coordinates (x, y) of the point of intersection and use these measurements to approximate the six trigonometric functions of a 20° angle.



76. HEIGHT A 20-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 85° with the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the balloon.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

(d) The breeze becomes stronger and the angle the balloon makes with the ground decreases. How does this affect the triangle you drew in part (a)?

(e) Complete the table, which shows the heights (in meters) of the balloon for decreasing angle measures θ.

Angle, θ	80°	70°	60°	50°
Height				

Angle, θ	40°	30°	20°	10°
Height				

(f) As the angle the balloon makes with the ground approaches 0°, how does this affect the height of the balloon? Draw a right triangle to explain your reasoning.

EXPLORATION

TRUE OR FALSE? In Exercises 77–82, determine whether the statement is true or false. Justify your answer.

77. $\sin 60° \csc 60° = 1$

78. $\sec 30° = \csc 60°$

79. $\sin 45° + \cos 45° = 1$

80. $\cot^2 10° - \csc^2 10° = -1$

81. $\dfrac{\sin 60°}{\sin 30°} = \sin 2°$

82. $\tan[(5°)^2] = \tan^2 5°$

83. THINK ABOUT IT

(a) Complete the table.

θ	0.1	0.2	0.3	0.4	0.5
$\sin\theta$					

(b) Is θ or $\sin\theta$ greater for θ in the interval $(0, 0.5]$?

(c) As θ approaches 0, how do θ and $\sin\theta$ compare? Explain.

84. THINK ABOUT IT

(a) Complete the table.

θ	0°	18°	36°	54°	72°	90°
$\sin\theta$						
$\cos\theta$						

(b) Discuss the behavior of the sine function for θ in the range from 0° to 90°.

(c) Discuss the behavior of the cosine function for θ in the range from 0° to 90°.

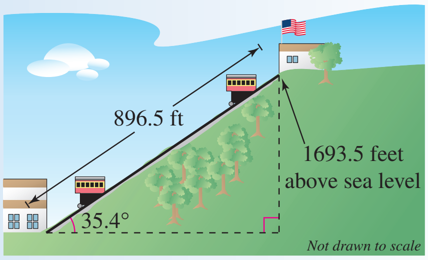
(d) Use the definitions of the sine and cosine functions to explain the results of parts (b) and (c).

85. WRITING In right triangle trigonometry, explain why $\sin 30° = \frac{1}{2}$ regardless of the size of the triangle.

86. GEOMETRY Use the equilateral triangle shown in Figure 4.29 and similar triangles to verify the points in Figure 4.23 (in Section 4.2) that do not lie on the axes.

87. THINK ABOUT IT You are given only the value $\tan\theta$. Is it possible to find the value of $\sec\theta$ without finding the measure of θ? Explain.

88. CAPSTONE The Johnstown Inclined Plane in Pennsylvania is one of the longest and steepest hoists in the world. The railway cars travel a distance of 896.5 feet at an angle of approximately 35.4°, rising to a height of 1693.5 feet above sea level.



(a) Find the vertical rise of the inclined plane.

(b) Find the elevation of the lower end of the inclined plane.

(c) The cars move up the mountain at a rate of 300 feet per minute. Find the rate at which they rise vertically.

4.4 Trigonometric Functions of Any Angle

What you should learn

- Evaluate trigonometric functions of any angle.
- Find reference angles.
- Evaluate trigonometric functions of real numbers.

Why you should learn it

You can use trigonometric functions to model and solve real-life problems. For instance, in Exercise 99 on page 318, you can use trigonometric functions to model the monthly normal temperatures in New York City and Fairbanks, Alaska.

James Urbach/SuperStock

Introduction

In Section 4.3, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. If θ is an *acute* angle, these definitions coincide with those given in the preceding section.

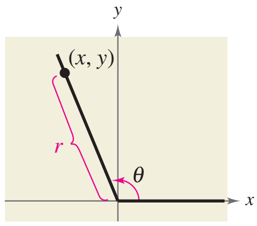
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

$$\sec\theta = \frac{r}{x},\ x \neq 0 \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$



Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of 90° is undefined. Similarly, if $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

Solution

Referring to Figure 4.36, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have the following.

$$\sin\theta = \frac{y}{r} = \frac{4}{5} \qquad \cos\theta = \frac{x}{r} = -\frac{3}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{4}{3}$$

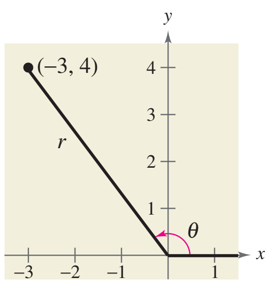


FIGURE 4.36

CHECKPoint Now try Exercise 9.

Algebra Help

The formula $r = \sqrt{x^2 + y^2}$ is a result of the Distance Formula. You can review the Distance Formula in Section 1.1.

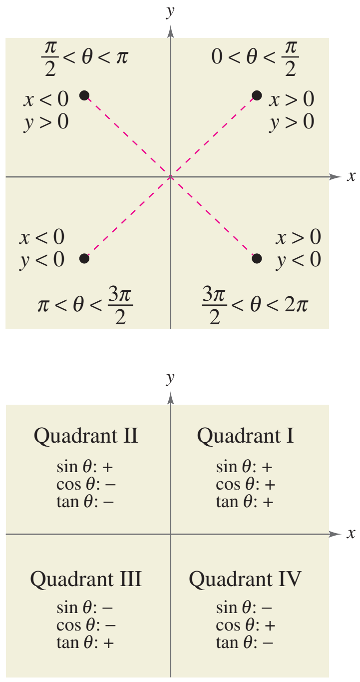


FIGURE 4.37

The *signs* of the trigonometric functions in the four quadrants can be determined from the definitions of the functions. For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.) In a similar manner, you can verify the results shown in Figure 4.37.

Example 2 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{5}{4}$ and $\cos\theta > 0$, find $\sin\theta$ and $\sec\theta$.

Solution

Note that θ lies in Quadrant IV because that is the only quadrant in which the tangent is negative and the cosine is positive. Moreover, using

$$\tan\theta = \frac{y}{x} = -\frac{5}{4}$$

and the fact that y is negative in Quadrant IV, you can let $y = -5$ and $x = 4$. So, $r = \sqrt{16 + 25} = \sqrt{41}$ and you have

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{41}} \approx -0.7809$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4} \approx 1.6008.$$

CHECKPoint Now try Exercise 23.

Example 3 Trigonometric Functions of Quadrant Angles

Evaluate the cosine and tangent functions at the four quadrant angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 4.38. For each of the four points, $r = 1$, and you have the following.

$$\cos 0 = \frac{x}{r} = \frac{1}{1} = 1 \qquad \tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad (x, y) = (1, 0)$$

$$\cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} \Longrightarrow \text{undefined} \qquad (x, y) = (0, 1)$$

$$\cos\pi = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \tan\pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad (x, y) = (-1, 0)$$

$$\cos\frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} \Longrightarrow \text{undefined} \qquad (x, y) = (0, -1)$$

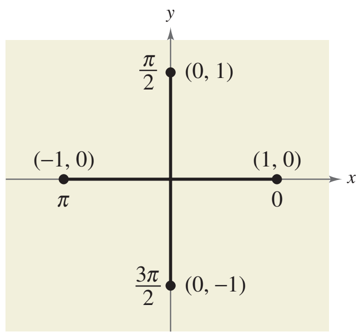


FIGURE 4.38

CHECKPoint Now try Exercise 37.

Reference Angles

The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 4.39 shows the reference angles for θ in Quadrants II, III, and IV.

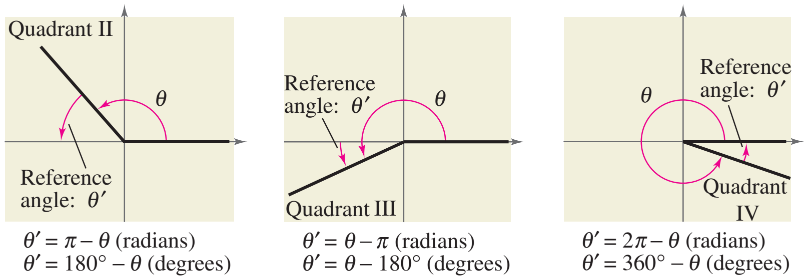


FIGURE 4.39

Example 4 Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution

a. Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

$$\theta' = 360° - 300°$$
$$= 60°. \quad \text{Degrees}$$

Figure 4.40 shows the angle $\theta = 300°$ and its reference angle $\theta' = 60°$.

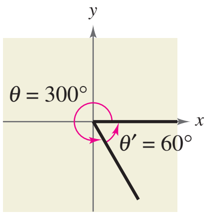


FIGURE 4.40

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$
$$\approx 0.8416. \quad \text{Radians}$$

Figure 4.41 shows the angle $\theta = 2.3$ and its reference angle $\theta' = \pi - 2.3$.

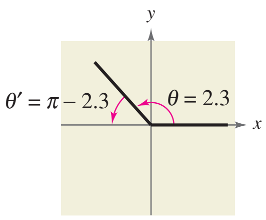


FIGURE 4.41

c. First, determine that $-135°$ is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$
$$= 45°. \quad \text{Degrees}$$

Figure 4.42 shows the angle $\theta = -135°$ and its reference angle $\theta' = 45°$.

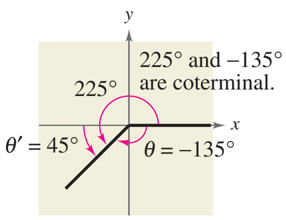
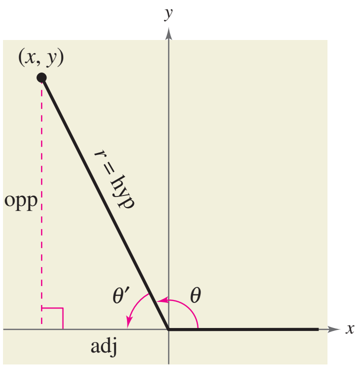


FIGURE 4.42

CHECKPoint Now try Exercise 45.



opp $= |y|$, adj $= |x|$

FIGURE 4.43

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ, as shown in Figure 4.43. By definition, you know that

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

So, it follows that $\sin\theta$ and $\sin\theta'$ are equal, *except possibly in sign*. The same is true for $\tan\theta$ and $\tan\theta'$ *and* for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.

2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

By using reference angles and the special angles discussed in the preceding section, you can greatly extend the scope of *exact* trigonometric values. For instance, knowing the function values of 30° means that you know the function values of all angles for which 30° is a reference angle. For convenience, the table below shows the exact values of the trigonometric functions of special angles and quadrant angles.

Trigonometric Values of Common Angles

θ (degrees)	0°	30°	45°	60°	90°	180°	270°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Study Tip

Learning the table of values at the right is worth the effort because doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

θ	0°	30°	45°	60°	90°
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Reverse the order to get cosine values of the same angles.

Example 5 Using Reference Angles

Evaluate each trigonometric function.

a. $\cos\dfrac{4\pi}{3}$ **b.** $\tan(-210°)$ **c.** $\csc\dfrac{11\pi}{4}$

Solution

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

as shown in Figure 6.41. Moreover, the cosine is negative in Quadrant III, so

$$\cos\frac{4\pi}{3} = (-)\cos\frac{\pi}{3}$$
$$= -\frac{1}{2}.$$

b. Because $-210° + 360° = 150°$, it follows that $-210°$ is coterminal with the second-quadrant angle $150°$. So, the reference angle is $\theta' = 180° - 150° = 30°$, as shown in Figure 4.45. Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210°) = (-)\tan 30°$$
$$= -\frac{\sqrt{3}}{3}.$$

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.46. Because the cosecant is positive in Quadrant II, you have

$$\csc\frac{11\pi}{4} = (+)\csc\frac{\pi}{4}$$
$$= \frac{1}{\sin(\pi/4)}$$
$$= \sqrt{2}.$$

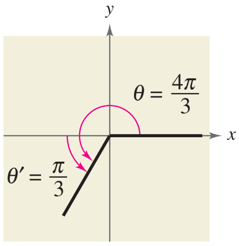


FIGURE 4.44

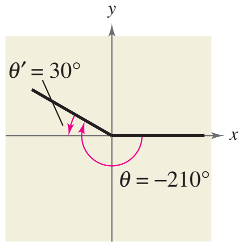


FIGURE 4.45

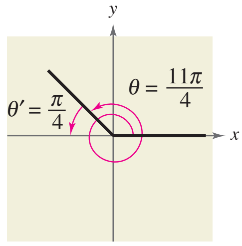


FIGURE 4.46

CHECKPoint Now try Exercise 59.

Example 6 Using Trigonometric Identities

Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find (a) $\cos\theta$ and (b) $\tan\theta$ by using trigonometric identities.

Solution

a. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, you obtain

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1 \qquad \text{Substitute } \tfrac{1}{3} \text{ for } \sin\theta.$$

$$\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}.$$

Because $\cos\theta < 0$ in Quadrant II, you can use the negative root to obtain

$$\cos\theta = -\frac{\sqrt{8}}{\sqrt{9}}$$

$$= -\frac{2\sqrt{2}}{3}.$$

b. Using the trigonometric identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, you obtain

$$\tan\theta = \frac{1/3}{-2\sqrt{2}/3} \qquad \text{Substitute for } \sin\theta \text{ and } \cos\theta.$$

$$= -\frac{1}{2\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{4}.$$

CHECKPoint Now try Exercise 69.

You can use a calculator to evaluate trigonometric functions, as shown in the next example.

Example 7 Using a Calculator

Use a calculator to evaluate each trigonometric function.

a. $\cot 410^\circ$ **b.** $\sin(-7)$ **c.** $\sec\dfrac{\pi}{9}$

Solution

Function	*Mode*	*Calculator Keystrokes*	*Display*
a. $\cot 410^\circ$	Degree	(TAN (410)) x⁻¹ ENTER	0.8390996
b. $\sin(-7)$	Radian	SIN ((−) 7) ENTER	−0.6569866
c. $\sec\dfrac{\pi}{9}$	Radian	(COS (π ÷ 9)) x⁻¹ ENTER	1.0641778

CHECKPoint Now try Exercise 79.

4.4 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

In Exercises 1–6, let θ be an angle in standard position, with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

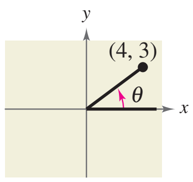
1. $\sin\theta =$ ________
2. $\dfrac{r}{y} =$ ________
3. $\tan\theta =$ ________
4. $\sec\theta =$ ________
5. $\dfrac{x}{r} =$ ________
6. $\dfrac{x}{y} =$ ________

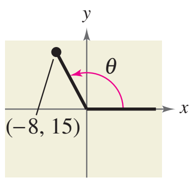
7. Because $r = \sqrt{x^2 + y^2}$ cannot be ________, the sine and cosine functions are ________ for any real value of θ.

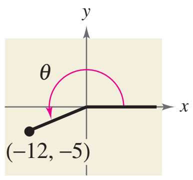
8. The acute positive angle that is formed by the terminal side of the angle θ and the horizontal axis is called the ________ angle of θ and is denoted by θ'.

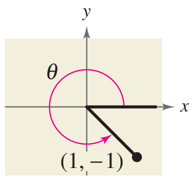
SKILLS AND APPLICATIONS

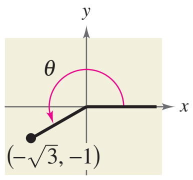
In Exercises 9–12, determine the exact values of the six trigonometric functions of the angle θ.

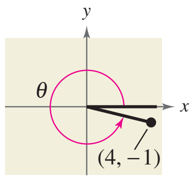
9. (a)


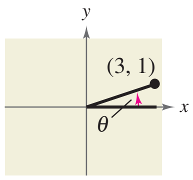
(b)


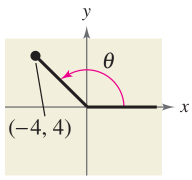
10. (a)


(b)


11. (a)


(b)


12. (a)


(b)


In Exercises 13–18, the point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

13. $(5, 12)$
14. $(8, 15)$
15. $(-5, -2)$
16. $(-4, 10)$
17. $(-5.4, 7.2)$
18. $\left(3\frac{1}{2}, -7\frac{3}{4}\right)$

In Exercises 19–22, state the quadrant in which θ lies.

19. $\sin\theta > 0$ and $\cos\theta > 0$
20. $\sin\theta < 0$ and $\cos\theta < 0$
21. $\sin\theta > 0$ and $\cos\theta < 0$
22. $\sec\theta > 0$ and $\cot\theta < 0$

In Exercises 23–32, find the values of the six trigonometric functions of θ with the given constraint.

	Function Value	*Constraint*
23.	$\tan\theta = -\frac{15}{8}$	$\sin\theta > 0$
24.	$\cos\theta = \frac{8}{17}$	$\tan\theta < 0$
25.	$\sin\theta = \frac{3}{5}$	θ lies in Quadrant II.
26.	$\cos\theta = -\frac{4}{5}$	θ lies in Quadrant III.
27.	$\cot\theta = -3$	$\cos\theta > 0$
28.	$\csc\theta = 4$	$\cot\theta < 0$
29.	$\sec\theta = -2$	$\sin\theta < 0$
30.	$\sin\theta = 0$	$\sec\theta = -1$
31.	$\cot\theta$ is undefined.	$\pi/2 \le \theta \le 3\pi/2$
32.	$\tan\theta$ is undefined.	$\pi \le \theta \le 2\pi$

In Exercises 33–36, the terminal side of θ lies on the given line in the specified quadrant. Find the values of the six trigonometric functions of θ by finding a point on the line.

	Line	*Quadrant*
33.	$y = -x$	II
34.	$y = \frac{1}{3}x$	III
35.	$2x - y = 0$	III
36.	$4x + 3y = 0$	IV

In Exercises 37–44, evaluate the trigonometric function of the quadrant angle.

37. $\sin \pi$

38. $\csc \frac{3\pi}{2}$

39. $\sec \frac{3\pi}{2}$

40. $\sec \pi$

41. $\sin \frac{\pi}{2}$

42. $\cot \pi$

43. $\csc \pi$

44. $\cot \frac{\pi}{2}$

In Exercises 45–52, find the reference angle θ', and sketch θ and θ' in standard position.

45. $\theta = 160°$

46. $\theta = 309°$

47. $\theta = -125°$

48. $\theta = -215°$

49. $\theta = \frac{2\pi}{3}$

50. $\theta = \frac{7\pi}{6}$

51. $\theta = 4.8$

52. $\theta = 11.6$

In Exercises 53–68, evaluate the sine, cosine, and tangent of the angle without using a calculator.

53. $225°$

54. $300°$

55. $750°$

56. $-405°$

57. $-150°$

58. $-840°$

59. $\frac{2\pi}{3}$

60. $\frac{3\pi}{4}$

61. $\frac{5\pi}{4}$

62. $\frac{7\pi}{6}$

63. $-\frac{\pi}{6}$

64. $-\frac{\pi}{2}$

65. $\frac{9\pi}{4}$

66. $\frac{10\pi}{3}$

67. $-\frac{3\pi}{2}$

68. $-\frac{23\pi}{4}$

In Exercises 69–74, find the indicated trigonometric value in the specified quadrant.

	Function	*Quadrant*	*Trigonometric Value*
69.	$\sin \theta = -\frac{3}{5}$	IV	$\cos \theta$
70.	$\cot \theta = -3$	II	$\sin \theta$
71.	$\tan \theta = \frac{3}{2}$	III	$\sec \theta$
72.	$\csc \theta = -2$	IV	$\cot \theta$
73.	$\cos \theta = \frac{5}{8}$	I	$\sec \theta$
74.	$\sec \theta = -\frac{9}{4}$	III	$\tan \theta$

In Exercises 75–90, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

75. $\sin 10°$

76. $\sec 225°$

77. $\cos(-110°)$

78. $\csc(-330°)$

79. $\tan 304°$

80. $\cot 178°$

81. $\sec 72°$

82. $\tan(-188°)$

83. $\tan 4.5$

84. $\cot 1.35$

85. $\tan \frac{\pi}{9}$

86. $\tan\left(-\frac{\pi}{9}\right)$

87. $\sin(-0.65)$

88. $\sec 0.29$

89. $\cot\left(-\frac{11\pi}{8}\right)$

90. $\csc\left(-\frac{15\pi}{14}\right)$

In Exercises 91–96, find two solutions of the equation. Give your answers in degrees $(0° \le \theta < 360°)$ and in radians $(0 \le \theta < 2\pi)$. Do not use a calculator.

91. (a) $\sin \theta = \frac{1}{2}$ (b) $\sin \theta = -\frac{1}{2}$

92. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

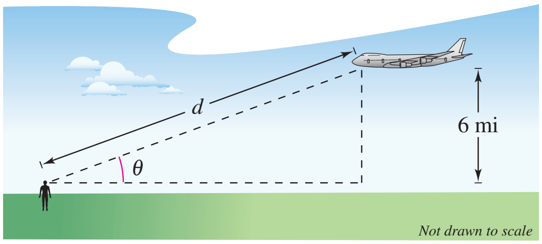
93. (a) $\csc \theta = \frac{2\sqrt{3}}{3}$ (b) $\cot \theta = -1$

94. (a) $\sec \theta = 2$ (b) $\sec \theta = -2$

95. (a) $\tan \theta = 1$ (b) $\cot \theta = -\sqrt{3}$

96. (a) $\sin \theta = \frac{\sqrt{3}}{2}$ (b) $\sin \theta = -\frac{\sqrt{3}}{2}$

97. DISTANCE An airplane, flying at an altitude of 6 miles, is on a flight path that passes directly over an observer (see figure). If θ is the angle of elevation from the observer to the plane, find the distance d from the observer to the plane when (a) $\theta = 30°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.



98. HARMONIC MOTION The displacement from equilibrium of an oscillating weight suspended by a spring is given by $y(t) = 2 \cos 6t$, where y is the displacement (in centimeters) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.