4. BEAMS: CURVED, COMPOSITE, UNSYMMETRICAL

Discussions of beams in bending are usually limited to beams with at least one longitudinal plane of symmetry with the load applied in the plane of symmetry or to symmetrical beams composed of longitudinal elements of similar material or to initially straight beams with constant cross section and longitudinal elements of the same length. If any of these assumptions are violated, the simple equations which describe the beam bending stress and strain are no longer applicable. The following sections discuss curved beams, composite beams and unsymmetrical beams.

Curved Beams

One of the assumptions of the development of the beam bending relations is that all longitudinal elements of the bean have the same length, thus restricting the theory to initially straight beams of constant cross section. Although considerable deviations from this restriction can be tolerated in real problems, when the initial curvature of the beams becomes significant, the linear variations of strain over the cross section is no longer valid, even though the assumption of plane cross sections remaining plane is valid. A theory for a beam subjected to pure bending having a constant cross section and a constant or slowly varying initial radius of curvature in the plane of bending is developed as follows. Typical examples of curved beams include hooks and chain links. In these cases the members are not slender but rather have a sharp curve and their cross sectional dimensions are large compared with their radius of curvature.

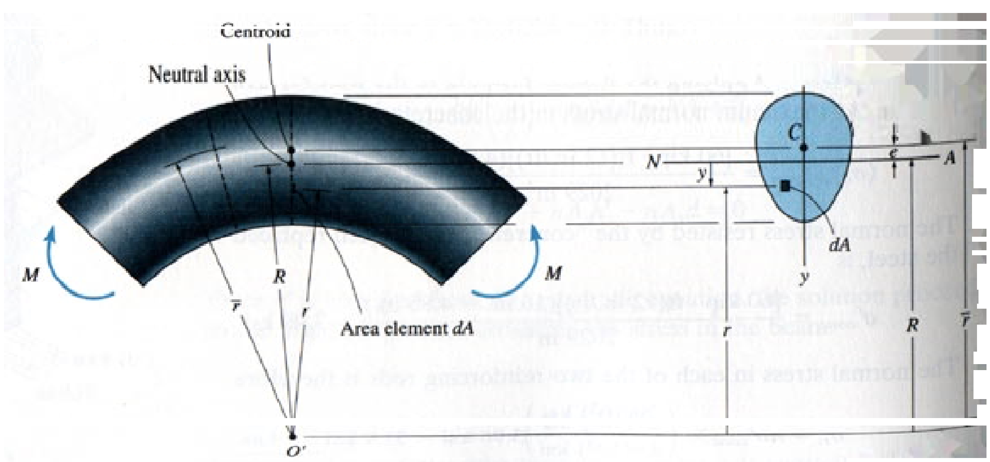


Fig 4.1 Curved beam element with applied moment, M

Fig 4.1 is the cross section of part of an initially curved beam. The x-y plane is the plane of bending and a plane of symmetry. Assumptions for the analysis are: cross sectional area is constant; an axis of symmetry is perpendicular to the applied moment; M, the material is homogeneous, isotropic and linear elastic; plane sections remain plane, and any distortions of the cross section within its own plane are neglected. Since a plane section before bending remains a plane after bending, the longitudinal deformation of any element will be proportional to the distance of the element from the neutral surface.

In developing the analysis, three radii, extending from the center of curvature, O', of the member are shown in Fig 4.1. The radii are: $\bar{r}$ that references the location of the centroid of the cross sectional area; R that references the location of the neutral axis; and r references some arbitrary point of area element dA on the cross section. Note that the neutral axis lies within the cross section since the moment M creates compression in the beams top fibers and tension in its bottom fibers. By definition, the neutral axis is al line of zero stress and strain.

If a differential segment is isolated in the beam (see Fig 4.2). The stress deforms the material in such a way that the cross section rotates through an angle of $\delta\theta/2$. The normal strain in an arbitrary strip at location r can be determined from the resulting deformation. This strip has an original length of L_o=r dθ. The strip's total change in length, $\Delta L = 2(R-r)\delta\theta/2$. The normal strain can be written as:

$$\varepsilon = \frac{\Delta L}{L_o} = \frac{(R-r)\delta\theta}{r\,d\theta} \tag{4.1}$$

To simplify the relation, a constant k is defined as $k = \delta\theta/d\theta$ such that the normal strain can be rewritten as:

$$\varepsilon = k\left(\frac{R-r}{r}\right) \tag{4.2}$$

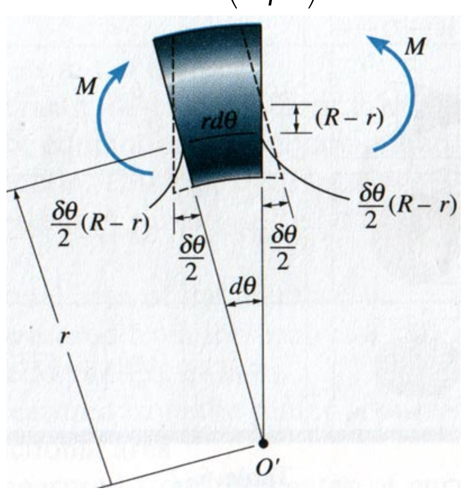


Fig 4.2 Isolated differential element is a curved beam

Note that Eq 4.2 shows that the normal strain is a nonlinear function of r varying in a hyperbolic fashion. This is in contrast to the linear variation of strain in the case of the straight beam (i.e., $\varepsilon = \frac{-y}{\rho}$). The nonlinear strain distribution for the curved beam occurs even though the cross section of the beam remains plane after deformation. The moment, M, causes the material to deform elastically and therefore Hooke's law applies resulting the following relation for stress:

$$\sigma = E\varepsilon = Ek\left(\frac{R-r}{r}\right) \tag{4.3}$$

Because of the linear relation between stress and strain, the stress relation is also hyperbolic. However, with the relation for stress determined, it is possible to determine the location of the neutral axis and thereby relate the applied moment, M to this resulting stress. First a relation for the unknown radius of the neutral axis from the center of curvature, R, is determined. Then the relation between the stress, σ, and the applied moment, M is determined.

Force equilibrium equations can applied to obtain the location of R (radius of the neutral axis). Specifically, the internal forces caused by the stress distribution acting over the cross section must be balanced such that the resultant internal force is zero:

$$\sum F_x = 0$$

Now since $\sigma = \frac{\mathrm{dF}}{\mathrm{dA}}$ then $\mathrm{dF} = \sigma\,\mathrm{dA}$ and

$$F = \sum \sigma\,\mathrm{dA} = \int \sigma dA = 0$$

$$\text{or } \int \mathrm{Ek}\left(\frac{\mathrm{R-r}}{\mathrm{r}}\right)\mathrm{dA} = 0 \tag{4.4}$$

Because Ek and R are constants Eq 4.4 can be rearranged such that:

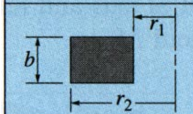
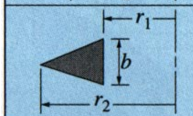
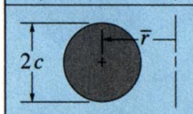
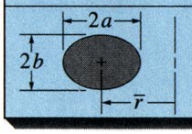
$$\mathrm{Ek}\left[\int\left(\frac{\mathrm{R}}{\mathrm{r}}\right)\mathrm{dA} - \int\left(\frac{\mathrm{r}}{\mathrm{r}}\right)\mathrm{dA}\right] = 0 \Rightarrow R\int\frac{\mathrm{dA}}{\mathrm{r}} - \int \mathrm{dA} = 0 \tag{4.5}$$

Solving Equation 4.5 for R results in:

$$\mathrm{R} = \frac{\int \mathrm{dA}}{\int \frac{\mathrm{dA}}{\mathrm{r}}} = \frac{\mathrm{A}}{\int \frac{\mathrm{dA}}{\mathrm{r}}} \tag{4.6}$$

where R is the location of the neutral axis referenced from the center of curvature, O', of the member, A is the cross sectional area of the member and r is the arbitrary position of

Table 4.1 Areas and Integrals for Various Cross Sections

Shape	Area	$\int_A \frac{dA}{r}$
	$b(r_2 - r_1)$	$b \ln \frac{r_2}{r_1}$
	$\frac{b}{2}(r_2 - r_1)$	$\frac{b\, r_2}{(r_2 - r_1)}\left(\ln \frac{r_2}{r_1}\right) - b$
	πc^2	$2\pi\left(\bar{r} - \sqrt{\bar{r}^2 - c^2}\right)$
	πab	$\frac{2\pi b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\right)$

the area element dA on the cross section and is referenced from the center of curvature, O', of the member. Equation 4.6 can be solved for various cross sections with examples of common cross sections listed in Table 4.1

Moment equilibrium equations can be applied to relate the applied moment, M, to the resulting stress, σ. Specifically, the internal moments caused by the stress distribution acting over the cross section about the neutral must be balanced such that the resultant internal moment balances the applied moment:

Recall that since $\sigma = \frac{dF}{dA}$ then $dF = \sigma\, dA$

if y is the distance from the neutral axis such that $y = R - r$

then $dM = y\ dF$ or $dM = y\ (\sigma\, dA)$

Applying moment equilibrium such that $\sum M = 0$ gives

$M - \sum y\ (\sigma\, dA) = 0$ or $M = \sum y\ (\sigma\, dA)$

Substituting the derived relations for y and σ gives

$$M = \sum y\ (\sigma\, dA) = \int (R - r) Ek \left(\frac{R - r}{r}\right) dA \qquad (4.7)$$

Again realizing that Ek and R are constants, Eq 4.7 can be expanded and grouped such that:

$$M = Ek\left[R^2\int\frac{dA}{r} - 2R\int dA + \int rdA\right] \text{ such that}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$R^2\left(\frac{A}{R}\right) \qquad 2R\,A \qquad \bar{r}A \qquad \text{which gives}$$

$$M = Ek\left[R^2\left(\frac{A}{R}\right) - 2R\,A \; + \; \bar{r}A\right] = EkA(\bar{r} - R) \tag{4.8}$$

Note that the third integral term in Eq 4.8, comes from the geometric determination of the centroid such that $\bar{r} = \int r\,dA/A$. Equation 4.8 can now be solved for Ek such that $Ek = \dfrac{M}{A(\bar{r} - R)}$. Substituting this relation for Ek into Eq 4.3 gives:

$$\sigma = E\varepsilon = Ek\left(\frac{R-r}{r}\right) = \frac{M}{A(\bar{r} - R)}\left(\frac{R-r}{r}\right) = \frac{M(R-r)}{Ar(\bar{r} - R)} \tag{4.9}$$

Substituting the relations involving y into Eqs 4.9 (y=R-r and r=R-y) along with an term for "eccentricity" e=$\bar{r}$-R gives:

$$\sigma = \frac{M(R-r)}{Ar(\bar{r} - R)} = \frac{My}{Ae(R - y)} \tag{4.10}$$

which is the so-called curved-beam flexure formula where σ is the normal stress, M is the applied moment, y is the distance from the neutral axis (y=R-r), A is the area of the cross section, e is the "eccentricity" (e=$\bar{r}$-R) and R is the radius of the neutral axis ($R = \dfrac{A}{\int \frac{dA}{r}}$).

Note that Eq. 4.10 gives the normal stress in a curved member that is in the direction of the circumference (a.k.a. circumferential stress) and is nonlinearly distributed across the cross section (see Fig 4.3). It is worth noting that due to the curvature of the beam a compressive radial stress (acting in the direction of r) will also be developed. Typically the radial stress is small compared to the circumferential stress and can be neglected, especially if the cross section of the member is a solid section. Sometimes, such as the case of thin plates or thin cross sections (e.g., I-beam), this radial stress can become large relative to the circumferential stress. If the beam is loaded with forces (instead of pure moments) then additional stresses will occur on radial planes. Because the action is elastic, the principle of superposition applies and the additional normal stresses can be added to the flexural stresses obtained in Eq. 4.10.

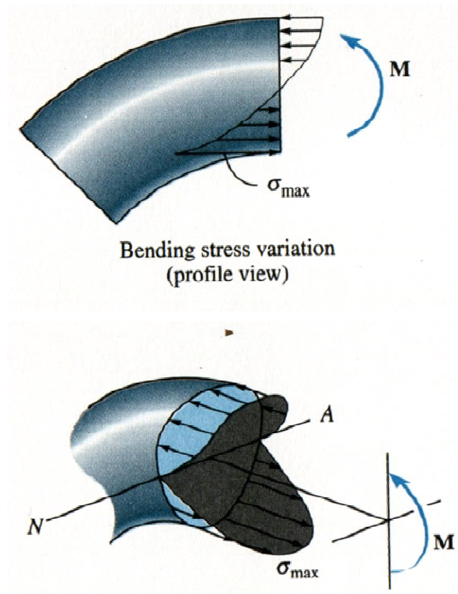


Fig 4.3 Nonlinear stress distribution across cross section is a curved beam

The curved beam flexure formula is usually used when curvature of the member is pronounced as in the cases of hooks and rings. A rule of thumb, for rectangular cross sections for which the ratio of radius of curvature to depth ($\bar{r}/h$) is >5, shows that the curved beam flexure formula agrees well with experimental, elasticity, and numerical results. If the flexure formula used, a difference of 7% from the maximum stress determined from the curved beam flexure formula can result at $\bar{r}/h$=5. As this ratio increases (i.e., at the beam becomes less curved and more straight), the difference of the maximum stress calculated from the flexure formula for the straight beam and the curved beam flexure formula becomes much less.

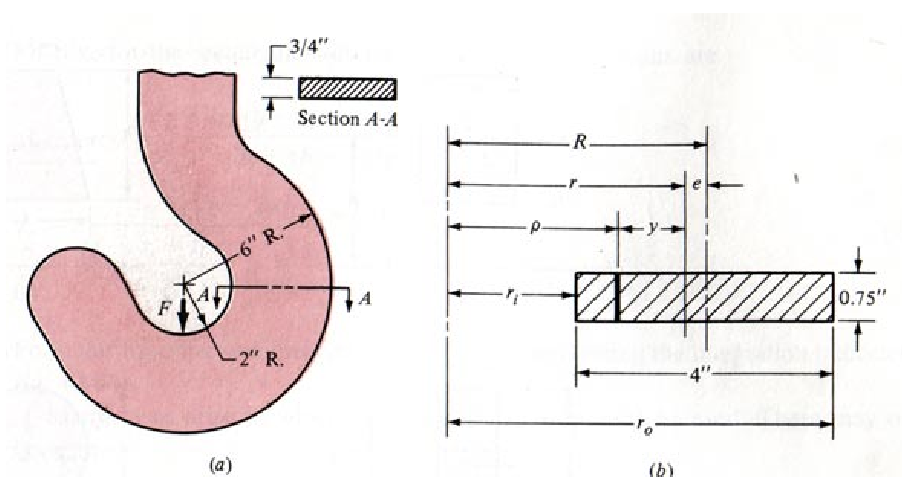


Fig. 4.4 Crane hook with rectangular cross section

A common machine element problem involving curved beams is the crane hook shown in Fig. 4.4. In this problem, the load, F is 22,240 N, the cross sectional thickness, t=b=19.05 mm and the cross sectional width, h=W= is 101.6 mm. Since A=bh, dA=bdr and from Eq. 4.6:

$$R = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r} dr} = \frac{h}{\ln\left(\frac{r_o}{r_i}\right)} \tag{4.11}$$

From Figs 4.4a and 4.4b, r_i=50.8 mm, r_o=152.4 mm, and A=5161.3 mm^2. Substituting the appropriate values into Eq. 4.11 gives R=92.5 mm. The eccentricity, e=(101.6-92.5)=9.1 mm. Since the moment, M is positive such that M = F $\bar{r}$ ($\bar{r}$ = radial distance to the centroid where $\bar{r}$ =(r_o+r_i)/2).

In the case, the axial force, F, superposes an axial stress on the bending stress such that

$$\sigma = \frac{F}{A} + \frac{My}{Ae(R-y)} = \frac{22{,}240}{5161.3} + \frac{(22{,}240*101.6)(92.5-r)}{5161.3(9.1)r} \tag{4.12}$$

Substituting values of r from r_i=50.8 mm to r_o=152.4 mm results in the stress distribution shown in Fig. 4.5 (in psi). The stresses on the inner and radii are 116.5 MPa and -38.6 MPa, respectively. Note that if the flexure stress relations for an initially straight beam are used such that:

$$\sigma = \frac{F}{A} + \frac{My}{I} = \frac{22{,}240}{5161.3} + \frac{(22{,}240*101.6)(101.6-r)}{\frac{19.05(101.6)^3}{12}} \tag{4.13}$$

The maximum and minimum stresses are ±80.4 MPa. A straight beam assumption thus underpredicts the maximum tensile stress and overpredicts the maximum compressive stress.

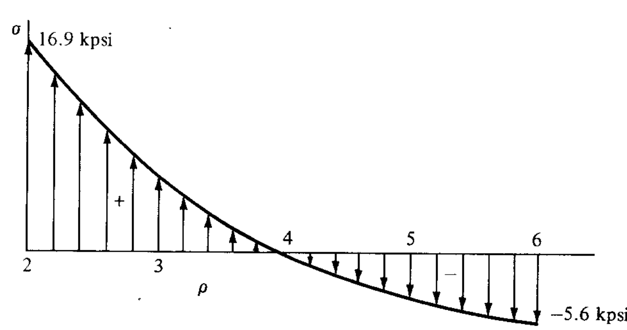


Fig 4.5 Stress distribution across the cross section of a crane hook

Unsymmetrical Bending

Another of the limitations of the usual development of beam bending equations is that beams are assumed to have at least one longitudinal plane of symmetry and that the load is applied in the plane of symmetry. The beam bending equations can be extended to cover pure bending (i.e., bent with bending moments only and no transverse forces) of 1) beams with a plane of symmetry but with the load (couple) applied not in or parallel to the plane of symmetry or 2) beams with no plane of symmetry.

Fig 4.6 depicts a beam of unsymmetrical cross section loaded with a couple, M, in a plane making an angle, α, with the xy plane, where the origin of coordinates is at the centroid of the cross section. The neutral axis, which passes through the centroid for linearly elastic action makes an unknown angle, β, with the z axis. The beam is straight and of uniform cress section and a plane cross section is assumed to remain plane after bending. Note that the following development is restricted to elastic action.

Since the orientation of the neutral axis is unknown, the usual flexural stress distribution function (i.e., $\sigma = E(\varepsilon_c / c)y = (\sigma_c / c)y$) cannot be expressed in terms of one variable.

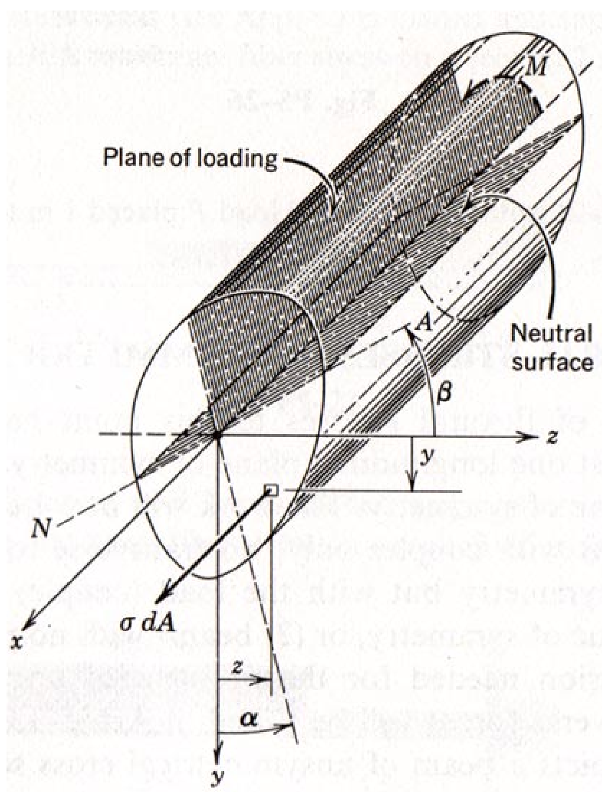


Fig 4.6 Beam undergoing unsymmetrical bending

However, since the plane section remains plane, the stress variation can be written as:

$$\sigma = k_1 y + k_2 z \tag{4.14}$$

The resisting moments with respect to the z and y axes can be written as

$$\begin{aligned} M_{rz} &= \int_A \sigma dAy = \int_A k_1 y^2 dA + \int_A k_2 zydA = k_1 \mathrm{I_z} + k_2 \mathrm{I_{yz}} \\ M_{ry} &= \int_A \sigma dAz = \int_A k_1 yzdA + \int_A k_2 z^2 dA = k_1 \mathrm{I_{yz}} + k_2 \mathrm{I_y} \end{aligned} \tag{4.15}$$

where $\mathrm{I_y}$ and $\mathrm{I_z}$ are the moment of inertia of the cross sectional area with respect to the z and y axes, respectively, and $\mathrm{I_{yz}}$ is the product of inertia with respect to these two axes. It will be convenient to let the y and z axes be principal axes, Y and Z; then $\mathrm{I_{yz}}$ is zero. Equating the applied moment to the resisting moment and solving for k_1 and k_2 gives:

$$\begin{aligned} M_{rZ} &= k_1 \mathrm{I_Z} = M\cos\alpha \qquad & k_1 &= \frac{M\cos\alpha}{\mathrm{I_Z}} \\ M_{rY} &= k_2 \mathrm{I_Y} = M\sin\alpha \qquad & k_2 &= \frac{M\sin\alpha}{\mathrm{I_Y}} \end{aligned} \tag{4.16}$$

Substituting the expressions for k given in Eq. 4.16 into Eq. 4.14 gives the elastic flexure formula for unsymmetrical bending.

$$\sigma = \frac{M\cos\alpha}{\mathrm{I_Z}} \mathrm{Y} + \frac{M\sin\alpha}{\mathrm{I_Y}} \mathrm{Z} \tag{4.17}$$

Since σ is zero at the neutral axis, the orientation of the neutral axis is found by setting Eq. 4.17 equal to zero, for which

$$\frac{\cos\alpha}{\mathrm{I_Z}} \mathrm{Y} = - \frac{\sin\alpha}{\mathrm{I_Y}} \mathrm{Z} \tag{4.18}$$

or

$$Y = -\tan\alpha \frac{\mathrm{I_Z}}{\mathrm{I_Y}} \mathrm{Z} \tag{4.19}$$

where Y is the equation of the neutral axis in the YZ plane. The slope of the line is the dY/dZ and since dY/dZ= $\tan\beta$, the orientation of the neutral axis is given by the expression

$$\tan\beta = -\frac{\mathrm{I_Z}}{\mathrm{I_Y}} \tan\alpha \tag{4.20}$$

The negative sign indicates that the angles, α and β are in adjacent quadrants.

Note that the neutral axis is not perpendicular to the plane of loading unless 1) the angle, α, is zero, in which case the plane of loading is (or is parallel to) a principal angle, or 2) the two principal moments of inertia are equal. This reduces to the special kind of symmetry where all centroidal moments of inertia are equal (e.g., square, rectangle, etc.)

Composite Beams

The method of "fibre" stress calculation for basic beam bending is sufficiently general to cover symmetrical beams composed of longitudinal elements (layers) of different materials. However, for many real beams of two materials (often referred to as reinforced beams) a method can be developed to allow the use of the elastic flexure formula, thus reducing the computational labor involved. Of course, the method is applicable to the elastic region only.

The assumption of plane sections remaining plane is still valid, provided that the different materials are securely bonded together so as to give the necessary resistance to longitudinal shearing stresses. Therefore, the usual linear transverse distribution of longitudinal strains is valid.

The beam shown in Fig. 4.7 is composed of a central portion of material A and two outer layers of material B. The beam will serve as the model for the development of the stress distribution. The section is assumed to be symmetrical with respect to the xy and xz planes and the moment is applied in the xy plane. As long as neither material is subjected to stresses greater than the proportional limit stress, then Hooke's law applies and the strain relation is:

$$\varepsilon_b = \varepsilon_a \frac{b}{a} \tag{4.21}$$

which, in terms of stress, becomes

$$\frac{\sigma_b}{E_B} = \frac{\sigma_a}{E_A} \left(\frac{b}{a}\right) \tag{4.22}$$

After rearranging, Eq. 4.22 becomes the relation for the stress distribution:

$$\sigma_b = \sigma_a \left(\frac{E_B}{E_A}\right) \left(\frac{b}{a}\right) \tag{4.23}$$

From this relation, it is evident that the junction between the two materials where distances a and b are equal, there is an abrupt change in the stress determined by the ratio, $n = \left(\frac{E_B}{E_A}\right)$, of the two elastic moduli (see Fig. 4.7). Using Eq. 4.23, the normal force on a differential end area of element B is given by the expression:

$$dF_B = \sigma_b dA = \sigma_a n \left(\frac{b}{a}\right) dA = \left(\frac{b}{a}\sigma_a\right) (nt)dy \quad \text{for } c_A \le y \le c_B \tag{4.24}$$

where t is the thickness (width) of the beam at a distance b from the neutral surface. The first term in parentheses represents the linear stress distribution in a homogeneous material A. The second term in the parentheses may be interpreted as the extended width of the beam from $y=c_A$ to $y=c_B$ if material B were replaced by material A, thus resulting in an equivalent or transformed cross section for a beam of homogeneous material. The transformed section is obtained by replacing either material by an equivalent amount of the other material as determined by the ratio, n, of their elastic moduli. The method is not limited to two materials: however the use of more than two materials in a beam might be unusual.

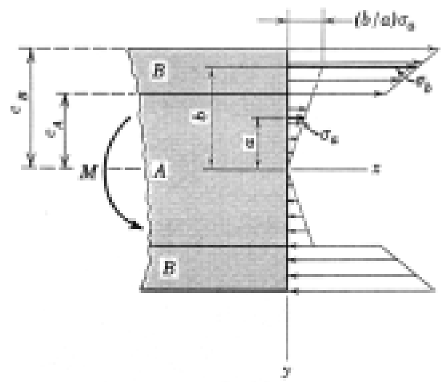


Fig 4.7 Composite beam with two different materials