4.5 Integration by Substitution

The Fundamental Theorem of Calculus tells us that in order to evaluate an integral, we need to find an antiderivative of the function we are integrating (the integrand). However, the list of antiderivatives we have is rather short, and does not cover all the possible functions we will have to integrate. For example, $\int xe^{x^2}dx$ is not in our list. Neither is $\int 2x\sqrt{1+x^2}dx$. What do we do then? One method, the one we will study in this section, involves changing the integral so that it looks like one we can do, by doing a change of variable, also called a substitution. Substitution for integrals corresponds to the chain rule for derivatives. We look at some simple examples to illustrate this.

Before we see how to do this, we need to review another concept, the differential.

4.5.1 The Differential

You will recall from differential calculus that the notation dx meant a small change in the variable x. It has a name, it is called the differential (of the variable x). Now, if $y = f(x)$ and f is a differentiable function, we may also be interested in finding the differential of y, denoted dy.

Definition 276 *The differential dy is defined by*

$$dy = f'(x)\,dx$$

Example 277 *Find dy if $y = x^2$*
By definition

$$\begin{aligned} dy &= \left(x^2\right)' dx \\ &= 2xdx \end{aligned}$$

Example 278 *Find dy if $y = \sin x$*
By definition

$$\begin{aligned} dy &= (\sin x)' dx \\ &= \cos xdx \end{aligned}$$

4.5.2 The Substitution Rule for Indefinite Integrals

Before we start, it is important to understand what you should know so far as well as what you do not know. In the previous sections, we reduced the problem of computing an integral to that of finding an antiderivative. Indeed, once we know an antiderivative of f, we can compute $\int_a^b f(x)\,dx$. If an antiderivative of f is F then $\int_a^b f(x)\,dx = F(b) - F(a)$. In the sections which follow, we will focus more on finding antiderivatives. It is also important to note that the name of the variable in the integral is not relevant. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt =$

$\int_a^b f(u)\,du$. What matters is that we use the same variable in the function as in dx. For example, we know that an antiderivative of $\cos x$ is $\sin x$. Hence, $\int_a^b \cos x dx = \int_a^b \cos u du = \int_a^b \cos t dt = \cos b - \cos a$. However, often, instead of having to compute $\int_a^b \cos x dx$ we have to compute an integral which involves $\cos u$ where u is a function of x. For example, the integral may involve $\cos x^2$, $\cos e^x$, ... This section will address such cases.

Substitution applies to integrals of the form $\int f(g(x))\,g'(x)\,dx$ where it is assumed we know an antiderivative of f. If we let $u = g(x)$, then $du = g'(x)\,dx$. Therefore, we have

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \tag{4.2}$$

This is the substitution rule formula. Note that the integral on the left is expressed in terms of the variable x. The integral on the right is in terms of u. The key when doing substitution is, of course, to know which substitution to apply. At the beginning, it is hard. With practice, it becomes easier. Also, looking at equation 4.2 and trying to understand the pattern will make things easier. In that formula, it is assumed that we can integrate the function f. Looking at the integral on the left, one sees the function f. But the integral also has extra expressions. Inside of f, there is an expression in terms of x. Outside of f, is the derivative of this expression. When this is the case, the expression will be the substitution. For example, given $\int 2x\sin\left(x^2\right)dx$, one would use $u = x^2$ as the substitution. Given $\int \cos x\sqrt{\sin x}dx$, one would use $u = \sin x$ as the substitution. Let us look at some examples.

Example 279 *Find* $\int 2x\sin\left(x^2\right)dx$
If $u = x^2$, *then* $du = 2xdx$, *therefore*

$$\begin{aligned}\int 2x\sin\left(x^2\right)dx &= \int \sin\left(x^2\right)2xdx\\ &= \int \sin udu\end{aligned}$$

The last integral is a known formula

$$\int \sin udu = -\cos u + C$$

The original problem was given in terms of the variable x, *you must give your answer in terms of* x. *Therefore,*

$$\int 2x\sin\left(x^2\right)dx = -\cos\left(x^2\right) + C$$

Example 280 *Find* $\int xe^{x^2}dx$

If $u = x^2$, then $du = 2xdx$, therefore

$$\begin{aligned}
\int xe^{x^2} dx &= \int e^{x^2} xdx \\
&= \int e^u \frac{du}{2} \\
&= \frac{1}{2} \int e^u du \\
&= \frac{1}{2} e^u + C \\
&= \frac{1}{2} e^{x^2} + C
\end{aligned}$$

Example 281 *Find* $\int x^3 \cos\left(x^4 + 1\right) dx$
If $u = x^4 + 1$, then $du = 4x^3 dx$, therefore

$$\begin{aligned}
\int x^3 \cos\left(x^4 + 1\right) dx &= \int \cos\left(x^4 + 1\right) x^3 dx \\
&= \int \cos u \frac{du}{4} \\
&= \frac{1}{4} \int \cos u du \\
&= \frac{1}{4} \sin u + C \\
&= \frac{1}{4} \sin\left(x^4 + 1\right) + C
\end{aligned}$$

Example 282 *Find* $\int \tan x dx$
If we think of $\tan x$ as $\dfrac{\sin x}{\cos x}$ and let $u = \cos x$, then $du = -\sin x dx$, therefore

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= \int \frac{1}{u} (-1) du \\
&= -\int \frac{1}{u} du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C
\end{aligned}$$

This is not the formula usually remembered. Since $\cos x = \dfrac{1}{\sec x}$, and, one of *the properties of logarithmic functions says that* $\ln \frac{a}{b} = \ln a - \ln b$*, we have*

$$\begin{aligned}\int \tan x dx &= -\ln|\cos x| + C \\ &= -\ln \frac{1}{|\sec x|} + C \\ &= -\ln 1 - (-\ln|\sec x|) + C \\ &= \ln|\sec x| + C\end{aligned}$$

Example 283 *Find* $\int 2x\sqrt{x^2+1}dx$
If $u = x^2 + 1$*, then* $du = 2xdx$*, therefore*

$$\begin{aligned}\int 2x\sqrt{x^2+1}dx &= \int \sqrt{u}du \\ &= \int u^{\frac{1}{2}}du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C\end{aligned}$$

4.5.3 The Substitution Rule for Definite Integrals

With definite integrals, we have to find an antiderivative, then plug in the limits of integration. We can do this one of two ways:

1. Use substitution to find an antiderivative, express the answer in terms of the original variable then use the given limits of integration.

2. Change the limits of integration when doing the substitution. This way, you won't have to express the antiderivative in terms of the original variable. More precisely,

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

We illustrate these two methods with examples.

Example 284 *Find* $\int_1^e \frac{\ln x}{x}dx$ *using the first method.*
First, we find an antiderivative of the integrand, and express it in term of x*. If*

$u = \ln x$, then $du = \dfrac{1}{x}dx$. Therefore

$$\begin{aligned}\int \frac{\ln x}{x}dx &= \int udu \\ &= \frac{u^2}{2} \\ &= \frac{(\ln x)^2}{2}\end{aligned}$$

It follows that

$$\begin{aligned}\int_1^e \frac{\ln x}{x}dx &= \left.\frac{(\ln x)^2}{2}\right|_1^e \\ &= \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} \\ &= \frac{1}{2}\end{aligned}$$

Example 285 *Same problem using the second method.*
The substitution will be the same, but we won't have to express the antiderivative in terms of x. Instead, we will find what the limits of integration are in terms of u. Since $u = \ln x$, when $x = 1$, $u = \ln 1 = 0$. When $x = e$, $u = \ln e = 1$. Therefore,

$$\begin{aligned}\int_1^e \frac{\ln x}{x}dx &= \int_0^1 udu \\ &= \left.\frac{u^2}{2}\right|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

Remark 286 *A special case of substitution is renaming a variable in an integral. You will recall that $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$. In this case, we just performed the trivial substitution $u = x$, in other words, we simply renamed the variable. This can always be done, however, it does not accomplish anything. Sometimes we do it for display purposes, as we will see in the next theorem.*

4.5.4 Integrating Even and Odd Functions

Definition 287 *A function f is even if $f(-x) = f(x)$. It is odd if $f(-x) = -f(x)$*

Example 288 *$f(x) = x^2$ is even. In fact $f(x) = x^n$ is even if n is even.*

Example 289 *$f(x) = x^3$ is odd. In fact $f(x) = x^n$ is odd if n is odd.*

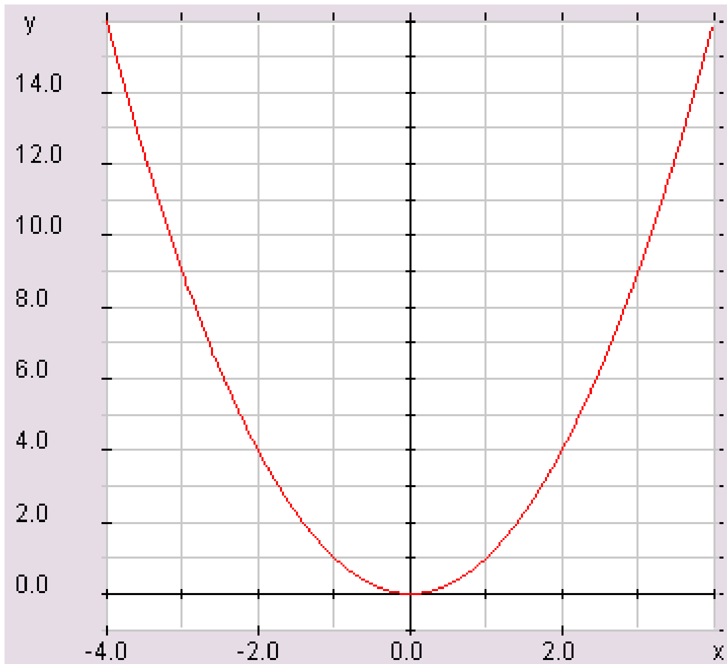

Figure 4.14: Even Function

Example 290 $\sin(-x) = -\sin x$, *therefore* $\sin x$ *is odd.*

Example 291 $\cos(-x) = \cos x$, *therefore* $\cos x$ *is even.*

Example 292 *From the previous two examples, it follows that* $\tan x$ *and* $\cot x$ *are odd*

The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin. Another way of thinking about it is the following. If f is even and (a, b) is on the graph of f, then $(-a, b)$ is also on the graph of f. If f is odd and (a, b) is on the graph of f, then $(-a, -b)$ is also on the graph of f. This is illustrated on figure 4.14 for even functions, and on figure 4.15 for odd functions.

Knowing if a function is even or odd can make integrating it easier.

Theorem 293 *Suppose that* f *is continuous on* $[-a, a]$ *then:*

1. *If* f *is even, then* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. *If* f *is odd, then* $\int_{-a}^{a} f(x)\,dx = 0$

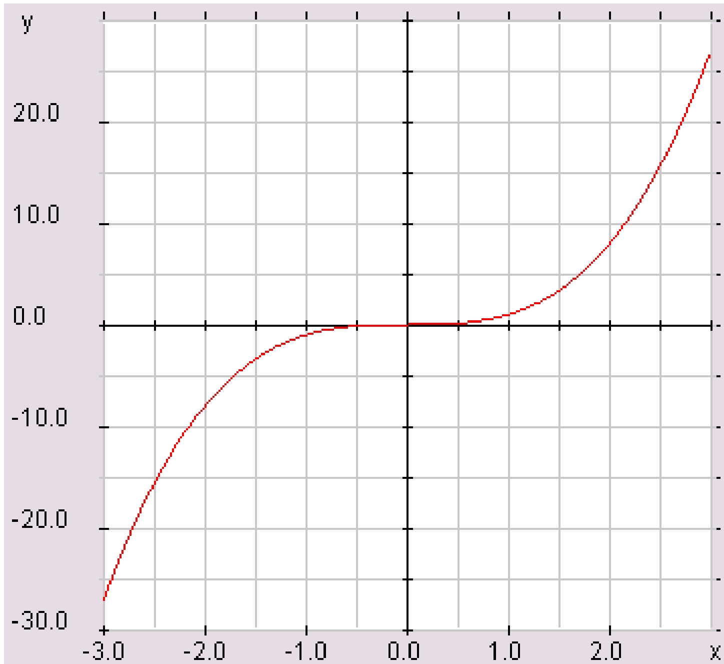

Figure 4.15: Odd Function

Proof. *Using the properties of integrals, we have:*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx\end{aligned}$$

In the first integral, we use the substitution $u = -x$ so that $du = -dx$, we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

For clarity, use the substitution $u = x$ to obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \tag{4.3}$$

We now consider the cases f is even and odd separately.

- *case 1: f is even. In this case, $f(-x) = f(x)$. Therefore, equation 4.3* *becomes*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx\end{aligned}$$

- *case 2: f is odd. In this case, $f(-x) = -f(x)$. Therefore, equation 4.3 becomes*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 0\end{aligned}$$

■

Remark 294 *The first part of the theorem does not save us a lot of work. We still have to be able to find an antiderivative in order to evaluate the integral. However, in the second part, we only need to know the function is odd. If it is, then the integral will be* 0*, there is no need to be able to find an antiderivative of the integrand.*

Example 295 *Find* $\int_{-1}^{1} \dfrac{\tan x}{x^4 + x^2 + 1} dx$

Let $f(x) = \dfrac{\tan x}{x^4 + x^2 + 1}$*. The reader can verify that* f *is an odd function, therefore*

$$\int_{-1}^{1} \frac{\tan x}{x^4 + x^2 + 1} dx = 0$$

4.5.5 Things to Know

- Be able to integrate using the substitution method. In particular, know how to identify the integrals for which substitution might work. They are integrals of the form $\int g'(x) f(g(x))\,dx$ where f is a function for which we know an antiderivative. Note that the function outside of f is the derivative of the function inside of f. For this this to work, the function outside of f does not have to be exactly the derivative of the function inside of f. Even if it is missing a constant, we can still do substitution. Examples include $\int 2x \sin\left(x^2\right) dx$, $\int x^2 \sin\left(x^3\right) dx$ (we are just missing a constant). Keep in mind that $g'(x)$ might involve a fraction as in $\int \dfrac{\sin(\ln x)}{x} dx$.
- Know what odd and even functions are, be able to recognize them and know how to integrate them on an interval of the form $[-a, a]$.

4.5.6 Problems

1. Evaluate the given integrals by making the given substitution.

   (a) $\int \cos 3x dx$, $u = 3x$

   (b) $\int x^2\sqrt{1+x^3}dx$, $u = 1 + x^3$

   (c) $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}dx$, $u = \sqrt{x}$

   (d) $\int \cos t e^{\sin t}dt$, $u = \sin t$

   (e) $\int \dfrac{1}{\sqrt{5x+8}}dx$, $u = 5x + 8$

   (f) $\int \dfrac{1}{\sqrt{5x+8}}dx$, $u = \sqrt{5x+8}$

2. Evaluate $\int \sqrt{3+2x}dx$

3. Evaluate $\int \dfrac{1}{\sqrt{3x+5}}dx$

4. Evaluate $\int (3x+5)^{10}\, dx$

5. Evaluate $\int 2x\left(x^2+1\right)^4 dx$

6. Evaluate $\int \dfrac{(\ln x)^2}{x}dx$

7. Evaluate $\int \dfrac{dx}{3-5x}$

8. Evaluate $\int e^x\sqrt{1+e^x}dx$

9. Evaluate $\int \cos t \sin^4 t dt$

10. Evaluate $\int \dfrac{\sin 2x}{1+\cos^2 x}dx$

11. Evaluate $\int \dfrac{1+x}{1+x^2}dx$

12. Evaluate $\int \dfrac{dx}{x\ln x}$

13. Evaluate $\int_0^1 x^2\left(1+2x^3\right)^4 dx$

14. Evaluate $\int_0^1 \dfrac{e^x+1}{e^x+x}dx$

15. Evaluate $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3 t dt$

16. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$. Hint: use substitution.

17. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf\left(x^2\right)dx$. Hint: use substitution.

18. If f is continuous on $\mathbb{R}$, prove that $\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$. Hint: use substitution.

19. Try this more challenging problem: Evaluate $\int \dfrac{xe^{2x}}{(2x+1)^2}dx$

4.5.7 Answers

1. Evaluate the given integrals by making the given substitution.

   (a) $\int \cos 3x dx,\ u = 3x$
   $\int \cos 3x dx = \dfrac{1}{3}\sin 3x + C$

   (b) $\int x^2\sqrt{1+x^3}dx,\ u = 1+x^3$
   $\int x^2\sqrt{1+x^3}dx = \sqrt{x^3+1}\left(\dfrac{2}{9}x^3 + \dfrac{2}{9}\right) + C$

   (c) $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}dx,\ u = \sqrt{x}$
   $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}dx = -2\cos\sqrt{x} + C$

   (d) $\int \cos te^{\sin t}dt,\ u = \sin t$
   $\int \cos te^{\sin t}dt = e^{\sin t} + C$

   (e) $\int \dfrac{1}{\sqrt{5x+8}}dx,\ u = 5x+8$
   $\int \dfrac{1}{\sqrt{5x+8}}dx = \dfrac{2}{5}\sqrt{5x+8} + C$

   (f) $\int \dfrac{1}{\sqrt{5x+8}}dx,\ u = \sqrt{5x+8}$
   $\int \dfrac{1}{\sqrt{5x+8}}dx = \dfrac{2}{5}\sqrt{5x+8} + C$

2. $\int \sqrt{3+2x}dx = \dfrac{1}{3}(2x+3)^{\frac{3}{2}} + C$

3. $\int \dfrac{1}{\sqrt{3x+5}}dx = \dfrac{2}{3}\sqrt{3x+5} + C$

4. $\int (3x+5)^{10}\,dx = \dfrac{(3x+5)^{11}}{33} + C$

5. $\int 2x\left(x^2+1\right)^4 dx = \dfrac{\left(x^2+1\right)^5}{5} + C$

6. $\int \dfrac{(\ln x)^2}{x} dx = \dfrac{1}{3}(\ln x)^3 + C$

7. $\int \dfrac{dx}{3-5x} = \dfrac{-1}{5} \ln|3-5x| + C$

8. $\int e^x\sqrt{1+e^x} dx = \dfrac{2}{3}\left(e^x+1\right)^{\frac{3}{2}} + C$

9. $\int \cos t \sin^4 t dt = \dfrac{\sin^5 t}{5} + C$

10. $\int \dfrac{\sin 2x}{1+\cos^2 x} dx = -\ln\left(1+\cos^2 x\right) + C$

11. $\int \dfrac{1+x}{1+x^2} dx = \dfrac{1}{2}\ln\left(x^2+1\right) + \arctan x + C$

12. $\int \dfrac{dx}{x\ln x} = \ln|\ln x| + C$

13. $\int_0^1 x^2\left(1+2x^3\right)^4 dx = \frac{121}{15}$

14. $\int_0^1 \dfrac{e^x+1}{e^x+x} dx = \ln(e+1)$

15. $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3 t dt = 0$

16. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$. Hint: use substitution.
$\int_0^2 f(2x)\,dx = 5$

17. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf\left(x^2\right)dx$. Hint: use substitution.
$\int_0^3 xf\left(x^2\right)dx = 2$

18. If f is continuous on $\mathbb{R}$, prove that $\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$. Hint: use substitution.

19. Try this more challenging problem: Evaluate $\int \dfrac{xe^{2x}}{(2x+1)^2} dx = \dfrac{e^{2x}}{8x+4} + C$