

4.4 Nonhomogeneous equations:  
method of undetermined coefficients  
a lesson for MATH F302 Differential Equations

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February 20, 2019

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

## general solutions to nonhomogeneous DEs

- for an  $n$ th-order, linear, and nonhomogeneous DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y \stackrel{*}{=} g(x)$$

- ... the general solution is a sum of the general solution of the associated *homogeneous* equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

plus one *particular solution*  $y_p(x)$  of \*

- the general solution of the homogeneous equation is called the *complementary function*  $y_c(x)$
- main structure:  $y(x) = y_c(x) + y_p(x)$  solves \*

## example 1

- *example 1*: find the general solution:

$$y'' + 4y = e^{-x}$$

## example 1, cont.

- verify that  $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^{-x}$  solves

$$y'' + 4y = e^{-x}$$

## example 1, cont.<sup>2</sup>

- solve the initial value problem:

$$y'' + 4y = e^{-x}, \quad y(0) = -1, \quad y'(0) = 1$$

## example 2

- the idea of “undetermined coefficients” is to try  $y_p(x)$  which has the same general form as the nonhomogeneity  $g(x)$
- *example 2* ( $\approx$  #5 in 4.4): find the general solution:

$$y'' + 4y' + 4y = x^2 - 2x$$

## example 3

- *example 3 (#8 in 4.4)*: find the general solution:

$$4y'' - 4y' - 3y = \cos 2x$$

## trial forms for the particular solution

- we need some guidance on how to guess!
- in words:

*For  $y_p$  try a linear combination of all linearly-independent functions generated by repeated differentiation of  $g(x)$ .*

- as a table:

**TABLE 4.4.1** Trial Particular Solutions

$g(x)$	Form of $y_p$
1. 1 (any constant)	$A$
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. $e^{5x}$	$Ae^{5x}$
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$



*example 4 shows we still have issues!*

- *example 4* ( $\approx$  #13 in 4.4): find the general solution:

$$y'' + 9y = 2 \cos 3x$$

## guidance on the hard case

- the problematic case happens when our guess for  $y_p$  “accidentally” contain terms which also appear in  $y_c$ 
  - because the left side then annihilates those terms
  - ... which blocks us from determining  $y_p$
- guidance in words:  
*If the trial form of  $y_p$  contains terms that duplicate terms in  $y_c$  then multiply the trial form by  $x^n$  where  $n$  is the smallest power that eliminates the duplication.*

## example 5

- *example 5 (#29 in 4.4)*: solve the initial value problem:

$$5y'' + y' = -6x, \quad y(0) = 0, \quad y'(0) = -10$$

## example 6

- *example 6 (#32 in 4.4):* solve the initial value problem:

$$y'' - y = \cosh x, \quad y(0) = 2, \quad y'(0) = 12$$

## example 6, cont.

- the last slide had an impressive calculation, so we should . . .
- verify that  $y(x) = 7e^x - 5e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}xe^{-x}$  solves  $y'' - y = \cosh x$ ,  $y(0) = 2$ ,  $y'(0) = 12$

clearly

- clearly **you** need to practice examples, not just me

## what we are skipping next

we are **skipping** the following sections:

- §4.5 *Undetermined Coefficients—Annihilator Approach*  
a more abstract view of undetermined coefficients . . . but no more powerful than our superposition method
- §4.6 *Variation of parameters*  
a general approach to nonhomogeneous linear equations but one may not be able to compute the integrals you get
  - it is somewhat like reduction of order in §4.2
- §4.7 *Cauchy-Euler equations*  
another class of homogeneous differential equations which can be solved via an auxiliary equation
- §4.8 *Green's Functions*  
mostly relevant to boundary value problems (not in Math 302)
- §4.9 *Solving Systems of Linear DEs by Elimination*  
a way of solving systems . . . which are important . . . but done generally and powerfully in chapter 8

## expectations

- just watching this video is *not* enough!
  - see “found online” videos at [bueler.github.io/math302/week7.html](https://bueler.github.io/math302/week7.html)
  - *read* section 4.4 in the textbook
  - *do* the WebAssign exercises for section 4.4