

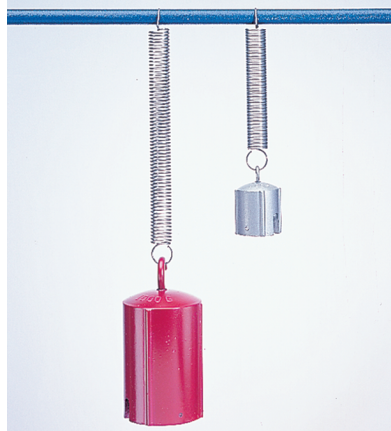
4.2 Trigonometric Functions: The Unit Circle

What you should learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use the domain and period to evaluate sine and cosine functions.
- Use a calculator to evaluate trigonometric functions.

Why you should learn it

Trigonometric functions are used to model the movement of an oscillating weight. For instance, in Exercise 57 on page 300, the displacement from equilibrium of an oscillating weight suspended by a spring is modeled as a function of time.



Richard Megna/Fundamental Photographs

Try demonstrating the wrapping function by using a spool and thread. Demonstrate the real number t as the length of the thread. Then wrap the thread around the spool to show the correspondence between t and the point (x, y) on the circle. For example, if $t = \pi/2$, the point on the unit circle to which it corresponds is $(0, 1)$.

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$$x^2 + y^2 = 1 \quad \text{Unit circle}$$

as shown in Figure 4.20.

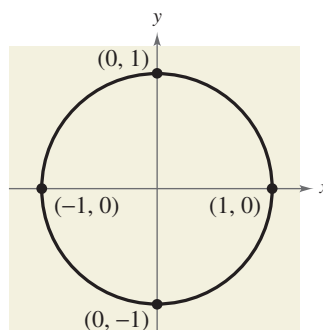


FIGURE 4.20

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown in Figure 4.21.

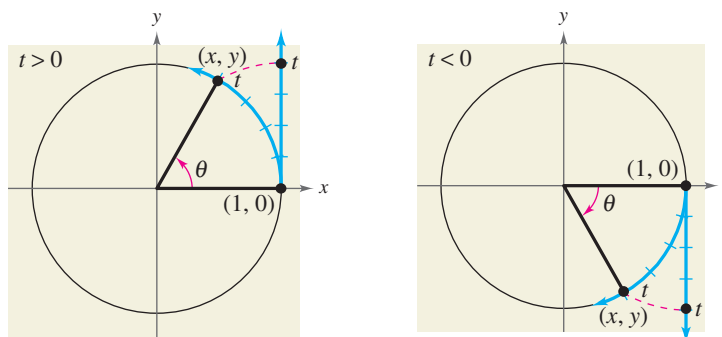


FIGURE 4.21

As the real number line is wrapped around the unit circle, each real number t corresponds to a point (x, y) on the circle. For example, the real number 0 corresponds to the point $(1, 0)$. Moreover, because the unit circle has a circumference of 2π , the real number 2π also corresponds to the point $(1, 0)$.

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t . With this interpretation of t , the arc length formula $s = r\theta$ (with $r = 1$) indicates that the real number t is the length of the arc intercepted by the angle θ , given in radians.

The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t . You can use these coordinates to define the six trigonometric functions of t .

sine cosecant cosine secant tangent cotangent

These six functions are normally abbreviated \sin , \csc , \cos , \sec , \tan , and \cot , respectively.

STUDY TIP

Note in the definition at the right that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t .

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$. For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*. Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, $\cot 0$ and $\csc 0$ are *undefined*.

In Figure 4.22, the unit circle has been divided into eight equal arcs, corresponding to t -values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

Similarly, in Figure 4.23, the unit circle has been divided into 12 equal arcs, corresponding to t -values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

To verify the points on the unit circle in Figure 4.22, note that $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ also lies on the line $y = x$. So, substituting x for y in the equation of the unit circle produces the following.

$$x^2 + x^2 = 1 \quad \Rightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm \frac{\sqrt{2}}{2}$$

Because the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$ and because $y = x$, you also have $y = \frac{\sqrt{2}}{2}$. You can use similar reasoning to verify the rest of the points in Figure 4.22 and the points in Figure 4.23.

Using the (x, y) coordinates in Figures 4.22 and 4.23, you can easily evaluate the trigonometric functions for common t -values. This procedure is demonstrated in Examples 1 and 2. You should study and learn these exact function values for common t -values because they will help you in later sections to perform calculations quickly and easily.

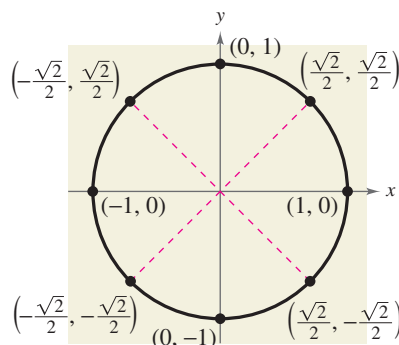


FIGURE 4.22

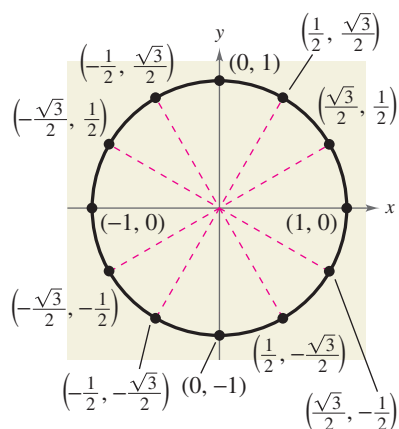


FIGURE 4.23

Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

$$\text{a. } t = \frac{\pi}{6} \quad \text{b. } t = \frac{5\pi}{4} \quad \text{c. } t = 0 \quad \text{d. } t = \pi$$

SolutionFor each t -value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 295.

$$\text{a. } t = \frac{\pi}{6} \text{ corresponds to the point } (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

$$\sin \frac{\pi}{6} = y = \frac{1}{2}$$

$$\csc \frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos \frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\sec \frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan \frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot \frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\text{b. } t = \frac{5\pi}{4} \text{ corresponds to the point } (x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

$$\sin \frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2}$$

$$\csc \frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos \frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$$

$$\sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan \frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\text{c. } t = 0 \text{ corresponds to the point } (x, y) = (1, 0).$$

$$\sin 0 = y = 0$$

$$\csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1$$

$$\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

$$\cot 0 = \frac{x}{y} \text{ is undefined.}$$

$$\text{d. } t = \pi \text{ corresponds to the point } (x, y) = (-1, 0).$$

$$\sin \pi = y = 0$$

$$\csc \pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos \pi = x = -1$$

$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

$$\cot \pi = \frac{x}{y} \text{ is undefined.}$$

Additional ExampleEvaluate the six trigonometric functions at $t = \frac{5\pi}{2}$.**Solution**Moving counterclockwise around the unit circle one and a quarter revolutions, you find that $t = 5\pi/2$ corresponds to the point $(x, y) = (0, 1)$.

$$\sin \frac{5\pi}{2} = y = 1 \quad \csc \frac{5\pi}{2} = \frac{1}{y} = 1$$

$$\cos \frac{5\pi}{2} = x = 0$$

$$\sec \frac{5\pi}{2} = \frac{1}{x} \text{ is undefined.}$$

$$\tan \frac{5\pi}{2} = \frac{y}{x} \text{ is undefined.}$$

$$\cot \frac{5\pi}{2} = \frac{x}{y} = 0$$

CHECKPOINT Now try Exercise 23.

Exploration

With your graphing utility in *radian* and *parametric* modes, enter the equations

$X1T = \cos T$ and $Y1T = \sin T$
and use the following settings.

$T_{\min} = 0$, $T_{\max} = 6.3$,
 $T_{\text{step}} = 0.1$
 $X_{\min} = -1.5$, $X_{\max} = 1.5$,
 $X_{\text{scl}} = 1$
 $Y_{\min} = -1$, $Y_{\max} = 1$,
 $Y_{\text{scl}} = 1$

- Graph the entered equations and describe the graph.
- Use the *trace* feature to move the cursor around the graph. What do the t -values represent? What do the x - and y -values represent?
- What are the least and greatest values of x and y ?

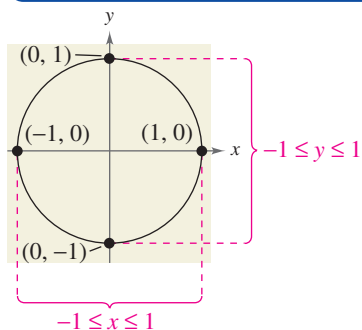


FIGURE 4.24

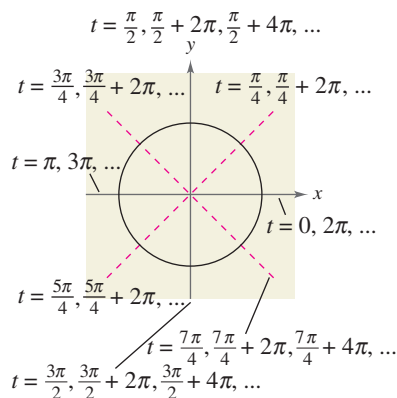


FIGURE 4.25

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\frac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = (1/2, -\sqrt{3}/2)$.

$$\begin{aligned}\sin\left(-\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2} & \csc\left(-\frac{\pi}{3}\right) &= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \\ \cos\left(-\frac{\pi}{3}\right) &= \frac{1}{2} & \sec\left(-\frac{\pi}{3}\right) &= 2 \\ \tan\left(-\frac{\pi}{3}\right) &= \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} & \cot\left(-\frac{\pi}{3}\right) &= \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\end{aligned}$$

CHECKPOINT Now try Exercise 25.

Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.24. Because $r = 1$, it follows that $\sin t = y$ and $\cos t = x$. Moreover, because (x, y) is on the unit circle, you know that $-1 \leq y \leq 1$ and $-1 \leq x \leq 1$. So, the values of sine and cosine also range between -1 and 1 .

$$\begin{aligned}-1 \leq y \leq 1 & \quad \text{and} \quad -1 \leq x \leq 1 \\ -1 \leq \sin t \leq 1 & \quad \text{and} \quad -1 \leq \cos t \leq 1\end{aligned}$$

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.25. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t . Functions that behave in such a repetitive (or cyclic) manner are called **periodic**.

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f . The smallest number c for which f is periodic is called the **period** of f .

Students may have difficulty evaluating cosecant, secant, and/or cotangent functions using a calculator. Try having them rewrite the expression in terms of sine, cosine, or tangent before evaluating.

Example: $\csc 1.3 = \frac{1}{\sin 1.3} \approx 1.0378$

STUDY TIP

From the definition of periodic function, it follows that the sine and cosine functions are periodic and have a period of 2π . The other four trigonometric functions are also periodic, and will be discussed further in Section 4.6.

Technology

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, if you want to evaluate $\sin \theta$ for $\theta = \pi/6$, you should enter

$\boxed{\text{SIN}} \boxed{(\pi)} \boxed{\div} \boxed{6} \boxed{)} \boxed{\text{ENTER}}$.

These keystrokes yield the correct value of 0.5. Note that some calculators automatically place a left parenthesis after trigonometric functions. Check the user's guide for your calculator for specific keystrokes on how to evaluate trigonometric functions.

Recall from Section 1.5 that a function f is *even* if $f(-t) = f(t)$, and is *odd* if $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \quad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \quad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \quad \cot(-t) = -\cot t$$

Example 3 Using the Period to Evaluate the Sine and Cosine

a. Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have $\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$.

b. Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.

 **CHECKPOINT** Now try Exercise 31.

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (*degree* or *radian*).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the $\boxed{x^{-1}}$ key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc \frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

$\boxed{(\text{SIN}} \boxed{(\pi)} \boxed{\div} \boxed{8} \boxed{)} \boxed{)} \boxed{x^{-1}} \boxed{\text{ENTER}}$ Display 2.6131259

Example 4 Using a Calculator

Function	Mode	Calculator Keystrokes	Display
a. $\sin \frac{2\pi}{3}$	Radian	$\boxed{\text{SIN}} \boxed{(\text{2}} \boxed{\pi} \boxed{\div} \boxed{3} \boxed{)} \boxed{\text{ENTER}}$	0.8660254
b. $\cot 1.5$	Radian	$\boxed{(\text{TAN}} \boxed{(\text{1.5}} \boxed{)} \boxed{)} \boxed{x^{-1}} \boxed{\text{ENTER}}$	0.0709148

 **CHECKPOINT** Now try Exercise 45.

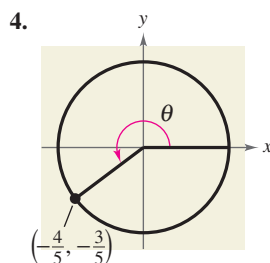
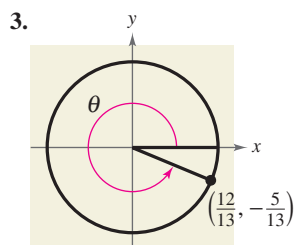
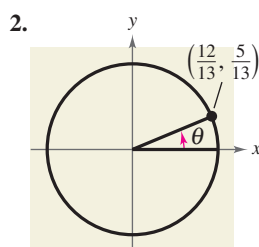
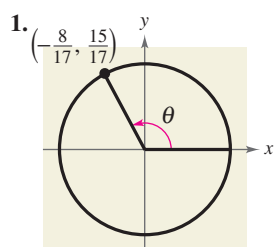
4.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

- Each real number t corresponds to a point (x, y) on the _____.
- A function f is _____ if there exists a positive real number c such that $f(t + c) = f(t)$ for all t in the domain of f .
- The smallest number c for which a function f is periodic is called the _____ of f .
- A function f is _____ if $f(-t) = -f(t)$ and _____ if $f(-t) = f(t)$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ .



In Exercises 5–12, find the point (x, y) on the unit circle that corresponds to the real number t .

- $t = \frac{\pi}{4}$
- $t = \frac{\pi}{3}$
- $t = \frac{7\pi}{6}$
- $t = \frac{5\pi}{4}$
- $t = \frac{4\pi}{3}$
- $t = \frac{5\pi}{3}$
- $t = \frac{3\pi}{2}$
- $t = \pi$

In Exercises 13–22, evaluate (if possible) the sine, cosine, and tangent of the real number.

- $t = \frac{\pi}{4}$
- $t = \frac{\pi}{3}$
- $t = -\frac{\pi}{6}$
- $t = -\frac{\pi}{4}$
- $t = -\frac{7\pi}{4}$
- $t = -\frac{4\pi}{3}$

- $t = \frac{11\pi}{6}$
- $t = \frac{5\pi}{3}$
- $t = -\frac{3\pi}{2}$
- $t = -2\pi$

In Exercises 23–28, evaluate (if possible) the six trigonometric functions of the real number.

- $t = \frac{3\pi}{4}$
- $t = \frac{5\pi}{6}$
- $t = -\frac{\pi}{2}$
- $t = \frac{3\pi}{2}$
- $t = \frac{4\pi}{3}$
- $t = \frac{7\pi}{4}$


In Exercises 29–36, evaluate the trigonometric function using its period as an aid.

- $\sin 5\pi$
- $\cos 5\pi$
- $\cos \frac{8\pi}{3}$
- $\sin \frac{9\pi}{4}$
- $\cos\left(-\frac{15\pi}{2}\right)$
- $\sin \frac{19\pi}{6}$
- $\sin\left(-\frac{9\pi}{4}\right)$
- $\cos\left(-\frac{8\pi}{3}\right)$

In Exercises 37–42, use the value of the trigonometric function to evaluate the indicated functions.

- $\sin t = \frac{1}{3}$
 - $\sin(-t)$
 - $\csc(-t)$
- $\sin(-t) = \frac{3}{8}$
 - $\sin t$
 - $\csc t$
- $\cos(-t) = -\frac{1}{5}$
 - $\cos t$
 - $\sec(-t)$
- $\cos t = -\frac{3}{4}$
 - $\cos(-t)$
 - $\sec(-t)$
- $\sin t = \frac{4}{5}$
 - $\sin(\pi - t)$
 - $\sin(t + \pi)$
- $\cos t = \frac{4}{5}$
 - $\cos(\pi - t)$
 - $\cos(t + \pi)$

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 In Exercises 43–52, use a calculator to evaluate the trigonometric function. Round your answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

43. $\sin \frac{\pi}{4}$ 44. $\tan \frac{\pi}{3}$
 45. $\csc 1.3$ 46. $\cot 1$
 47. $\cos(-1.7)$ 48. $\cos(-2.5)$
 49. $\csc 0.8$ 50. $\sec 1.8$
 51. $\sec 22.8$ 52. $\sin(-0.9)$

Estimation In Exercises 53 and 54, use the figure and a straightedge to approximate the value of each trigonometric function. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

53. (a) $\sin 5$ (b) $\cos 2$ 54. (a) $\sin 0.75$ (b) $\cos 2.5$

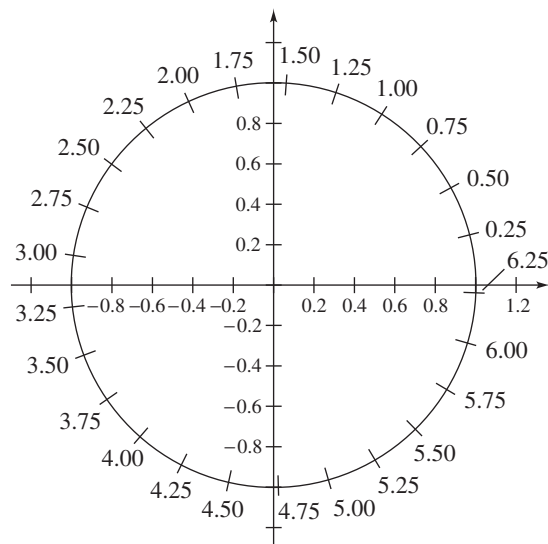


FIGURE FOR 53–56

Estimation In Exercises 55 and 56, use the figure and a straightedge to approximate the solution of each equation, where $0 \leq t < 2\pi$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

55. (a) $\sin t = 0.25$ (b) $\cos t = -0.25$
 56. (a) $\sin t = -0.75$ (b) $\cos t = 0.75$


Model It

57. **Harmonic Motion** The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by $y(t) = \frac{1}{4}e^{-t} \cos 6t$ where y is the displacement (in feet) and t is the time (in seconds).

Model It (continued)

- (a) Complete the table.

t	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y					

-  (b) Use the *table* feature of a graphing utility to approximate the time when the weight reaches equilibrium.
 (c) What appears to happen to the displacement as t increases?

58. **Harmonic Motion** The displacement from equilibrium of an oscillating weight suspended by a spring is given by $y(t) = \frac{1}{4} \cos 6t$, where y is the displacement (in feet) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

Synthesis

True or False? In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.

59. Because $\sin(-t) = -\sin t$, it can be said that the sine of a negative angle is a negative number.
 60. $\tan a = \tan(a - 6\pi)$
 61. **Exploration** Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = \pi - t_1$, respectively.
 (a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2) .
 (b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(\pi - t_1)$.
 (c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(\pi - t_1)$.
 62. Use the unit circle to verify that the cosine and secant functions are even and that the sine, cosecant, tangent, and cotangent functions are odd.

Skills Review

In Exercises 63–66, find the inverse function f^{-1} of the one-to-one function f .

63. $f(x) = \frac{1}{2}(3x - 2)$ 64. $f(x) = \frac{1}{4}x^3 + 1$
 65. $f(x) = \sqrt{x^2 - 4}$, $x \geq 2$ 66. $f(x) = \frac{x + 2}{x - 4}$

In Exercises 67–70, sketch the graph of the rational function by hand. Show all asymptotes.

67. $f(x) = \frac{2x}{x - 3}$ 68. $f(x) = \frac{5x}{x^2 + x - 6}$
 69. $f(x) = \frac{x^2 + 3x - 10}{2x^2 - 8}$ 70. $f(x) = \frac{x^3 - 6x^2 + x - 1}{2x^2 - 5x - 8}$