4.7 Inverse Trigonometric Functions

What you should learn

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate and graph the compositions of trigonometric functions.

Why you should learn it

You can use inverse trigonometric functions to model and solve real-life problems. For instance, in Exercise 92 on page 351, an inverse trigonometric function can be used to model the angle of elevation from a television camera to a space shuttle launch.

NASA

Inverse Sine Function

Recall from Section 1.9 that, for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test. From Figure 4.71, you can see that $y = \sin x$ does not pass the test because different values of x yield the same y-value.

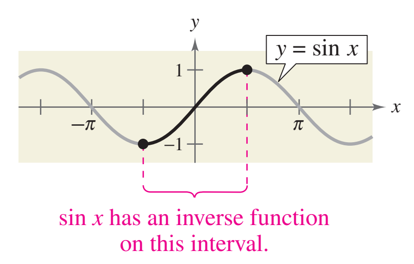


FIGURE 4.71

However, if you restrict the domain to the interval $-\pi/2 \le x \le \pi/2$ (corresponding to the black portion of the graph in Figure 4.71), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

So, on the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by

$$y = \arcsin x \qquad \text{or} \qquad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The $\arcsin x$ notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle. So, $\arcsin x$ means the angle (or arc) whose sine is x. Both notations, $\arcsin x$ and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of $\arcsin x$ lie in the interval $-\pi/2 \le \arcsin x \le \pi/2$. The graph of $y = \arcsin x$ is shown in Example 2.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

STUDY TIP

When evaluating the inverse sine function, it helps to remember the phrase "the arcsine of x is the angle (or number) whose sine is x."

STUDY TIP

As with the trigonometric functions, much of the work with the inverse trigonometric functions can be done by *exact* calculations rather than by calculator approximations. Exact calculations help to increase your understanding of the inverse functions by relating them to the right triangle definitions of the trigonometric functions.

Example 1 Evaluating the Inverse Sine Function

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$ Angle whose sine is $-\frac{1}{2}$

b. Because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, it follows that

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$ Angle whose sine is $\sqrt{3}/2$

c. It is not possible to evaluate $y = \sin^{-1} x$ when $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

CHECKPOINT Now try Exercise 1.

You may wish to illustrate the reflections of $y = \sin x$ and $y = \arcsin x$ about the line $y = x$. Consider using a graphing utility to do this.

Example 2 Graphing the Arcsine Function

Sketch a graph of

$$y = \arcsin x.$$

Solution

By definition, the equations $y = \arcsin x$ and $\sin y = x$ are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same. From the interval $[-\pi/2, \pi/2]$, you can assign values to y in the second equation to make a table of values. Then plot the points and draw a smooth curve through the points.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

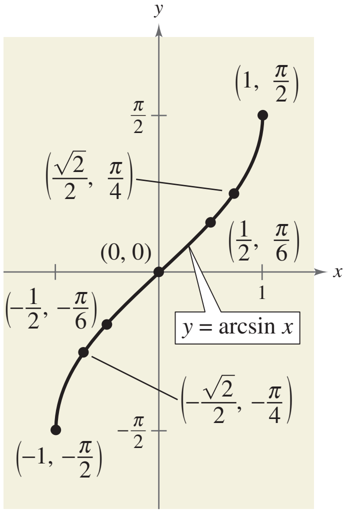


FIGURE 4.72

The resulting graph for $y = \arcsin x$ is shown in Figure 4.72. Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 4.71. Be sure you see that Figure 4.72 shows the *entire* graph of the inverse sine function. Remember that the domain of $y = \arcsin x$ is the closed interval $[-1, 1]$ and the range is the closed interval $[-\pi/2, \pi/2]$.

CHECKPOINT Now try Exercise 17.

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \le x \le \pi$, as shown in Figure 4.73.

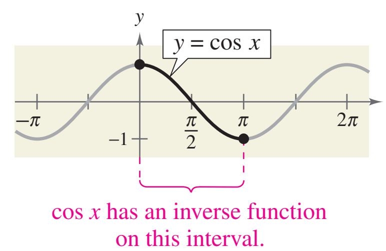


FIGURE 4.73

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

$$y = \arccos x \qquad \text{or} \qquad y = \cos^{-1} x.$$

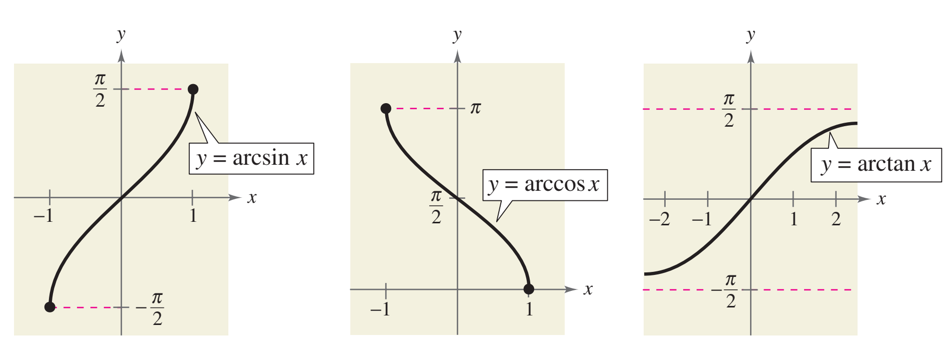
Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The following list summarizes the definitions of the three most common inverse trigonometric functions. The remaining three are defined in Exercises 101–103.

You may need to point out to your students that the range for each of these functions is different. Students should know these ranges well to ensure that their answers are within the correct range. Referencing the graphs of the inverse trigonometric functions may also be helpful.

Definitions of the Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these three inverse trigonometric functions are shown in Figure 4.74.



DOMAIN: $[-1, 1]$
RANGE: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

DOMAIN: $[-1, 1]$
RANGE: $[0, \pi]$

DOMAIN: $(-\infty, \infty)$
RANGE: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

FIGURE 4.74

Example 3 Evaluating Inverse Trigonometric Functions

Find the exact value.

a. $\arccos \frac{\sqrt{2}}{2}$ **b.** $\cos^{-1}(-1)$

c. $\arctan 0$ **d.** $\tan^{-1}(-1)$

Solution

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \qquad \text{Angle whose cosine is } \sqrt{2}/2$$

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$, it follows that

$$\cos^{-1}(-1) = \pi. \qquad \text{Angle whose cosine is } -1$$

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0. \qquad \text{Angle whose tangent is } 0$$

d. Because $\tan(-\pi/4) = -1$, and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}. \qquad \text{Angle whose tangent is } -1$$

✓CHECKPOINT Now try Exercise 11.

Example 4 Calculators and Inverse Trigonometric Functions

Use a calculator to approximate the value (if possible).

a. $\arctan(-8.45)$

b. $\sin^{-1} 0.2447$

c. $\arccos 2$

Solution

	Function	*Mode*	*Calculator Keystrokes*
a.	$\arctan(-8.45)$	Radian	[TAN⁻¹] [(] [(–)] 8.45 [)] [ENTER]

From the display, it follows that $\arctan(-8.45) \approx -1.453001$.

b.	$\sin^{-1} 0.2447$	Radian	[SIN⁻¹] [(] 0.2447 [)] [ENTER]

From the display, it follows that $\sin^{-1} 0.2447 \approx 0.2472103$.

c.	$\arccos 2$	Radian	[COS⁻¹] [(] 2 [)] [ENTER]

In *real number* mode, the calculator should display an *error message* because the domain of the inverse cosine function is $[-1, 1]$.

✓CHECKPOINT Now try Exercise 25.

STUDY TIP

It is important to remember that the domain of the inverse sine function and the inverse cosine function is $[-1, 1]$, as indicated in Example 4(c).

In Example 4, if you had set the calculator to *degree* mode, the displays would have been in degrees rather than radians. This convention is peculiar to calculators. By definition, the values of inverse trigonometric functions are *always in radians*.

Compositions of Functions

Recall from Section 1.9 that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Activities

1. Evaluate $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

   Answer: $\frac{5\pi}{6}$

2. Use a calculator to evaluate arctan 3.2.

   Answer: 1.268

3. Write an algebraic expression that is equivalent to $\sin(\arctan 3x)$.

   Answer: $\frac{3x}{\sqrt{1 + 9x^2}}$

Keep in mind that these inverse properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \ne \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Example 5 Using Inverse Properties

If possible, find the exact value.

a. $\tan[\arctan(-5)]$ **b.** $\arcsin\left(\sin\frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1}\pi)$

Solution

a. Because -5 lies in the domain of the arctan function, the inverse property applies, and you have

$$\tan[\arctan(-5)] = -5.$$

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$. However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

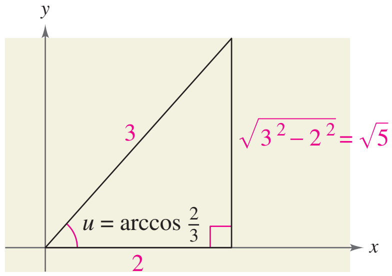
which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin\frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

c. The expression $\cos(\cos^{-1}\pi)$ is not defined because $\cos^{-1}\pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

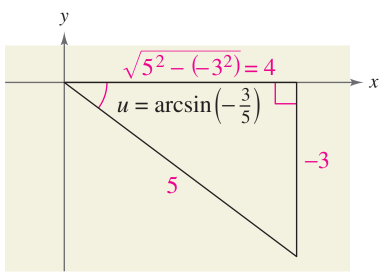
CHECKPOINT Now try Exercise 43.

Example 6 shows how to use right triangles to find exact values of compositions of inverse functions. Then, Example 7 shows how to use right triangles to convert a trigonometric expression into an algebraic expression. This conversion technique is used frequently in calculus.



Angle whose cosine is $\frac{2}{3}$

FIGURE 4.75



Angle whose sine is $-\frac{3}{5}$

FIGURE 4.76

Example 6 Evaluating Compositions of Functions

Find the exact value.

a. $\tan\left(\arccos \dfrac{2}{3}\right)$ **b.** $\cos\left[\arcsin\left(-\dfrac{3}{5}\right)\right]$

Solution

a. If you let $u = \arccos \frac{2}{3}$, then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a *first*-quadrant angle. You can sketch and label angle u as shown in Figure 4.75. Consequently,

$$\tan\left(\arccos \frac{2}{3}\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

b. If you let $u = \arcsin\left(-\frac{3}{5}\right)$, then $\sin u = -\frac{3}{5}$. Because $\sin u$ is negative, u is a *fourth*-quadrant angle. You can sketch and label angle u as shown in Figure 4.76. Consequently,

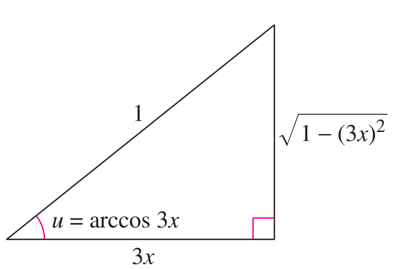
$$\cos\left[\arcsin\left(-\frac{3}{5}\right)\right] = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

✓CHECKPOINT Now try Exercise 51.

Example 7 Some Problems from Calculus

Write each of the following as an algebraic expression in x.

a. $\sin(\arccos 3x), \quad 0 \le x \le \dfrac{1}{3}$ **b.** $\cot(\arccos 3x), \quad 0 \le x < \dfrac{1}{3}$



Angle whose cosine is 3x

FIGURE 4.77

Solution

If you let $u = \arccos 3x$, then $\cos u = 3x$, where $-1 \le 3x \le 1$. Because

$$\cos u = \frac{\text{adj}}{\text{hyp}} = \frac{3x}{1}$$

you can sketch a right triangle with acute angle u, as shown in Figure 4.77. From this triangle, you can easily convert each expression to algebraic form.

a. $\sin(\arccos 3x) = \sin u = \dfrac{\text{opp}}{\text{hyp}} = \sqrt{1 - 9x^2}, \quad 0 \le x \le \dfrac{1}{3}$

b. $\cot(\arccos 3x) = \cot u = \dfrac{\text{adj}}{\text{opp}} = \dfrac{3x}{\sqrt{1 - 9x^2}}, \quad 0 \le x < \dfrac{1}{3}$

✓CHECKPOINT Now try Exercise 59.

You may want to review with students how to rationalize the denominator of a fractional expression.

In Example 7, similar arguments can be made for x-values lying in the interval $\left[-\frac{1}{3}, 0\right]$.

4.7 Exercises

VOCABULARY CHECK: Fill in the blanks.

	Function	*Alternative Notation*	*Domain*	*Range*
1.	$y = \arcsin x$	________	________	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	________	$y = \cos^{-1} x$	$-1 \le x \le 1$	________
3.	$y = \arctan x$	________	________	________

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–16, evaluate the expression without using a calculator.

1. $\arcsin \frac{1}{2}$

2. $\arcsin 0$

3. $\arccos \frac{1}{2}$

4. $\arccos 0$

5. $\arctan \frac{\sqrt{3}}{3}$

6. $\arctan(-1)$

7. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

8. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

9. $\arctan(-\sqrt{3})$

10. $\arctan \sqrt{3}$

11. $\arccos\left(-\frac{1}{2}\right)$

12. $\arcsin \frac{\sqrt{2}}{2}$

13. $\sin^{-1} \frac{\sqrt{3}}{2}$

14. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

15. $\tan^{-1} 0$

16. $\cos^{-1} 1$

In Exercises 17 and 18, use a graphing utility to graph *f*, *g*, and $y = x$ in the same viewing window to verify geometrically that *g* is the inverse function of *f*. (Be sure to restrict the domain of *f* properly.)

17. $f(x) = \sin x, \quad g(x) = \arcsin x$

18. $f(x) = \tan x, \quad g(x) = \arctan x$

In Exercises 19–34, use a calculator to evaluate the expression. Round your result to two decimal places.

19. $\arccos 0.28$

20. $\arcsin 0.45$

21. $\arcsin(-0.75)$

22. $\arccos(-0.7)$

23. $\arctan(-3)$

24. $\arctan 15$

25. $\sin^{-1} 0.31$

26. $\cos^{-1} 0.26$

27. $\arccos(-0.41)$

28. $\arcsin(-0.125)$

29. $\arctan 0.92$

30. $\arctan 2.8$

31. $\arcsin \frac{3}{4}$

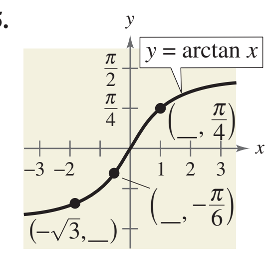
32. $\arccos\left(-\frac{1}{3}\right)$

33. $\tan^{-1} \frac{7}{2}$

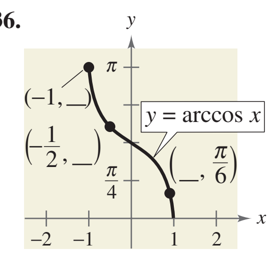
34. $\tan^{-1}\left(-\frac{95}{7}\right)$

In Exercises 35 and 36, determine the missing coordinates of the points on the graph of the function.

35.

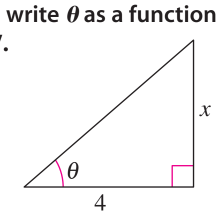


36.

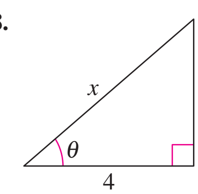


In Exercises 37–42, use an inverse trigonometric function to write θ as a function of *x*.

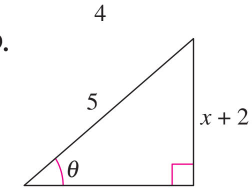
37.



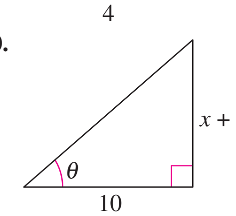
38.



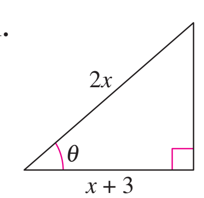
39.



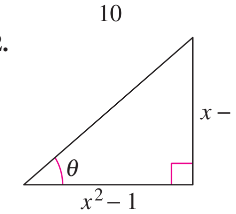
40.



41.



42.



In Exercises 43–48, use the properties of inverse trigonometric functions to evaluate the expression.

43. $\sin(\arcsin 0.3)$

44. $\tan(\arctan 25)$

45. $\cos[\arccos(-0.1)]$

46. $\sin[\arcsin(-0.2)]$

47. $\arcsin(\sin 3\pi)$

48. $\arccos\left(\cos \frac{7\pi}{2}\right)$

In Exercises 49–58, find the exact value of the expression. (*Hint:* Sketch a right triangle.)

49. $\sin\left(\arctan \frac{3}{4}\right)$

50. $\sec\left(\arcsin \frac{4}{5}\right)$

51. $\cos(\tan^{-1} 2)$

52. $\sin\left(\cos^{-1} \frac{\sqrt{5}}{5}\right)$

53. $\cos\left(\arcsin \frac{5}{13}\right)$

54. $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

55. $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$

56. $\tan\left[\arcsin\left(-\frac{3}{4}\right)\right]$

57. $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

58. $\cot\left(\arctan \frac{5}{8}\right)$

In Exercises 59–68, write an algebraic expression that is equivalent to the expression. (*Hint:* Sketch a right triangle, as demonstrated in Example 7.)

59. $\cot(\arctan x)$

60. $\sin(\arctan x)$

61. $\cos(\arcsin 2x)$

62. $\sec(\arctan 3x)$

63. $\sin(\arccos x)$

64. $\sec[\arcsin(x - 1)]$

65. $\tan\left(\arccos \frac{x}{3}\right)$

66. $\cot\left(\arctan \frac{1}{x}\right)$

67. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

68. $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 69 and 70, use a graphing utility to graph *f* and *g* in the same viewing window to verify that the two functions are equal. Explain why they are equal. Identify any asymptotes of the graphs.

69. $f(x) = \sin(\arctan 2x), \quad g(x) = \frac{2x}{\sqrt{1 + 4x^2}}$

70. $f(x) = \tan\left(\arccos \frac{x}{2}\right), \quad g(x) = \frac{\sqrt{4 - x^2}}{x}$

In Exercises 71–74, fill in the blank.

71. $\arctan \frac{9}{x} = \arcsin(\square), \quad x \neq 0$

72. $\arcsin \frac{\sqrt{36 - x^2}}{6} = \arccos(\square), \quad 0 \leq x \leq 6$

73. $\arccos \frac{3}{\sqrt{x^2 - 2x + 10}} = \arcsin(\square)$

74. $\arccos \frac{x - 2}{2} = \arctan(\square), \quad |x - 2| \leq 2$

In Exercises 75 and 76, sketch a graph of the function and compare the graph of *g* with the graph of $f(x) = \arcsin x$.

75. $g(x) = \arcsin(x - 1)$

76. $g(x) = \arcsin \frac{x}{2}$

In Exercises 77–82, sketch a graph of the function.

77. $y = 2 \arccos x$

78. $g(t) = \arccos(t + 2)$

79. $f(x) = \arctan 2x$

80. $f(x) = \frac{\pi}{2} + \arctan x$

81. $h(v) = \tan(\arccos v)$

82. $f(x) = \arccos \frac{x}{4}$

In Exercises 83–88, use a graphing utility to graph the function.

83. $f(x) = 2 \arccos(2x)$

84. $f(x) = \pi \arcsin(4x)$

85. $f(x) = \arctan(2x - 3)$

86. $f(x) = -3 + \arctan(\pi x)$

87. $f(x) = \pi - \sin^{-1}\left(\frac{2}{3}\right)$

88. $f(x) = \frac{\pi}{2} + \cos^{-1}\left(\frac{1}{\pi}\right)$

In Exercises 89 and 90, write the function in terms of the sine function by using the identity

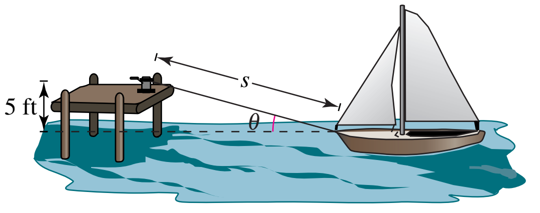
$$A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \sin\left(\omega t + \arctan \frac{A}{B}\right).$$

Use a graphing utility to graph both forms of the function. What does the graph imply?

89. $f(t) = 3 \cos 2t + 3 \sin 2t$

90. $f(t) = 4 \cos \pi t + 3 \sin \pi t$

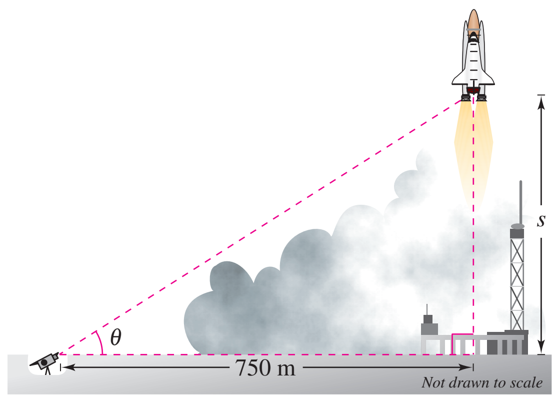
91. ***Docking a Boat*** A boat is pulled in by means of a winch located on a dock 5 feet above the deck of the boat (see figure). Let θ be the angle of elevation from the boat to the winch and let s be the length of the rope from the winch to the boat.



(a) Write θ as a function of s.

(b) Find θ when $s = 40$ feet and $s = 20$ feet.

92. ***Photography*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad (see figure). Let θ be the angle of elevation to the shuttle and let s be the height of the shuttle.



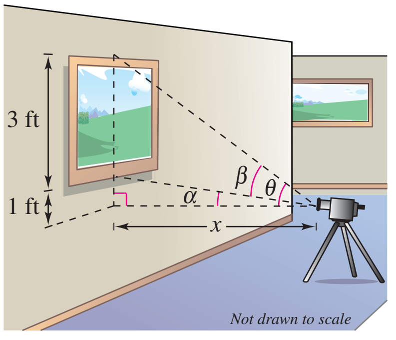
(a) Write θ as a function of s.

(b) Find θ when $s = 300$ meters and $s = 1200$ meters.

Model It

93. ***Photography*** A photographer is taking a picture of a three-foot-tall painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting (see figure). The angle β subtended by the camera lens x feet from the painting is

$$\beta = \arctan \frac{3x}{x^2 + 4}, \quad x > 0.$$

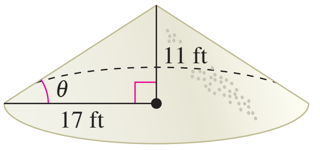


(a) Use a graphing utility to graph β as a function of x.

(b) Move the cursor along the graph to approximate the distance from the picture when β is maximum.

(c) Identify the asymptote of the graph and discuss its meaning in the context of the problem.

94. ***Granular Angle of Repose*** Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose* (see figure). When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. (Source: Bulk-Store Structures, Inc.)



(a) Find the angle of repose for rock salt.

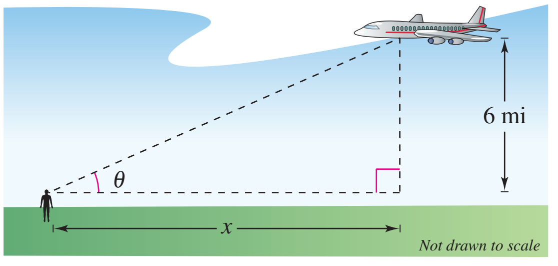
(b) How tall is a pile of rock salt that has a base diameter of 40 feet?

95. ***Granular Angle of Repose*** When whole corn is stored in a cone-shaped pile 20 feet high, the diameter of the pile's base is about 82 feet.

(a) Find the angle of repose for whole corn.

(b) How tall is a pile of corn that has a base diameter of 100 feet?

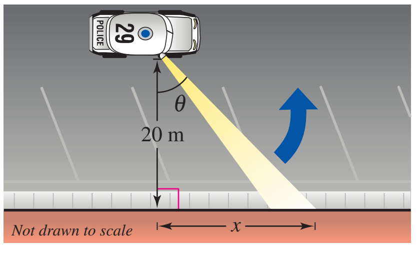
96. ***Angle of Elevation*** An airplane flies at an altitude of 6 miles toward a point directly over an observer. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) Find θ when $x = 7$ miles and $x = 1$ mile.

97. ***Security Patrol*** A security car with its spotlight on is parked 20 meters from a warehouse. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) Find θ when $x = 5$ meters and $x = 12$ meters.

Synthesis

True or False? **In Exercises 98–100, determine whether the statement is true or false. Justify your answer.**

98. $\sin \dfrac{5\pi}{6} = \dfrac{1}{2}$ $\Rightarrow$ $\arcsin \dfrac{1}{2} = \dfrac{5\pi}{6}$

99. $\tan \dfrac{5\pi}{4} = 1$ $\Rightarrow$ $\arctan 1 = \dfrac{5\pi}{4}$

100. $\arctan x = \dfrac{\arcsin x}{\arccos x}$

101. Define the inverse cotangent function by restricting the domain of the cotangent function to the interval $(0, \pi)$, and sketch its graph.

102. Define the inverse secant function by restricting the domain of the secant function to the intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, and sketch its graph.

103. Define the inverse cosecant function by restricting the domain of the cosecant function to the intervals $[-\pi/2, 0)$ and $(0, \pi/2]$, and sketch its graph.

104. Use the results of Exercises 101–103 to evaluate each expression without using a calculator.

(a) $\operatorname{arcsec} \sqrt{2}$ (b) $\operatorname{arcsec} 1$

(c) $\operatorname{arccot}(-\sqrt{3})$ (d) $\operatorname{arccsc} 2$

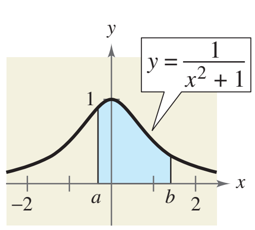
105. ***Area*** In calculus, it is shown that the area of the region bounded by the graphs of $y = 0$, $y = 1/(x^2 + 1)$, $x = a$, and $x = b$ is given by

$$\text{Area} = \arctan b - \arctan a$$

(see figure). Find the area for the following values of a and b.

(a) $a = 0, b = 1$ (b) $a = -1, b = 1$

(c) $a = 0, b = 3$ (d) $a = -1, b = 3$



106. ***Think About It*** Use a graphing utility to graph the functions

$$f(x) = \sqrt{x} \text{ and } g(x) = 6 \arctan x.$$

For $x > 0$, it appears that $g > f$. Explain why you know that there exists a positive real number a such that $g < f$ for $x > a$. Approximate the number a.

107. ***Think About It*** Consider the functions given by

$$f(x) = \sin x \quad \text{and} \quad f^{-1}(x) = \arcsin x.$$

(a) Use a graphing utility to graph the composite functions $f \circ f^{-1}$ and $f^{-1} \circ f$.

(b) Explain why the graphs in part (a) are not the graph of the line $y = x$. Why do the graphs of $f \circ f^{-1}$ and $f^{-1} \circ f$ differ?

108. ***Proof*** Prove each identity.

(a) $\arcsin(-x) = -\arcsin x$

(b) $\arctan(-x) = -\arctan x$

(c) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

(d) $\arcsin x + \arccos x = \dfrac{\pi}{2}$

(e) $\arcsin x = \arctan \dfrac{x}{\sqrt{1 - x^2}}$

Skills Review

In Exercises 109–112, evaluate the expression. Round your result to three decimal places.

109. $(8.2)^{3.4}$ **110.** $10(14)^{-2}$

111. $(1.1)^{50}$ **112.** $16^{-2\pi}$

In Exercises 113–116, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side. Then find the other five trigonometric functions of θ.

113. $\sin \theta = \frac{3}{4}$ **114.** $\tan \theta = 2$

115. $\cos \theta = \frac{5}{6}$ **116.** $\sec \theta = 3$

117. ***Partnership Costs*** A group of people agree to share equally in the cost of a \$250,000 endowment to a college. If they could find two more people to join the group, each person's share of the cost would decrease by \$6250. How many people are presently in the group?

118. ***Speed*** A boat travels at a speed of 18 miles per hour in still water. It travels 35 miles upstream and then returns to the starting point in a total of 4 hours. Find the speed of the current.

119. ***Compound Interest*** A total of \$15,000 is invested in an account that pays an annual interest rate of 3.5%. Find the balance in the account after 10 years, if interest is compounded (a) quarterly, (b) monthly, (c) daily, and (d) continuously.

120. ***Profit*** Because of a slump in the economy, a department store finds that its annual profits have dropped from \$742,000 in 2002 to \$632,000 in 2004. The profit follows an exponential pattern of decline. What is the expected profit for 2008? (Let $t = 2$ represent 2002.)