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MATRICES AND THEIR APPLICATIONS

1.1 INTRODUCTION: DEFINITION INVOLVING MATRICES

Matrices: A rectangular array of mn numbers consisting of m rows and n columns bounded by the commonly accepted notations [ ] or || is termed a matrix of order m by n (or $m \times n$). It is also denoted by a single capital letter.

Thus
$$\text{'}A\text{'} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A matrix is also briefly denoted as 'A' = $[a_{ij}]$, ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) where a_{ij} are the entries of the matrix locating an individual element in the ith row and jth column.

If the rows and columns of a matrix are equal (say $m = n$) then it is called a square matrix.

Significance of Matrices

Though as such the above arrangement of elements has no value of its own but it has a unique utility of summarising or expressing the information in terse and succinct way.

Suppose a builder has bidding for construction of 2 'Cape Cod' type of houses, 'Ranch Type' (cattle farm) where 3, 'Colonial Type' of houses using raw materials as wood, iron, glass, cement and paint.

The numbers in the matrix below, at a glance gives the amount of each raw material required or used (as the case may be) in each type of house in their conventional units:

$$\begin{array}{l} \text{(Type of House)} \\ \text{Cape code } (x) \\ \text{Ranch Type } (y) \\ \text{Colonial Type } (z) \end{array} \quad \begin{array}{c} \begin{array}{ccccc} \text{Wood} & \text{Iron} & \text{Glass} & \text{Cement} & \text{Paint} \end{array} \\ \begin{bmatrix} 13 & 10 & 3 & 4 & 3 \\ 20 & 18 & 2 & 5 & 1 \\ 16 & 14 & 12 & 10 & 8 \end{bmatrix} \end{array}$$

Row Matrix: A matrix consisting of a single row of elements is termed as 'row matrix' or row vector.

e.g. $[1\ 2\ 4\ 5]$ is a row matrix.

Column Matrix: A matrix composed of a single column is called a column matrix or column vector.

e.g. $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a column matrix where we have a single column.

Considering the case of contractor given just, if we represent his orders by a row matrix (2, 2, 3) and the prices of raw materials wood, iron, glass, cement and paint by (5, 4, 3, 2, 1) rupees per unit respectively, we can find the cost of each type of house as follows:

$$(x, y, z) = \begin{bmatrix} 13 & 10 & 3 & 4 & 3 \\ 20 & 18 & 2 & 5 & 1 \\ 16 & 14 & 12 & 10 & 8 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \times 5 + 10 \times 4 + 3 \times 3 + 4 \times 2 + 3 \times 1 \\ 20 \times 5 + 18 \times 4 + 2 \times 3 + 5 \times 2 + 1 \times 1 \\ 16 \times 5 + 14 \times 4 + 12 \times 3 + 10 \times 2 + 8 \times 1 \end{bmatrix} = \begin{bmatrix} 125 \\ 189 \\ 185 \end{bmatrix}$$

Hence, we have calculated the cost of each type of house at the same time we have explained the multiplication of two matrices.

Square Matrix: A matrix having equal number of rows and columns is termed as 'square matrix'.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is a square matrix of order 3 × 3.

The elements a_{ii} in a square matrix 'A' form the **Principal Diagonal** (or Mean Diagonal) and their sum $a_{11} + a_{22} + a_{33}$ is called the **Trace** or **Spur** of 'A'.

For eaxmple, a matrix 'A' for which $A^{k+1} = A$, where k is a positive integer, is called ***'periodic matrix'***. Whereas if k is the least positive integer for which $A^{k+1} = A$, then A is said to be of 'Period' k. If $k = 1$, so that $A^2 = A$, then A is called ***'Idempotent'***. However, if $A^k = 0$ (for positive integer k) A is termed ***'Nilpotent'***. Furthermore, if k is least, A is said to Nilpotent of index 'k'.

e.g. The matrix $'A' = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}$, where p is any integer.

Clearly, $A^2 = A \cdot A = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Whence 'A' is a nilpotent matrix.

Involuntary Matrix: A matrix A will be called an involuntary matrix, if $A^2 = I$ (unit matrix). Since I^2 always is equal to I, therefore unit matrix is involuntary.

Singular Matrix: If the determinant of a matrix 'A' is zero, i.e. $|A| = 0$ then A is called 'singular matrix'. Otherwise, 'non-singular'.

e.g. $$\begin{bmatrix} 2 & 1 & -2 \\ 3 & 0 & 5 \\ 4 & 2 & -4 \end{bmatrix} \text{ is a singular matrix, since } |A| = 0$$

Diagonal Matrix: A square matrix of whose all elements except those in the leading diagonal are zero, i.e. $a_{ij} = 0$ when $i \neq j$ is called a 'diagonal matrix'.

e.g. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Scalar Matrix: A diagonal matrix whose diagonal elements are all equal is termed as 'scalar matrix'.

e.g. $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Unit or Identity Matrix: A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero.

e.g. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix (Echelon Form): A matrix whose elements are all zero is known as 'Null Matrix' or zero matrix'.

Triangular Matrix: A square matrix whose elements either above or below the leading diagonal are all zero is known as a 'triangular matrix'.

e.g. $$\underset{\text{Lower Triangular}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}} \quad \underset{\text{Upper Triangular}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}}$$

Transpose: The matrix obtained from given matrix 'A' by interchange of its row and column is termed as transpose of 'A' and more commonly denoted by A'.

Symmetric Matrix: A square matrix 'A' is said to be symmetric (about the principal diagonal) if $a_{ij} = a_{ji}$. Hence it is clear that transpose of a symmetric matrix is the given matrix itself. Whereas in case of **skew-symmetric** matrix, $A' = -A$.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & -5 \\ -3 & 5 & 4 \end{bmatrix}$

Symmetric Matrix Skew-symmetric

Boolean Matrix: A rectangular array of zeros and ones is called 'Boolean Matrix'.

The rows are labelled by successive integers starting with zero from top to bottom whereas in column from left to right.

e.g.
$$\begin{array}{c|c} i \backslash j & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Zeros and unities are called the elements of the matrix. These elements in general can be denoted by a_{ij} indicating the position of an individual item in ith row and jth column. For instance, $a_{01} = 1$ and $a_{24} = 1$.

Sub-Matrix: A matrix obtained by striking out some rows and some columns of a given matrix 'A'. A is also a sub-matrix of itself.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ contains one, 2×3 sub-matrix, i.e. A itself along with three, 2×2 sub-matrices $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$; two 1×3 sub-matrices, viz. $[1\ 2\ 3]$ and $[4\ 5\ 6]$ likewise, six 1×2 sub-matrices and hence total 21 sub matrices of 'A'.

Equal Matrix: Two matrices are said to be equal if and only if they are of the same order and their respective elements are equal (exactly same).

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Thus A is equal to B only if $a = 1$, $b = 2$, $c = 3$, $d = 4$.

Trace of a Matrix: Sum of principal diagonal elements of a matrix $(n \times n)$ is called the trace of the matrix, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$. Equivalently, the trace of a matrix is the sum of its eigen values, making it an invariant with respect to a change of basis.

Adjoint Matrix: Adjoint of a square matrix 'A' is the transpose of the matrix formed by cofactors of the respective elements of the given square matrix 'A', e.g.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix with determinant $|A|$, then

Adjoint $'A' = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

whereas $A_{11}, A_{12}, A_{13}; A_{21}, A_{22}, A_{23}; A_{31}, A_{32}, A_{33}$ are the cofactors of $a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33}$ respectively.

Addition and Subtraction of Matrices

If 'A' and 'B' are two matrices having equal number of rows and columns, then the sum of 'A' and 'B' is defined as the matrix, each element of which is the sum of the corresponding elements of 'A' and 'B'.

$$\text{Thus for} \quad 'A' = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 + \alpha_2 & b_2 + \beta_2 & c_2 + \gamma_2 \\ a_3 + \alpha_3 & b_3 + \beta_3 & c_3 + \gamma_3 \end{bmatrix}$$

Multiplication of Matrix by a Scalar

If we multiply a matrix 'A' by a scalar k, then 'kA' is defined as the matrix, each element of which is k times the corresponding elements of the matrix 'A', viz.

$$k \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ kb_1 & kb_2 \\ kc_1 & kc_2 \end{bmatrix}$$

Note: As the addition and subtraction of matrices are based on addition of their elements, it follows that in addition of matrices, the law of commutativity and associativity holds viz.,

$A + B = B + A$ and $(A + B) + C = A + (B + C)$, and also holds good for the distributive law viz.,

$k(A + B) = kA + kB$

Multiplication of Two Matrices

Two matrices 'A' and 'B' can be multiplied only if number of columns of 'A' and the number of rows of 'B' are equal.

e.g. If A is a matrix of order (4×3) and 'B' is a matrix of order (3×2), then the product AB will be order (4×2), illustrated thus

$$A \times B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 \\ a_4\alpha_1 + b_4\alpha_2 + c_4\alpha_3 & a_4\beta_1 + b_4\beta_2 + c_4\beta_3 \end{bmatrix}$$

Remarks: Addition and multiplication of two matrices 'A' and 'B' have been defined under certain restrictions. 'A' and 'B' can be added only when 'A' has the same number of rows and columns as 'B' while the product AB can be performed only when the number of columns in 'A' are equal to the number of rows in 'B' or in other words 'A' and 'B' are conformable for addition and, or conformable for the product AB. Further, the two matrices 'A' and 'B' may not be conformable for both the products 'AB' and 'BA', and even if they are then not necessarily, $AB = BA$. Means, in general, multiplication of matrices is not commutative, i.e. $AB \neq BA$.

Illustration of Above Facts with Examples

Case I: AB is defined but BA is not defined. Take matrix 'A' of order 2×3 and 'B' of order 3×4, then AB is defined and it is a matrix of order 2×4 whereas BA is not defined.

Case II: AB and BA are both defined but their orders are different.

Take matrix 'A' of order 2×3 and 'B' of order 3×2, then AB and BA are both defined but their orders are different, viz. 2×2 and 3×3 respectively.

Case III: AB and BA are both defined and are matrices of the same order, still $AB \neq BA$.

Take $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Clearly, $AB \neq BA$.

Inverse or Reciprocal Matrix

If A and B be the two square matrices of the same order such that $AB = I = BA$, then matrix 'B' is called the Inverse (or reciprocal) of matrix 'A' and more often denoted by A^{-1}, i.e. $B = A^{-1}$.

Hence it follows that inverse of the inverse is the matrix itself.

i.e., $(A^{-1})^{-1} = (B)^{-1} = A$

Further, multiplication of 'A' with its adjoint is the determinant value of the matrix 'A'.

$$\text{or} \quad A \times \text{Adjoint } 'A' = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}' = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Delta I$$

In other words,

$$A^{-1} = \frac{\text{Adjiont } 'A'}{\Delta}$$

Hence inverse is possible for a non-singular matrix only.

Note:

1. If 'A' and 'B' are two square matrices of same order with inverses, A^{-1} and B^{-1} respectively, then $(AB)^{-1} = B^{-1} A^{-1}$, i.e. the inverse of the product of two matrices, having inverses, is the product in reverse order of these inverses.
2. The inverse of a diagonal matrix is also diagonal.
3. The inverse of an upper triangular matrix (lower triangular) matrix is an upper triangular (lower triangular).

Involutary matrix: If a square matrix 'A' is such that $A^2 = I$, then A is called an Involutary Matrix, e.g. An identity matrix is involutary. Thus an involutary matrix is its own inverse.

Power Matrix: For a square matrix 'A', the product AA, AAA, A … m times (i.e. $A^2, A^3, \ldots A^m$) are called Power Matrices.

For non-singular 'A', we know that $A^{-1}A = I = AA^{-1}$, i.e. $A^{-1}A' = I = A^1A^{-1}$ (since $A^mA^n = A^{m+n}$, mn are positive integers).

Therefore, with above contention, we can write $A^0 = I$ and $A^{-m} = (A^{-1})^m$

Also with the help of all above derived relations, we define

$(A^m)^n = (A^n)^m = A^{mn}$, where m and n are any integers.

Few Examples on Multiplication, Adjoint and Inverse of Matrices

Example 1: By mathematical induction, prove that if

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \text{then } A^n = \begin{bmatrix} 1 + 10n & -25n \\ 4n & 1 - 10n \end{bmatrix}$$

Solution: Mathematical Induction is very useful technique for providing results for all positive integers, under which we verify the result for $n = 1$, and then assume it true for $n = 1$. For proving it true for $n = m$, prove it is true for $n = m + 1$.

Therefore, when $n = 1$,

$$A^n = A = \begin{bmatrix} 1+10 & -25 \\ 4 & 1-10 \end{bmatrix} = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \qquad \ldots(1)$$

Hence the result is true for $n = 1$.

Now assume that the result is true for $n = m$ (any positive integer)

i.e.,
$$A^m = \begin{bmatrix} 1+10m & -25m \\ 4m & 1-10m \end{bmatrix} \qquad \ldots(2)$$

So that
$$A^{m+1} = A^m \cdot A = \begin{bmatrix} 1+10m & -25m \\ 4m & 1-10m \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} (1+10m)11+(-25m)4 & (1+10m)+(-25)+(-25m)(-9) \\ (4m)11+(1-10m)4 & (4m)(-25)+(1-10m)(-9) \end{bmatrix}$$

$$= \begin{bmatrix} 11+10m & -25-25m \\ 4+4m & -9-10m \end{bmatrix}$$

$$A^{m+1} = \begin{bmatrix} 1+10(m+1) & -25(m+1) \\ 4(m+1) & 1-10(m+1) \end{bmatrix} \qquad \ldots(3)$$

Example 2: Show that the product of matrices

$$\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\psi & \cos\psi\sin\psi \\ \cos\psi\sin\psi & \sin^2\psi \end{bmatrix}$$

is a null matrix, where ϕ and ψ differ by an odd multiple of $\pi/2$.

Solution:
$$\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\psi & \cos\psi\sin\psi \\ \cos\psi\sin\psi & \sin^2\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\phi\cos^2\psi + \cos\phi\sin\phi\cos\psi\sin\psi & \cos^2\phi\cos\psi\sin\psi + \cos\phi\sin\phi\sin^2\psi \\ \cos\phi\sin\phi\cos^2\psi + \sin^2\phi\cos\psi\sin\psi & \cos\phi\sin\phi\cos\psi\sin\psi + \sin^2\phi\sin^2\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\psi(\cos\phi\cos\psi + \sin\phi\sin\psi) & \cos\phi\sin\psi(\cos\phi\cos\psi + \sin\phi\sin\psi) \\ \sin\phi\cos\psi(\cos\phi\cos\psi + \sin\phi\sin\psi) & \sin\phi\sin\psi(\cos\phi\cos\psi + \sin\phi\sin\psi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\psi\cos(\phi-\psi) & \cos\phi\sin\psi\cos(\phi-\psi) \\ \sin\phi\cos\psi\cos(\phi-\psi) & \sin\phi\sin\psi\cos(\phi-\psi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}.$$ [For $(\phi - \psi)$ = an odd multiple of $\pi/2$, $\cos(\phi - \psi) = 0$]

Example 3: If 'A' is the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$ and 'I' is the unit matrix of order 3, show that $A^3 = pI + qA + rA^2$.

Solution: For $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$,

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix} \qquad \ldots(1)$$

Similarly,

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ rp & p+rq & q+r^2 \\ pq+r^2p & q^2+pr+r^2q & p+2qr+r^3 \end{bmatrix}; \qquad \ldots(2)$$

$$pI = p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}; \qquad \ldots(3)$$

$$qA = q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} = \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & rq \end{bmatrix}; \qquad \ldots(4)$$

and
$$rA^2 = r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & r \\ pr & rq & r^2 \\ r^2p & pr+r^2q & rq+r^3 \end{bmatrix} \qquad \ldots(5)$$

Now
$$pI + qA + rA^2 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & rq \end{bmatrix} + \begin{bmatrix} 0 & 0 & r \\ pr & qr & r^2 \\ r^2p & pr+r^2q & rq+r^3 \end{bmatrix}$$

$$= A^3.$$

Example 4: Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{bmatrix}^{-1}$

Solution: In this problem, we need to prove that the product of

$$\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1} \text{ equal to } \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We first find the inverse of $\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$

Let $\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

so that matrix of cofactors $= \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$

Thus $\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1} = \dfrac{\text{adjoint}}{\begin{vmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{vmatrix}} = \dfrac{\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}}{\left(1+\tan^2\theta/2\right)}$

Whence the product,

$$\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}}{\left(1+\tan^2\theta/2\right)}$$

$$= \frac{\begin{bmatrix} 1-\tan^2\theta/2 & -\tan\theta/2-\tan\theta/2 \\ \tan\theta/2+\tan\theta/2 & -\tan^2\theta/2+1 \end{bmatrix}}{\left(1+\tan^2\theta/2\right)}$$

$$= \begin{bmatrix} \dfrac{\left(1-\tan^2\theta/2\right)}{\left(1+\tan^2\theta/2\right)} & -\dfrac{2\tan\theta/2}{\left(1+\tan^2\theta/2\right)} \\ \dfrac{2\tan\theta/2}{\left(1+\tan^2\theta/2\right)} & +\dfrac{\left(1-\tan^2\theta/2\right)}{\left(1+\tan^2\theta/2\right)} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

ASSIGNMENT 1

1. Prove that the product of two upper (lower) triangular matrices is an upper (lower) triangular matrix.

2. If e^A is defined as $I + A + \dfrac{A^2}{2!} + \dfrac{A^3}{3!} + \ldots +$. Show that $e^A = e^x \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$,

   when $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$.

3. If A and B are square matrices of the same order and A is symmetrical, show that $B'AB$ is also symmetrical.
4. If Δ = diag. $[d_1, d_2, \ldots, d_n]$, $d_1, d_2, \ldots, d_n \neq 0$, prove that Δ^{-1} = diag. $[d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1}]$.
5. If $A = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cosh nx & \sinh nx \\ \sinh nx & \cosh nx \end{bmatrix}$.

1.2 ELEMENTARY TRANSFORMATIONS, RANK, NORMAL FORMS AND GAUSS-JORDAN METHOD

The operations (referring to either rows or columns), viz.

(*a*) interchange of any two rows (columns)
(*b*) multiplication of any given row (columns) by a non-zero number
(*c*) addition of a constant multiple of elements of any row (column) to the respective elements of any other row (column)

are called elementary transformations on matrices.

Mathematically,

(*i*) R_{ij} denotes interchange of elements of ith and jth rows.
(*ii*) pR_i denotes multiplication by p to the elements of ith row.
(*iii*) $R_i + pR_j$ denotes addition of p times the elements of jth row to the respective elements of ith row.

Likewise, C_{ij}, pC_i and $C_i + pC_j$ respectively denote elementary column transformations.

1. **Elementary Matrices:** The matrices obtained by subjecting the unit matrix to the above stated elementary transformation are called elementary matrices.

e.g. If $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_{21}$,

$$pR_1 = \begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = pC_1;$$

$$R_2 + pR_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}, \quad C_2 + pC_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 1 \end{bmatrix} \text{ etc.}$$

Matrices R_{12}, pR_1, $R_2 + pR_3$ are elementary row matrices while C_{21}, pC_1 and $C_2 + pC_3$ are the examples of elementary column matrices.

Observations: Pre-multiplications of a matrix (say A) by an elementary row matrix results in row transfer matrix of the given matrix itself while post-multiplication to this by an elementary column matrix results in the respective column transformation in the given matrix itself.

e.g. For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$,

$$R_{12} \times A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

and
$$A \times C_{12} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+a_2+0 & a_1+0+0 & 0+0+a_3 \\ 0+b_2+0 & b_1+0+0 & 0+0+b_3 \\ 0+c_2+0 & c_1+0+0 & 0+0+c_3 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{bmatrix}$$

Clearly the pre-multiplication of A with R_{12} results in the interchange of Ist and 2nd row in 'A' while the post-multiplication of A with C_{12} results in the interchange of Ist and 2nd columns in 'A'.

2. **Equivalent Matrix**: Two matrices 'A' and 'B' are said to be 'equivalent' if one is obtained from the other by a set of elementary transformations. Mathematically, it is denoted as $A \sim B$.

Minor of Matrix: Minor of a matrix is the determinant composed of elements of the matrix left after striking out certain rows and columns.

e.g. Suppose we have a matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{(3\times 4)}$

IIIrd order minors of this matrix are obtained striking out one column and replacing the sign [ ] by | |. These are 3 in number.

IInd order minors are obtained by striking out two columns and one row. These are 18 in numbers.

Ist order minors obtained, likewise, are 12 in number.

However, in general, for $m \times n$ matrix $(m \geq n)$, there will be

$({}^nC_0)^2 = 1$ minor of order n;

$({}^nC_1)^2 = n^2$ minors of order $(n - 1)$;

$$\left({}^nC_2\right)^2 = \frac{n^2\,(n-1)^2}{(2!)^2}$$ minors of order $(n - 2)$ and so on.

Minors with proper sign are called '**co-factors**' of the respective a_{ij}'s.

'**Remarks**': For a matrix 'A', if the minors of order r are zero, then all the minors of higher order will also be zero. Further if 'A' is a square matrix of order n, then the largest order minor of 'A' is the determinant of the matrix itself.

3. **Rank of a Matrix**: The rank of a matrix 'A' (say) is the order of the highest non-zero minor of 'A'. **[PTU, 2005, 2006]**

e.g. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$ is 1.

For a square matrix 'A' of order n, rank r satisfies the relation $r \leq n$.

If $r = n$, the matrix is non-singular and if $r < n$ the matrix is singular.

For instance, the matrix $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is singular matrix, since $r(= 2) < n(= 3)$ with $\Delta A = 0$.

While the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is non-singular, since $\Delta A = 1 \neq 0$.

Hence $r = n = 3$ in this case.

Observation: Elementary transformations on a matrix do not change either its '*order*' or '*rank*' whereas the value of *minors* may get changed by applying any elementary (*Row* or *Column*) transformations on the same matrix (with no change in its zero or non-zero character).

4. **Echelon Form of a Matrix:** A matrix is said to possess echelon form subject to

(*i*) all its non-zero row, if any, proceeding the zero rows
(*ii*) the number of zero in all succeeding rows are higher than its proceeding one
(*iii*) the first non-zero entry in each of its rows is unity.

e.g. $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly, in the above matrix, the non-zero row proceeds the zero row, the number of zeros in IInd, IIIrd, IVth rows are 1, 2, 4 in number, i.e. in an ascending order and the first entry in each row is 1.

5. **Normal Form of a Matrix:** Every non-zero matrix 'A' (order $m \times n$) of rank $r > 0$ can be reduced by a sequence of elementary transformations to one of the form I_r, $[I_r, 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, etc. are called normal form (**Ist canonical form**) of the matrix 'A'.

Note: For a matrix 'A' (order $m \times n$) of rank $r > 0$, there corresponds two non-singular matrices P and Q such $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Further, normal form of a matrix indicates the rank of that matrix.

6. **Gauss-Jordan Method for Inverse of a Matrix:** If a set of certain elementary row transformations reduces a given square matrix 'A' (say, order n) to the unit matrix (I_n) when applied to the unit matrix give the inverse of 'A'.

Working Rule: For finding A^{-1}, write A and I, the two matrices side by side and apply certain row operations to reduce 'A' to unit matrix I, so unit I in turn reduces into A^{-1}.

Example 5: Find the rank of the following matrices:

(*i*) $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ [PTU, 2007; NIT Kurukshetra, 2005; KUK, 2004]

(*v*) $\begin{bmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ [Kottayam, 2005]

(*vii*) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

Solution: (*i*) $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

Operate $(R_2 + 2R_1)$, $(R_3 + R_1)$; $\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix}$

As $\begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$

Clearly highest non-zero minor is of order 2 and, therefore, the rank of this matrix is 2.

(*ii*) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

Operate $(R_2 + 2R_1)$, $(R_3 + R_1)$, $\sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix}$

Operate $(R_3 - 2R_2)$, $\sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $\frac{1}{5}R_2$, $\sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the highest non-zero minor is of order 2 with $\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$

$\therefore$ Rank of the given matrix is 2.

(*iii*) $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

Operate $(C_1 - 2C_4)$, $(C_2 + C_4)$ and $(C_3 - 3C_4)$ $\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 5 & -5 & 1 \\ -1 & 5 & -5 & 3 \end{bmatrix}$

Operate $(C_2 + 5C_1)$, $(C_3 - 5C_1)$ $\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix}$

Clearly all the minors of order 3 are zero.

The highest non-zero minor of order 2, $\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1 \neq 0$

Hence the rank of the matrix is 2.

(*iv*) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operate R_{12} $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operate $(C_2 + C_1)$, $(C_3 + 2C_1)$ and $(C_4 + 4C_1)$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix}$

Operate $R_4 - (R_1 + R_2 + R_3)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the highest non-zero minor is of order 3 with $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 5 & 3 \\ 3 & 4 & 9 \end{vmatrix} = 33 \neq 0$

Hence the rank of given matrix is 3.

(*v*) $\begin{bmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Operate $(R_2 + 2R_1)$, $(R_3 + R_1)$ $\sim \begin{bmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly the rank of the matrix is 1 as minor of order 1, i.e. $|3| = 3 \neq 0$.

(*vi*) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Operate $R_4 - (R_1 + R_2 + R_3)$ $\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $(C_2 - 2C_1)$, $(C_3 - 3C_1)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 3 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Further $(C_1 - C_4)$, $(C_3 - 2C_2)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

As $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -4 & 0 \end{vmatrix} = -12 \neq 0$.

The highest non-zero minor is of order 3. Hence the rank of the matrix is 3.

(*vii*) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

Operate $(C_2 - C_1)$, $(C_4 - C_3)$, $\sim \begin{bmatrix} 5 & 1 & 7 & 1 \\ 6 & 1 & 8 & 1 \\ 11 & 1 & 13 & 1 \\ 16 & 1 & 18 & 1 \end{bmatrix}$

Operate $(C_4 - C_2)$, $C_3 - (C_1 + 2C_2)$ $\sim \begin{bmatrix} 5 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 11 & 1 & 0 & 0 \\ 16 & 1 & 0 & 0 \end{bmatrix}$

Clearly the rank of the matrix is 2.

Example 6: Find the rank of the matrix,

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$ by reducing it to canonical form.

Solution: Applying $\frac{1}{2}R_1$ and $\frac{1}{2}R_2$, $A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Applying $(R_2 - 2R_1)$, $(R_3 - R_1)$, $(R_4 - R_1)$, $A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Applying $(C_2 + C_1)$, $(C_4 - 3C_1)$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$R_2 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \end{bmatrix}$

Now applying $(R_4 + 3R_2)$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix}$

On applying $(C_3 + C_2)$, $(C_4 - C_2)$,. $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix}$

$-R_2$, $R_4 \leftrightarrow R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Next applying $\frac{1}{3}C_3$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Finally, applying $(C_4 + 8C_3)$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad A \sim \begin{vmatrix} I_3 & 0 \\ 0 & 0 \end{vmatrix}$$

Hence the rank of 'A' is = 3.

Example 7: Using Gauss Jordan method find the inverse of the matrices:

$$*(i) \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad **(ii) \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

*[NIT Kurukshetra, 2008]
**[KUK, 2006]

$$(iii) \begin{bmatrix} 8 & 4 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad (iv) \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution:

(*i*) On taking the given matrix side by side with a unit matrix and performing elementary row operations, we have

$$\begin{bmatrix} 1 & 3 & 3 & : & 1 & 0 & 0 \\ 1 & 4 & 3 & : & 0 & 1 & 0 \\ 1 & 3 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

$$(R_2 - R_1),(R_3 - R_1) \sim \begin{bmatrix} 1 & 3 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix},$$

$$(R_1 - 3R_2 - 3R_3) \sim \begin{bmatrix} 1 & 0 & 0 & : & 7 & -3 & -3 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Hence the inverse matrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(*ii*) We have $\begin{bmatrix} 2 & 1 & -1 & : & 1 & 0 & 0 \\ 0 & 2 & 1 & : & 0 & 1 & 0 \\ 5 & 2 & -3 & : & 0 & 0 & 1 \end{bmatrix}$

$$\text{Operate } (2R_3 - 5R_1) \sim \begin{bmatrix} 2 & 1 & -1 & : & 1 & 0 & 0 \\ 0 & 2 & 1 & : & 0 & 1 & 0 \\ 0 & -1 & -1 & : & -5 & 0 & 2 \end{bmatrix}$$

$$\text{Operate } (R_2 + R_3) \sim \begin{bmatrix} 2 & 1 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -5 & 1 & 2 \\ 0 & -1 & -1 & : & -5 & 0 & 2 \end{bmatrix}$$

Operate $(R_1 - 2R_2 - R_3)$, $\sim \begin{bmatrix} 2 & 0 & 0 & : & 16 & -2 & -6 \\ 0 & 1 & 0 & : & -5 & 1 & 2 \\ 0 & -1 & -1 & : & -5 & 0 & 2 \end{bmatrix}$

Operate $\frac{R_1}{2}, -R_3$, $\sim \begin{bmatrix} 1 & 0 & 0 & : & 8 & -1 & -3 \\ 0 & 1 & 0 & : & -5 & 1 & 2 \\ 0 & 1 & 1 & : & 5 & 0 & -2 \end{bmatrix}$

Operate $(R_3 - R_2)$, $\sim \begin{bmatrix} 1 & 0 & 0 & : & 8 & -1 & -3 \\ 0 & 1 & 0 & : & -5 & 1 & 2 \\ 0 & 0 & 1 & : & 10 & -1 & -4 \end{bmatrix}$

(*iii*) Write $\begin{bmatrix} 8 & 4 & -3 & : & 1 & 0 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$

Operate $(R_1 + 3R_2)$, $\sim \begin{bmatrix} 14 & 7 & 0 & : & 1 & 3 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$

Operate $\frac{R_1}{7}, (R_2 - R_3), \sim \begin{bmatrix} 2 & 1 & 0 & : & \frac{1}{7} & \frac{3}{7} & 0 \\ 1 & -1 & 0 & : & 0 & 1 & -1 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$

Operate $(R_1 + R_2)$, $\sim \begin{bmatrix} 3 & 0 & 0 & : & \frac{1}{7} & \frac{10}{7} & -1 \\ 1 & -1 & 0 & : & 0 & 1 & -1 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$

Operate $\frac{R_1}{3}, -R_2$, $\sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{21} & \frac{10}{21} & -\frac{1}{3} \\ -1 & 1 & 0 & : & 0 & -1 & 1 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$

Operate $R_3 - (3R_1 + 2R_2)$, $\sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{21} & \frac{10}{21} & \frac{-1}{3} \\ -1 & 1 & 0 & : & 0 & -1 & 1 \\ 0 & 0 & 1 & : & \frac{-1}{7} & \frac{4}{7} & 0 \end{bmatrix}$

Operate $(R_2 + R_1)$, $\sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{21} & \frac{10}{21} & -\frac{1}{3} \\ 0 & 1 & 0 & : & \frac{1}{21} & -\frac{11}{21} & -\frac{2}{3} \\ 0 & 0 & 1 & : & -\frac{1}{7} & \frac{4}{7} & 0 \end{bmatrix}$

Hence the desired inverse is $\frac{1}{21}\begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & -2 \\ -1 & 4 & 0 \end{bmatrix}$.

(*iv*) $\begin{bmatrix} 2 & 1 & 2 & : & 1 & 0 & 0 \\ 2 & 2 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$

$$(R_1 - R_3),\ (R_2 - 2R_3) \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 & : & 1 & 0 & -1 \\ 0 & -2 & -3 & : & 0 & 1 & -2 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate $\dfrac{R_3 - (R_1 + R_2)}{5} \sim \begin{bmatrix} 1 & -1 & 0 & : & 1 & 0 & -1 \\ 0 & -2 & -3 & : & 0 & 1 & -2 \\ 0 & 1 & 1 & : & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$

Operate $(R_2 + 3R_3)$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 0 & : & 1 & 0 & -1 \\ 0 & 1 & 0 & : & -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 1 & 1 & : & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

$$\Rightarrow \qquad (R_1 + R_2), (R_3 - R_2) \sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & : & -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 & : & \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}.$$

Example 8: Find the singular matrices P and Q such that PAQ is the normal form of the matrix A and hence find the inverse of A.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}. \qquad \textbf{[JNTU, 2002]}$$

Solution: Write $A = I\,A\,I.$ i.e.

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $\begin{matrix}(R_2 - R_1) \\ (R_3 - 3R_1)\end{matrix}$ $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $(C_3 - C_2)$, $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $(R_3 - 2R_2)$, $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $(C_2 + C_1)$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $\dfrac{R_2}{2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \qquad \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $|P| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{vmatrix} = \frac{1}{2} \neq 0, \; |Q| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

i.e. both P and Q are non-singular matrices

Now $PAQ = I_r \;\Rightarrow\; P^{-1}PAQQ^{-1} = P^{-1} I_r Q^{-1}$ or $A = (QP)^{-1}$

Taking inverses, $A^{-1} = PQ = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & -3 & 3 \end{bmatrix}.$

Example 9: Reduce $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ to its first canonical form (*Normal form*) N and compute the matrix $PAQ = N$. **[NIT Kurukshetra, 2008]**

Solution: Since the matrix A is 3×4 i.e. with 3 rows and 4 columns, therefore, we shall take $I_{3\times 3} A I_{4\times 4}$ in such a way that $I_{3\times 3}$ is employed for elementary row operations and $I_{4\times 4}$ for elementary column operations.

Write $A = IAI$

i.e., $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operate $(R_2 - 2R_1),(R_3 - 3R_1)$, $\sim \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operate $(C_2 - 2C_1),(C_3 - 3C_1),(C_4 + 2C_1)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operate $(R_3 - R_2)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operate $-\frac{1}{6}C_2$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -3 & 2 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operate $(C_3 + 5C_2),(C_4 - 7C_2)$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$, , where

$P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are two non-singular matrices.

Example 10: Prove that the row equivalent matrices have the same rank.

OR

Show that the elementary row operations do not alter the rank of a given matrix.

Solution: If the matrices A and B are row equivalent, then B can be obtained from A by elementary row operations. It follows that each row vector of B must be a linear combination of the row vectors of A. So the row space of B must be a sub-space of row space of A. Similarly, the row space A must be a row sub-space of the row space of B.

Thus, the row space of A is identical to the row space of B, and hence the dimension of the row space of A (i.e. rank, $r(A)$) must be equal to row space of B (i.e. rank, $r(B)$).

1.3 PORTIONING OF MATRICES FOR ADDITION, MULTIPLICATION AND INVERSE

Definition: For convenience, matrices are divided into sub-matrices by drawing lines parallel to the rows and columns of the given matrices. Thus, the process of dividing a matrix into sub-matrices enclosed into rectangular boxes formed by the intersection of lines drawn parallel to the rows and columns of the given matrix is called 'partitioning' of matrix.

Addition and Multiplication by Partioning Method

If A and B are the two matrices of the same order and are conformable for addition and product, then their sum and product can also be obtained by partitioning method as explained below

Let $$A = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] \text{ and } B = \left[\begin{array}{cc|c} 10 & 11 & 12 \\ 13 & 14 & 15 \\ \hline 16 & 17 & 18 \end{array}\right]$$

or if $$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \text{ and } B = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

We can add two matrices A and B identically partitioned provided the corresponding sub-matrices A_1 and B_1, A_2 and B_2, etc. of A and B respectively having the same order.

Thus, $$A + B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1 + B_1 & A_2 + B_2 \\ A_3 + B_3 & A_4 + B_4 \end{bmatrix}$$

whereas $$A_1 + B_1 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 10 & 11 \\ 13 & 14 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 17 & 19 \end{bmatrix},$$

$$A_2 + B_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \end{bmatrix},$$

$$A_3 + B_3 = [7 \quad 8] + [16 \quad 17] = [23 \quad 25]$$

$$A_4 + B_4 = [9] + [18] = [27]$$

$\therefore$ $$A + B = \begin{bmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{bmatrix}$$ which is same as if A and B are directly added.

For multiplication

Let $$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

then
$$AB = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 3 & 1 & 2 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$= [A_1B_1 + A_2B_2]$$

$$= \left[\begin{bmatrix} I_3 & I_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 9 & 3 & 6 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 4 & 1 & 2 \\ 6 & 3 & 4 \\ 9 & 3 & 7 \end{bmatrix}$$

which is same as if A and B were multiplied without partitioning.

Inverse of a Matrix by Partition Method

By partition method, the inverse of a matrix of order $(n + 1)$ can be obtained if the inverse of the matrix of order n is known simply by adding one more row and one more column to this nth order matrix.

Let the matrix $A = [a_{ij}]$ of order n and its inverse $B = [b_{ij}]$ be partitioned into submatrices of indicated orders:

$$\left[\begin{array}{c|c} A_1 & A_2 \\ (p \times p) & (p \times q) \\ \hline A_3 & A_4 \\ (q \times p) & (q \times q) \end{array}\right] \text{ and } \left[\begin{array}{c|c} B_1 & B_2 \\ (p \times p) & (p \times q) \\ \hline B_3 & B_4 \\ (q \times p) & (q \times q) \end{array}\right] \text{ where } p + q = n$$

Since $AB = I_n = BA$, we have

$$\left.\begin{array}{ll} (i)\ A_1B_1 + A_2B_3 = I_n & (iii)\ A_3B_1 + A_4B_3 = 0 \\ (ii)\ A_1B_2 + A_2B_4 = 0, & (iv)\ A_3B_2 + A_4B_4 = 1 \end{array}\right\} \quad \ldots(1)$$

Then provided A_1 is non-singular,

$$\left.\begin{array}{ll} B_1 = A_1^{-1} + (A_1^{-1} A_2)\eta^{-1} (A_3A_1^{-1}), & B_3 = -(A_3A_1^{-1})\eta^{-1} \\ B_2 = -(A_1^{-1} A_2)\eta^{-1}; & B_4 = \eta^{-1} \end{array}\right\} \quad \ldots(2)$$

where $\eta = B_4^{-1} = A_4 - A_3\,(A_1^{-1} A_2)$

Practically, 'A_1' is taken of order $(n - 1)$ and if inverse of A_1, i.e. A_1^{-1} is made known then the inverse of A, i.e. a matrix of order $(n - 1) + 1 = n$ can be made known.

To obtain A_1^{-1}, the following procedure is used.

Let $$D_2 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},\ D_3 = \left[\begin{array}{cc|c} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c \end{array}\right],\ D_4 = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ \hline d_1 & d_2 & d_3 & d_4 \end{array}\right]$$

On computing D_2^{-1}, partition D_3 so that $A_4 = [C_3]$ and use (2) to obtain D_3^{-1}. Repeat the process on D_4 after partitioning it so that $A_4 = [d_4]$ and so on.

Example 11: Find the inverse of $\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & 4 & 3 \\ \hline 1 & 3 & 4 \end{array}\right]$, **using partition.**

Solution: Take $A_1 = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix},\quad A_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix},\quad A_3 = [1 \quad 3],\quad A_4 = [4]$

Now $$A_1^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix},\quad A_1^{-1}A_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A_3A_1^{-1} = \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\eta = A_4 - A_3\,(A_1^{-1}A_2) = [4] - \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}[1]$$

and $\eta^{-1} = [1]$

Then, $$B_1 = A_1^{-1} + (A_1^{-1}A_2)\,\eta^{-1}\,(A_3A_1^{-1})$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}[1]\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix}$$

$$B_2 = -\left(A_1^{-1}A_2\right)\eta^{-1} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$B_3 = -\eta^{-1}(A_3A_1^{-1}) = [-1 \quad 0]$$

$$B_4 = \eta^{-1} = [1].$$

Thus $$A^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Example 12: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$

Solution: Step (*i*) Take $D_3 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ and make partitions so that

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 4 \end{bmatrix}, \quad A_4 = [3]$$

Now $$A_1^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad A_1^{-1} A_2 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

$$A_3 A_1^{-1} = \begin{bmatrix} 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$\eta = A_4 - A_3\left(A_1^{-1}A_2\right) = [3] - \begin{bmatrix} 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = [-3]$$

and $$\eta^{-1} = \left[-\frac{1}{3}\right]$$

Then, $$B_1 = A_1^{-1} + (A_1^{-1}A_2)\,\eta^{-1}\,(A_3A_1^{-1})$$

$$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}\left[-\frac{1}{3}\right]\begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$B_2 = -\left(A_1^{-1} A_2\right)\eta^{-1} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_3 = -\eta^{-1} - \left(A_3 A_1^{-1}\right) n^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$B_4 = \eta^{-1} = \left[-\frac{1}{3}\right] = \frac{1}{3}[-1]$$

and $$D_3^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Step (*ii*) Partition A so that

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad A_4 [1]$$

Now $$A_1^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}, \quad A_1^{-1}A_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad A_3A_1^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -3 & 2 \end{bmatrix},$$

$$\eta = A_4 - A_3\,(A_1^{-1}\,A_2) = [1] - \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = \left[\frac{1}{3}\right]$$

and $\eta^{-1} = [3]$

Then, $B_1 = A_1^{-1} + (A_1^{-1}A_2)\eta^{-1}(A_3A_1^{-1})$

$$= \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}[3]\frac{1}{3}\begin{bmatrix} 2 & -3 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 6 & -9 & 6 \\ -2 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

$$B_2 = -\left(A_1^{-1}A_2\right)\eta^{-1} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

$$B_3 = -\left(A_3A_1^{-1}\right)\eta^{-1} = -\begin{bmatrix} 2 & -3 & 2 \end{bmatrix}$$

$$\mathrm{B}_4 = \eta^{-1} = [3]$$

and therefore $A^{-1} = B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

Example 13: Compute the inverse of the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$$

Solution: Step (*i*) Consider the first symmetric matrix

$$D_3 = \left[\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 3 & 2 \\ \hline -1 & 2 & 1 \end{array}\right] \text{ partitioned, such that}$$

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ A_3 = \begin{bmatrix} -1 & 2 \end{bmatrix},\ A_4 = [1]$$

$\therefore$ $$A_1^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix},\ A_1^{-1}A_2 = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\eta = A_4 - A_3\,(A_1^{-1} A_2)$$

$$= [1] - \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [-2] \quad \text{and} \quad \eta^{-1} = \left[-\frac{1}{2} \right],$$

Then $$B_1 = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left[-\frac{1}{2} \right] \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \; B_3 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \; B_4 = \left[-\frac{1}{2} \right]$$

and $$D_3^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$$

Step (*ii*): Now consider the matrix A partitioned, such that

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2 & -3 & -1 \end{bmatrix}, \; A_4 = [4]$$

$\therefore$ $$A_1^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$$

$$A_1^{-1} A_2 = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ -2 \end{bmatrix}, \; \eta = \left[\frac{18}{5} \right], \quad \eta^{-1} = \left[\frac{5}{18} \right]$$

then $$B_1 = \frac{1}{18} \begin{bmatrix} 2 & 5 & -7 \\ 5 & -1 & 5 \\ -7 & 5 & 11 \end{bmatrix}, \; B_2 = \frac{1}{18} \begin{bmatrix} -1 \\ -2 \\ 10 \end{bmatrix}$$

$$B_3 = \frac{1}{18} \begin{bmatrix} 1 & -2 & 10 \end{bmatrix}, \; B_4 = \left[\frac{5}{18} \right]$$

$\therefore$ $$A^{-1} = B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}.$$

Example 14: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ by partitioning.

Solution: We can't take $A_{11} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ since it is singular.

Take $\quad R_{23}\,A = B$ (say),

where R_{23} is an elementary matrix obtained by elementary row transformation of unit matrix. On applying B^{-1} on both sides,

$$B^{-1} R_{23} A = B^{-1} B = 1 \quad \text{or} \quad B^{-1} R_{23} = A^{-1}$$

On finding, $\quad B^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, we get

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Thus, if the $(n-1)$th order square minor, A_{11} of n-square non-singular matrix A is singular, we first bring a non-singular $(n-1)$-square matrix into the upper left corner to obtain B, find the inverse of B, and by the proper transformation on B^{-1}, obtain A^{-1}.

1.4 TRIANGULARIZATION OF MATRICES *(FACTORIZATION OF MATRICES)*

The process of factorization of a square matrix A (say) into the product of lower triangular (with unit diagonal elements) and upper triangular matrices, provided all principal minors of A are non-zero is called as **Triangularization** of matrices.

E.g., if $A = [a_{ij}]$ is a square matrix of order 3 with

$$a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \text{ and } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{vmatrix} \neq 0$$

where $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then $\quad A = LU,$

Inverse By Doolittle Triangularization Method

As defined above, a square matrix A can be written as $A = LU$ …(1)

where L is the lower triangular matrix and U is the upper triangular matrix.

From relation (1), we can write

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} \quad \text{…(2)}$$

We also know that $LL^{-1} = 1$, i.e. if we take $L^{-1} = B$ (which is also a lower triangular matrix)

then $$LL^{-1} = LB = I \quad \ldots(3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b_{11} & 0 & 0 \\ l_{21}b_{11} + b_{21} & b_{22} & 0 \\ l_{31}b_{11} + l_{32}b_{21} + b_{31} & l_{32}b_{22} + b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On equating, we have

$$b_{11} = b_{22} = b_{33} = 1 \quad \ldots(4)$$

$$l_{21}b_{11} + b_{21} = 0 \Rightarrow b_{21} = -l_{21} \quad \ldots(5)$$

$$l_{31}b_{11} + l_{32}\, b_{21} + b_{31} = 0 \Rightarrow l_{31} \cdot 1 + l_{32} \cdot (-l_{21}) + b_{31} = 0 \Rightarrow b_{31} = -l_{31} + l_{21}l_{32} \quad \ldots(6)$$

and $$l_{32}b_{22} + b_{32} = 0 \Rightarrow b_{32} = -l_{32} \quad \ldots(7)$$

Hence $L^{-1} = B = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} + l_{21}l_{32} & -l_{32} & 1 \end{bmatrix}$ is completely made known. …(8)

Again, if we take $U^{-1} = C$ (an upper triangular matrix) then

$$UU^{-1} = UC = I \quad \ldots(9)$$

i.e. $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & 0 & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} u_{11}C_{11} & u_{11}C_{12} + u_{12}C_{22} & u_{11}C_{13} + u_{12}C_{23} + u_{13}C_{33} \\ 0 & u_{22}C_{22} & u_{22}C_{23} + u_{22}C_{33} \\ 0 & 0 & u_{33}C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On equating, we get

$$\left.\begin{aligned} u_{11}C_{11} = 1 &\Rightarrow C_{11} = \frac{1}{u_{11}} \\ u_{22}C_{22} = 1 &\Rightarrow C_{22} = \frac{1}{u_{22}} \\ u_{33}C_{33} = 1 &\Rightarrow C_{33} = \frac{1}{u_{33}}; \end{aligned}\right\} \quad \ldots(10)$$

$$u_{11}C_{12} + u_{12}C_{22} = 0 \Rightarrow u_{11}C_{12} = -\frac{u_{12}}{u_{22}} \Rightarrow C_{12} = -\frac{u_{12}}{u_{11}u_{22}} \quad \ldots(11)$$

$$u_{22}C_{23} + u_{23}C_{33} = 0 \Rightarrow u_{22}C_{23} = -\frac{u_{23}}{u_{33}} \Rightarrow C_{23} = -\frac{u_{23}}{u_{22}u_{23}} \quad \ldots(12)$$

$$u_{11}C_{13} + u_{12}C_{23} + u_{13}C_{33} = 0 \Rightarrow C_{13} = -\frac{1}{u_{33}}\left(\frac{u_{13}}{u_{11}} - \frac{u_{12}u_{23}}{u_{11}u_{22}}\right) \qquad \ldots(13)$$

Thus $U^{-1} = C$ is completely known and hence we can find A^{-1} by putting the values of L^{-1} and U^{-1}.

Note: This technique of finding inverse is also called **Crout's Method**, if we take

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Example 15: Use Crout's triangularization (*Factorization*) method for finding the inverse for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

Solution: Let the given matrix be denoted by $A = [a_{ij}]$ so that

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \qquad \ldots(1)$$

Then by definition of triangularization, we can write

$$A = LU \qquad \ldots(2)$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

are lower triangular (with unit diagonal elements) and upper triangular matrices respectively.

or $$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad \ldots(3)$$

$$\left.\begin{array}{lll} u_{11} = 1, & u_{12} = 2, & u_{13} = 3 \\ 3 = l_{21}u_{11}, & 2 = l_{21}u_{12} + u_{22}, & 1 = l_{21}u_{13} + u_{23} \\ 2 = l_{31}u_{11}, & 1 = l_{31}u_{12} + l_{32}u_{22}, & 3 = l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{array}\right\} \qquad \ldots(4)$$

Solve these equations for $l_{21}, l_{31}, l_{32}, u_{22}, u_{23}, u_{33}$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{3}{4} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 3 \end{bmatrix}$$

Now if $$L^{-1} = B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \text{ and } U^{-1} = C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{bmatrix}$$

then $$LL^{-1} = I \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing both sides and then solving for b_{ij}'s,

$$L^{-1} = B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{1}{4} & -\frac{3}{4} & 1 \end{bmatrix} \qquad \ldots(5)$$

Similarly $\quad UU^{-1} = 1 = U^{-1}U$

or $$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_{11} & 2c_{11} - 4c_{12} & 3c_{11} - 8c_{12} + 3c_{13} \\ 0 & -4c_{22} & -8c_{22} + 3c_{23} \\ 0 & 0 & 3c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing respective elements on both sides and then for solving c_{ij}'s, we get

$$U^{-1} = C = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{4} & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Now $\quad A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = C \cdot B$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{4} & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{1}{4} & -\frac{3}{4} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} & \frac{1}{4} & \frac{1}{3} \\ \frac{7}{12} & \frac{1}{4} & -\frac{2}{3} \\ \frac{1}{12} & -\frac{1}{4} & \frac{1}{3} \end{bmatrix}.$$

1.5 VECTORS [PTU, 2006]

Definition: Any physical entity having n components say $x_1, x_2, \ldots, x_n$ written in a certain definite order is called a vector. Vector is briefly, in general, denoted by a single capital letter X.

Thus, by an n-dimensional vector X over F we meant an ordered set of n elements x_i of F, as

$$X = [x_1, x_2, \ldots, x_n]' = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

denoting row vector or column vector with $x_1, x_2, \ldots, x_n$ as Ist, 2nd, ..., nth elements respectively.

The vectors $X_1, X_2, \ldots, X_r$ are said to be **Linearly Dependent** if there exist numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$, not all zeros, such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_r X_r = 0 \qquad \ldots(1)$$

If no such number, other than zero, can be found, the vectors are said to be **Linearly Independent**.

If $\lambda_1 \neq 0$, we can write the above equation (1) as

$$X_1 = \mu_2 X_2 + \mu_3 X_3 + \ldots + \mu_r X_r \qquad \ldots(2)$$

Clearly, the vector X_1 is the linear combination of the vectors $X_2, X_3, \ldots, X_r$

Inner Product of Vectors: In general, all vectors are real and $V_n(R)$ denote the space of all real n-vectors.

If $X = [x_1, x_2, \ldots, x_n]'$ and $Y = [y_1, y_2, \ldots, y_n]'$

are two vectors of $V_n(R)$, their inner product is defined as a scalar

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \qquad \ldots(3)$$

which in actual practice is carried out, thus,

$$X \cdot Y = X'Y = Y'X$$

In vector analysis, the inner product is called the dot product.

E.g. for vectors $X_1 = [1, 1, 1]'$, $X_2 = [2, 1, 2]'$, $X_3 = [1, -2, 1]'$,

we have
$$\left.\begin{aligned} X_1 \cdot X_2 &= 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5 \\ X_1 \cdot X_3 &= 1\cdot 1 + 1\cdot(-2) + 1\cdot 1 = 0 \\ X_1 \cdot X_1 &= 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3 \end{aligned}\right\} \qquad \ldots(4)$$

Orthogonal Vectors: Vectors X and Y are said to be orthogonal if their inner product is zero. Clearly, vectors X_1 and X_3 of the above example are orthogonal.

Normalization of a Vector: If we associate a non-zero vector X to a unique unit vector U obtained by dividing the components of X by $||X||$. This operation is called normalization of a vector.

Thus to normalize a vector $X = [1, 2, 3]'$, divide each component by $\|X\| = \sqrt{1+4+9} = \sqrt{14}$

and obtain the unit vector $\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]$, where $||X||$ denotes modulus of vector X.

Example 16: Are the vectors $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3,)$ $x_3 = (0, 1, 2)$, $x_4 = (-3, 7, 2)$ are linearly dependent? If so, find the relation between them.

Solution: It the given vectors are linearly dependent then there exist scalars $\lambda_1, \lambda_2, \lambda_3$, not all zero, such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \qquad \ldots(1)$$

implying
$$\left.\begin{aligned} \lambda_1 + 2\lambda_2 - 3\lambda_4 &= 0 \qquad \ldots(i) \\ 2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 &= 0 \qquad \ldots(ii) \\ 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 &= 0 \qquad \ldots(iii) \end{aligned}\right\} \qquad \ldots(2)$$

Thus we get three homogeneous equations in 4 unknowns.

For solving them, operate $2(ii) - (iii)$, we get

$$\lambda_2 = \frac{12}{5}\lambda_4 \qquad \ldots(iv)$$

Again take $4\,(i) - (iii)$,

$$5\lambda_2 - 2\lambda_3 - 14\lambda_4 = 0 \quad \text{or} \quad \lambda_3 - = \lambda_4 \qquad \ldots(v)$$

Lastly, on substituting values of λ_2 and λ_3 from (iv) and (v) respectively in (i), we get

$$\lambda_1 = -\frac{9}{5}\lambda_4 \qquad \ldots(vi)$$

Thus on solving for non-trivial solution, we get proportional values of the scalars as 9, –12, 5, –5 respectively and get the desired relation as:

$$9x_1 - 12x_2 + 5x_3 - 5x_4 = 0 \qquad \ldots(3)$$

Alternately: Using x_1 to reduce the first component to zero, we get

$$(2x_1 - x_2) = (2, 4, 8) - (2, -1, 3) = (0, 5, 5)$$

and $$(3x_1 + x_4) = (3, 6, 12) + (-3, 7, 2) = (0, 13, 14)$$

Now using x_3 to reduce the second component to zero, we get

$$(2x_1 - x_2 - 5x_3) = (0, 5, 5) - (0, 5, 10) = (0, 0, -5) \qquad \ldots(4)$$

and $$(3x_1 + x_4 - 13x_3) = (0, 13, 14) - (0, 13, 26) = (0, 0, -12) \qquad \ldots(5)$$

Now multiplying (4) by 12 and (5) by 5 and take the difference of the two, we have

$$12(2x_1 - x_2 - 5x_3) - 5(3x_1 + x_4 - 13x_3) = (0, 0, 0)$$

$$\Rightarrow \quad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Hence the given vectors are Linearly dependent.

Observations: On applying elementary row operations to the vectors x_1, x_2, x_3, x_4; we see that the matrices

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix};\ B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 2x_1 - x_2 - 5x_3 \end{bmatrix} \text{ and } C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 9x_1 - 12x_2 + 5x_3 - 5x_4 \end{bmatrix}$$

have the same rank, as we have been able to obtain a null vector $(9x_1 - 12x_2 + 5x_3 - 5x_4)$ only because x_1, x_2, x_3, x_4 are linearly dependent and x_1 can be expressed as a linear combination of x_2, x_3, x_4 viz. $\frac{1}{9}(12x_2 - 5x_3 + 5x_4)$.

Similar results will hold for column operations and for any matrix.

Note: It should be noted that if we have n-component vectors, at the most n could be linearly independent, as illustrated below:

$$\underset{\text{Ist}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}}, \quad \underset{\text{IInd}}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 4 \end{bmatrix}}$$

The rows of the Ist matrix are linearly dependent while that of IInd are linearly independent.

Since the Ist matrix is formed from the Ist three columns of the IInd matrix, we shall apply the row operations only to the IInd matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 4 \end{bmatrix}, (R_3 - R_2) \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}, \begin{matrix} (R_3 - R_1) \\ (R_2 - 2R_1) \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that in the given matrices, Ist is of rank 2 and the IInd is of rank 3. Hence the rows of the Ist matrix are linearly dependent, while those of IInd are not.

It follows that if a given matrix has r linearly independent rows and the remaining rows are linear combination of these rows, then the rank of the matrix is r. Conversely, if a matrix 'A' is of rank r, it contains r linearly independent rows. The remaining rows of 'A' (if any) can by expressed as linear combination of these rows.

Example 17: Are the following vectors linearly dependent? If so, find the relation between them:

(1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9).

Solution: For linearly dependence of the vector x_1, x_2, x_3 we have the relation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

implying
$$\left.\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \quad \ldots(i) \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \quad \ldots(ii) \\ \lambda_1 + 3\lambda_2 + 4\lambda_3 &= 0 \quad \ldots(iii) \\ 3\lambda_1 + 4\lambda_2 + 9\lambda_3 &= 0 \quad \ldots(iv) \end{aligned}\right\}$$

From (i), we have $\lambda_1 = -(\lambda_2 + 2\lambda_3)$ ... (v)

Putting (v) into eqns. (iii) and (iv), we get

$$\lambda_2 + \lambda_3 = 0 \quad \ldots(vi)$$

and
$$\lambda_2 + 3\lambda_3 = 0 \quad \ldots(vii)$$

From (vi) and (vii), we see $\lambda_2 = 0 = \lambda_3$ ...($viii$)

Further, on using ($viii$) in (v), we see $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Thus x_1, x_2, x_3 are not linearly dependent as there are no such non-zero λ_i's which put

$$\lambda x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0.$$

1.6 CONSISTENCY AND SOLUTIONS OF LINEAR EQUATIONS: ROUCHE'S THEOREM

Definition: Consider a system of m linear equations in the n unknowns $x_1, x_2, \ldots, x_n$ i.e.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= \alpha_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= \alpha_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= \alpha_n \end{aligned}\right\} \quad \ldots(1)$$

in which the coefficients (a_{ij}'s) and the constants (α_i's) are in F.

By a solution of the system in F, meant any set of values of $x_1, x_2, \ldots, x_n$ in F which satisfy simultaneously these m equations.

When the system has a solution it is said to be 'Consistent', otherwise 'Inconsistent'.

A consistant system has either just one i.e., unique solution or infinite many solutions. The two systems of linear equations over F in the same number of unknowns are called '**equivalent**' if every solution of either system is a solution of the other.

In matrix notations, the system of linear equations (1) may be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

or, more precisely, as $AX = \alpha$ ...(2)

where $A = [a_{ij}]$ is the coefficient matrix and $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]'$

Now consider the augmented matrix 'K' (say)

$$K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & \alpha_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & \alpha_n \end{bmatrix} = [A : \alpha]$$

Rouche's Theorem: The systems of equations (1) is said to be 'consistent' if and only if the coefficient matrix 'A' and the augmented matrix 'K' are of the same rank, otherwise, 'inconsistent'. This is known as "Rouche' Theorem". **[NIT Kurukshetra, 2008]**

We consider the following two possible cases:

Case (*i*) Rank of A = rank of $K = r$ ($r \geq m$ or n whichever is smaller) means set of equation (1) can, by suitable row operations, be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= \beta_1 \\ 0x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= \beta_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0x_1 + 0x_2 + \dots + b_{rn}x_n &= \beta_r \end{aligned}\right\} \quad \dots(3)$$

and the remaining $(m - r)$ equations being all of the form.

$$0x_1 + 0x_2 + \dots + 0x_n = 0$$

The equation (3) will have a solution, through $(n - r)$ of the unknowns, may be chosen arbitrarily.

The solution will be unique only when $r = n$ $(= m)$

Hence the equations (1) are consistent.

Case (*ii*) Rank of A (i.e. r) < rank of K.

Let the rank of K be $(r + 1)$. In this cases, the equations (1) will reduce by suitable row operations to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= b_1, \\ 0x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0x_1 + 0x_2 + \dots + b_{rn}x_n &= b_r, \\ 0x_1 + 0x_2 + \dots + 0x_n &= b_{r+1}, \end{aligned}\right\} \quad \dots(4)$$

and the remaining $m - (r + 1)$ equations are of the form

$$0x_1 + 0x_2 + \dots + 0x_n = 0.$$

Clearly, the $(r + 1)$ equation cannot be satisfied by any set of values for the unknowns. Hence the equations (1) are inconsistent.

Working Rule for Testing the Consistency of System of Linear Equations

Find the ranks of the coefficient matrix 'A' and the augmented matrix 'K' by reducing A to the triangular form by elementary row or column operations. Let the rank of A be r and that of K be r'.

(*i*) If $r \neq r'$, the equations are inconsistent, i.e. there is no solution.

(*ii*) If $r = r' < n$, the equations are consistent and there are infinite many number of solutions. [Giving arbitrary value to $(n - r)$ of the unknowns, we may express the other r unknowns in terms of these.]

(*iii*) If $r = r' = n$ (the number of unknowns), the system possesses a unique solution.

Consistency of System of Linear Homogeneous Equations

If $m = n$ and $\beta_1 = \beta_2 = \ldots = \beta_m = 0$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \qquad \ldots(5)$$

This system is always consistant and have either unique or infinite many set of solutions. Since here is no K (augmented matrix) and, therefore, no question of inconsistency.

Thus if 'A' is non-singular, i.e. $|A| \neq 0$, the only solution will be trivial one (i.e. unique solution), viz. $x_1, x_2, \ldots = x_n = 0$

But if 'A' is singular, i.e. $|A| = 0$, the system of equations given in (4) will have infinite many solutions.

Example 18: Show that the equations $\left.\begin{aligned} 3x + 4y + 5z &= a, \\ 4x + 5y + 6z &= b, \\ 5x + 6y + 7z &= c \end{aligned}\right\}$ **do not have a solution unless** $\boldsymbol{a + c = 2b}$. **[Raipur, UP Tech, 2004; NIT Jalandhar, 2005; KUK, 2006]**

Solution: The above system of equations in matrix form can be represented as

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ or more precisely } AX = D$$

For the above system to possess a solution, we must have the rank of 'A' and that of 'K' equal.

Therefore, to test the rank of A and K, we write A and K collectively as:

$$K = [A : D] = \begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & : & (a+c)-2b \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix}$$

by operation $(R_1 + R_3 - 2R_2)$

Clearly, from above, the rank of the matrix 'A' is 2. So for the system to be consistent the rank of 'K' should be 2, which is only possible if $(a + c) - 2b = 0$, i.e., if $(a + c) = 2b$.

With above condition, the system of equations will have infinite many solutions, since $r_A = r_K = 2 < n\ (= 3)$.

Example 19: Investigate for what values of λ and μ, the simultaneous equations

$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}\right\}$$

have (*i*) no solution, (*ii*) unique solution, (*iii*) infinite many number of solutions.

[UPTech, 2006; NIT Jalandhar, 2004; PTU, 2005, 2007, Sambalpur, 2002]

Solution: Express the above system of equations in matrix form, $AX = D$,
where A is the coefficient matrix.

The system admits a unique solution if and only if the matrix 'A' is non-singular, i.e. has the same rank as the number of variables, viz. 3.

or $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 1(2\times\lambda - 2\times 3) + 1(3-\lambda) \neq 0, \quad \text{i.e.} \quad \lambda \neq 3$$

Thus for unique solution, $\lambda \neq 3$ and μ may have any value.

If $\lambda = 3$, the system will not possess any solution for the values of μ other than 10 for which the matrices 'A' and 'K' are not of the same rank.

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & 3 & : & \mu \neq 10 \end{bmatrix}$$

Clearly, for $\lambda = 3$, $\mu \neq 10$ the systems does not possess any solution, since the rank of 'A' is 2 whereas that of 'K' is 3.

For $\lambda = 3$ and $\mu = 10$, the rank 'A' and that of 'K' is the same, viz. 2. Hence in this case system possesses an infinite many solutions.

Example 20: Show that if $\lambda \neq -5$, the system of equations

$$\left.\begin{aligned} 3x - y + 4z &= 3 \\ x + 2y - 3z &= -2 \\ 6x + 5y + \lambda z &= -3 \end{aligned}\right\} \text{ has a unique solution}$$

If $\lambda = -5$, show that the equations are consistent. Determine the solution in each case.

[KUK, 2001; UPTech, 2004]

Solution: The system of equations is consistent if the rank of 'A', the coefficient matrix and the augmented matrix 'K' are the same, and will have a unique solution if rank of 'A' = rank of 'K' = $n = 3$ (the number of variables).

So in order to have the rank of 'A' = 3, $|A| \neq 0$

$$\Rightarrow \quad \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} \neq 0 \quad \Rightarrow \quad 1(2\lambda + 10) \neq 0 \quad \Rightarrow \quad \lambda \neq -5$$

For infinite many solutions, we must have rank 'A' = rank 'K' (augmented matrix) $\leq n = 3$.

$$\therefore \quad \text{Check the rank of } 'A : K, \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 1 & 2 & -3 & : & -2 \\ 6 & 5 & -5 & : & -3 \end{bmatrix}$$

$$\text{Operate} \quad \begin{matrix} (R_1 - 3R_2), \\ (R_3 - 6R_2), \end{matrix} \sim \begin{bmatrix} 0 & -7 & 13 & : & 9 \\ 1 & 2 & -3 & : & -2 \\ 0 & -7 & 13 & : & 9 \end{bmatrix}$$

$$\text{Operate} \quad (R_3 - R_1), \sim \begin{bmatrix} 0 & -7 & 13 & : & 9 \\ 1 & 2 & -3 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Clearly, rank 'A' = Rank 'K' = 2 < 3 = n

Now from above, we have $\left.\begin{matrix} -7y + 13z = 9 \\ x + 2y - 3y = -2 \end{matrix}\right\}$

Further, if $z = 0$ then $x = \frac{4}{7}$ and $y = -\frac{9}{7}$

let $z = k$ which $\Rightarrow$ $y = \left(\frac{13k - 9}{7}\right)$ and $x = \frac{1}{7}(4 - 5k)$

$$\textbf{Alternately} \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 1 & 2 & -3 & : & -2 \\ 6 & 5 & \lambda & : & -3 \end{bmatrix}$$

$$\text{Operate} \quad (R_3 - R_1 - 3R_2), \sim \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 1 & 2 & -3 & : & -2 \\ 0 & 0 & \lambda + 5 & : & 0 \end{bmatrix}$$

For unique solution; (λ + 5) should not be equal to zero, i.e. $\lambda \neq 5$.

Clearly if (λ + 5) = 0, i.e. the rank of the coefficient matrix 'A' and 'K' is 2 which is less than n = 3, the system will possess infinite many solutions.

Thus if (λ + 5) = 0, then we have nearly two equations

$$\left.\begin{matrix} 3x - y + 4z = 3 \\ x + 2y - 3z = -2 \end{matrix}\right\}$$

From these two equations, three variables x, y, z are to be found

Let $z = 0$, then $\left.\begin{array}{r} 3x - y = 3 \\ x + 2y = -2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = \frac{4}{7} \\ y = -\frac{9}{7} \end{array}\right\}$

Thus, the desired solution is $\left(\frac{4}{7}, -\frac{9}{7}, 0\right)$

But, if we take $z = k$ (some arbitrary constant), we get infinite many sets of values satisfying the given system of equations.

Example 21: For what values of k the equations $\left.\begin{array}{r} x + y + z = 1 \\ 2x + y + 4z = k \\ 4x + y + 10z = k^2 \end{array}\right\}$ **have a solution? Solve them completely in each case.** [KUK, 2005; PTU, 2005]

Solution: The system of given equations in matrix form is written as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Precisely $AX = B$, where A is the coefficient matrix.

The above given system will possess a solution if it is consistent, i.e. if the rank of A and B are same, and if equal to the number of variables involved, there will be a unique solution.

In order to check the rank of A and K, write

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix}$$

Operate $(R_3 - 3R_2 + 3R_1) \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 1 & 1 & 1 & : & k^2 - 3k + 3 \end{bmatrix}$

Hence clearly the rank of 'A' is 2 whereas that of 'K' = 3

But if $k^2 - 3k + 3$ is taken equal to 1, then Ist and IIIrd row of 'K' becomes the same and thus the rank of 'K' reduces to 2.

In **this case**, the system possesses an infinite many solutions.

So, $k^2 - 3k + 3 = 1$ or $k^2 - 2k - k + 2 = 0$, i.e. $k = 2, 1$

Case I: when $k = 1$, then from above,

$\left.\begin{array}{r} x + y + z = 1 \\ 2x + y + 4z = 1 \end{array}\right\}$ which on solving for x and y in terms of z gives $x = -3z$, $y = (1 + 2z)$

Case II: when $k = 2$, then from above system of equations

$\left.\begin{array}{r} x + y + z = 1 \\ 2x + y + 4z = 2 \end{array}\right\}$ which on solving for x and y in terms of z gives $x = (1 - 3z)$; $y = 2z$

Example 22: Examine the consistency of the system of equations $\left.\begin{aligned} 2x+6y+11&=0\\ 6x+20y-6z&=-3\\ 6y-18z&=-1 \end{aligned}\right\}$

Solution: Write the system of equations in the matrix form, i.e. $AX = B$ where A is the coefficient matrix.

$$\begin{bmatrix} 2 & 6 & 0\\ 6 & 20 & -6\\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} -11\\ -3\\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 6 & 0 & : & -11\\ 6 & 20 & -6 & : & -3\\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

Operate $(R_2 - 3R_1)$, $\sim \begin{bmatrix} 2 & 6 & 0 & : & -11\\ 0 & 2 & -6 & : & 30\\ 0 & 6 & -18 & : & -1 \end{bmatrix}$

Operate $(R_3 - 3R_2)$, $\sim \begin{bmatrix} 2 & 6 & 0 & : & -11\\ 0 & 2 & -6 & : & 30\\ 0 & 0 & 0 & : & 91 \end{bmatrix}$

Clearly, the rank of A is 2 whereas that of 'K' (the augmented matrix) is 3.
Hence the given system **does not possess any solution.**

Example 23: Solve the system to equations $\left.\begin{aligned} x_1 + 2x_3 - 2x_4 = 0, \quad & 2x_1 - x_2 - x_4 = 0\\ x_1 + 2x_3 - x_4 = 0, \quad & 4x_1 - x_2 + 3x_3 - x_4 = 0 \end{aligned}\right\}$

Solution: The coefficient matrix A is given by $\begin{bmatrix} 1 & 0 & 2 & -2\\ 2 & -1 & 0 & -1\\ 1 & 0 & 2 & -1\\ 4 & -1 & 3 & -1 \end{bmatrix}$

Operating $(R_2 - 2R_1)$, $(R_3 - R_1)$ and $(R_4 - 4R_1)$, $\sim \begin{bmatrix} 1 & 0 & 2 & -2\\ 0 & -1 & -4 & 3\\ 0 & 0 & 0 & 1\\ 0 & -1 & -5 & 7 \end{bmatrix}$

Operating $(-1)R_2$ and $(R_3 \leftrightarrow R_4)$, $\sim \begin{bmatrix} 1 & 0 & 2 & -2\\ 0 & 1 & 4 & -3\\ 0 & -1 & -5 & 7\\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operating $(R_3 + R_2) \sim \begin{bmatrix} 1 & 0 & 2 & -2\\ 0 & 1 & 4 & -3\\ 0 & 0 & -1 & 4\\ 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly, $r(A) = 4$ = number of unknowns.

Hence the equations have a unique solution which is trivial one, i.e.

$$x_1 = x_2 = x_3 = x_4 = 0.$$

Example 24: Solve the homogeneous equations

$$3x + 4y - z - 6w = 0, \quad 2x + 3y + 2z - 3w = 0$$

$$2x + y - 14z - 9w = 0, \quad x + 3y + 13z + 3w = 0$$

[JNTU, 2002]

Solution: Let A be the coefficient matrix then $A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

Operating $(R_1 \leftrightarrow R_4) \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}$

Operating $(R_2 - 2R_1)$, $(R_3 - 2R_1)$ and $(R_4 - 3R_1)$, we get $\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$

Operating $\left(-\frac{1}{3}\right) R_2$ and $(R_4 - R_3)$, $\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & -5 & -40 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operating $(R_3 + 5R_2)$, we get $A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly, $r(A) = 2 <$ the number of unknowns which is 4. Thus, the system of the equations will have infinite sets of solutions including the trivial solution $x = y = z = w = 0$.

The reduced system of equations is

$$\left.\begin{aligned} x + 3y + 13z + 3w &= 0 \\ y + 8z + 3w &= 0 \end{aligned}\right\}$$

By giving arbitrary value to any two variables, say $z = c_1$ and $w = c_2$ and solving the equations for the remaining variables x and y, we have

$$\left.\begin{aligned} x &= 11c_1 + 6c_2 \\ y &= -8c_1 - 3c_2 \\ z &= c_1 \\ w &= c_2 \end{aligned}\right\}, \text{ where } c_1 \text{ and } c_2 \text{ can take any value.}$$

Example 25: Solve completely the system of equation $\left.\begin{aligned} x+y-2z+3w=0, \\ x-2y+z-w=0, \\ 4x+y-5z+8w=0, \\ 5x-7y+2z-w=0 \end{aligned}\right\}$

[NIT Kurukshetra 2005, 02]

Solution: Find out the rank of A (For non-trivial solutions)

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4-(R_3+R_1), R_2-R_1 \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 4 & 1 & -5 & 8 \\ 0 & -9 & 9 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(R_4-3R_4), (R_3-4R_1) \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the rank of the matrix is '2' and it implies

$$\left.\begin{aligned} x+y-2z+3w=0 \quad &\ldots(i) \\ 3y+3z-4w=0 \quad &\ldots(ii) \end{aligned}\right\}$$

Let $z=\lambda$ and $w=\mu$; then from (*ii*), $y=\left(\frac{4}{3}\mu-\lambda\right)$

from (*i*), $x=\left(3\lambda-\frac{13}{3}\mu\right)$

giving infinite many sets of values of (x, y, z, w) for all possible values of λ and μ.

Example 26: Show that the system of equations $\left.\begin{aligned} 2x_1-2x_2+x_3=\lambda x_1 \\ 2x_1-3x_2+2x_3=\lambda x_2 \\ -x_1+2x_2=\lambda x_3 \end{aligned}\right\}$ can possess a non-trivial solution only if $\lambda=1$, $\lambda=-3$, obtain the general solution.

[NIT KURUKSHETRA, 2005, 03, 02]

Solution: The given system of homogeneous equations can be written as:

$$\left.\begin{aligned} (2-\lambda)x_1-2x_2+x_3=0 \\ 2x_1-(3+\lambda)x_2+2x_3=0 \\ -x_1+2x_2-\lambda x_3=0 \end{aligned}\right\}$$

In order to have non-trivial solution, $|A|$ should be zero.

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2 - \lambda)\ [-(3 + \lambda)(-\lambda) - 2 \times 2] + (-1)^{1+2}\ (-2)\ [-2\lambda + 2] + (-1)^{1+3} \times 1\ [4 - (3 + \lambda)] = 0$

$(-1)^{m+n}$, i.e. sign of cofactor of an element in a matrix when m denotes number of rows and n denotes the number of columns.

$\Rightarrow$ $\lambda^3 + \lambda^2 - 5\lambda + 3 = 0$

$\Rightarrow$ $\lambda = 1, 1, -3$

Hence the system possesses a non-trivial solution only if $\lambda = 1, -3$

Now for $\lambda = 1$, we have $[A]\ [X] = 0$

$$\Rightarrow \left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 4x_2 + 2x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \end{aligned}\right\} \text{ All the three equation are nearly same, viz. } x_1 - 2x_2 + x_3 = 0$$

If $x_3 = s$, $x_2 = t$ then $x_1 = (2t - s)$ $\quad \therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t - s \\ t \\ s \end{bmatrix}$

Further, for $\lambda = -3$, we have $AX = 0$

$$\Rightarrow \left.\begin{aligned} 5x_1 - 2x_2 + x_3 &= 0 & \ldots(i) \\ 2x_1 - 0x_2 + 2x_3 &= 0 & \ldots(ii) \\ -x_1 + 2x_2 + 3x_3 &= 0 & \ldots(iii) \end{aligned}\right\}$$

By (i) and (iii), $\dfrac{x_1}{(-6-2)=-8} = \dfrac{x_2}{(-1-15)=-16} = \dfrac{x_3}{(10-2)=8}$

$$\Rightarrow \qquad x_1 = -t, \quad x_2 = -2t, \quad x_3 = t \quad \therefore \quad X = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}.$$

1.7 SOLUTION OF LINEAR EQUATIONS BY CRAMER'S RULE AND ADJOINT METHOD

I. Method of Determinants—Cramer's Rule

Consider the system of non-homogeneous equations,

$$\left.\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n &= \alpha_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2n}\, x_n &= \alpha_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \ldots + a_{nn}\, x_n &= \alpha_n \end{aligned}\right\} \quad \text{or} \quad AX = \alpha$$

where $A = [a_{ij}]$ is the coefficient matrix and $\alpha = [\alpha_i]$ is matrix form of the scalars.

If the determinant of the coefficients be

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

then $$x_1 \Delta = \begin{vmatrix} x_1 a_{11} & a_{12} & \dots & a_{1n} \\ x_1 a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ x_1 a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}, \text{ where } \Delta = |A|$$

On operating $(C_1 + x_2C_2 + x_3C_3 + \dots)$, we get

$$x_1 \Delta = \begin{vmatrix} x_1 a_{11} + x_2 a_{12} + \dots + x_n a_{1n} & a_{12} \dots a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \dots + x_n a_{2n} & a_{22} \dots a_{2n} \\ \dots & \dots \\ x_1 a_{n1} + x_2 a_{n2} + \dots + x_n a_{nn} & a_{n2} \dots a_{nn} \end{vmatrix}$$

$$x_1 \Delta = \begin{vmatrix} \alpha_1 & a_{12} \dots a_{1n} \\ \alpha_2 & a_{22} \dots a_{2n} \\ \dots & \dots \\ \alpha_n & a_{n2} \dots a_{nn} \end{vmatrix}$$

or $\quad x_1 |A| = |A_1|$

where A_1 is the matrix obtained from A by replacing its Ist column with the column of constants, i.e. by $[\alpha]$.

Similarly, $$x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|}, \dots \text{ so on.}$$

In general, $x_i = \frac{|A_i|}{|A|}$, where A_i $(i = 1, 2, \dots, n)$ denotes the matrix obtained from A by replacing its ith column with the column of constants $[\alpha_i]$.

Example 27: Solve the system $\left.\begin{aligned} 2x_1 + x_2 + 5x_3 + x_4 &= 5 \\ x_1 + x_2 - 3x_3 - 4x_4 &= -1 \\ 3x_1 + 6x_2 - 2x_3 + x_4 &= 8 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 &= 2 \end{aligned}\right\}$ using Cramer's Rule.

Solution: We find $|A| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = 120,$

where A is the coefficient matrix obtained form the above system of equations.

$$|A_1| = \begin{vmatrix} 5 & 1 & 5 & 1 \\ -1 & 1 & -3 & -4 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -240,$$

where A_1 is the matrix obtained from the matrix A by replacing its Ist column by column of constants.

and
$$|A_2| = \begin{vmatrix} 2 & 5 & 5 & 1 \\ 1 & -1 & -3 & -4 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -24,$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -1 & -4 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = 0,$$

$$|A_4| = \begin{vmatrix} 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \\ 2 & 2 & 2 & 2 \end{vmatrix} = -96$$

where A_2, A_3 and A_4 are the matrices obtained from the matrix A by replacing elements of column 2, column 3 and column 4 by column of scalars (constants) respectively.

Thus,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-240}{-120} = 2, \quad x_2 = \frac{|A_2|}{|A|} = \frac{-24}{-120} = \frac{1}{5}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{0}{-120} = 0, \quad x_4 = \frac{|A_4|}{|A|} = \frac{-96}{-120} = \frac{4}{5}.$$

Example 28: In a given electrical network, the equations of the currents i_1, i_2, i_3 are

$$\left.\begin{aligned} 3i_1 + i_2 + i_3 &= 8 \\ 2i - 3i_2 - 2i_3 &= -5 \\ 7i_1 + 2i_2 - 5i_3 &= 0 \end{aligned}\right\} \text{ Calculate } i_1 \text{ and } i_3 \text{ by Cramer's rule.}$$

Solution: In matrix notations the above system of equations is written as below:

$$AX = B \text{ or } \begin{bmatrix} 3 & 1 & 1 \\ 2 & -3 & -2 \\ 7 & 2 & -5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix}$$

with A as coefficient matrix and B as scalar matrix.

Then by Cramer's rule, matrices A_1, A_2, A_3, are

$$A_1 = \begin{bmatrix} 8 & 1 & 1 \\ -5 & -3 & -2 \\ 0 & 2 & -5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & 8 & 1 \\ 2 & -5 & -2 \\ 7 & 0 & -5 \end{bmatrix}, \; A_3 = \begin{bmatrix} 3 & 1 & 8 \\ 2 & -3 & -5 \\ 7 & 2 & 0 \end{bmatrix},$$

obtained from the matrix replacing its Ist, IInd and IIIrd columns respectively by column of scalars.

Now
$$|A| = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -3 & -2 \\ 7 & 2 & -5 \end{vmatrix} = 78, \quad |A_1| = \begin{vmatrix} 8 & 1 & 1 \\ -5 & -3 & -2 \\ 0 & 2 & -5 \end{vmatrix} = 117$$

$$|A_2| = \begin{vmatrix} 3 & 8 & 1 \\ 2 & -5 & -2 \\ 7 & 0 & -5 \end{vmatrix} = 78, \quad |A_3| = \begin{vmatrix} 3 & 1 & 8 \\ 2 & -3 & -5 \\ 7 & 2 & 0 \end{vmatrix} = 195$$

Hence, $i_1 = \dfrac{|A_1|}{|A|} = \dfrac{117}{78} = 1.50$ units and $i_3 = \dfrac{|A_3|}{|A|} = \dfrac{195}{78} = 2.5$ units.

II. Matrix Inversion Method or Method of Adjoint

If $|A| \neq 0$, A^{-1} exists, then the solution of the system of equations given as

$$AX = \alpha, \quad A = [a_{ij}]$$

implying $\quad A^{-1}AX = A^{-1}\alpha.$

or $$X = A^{-1}\alpha = \frac{\text{adj A}}{\Delta}\alpha., \text{ with } X = [x_i] \text{ and } \alpha = [\alpha_i]$$

For example, 'A' is a matrix of order 3×3, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},$$

where A_{11}, A_{12}, etc. are the co-factors of a_{11}, a_{12}, etc. and Δ is non-zero value of the determinant of A.

Hence on equating the values of x_1, x_2, x_3 to the corresponding elements in the product on the right hand side of the above expression, we get the desired solution.

Note: The above method fails if 'A' is singular, i.e., if $|A| = 0$. It is also inapplicable when the number of equations and the number of unknowns are unequal as in such situation A^{-1} does not exist. Matrices can be usefully employed to the theory of such system of equations.

Example 29: Solve the following simultaneous equations by matrix inversion method:

$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 6 \end{aligned}\right\} \qquad \ldots(1)$$

Solution: The above system of equations in matrix notations is expressed as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \quad \text{or} \quad AX = \alpha \qquad \ldots(2)$$

From (2), we can have

$$X = A^{-1}\alpha \qquad \ldots(3)$$

where A^{-1} exists if and only if $|A| \neq 0$

Now $\quad |A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ on expanding by 1st row

$$= 1\,(18 - 12) - 1\,(9 - 3) + 1\,(4 - 2) = 2 \qquad \ldots(4)$$

Find adjoint A, which is the matrix obtained form the transpose of the matrix consisting of cofactors of the matrix A.

$$\text{Adjoint } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} = \begin{vmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{vmatrix} \qquad \ldots(5)$$

$$\therefore \quad A^{-1} = \frac{\text{Adj} A}{|A|} = \frac{1}{2}\begin{vmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{vmatrix} \qquad \ldots(6)$$

Thus, $X = A^{-1}B$ implies $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 3\times 3 & +(-5/2)4 & +(1/2)6 \\ -3\times 3 & +4\times 4 & +(-1)6 \\ 1\times 3 & +(-3/2)4 & +(1/2)6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Hence $x = 2$, $y = 1$, $z = 0$.

Example 30: By method of matrices, solve the following equations for x, y, z and w

$$\left.\begin{aligned} x - 3y + z &= a \\ 2x + y - w &= b \\ 3x - 2y - z - 2w &= c \\ 4x - y + 3w &= d. \end{aligned}\right\}$$

Solution: The given equations can be expressed in the matrix form, $AX = B$

or $$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 3 & -2 & -1 & -2 \\ 4 & -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Now $$|A| = \begin{vmatrix} 1 & -3 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 3 & -2 & -1 & -2 \\ 4 & -1 & 0 & 3 \end{vmatrix} = -70,$$

$$A^{-1} = \frac{\text{Adjoint } A}{|A|} = \frac{1}{70}\begin{bmatrix} 2 & 17 & 2 & 7 \\ -10 & 20 & -10 & 0 \\ 38 & 43 & -32 & -7 \\ -6 & -16 & -6 & 14 \end{bmatrix}$$

$$\therefore \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = X = A^{-1}B = \frac{1}{70}\begin{bmatrix} 2 & 17 & 2 & 7 \\ -10 & 20 & -10 & 0 \\ 38 & 43 & -32 & -7 \\ -6 & -16 & -6 & 14 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{70}(2a+17b+2c+7d) \\ -\frac{1}{7}(-a+2b-c) \\ \frac{1}{70}(38a+43b-32c-7d) \\ \frac{1}{35}(3a+8b+3c-7d) \end{bmatrix}.$$

Example 31: Using the loop current method on a circuit, the following equations are obtained:

$$\left.\begin{aligned} 7i_1 - 4i_2 &= 12 \\ -4i_1 + 12i_2 - 6i_3 &= 0 \\ -6i_2 + 14i_3 &= 0 \end{aligned}\right\}$$

By matrix method, solve for i_1, i_2 and i_3.

Solution: Under this method the solution is possible only if the coefficient matrix is non-singular, i.e. $|A| \neq 0$.

Find, $$|A| = \begin{vmatrix} 7 & -4 & 0 \\ -4 & 12 & -6 \\ 0 & -6 & 14 \end{vmatrix} = 7(168-36)+4(-56-0) = 700 \neq 0$$

Now $$A' = \begin{bmatrix} 7 & -4 & 0 \\ -4 & 12 & -6 \\ 0 & -6 & 14 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Co-factors of $a_{ij's}$ in A' are found $\begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix}$

$$\therefore \qquad A^{-1} = \frac{\text{Adjoint } A}{|A|} = \frac{1}{700}\begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix}$$

Hence $X = A^{-1}B$, where B is a scalar matrix $\begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$

implying $$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \frac{1}{700}\begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix}\begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{700}\begin{bmatrix} 132\times 12+0+0 \\ 56\times 12+0+0 \\ 24\times 12+0+0 \end{bmatrix} = \frac{1}{700}\begin{bmatrix} 132\times 12 \\ 56\times 12 \\ 24\times 12 \end{bmatrix}$$

$\therefore$ $$i_1 = \frac{132\times 12}{700} = \frac{396}{175},\ i_2 = \frac{56\times 12}{700} = \frac{168}{175},\ i_3 = \frac{24\times 12}{700} = \frac{72}{75}.$$

Example 32: Solve the set of simultaneous equations $$\left.\begin{aligned} x+y+z &= 3 \\ x+2y+3z &= 4 \\ 2x+3y+4z &= 7 \end{aligned}\right\}$$

Solution: Here the coefficient matrix is singular in nature, hence the method of inversion is inapplicable. Clearly, out of the above 3 equations, only two equations are independent, as the equation at serial number IIIrd is the sum of the first two.

Hence the given set of equations can be replaced by the set

$$\left.\begin{aligned} x+y+z &= 3 \\ x+2y+3z &= 4 \end{aligned}\right\}$$

The above two equations, which are in three variables cannot give a unique solution.

But, if we assume any one of the unknown arbitrarily, say $z = k$, then we write the equations as

$$\left.\begin{aligned} x+y &= 3-k \\ x+2y &= 4-3k \end{aligned}\right\}$$

The coefficient matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ of the above equations is non-singular, and its reciprocal is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Therefore, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3-k \\ 4-3k \end{bmatrix} = \begin{bmatrix} 2+k \\ 1-2k \end{bmatrix},\ (\text{since } X = A^{-1}B)$$

So $x = (2 + k),\ y = (1 - 2K),\ z = k$

Hence by giving different values to k, get different sets of solutions.

Note: If we replace the 3rd equation, viz. $2x + 3y + 4z = 7$ by $2x + 3y + 4z = 9$, then we see that the coefficient matrix is still singular, and the above set of values of x, y, z satisfies them, putting these values in the 3rd equation viz.

$$2x + 3y + 4z = 9, \text{ we get}$$

$$2(2 + k) + 3(1 - 2k) + 4k = 9, \text{ i.e. } 7 = 9$$

which is impossible. Hence no set of values can be found satisfying all these equations. The reason is that while the left-hand side of the 3rd equation is a combination of the first two (their sum), the right hand side does not follow the same combination. Such equation is said to be inconsistent.

1.8 EIGEN VALUES AND EIGEN VECTORS

Characteristic Equation: For every square matrix A of order n, we can form a matrix $[A - \lambda I]$ with I as the unit matrix of order n. The determinant of this matrix equated to zero, namely

$$|A - \lambda I| = 0, \quad \text{or} \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \dots(1)$$

is called the 'characteristic equation' of A

On expanding the determinant, we may write this equation as

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0 \qquad \dots(2)$$

where $k_1, k_2, \dots, k_n$ may be expressed in terms of the elements a_{ij}.

The roots of the 'characteristic equation' are called **Characteristic Roots** or **Latent-Roots** or **Eigen values** of the matrix A.

Note: A square matrix of order n will have n latent roots.

Characteristic Vector: Consider the linear transformation $Y = AX$ ...(3)
which carries the transformation of a column vector X into another column vector Y by means of a square matrix A.

In practice, several times, we need to find the particular vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms to its multiple λX by the transformation (3).

Then $\qquad \lambda X = AX \quad \text{or} \quad AX - \lambda IX = 0$

i.e. $\qquad (A - \lambda I)X = 0 \qquad \dots(4)$

The above matrix equation represents n homogeneous equations in n unknown say $x_1, x_2, \dots, x_n$

i.e.
$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0 \end{aligned}\right\} \qquad \dots(5)$$

These equations will have a non-zero solution only if the coefficient matrix is singular, i.e. if

$$|A - \lambda I| = 0 \qquad \dots(6)$$

This is known as the characteristic equation of the transformation, and is the same as the characteristic equation of the matrix A. This has n roots and corresponding to each root, there exists non-zero solution,

$$X = [x_1, x_2, \dots, x_n]'$$

which is known as **Characteristic Vector** or **Eigen Vector** or **Invariant Vector** or **Latent Vector.**

Note: For n distinct eigen values, there exist n independent eigen vectors. However, corresponding to two or more repeated eigen values, it may or may not be possible to get linearly independent eigen vectors.

Further, if X_i is the eigen vector corresponding to the eigen value, λ_i, then it follows from (4) that cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to a root is not unique, but may be one of the vectors cX_i.

Properties of Characteristic Roots (Eigen Values)

(I) The sum of the n-characteristic values of an n-square matrix A is the sum of the elements in the principal diagonal, i.e. if $\lambda_1, \lambda_1, \ldots, \lambda_n$ are the n-characteristic roots of an n-square matrix $A = [a_{ij}]$, $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ then

$(\lambda_1 + \lambda_2 + \ldots + \lambda_n) = (a_{11} + a_{22} + \ldots + a_{nn})$.

We prove the result physically for a matrix of order 3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with characteristic roots $\lambda_1, \lambda_2, \lambda_3$

then the corresponding characteristic equation

$$|A - \lambda I| = \begin{vmatrix} a_{11-\lambda} & a_{12} & a_{13} \\ a_{21} & a_{22-\lambda} & a_{23} \\ a_{31} & a_{32} & a_{33-\lambda} \end{vmatrix} = (-\lambda)^3 + \lambda^2 (a_{11} + a_{22} + a_{33}) + \ldots \qquad \ldots(1)$$

Also if $\lambda_1, \lambda_2, \lambda_3$ be the characteristic roots of A, then

$$\begin{aligned} |A - \lambda I| &= (-1)^3 [(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)] \\ &= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \ldots \qquad \ldots(2) \end{aligned}$$

Thus, on equating the coefficients of equal powers of λ on both sides of (1) and (2), we have

$$(\lambda_1 + \lambda_2 + \lambda_3) = (a_{11} + a_{22} + a_{33})$$

(II) If l_i be the characteristic roots of a matrix A, then $\frac{1}{\lambda_i}$ are the characteristic roots of the matrix A^{-1}. **[PTU, 2005]**

Let X_i be the characteristic vector of A corresponding to characteristic value λ_i then linear transformation,

$$AX_i = \lambda_i X_i \qquad \ldots(3)$$

Operating A^{-1} on both sides, $A^{-1}AX_i = A^{-1}\lambda_i X_i$

or $\quad IX_i = \lambda_i (A^{-1} X_i)$

$$\Rightarrow \quad A^{-1} X_i = \left(\frac{1}{\lambda_i}\right) X_i \qquad \ldots(4)$$

which is alike equation (3).

Hence $\frac{1}{\lambda_i}$ represents the characteristic roots of A^{-1}.

(III) If λ_i are the characteristic values of an orthogonal matrix A, then $\frac{1}{\lambda_i}$ are also the characteristic values of A.

As we have just proved in the II case that if λ_i are the characteristic values of A, $\frac{1}{\lambda_i}$ are the characteristic values of A^{-1}

Since the matrix A is orthogonal, i.e. $A^{-1} = A'$.

$\therefore \quad \dfrac{1}{\lambda_i}$ are the characteristic roots of A′.

Again, the matrices A and A' have the same characteristic roots since the determinant $|A - \lambda I|$ and $|A' - \lambda I|$ are the same.

Hence $\dfrac{1}{\lambda_i}$ also represents the characteristic roots of A.

(IV) If λ_i's $(i = 1, 2, \ldots, n)$ are the characteristic roots of a matrix A, then λ_i^m are the characteristic roots of A^m.

Let X_i be the characteristic vector of the matrix A corresponding to the characteristic roots λ_i, then by the linear transformation (3), we have

$$AX_i = \lambda_i X_i$$

Multiplying both sides by A,

$$A^2 X_i = A \cdot \lambda_i X_i = \lambda_i (AX_i) = \lambda_i (\lambda_i X_i), \text{ (By above equation)}$$

or $$A^2 X_i = \lambda_i^2 X_i \qquad \ldots(5)$$

Again multiplying by A on both sides,

$$A^3 X_i = A(\lambda_i^2 X_i) = \lambda_i^2 (AX_i) = \lambda_i^3 X_i \qquad \ldots(6)$$

and so on $$A^m X_i = \lambda_i^m X_i \qquad \ldots(7)$$

Hence λ_i^m are the characteristic roots of the matrix A^m.

(V) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic roots of an n-square matrix A and if k is a scalar then $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the characteristic roots of the matrix $(A - kI)$.

If $\lambda_1, \lambda_2, \ldots., \lambda_n$ are the n characteristic roots of n-square matrix A, then the corresponding characteristic equation of A is given by

$$|A - \lambda I| = 0 \qquad \ldots(8)$$

and the determinant value of the $[A - \lambda I]$ is

$$|A - \lambda I| = (-1)^n [(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)] \qquad \ldots(9)$$

Now on replacing λ by $(\lambda + k)$, we have

$$|A - (\lambda + k) I| = (-1)^n [((\lambda + k) - \lambda_1)((\lambda + k) - \lambda_2)\ldots((\lambda + k) - \lambda_n)]$$

or $$|(A - Ik) - \lambda I| = (-1)^n [(\lambda - (\lambda_1 - k)(\lambda - (\lambda_2 - k)) \ldots (\lambda - (\lambda_n - k))] \qquad \ldots(10)$$

Clearly, equation (10) is identical to the equation (9), which represents the determinant value in characteristic equation (8) of the matrix X correspondingly to its eigen values $\lambda_1, \lambda_2, \ldots.$

Hence $(\lambda_1 - k), (\lambda_2 - k), \ldots, (\lambda_n - k)$ would be representing the characteristic roots of the matrix $[A - Ik]$ with characteristic equation $|(A - Ik) - \lambda I| = 0$

Theorem 1: Show that characteristic vectors corresponding to real and distinct characteristic roots are linearly independent.

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ be three real and distinct characteristic (eigen) values and X_1, X_2, X_3 be the corresponding characteristic (invariant) vectors of the matrix A.

Let us assume, contrary, that there exist scalars a, b, c not all zero, such that

$$aX_1 + bX_2 + cX_3 = 0 \qquad \ldots(1)$$

Multiplying (1) by A, and recall that $AX_i = \lambda_i X_i$, we have

$$aAX_1 + bAX_2 + cAX_3 = a\lambda_1 X_1 + b\lambda_2 X_2 + c\lambda_3 X_3 = 0, \text{ as } [A - \lambda_i I]X_i = 0 \qquad \ldots(2)$$

Multiply (2) by A, again, and obtain

$$a\lambda_1^2 X_1 + b\lambda_2^2 X_2 + c\lambda_3^2 X_3 = 0 \qquad \ldots(3)$$

Now writing (1), (2), (3) together as

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} aX_1 \\ bX_2 \\ cX_3 \end{bmatrix} = 0 \qquad \ldots(4)$$

Now we see that

$$|B| = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = -(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq 0$$

Hence B^{-1} exists.

Multiplying (4) by B^{-1} results in, $\begin{bmatrix} aX_1 \\ bX_2 \\ cX_3 \end{bmatrix} = 0.$

But this requires $a = b = c = 0$ which is contrary to the hypothesis.

Thus X_1, X_2, X_3 are linearly independent.

Theorem 2: If λ be a non-zero characteristic root (eigen value) of the non-singular n-square matrix A, then $\dfrac{|A|}{\lambda}$ is a charactristic polynomial of adjoint A.

Proof: For non-singular n-square matrix A, the 'characteristic polynomial'

$$\phi(\lambda) = |\lambda I - A| = \lambda^n + s_1\lambda^{n-1} + s_2\lambda^{n-2} + \ldots + s_{n-1}\lambda^1 + (-1)^n |A| \qquad \ldots(1)$$

where s_r ($r = 1, 2, \ldots, n-1$) is $(-1)^r$ times the sum of all the r-square principal minors of A.

Corresponding characteristic equation is given by

$$\lambda^n + s_1\lambda^{n-1} + s_2\lambda^{n-2} + \ldots + (-1)^n |A| = 0 \qquad \ldots(2)$$

and on the same lines

$$|\mu I - \text{Adj} \cdot A| = \mu^n + s_1\mu^{n-1} + s_2\mu^{n-2} + \ldots + s_{n-1}\mu + (-1)^n |\text{Adj} \cdot A| \qquad \ldots(3)$$

where s_r ($r = 1, 2, \ldots, n-1$) is $(-1)^r$ times the sum of the r-square principal minors of Adj $\cdot A$.

Thus by the property $|\text{adj } A| = |A|^{n-1}$ and definition of s_r

we have
$$\left.\begin{aligned} s_1 &= (-1)^n s_{n-1} \\ s_2 &= (-1)^n |A| s_{n-2}, \\ &\vdots \\ s_{n-1} &= (-1)^n |A| s_1; \end{aligned}\right\} \qquad \ldots(4)$$

then
$$|\mu I - \text{adj} \cdot A| = (-1)^n \{(-1)^n \mu^n + s_{n-1}\mu^{n-1} + s_{n-2}\mu^{n-2} |A| + \ldots + s_2 |A|^{n-3}\mu^2 + s_1 |A|^{n-2}\mu + |A|^{n-1}\}$$

$$= (-1)^n \left\{ 1 + s_1\left(\frac{\mu}{|A|}\right) + \ldots + s_{n-1}\left(\frac{\mu}{|A|}\right)^{n-1} + (-1)^n \left(\frac{\mu}{|A|}\right)^n |A| \right\} = f(\mu) \qquad \ldots(5)$$

Now

$$f\left(\frac{|A|}{\lambda}\right) = (-1)^n \left\{ 1 + s_1\left(\frac{1}{\lambda}\right) + \ldots + s_{n-1}\left(\frac{1}{\lambda}\right)^{n-1} + (-1)^n \left(\frac{1}{\lambda}\right)^n |A| \right\} \qquad \ldots(6)$$

and by equation (2), we have

$$\lambda^n f\left(\frac{|A|}{\lambda}\right) = (-1)^n \left\{ \lambda^n + s_1 \lambda^{n-1} + \ldots + s_{n-1}\lambda + (-1)^n |A| \right\} = 0$$

Hence, $\dfrac{|A|}{\lambda}$ is a characteristic root of adjoint A.

Theorem 3: Eigen values (characteristic roots) of orthogonal matrix *A* are of absolute value 1.

Proof: Let λ_i, X_i be characteristic roots and associated (characteristic vectors) invariant vectors of an orthogonal matrix A, then

$$X_i'\, X_i = X_1'\, (A'\, A)\, X_i = (AX_i)'\, (AX_i), \text{ since for orthogonal } A,\ A'A = I$$

$$\Rightarrow \quad X_i'\, X_i = (\lambda_i X_i)'\, (\lambda_i X_i) = (\lambda_i' X_c')\, (\lambda_i X_i) = \lambda_i \lambda_i\, X_i'\, X_i$$

or $\quad (1 - \lambda_i\lambda_i)\, X_i' X_i = 0 \quad$ implies $\quad (1 - \lambda_i\lambda_i) = 0, \quad$ since $\quad x_i' x_i \neq 0$

Thus $\qquad |\lambda_i| = 1.$

Theorem 4: Prove if $\lambda_i \neq \pm 1$ is a characteristic root and X_i is the associated invariant vector of an orthogonal matrix *A*, then $X_i'\, X_i = 0$.

Proof: For characteristic value λ_i and corresponding characteristic vector X_i of the orthogonal matrix A, we have

$$X_i'\, X_i = X_i'\, (A'\, A)\, X_i = (AX_i)'(AX_i), \quad \text{(as A is given orthogonal)}$$

$$\Rightarrow \quad X_i' X_i = (\lambda_i X_i)'\, (\lambda_i X_i) = \lambda_i \lambda_i X_i' X_i, \quad \text{Using the transformation, } AX_i = \lambda_i X_i$$

$$\Rightarrow \quad (1 - \lambda_i\lambda_i) X_i'\, X_i = 0$$

$$\Rightarrow \quad \text{Either } (1 - \lambda_i\lambda_i) = 0 \quad \text{or} \quad X_i'\, X_i = 0 \quad \text{But} \quad \lambda_i \neq \pm 1$$

Hence $\qquad X_i' X_i = 0.$

Theorem 5: For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal. [NIT Kurukshetra, 2004; KUK, 2004, 2006]

Proof: Let A be any symmetric matrix i.e., $A' = A$ and λ_1 and λ_2 two unequal eigen values, i.e., $\lambda_1 \neq \lambda_2$

Let X_1 and X_2 be the two corresponding eigen vectors.

Now for λ_1, $\quad (A - \lambda_1 I)\, X_1 = 0$

or $$AX_1 = \lambda_1 X_1 \quad \ldots(i)$$

Similarly $$AX_2 = \lambda_2 X_2 \quad \ldots(ii)$$

Taking the transpose of (*ii*), we get

$$(AX_2)' = (\lambda_2 X_2)'$$

$$X_2' A' = \lambda_2 X_2' \qquad \text{(as } \lambda_2 \text{ is an arbitrary constant)}$$

$$X_2' A = \lambda_2 X_2' \qquad \text{(Since } A' = A\text{)}$$

$$X_2' A X_1 = \lambda_2 X_2' X_1$$

$$X_2' (\lambda_1 X_1) = \lambda_2 X_2' X_1 \qquad \text{(As } AX_1 = \lambda_1 X_1\text{)}$$

$$\lambda_1 X_2' X_1 = \lambda_2 X_2' X_1$$

$$(\lambda_1 - \lambda_2)\, X_2'\, X_1 = 0 \quad \text{But} \quad \lambda_1 - \lambda_2 \neq 0$$

$\therefore$ $$X_2' X_1 = 0$$

If $$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore$ $$X_2' X_1 = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y_1 x_1 + y_2 y_2 + y_3 y_3$$

Clearly, $(y_1 x_1 + y_2 x_2 + y_3 x_3) = 0$

This means, the two system of co-ordinates are orthogonal.

$\therefore$ Hence the transformation is an orthogonal transformation.

Example 33: Determine the eigen values and eigen vectors of $\boldsymbol{A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}}$

[NIT Kurukshetra, 2008]

Solution: The characteristic equation,

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$\Rightarrow$ The roots of above equation are 5, –3, –3.

Putting $\lambda = 5$, the equations to be solved for x_1, x_2, x_3 are $[A - \lambda I]x = 0$

i.e. $-7x + 2y - 3z = 0$, $2x - 4y - 6z = 0$, $-x - 2y - 5z = 0$.

Note that third equation is dependent on first two i.e. $R_1 + 2R_2 \simeq R_3$

Solving them, we get $x = k$, $y = 2k$, $z = -k$

Similarly for $\lambda = -3$, the equations are

$$x + 2y - 3z = 0, \quad 2x + 4y - 6z = 0, \quad -x - 2y + 3z = 0$$

Second and third equations are derived from the first. Therefore, only one equation is independent in this case.

Taking $z = 0$, $y = 1$, we get $x = -2$. Again taking $y = 0$, $z = 1$, we get $x = 3$. Two linearly independent eigen vectors are $(-2, 1, 0)$ and $(3, 0, 1)$. A linear combination of these viz. $(-2 + 3k, 1, k)$ is also an eigen vector.

Example 34: Find Eigen values and Eigen vectors for $\mathbf{A} = \begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$.

Solution: The characteristic equation,

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0,$$

$\Rightarrow$ $\lambda = 2, 2, 8$ are the **characteristic roots (latent roots)**.

Considering $[A - 8I]X = 0$, we may show that there exists only one linearly independent solution

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

so that every non-zero multiple of the same is a characteristic vector for the characteristic root 8.

For the characteristic root 2, we have

$$[A - 2I]X = 0 \quad \Rightarrow \quad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or $\quad 4x - 2y + 2z = 0 \qquad \ldots(i)$

$\quad -2x + y - z = 0 \qquad \ldots(ii)$

$\quad 2x - y + z = 0 \qquad \ldots(iii)$

which are equivalent to a single equation.

Thus we obtain two linearly independent solutions, may take as

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The sub-space of V_2 possessed by these two vectors is the characteristic space for the root 2.

ASSIGNMENT 2

1. The characteristic roots of A and A' are the same.

2. The characteristic roots of $\bar{A}$ and $\bar{A}'$ are the conjugates of the characteristic roots of A.

3. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic roots of an n-square matrix A and if k is a scalar, then $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the characteristic roots of kA.
4. If A is a square matrix, show that the latent roots of 'A' are identical.

1.9 LINEAR TRANSFORMATIONS AND ORTHOGONAL TRANSFORMATIONS

I. Linear Transformations

Let P be a point with co-ordinates (x, y) to a set of rectangular axes in the plane-xy. If we take another set of rectangular axes inclined to the former at an angle θ, then the new coordinates (x', y') referred to the new system (see the geometry) are related with x and y by

$$\left.\begin{aligned} x' &= ON' = ON + NN' = (x\cos\theta + y\sin\theta) \\ y' &= MP = M'P - M'M = (-x\sin\theta + y\cos\theta) \end{aligned}\right\} \quad \ldots(1)$$

A more general transformation than (1) will be obtained when the new axes are rotated through different angles θ and ϕ, and then angle does not remain a right angle.

So, the most general linear transformation in two dimensions is

$$\left.\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned}\right\} \quad \ldots(2)$$

Expressed in matrix notation, thus

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \ldots(3)$$

More precisely, Y = AX, where X is transformed into Y.

More general, the relation $Y = AX$,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_1\, b_1 \ldots k_1 \\ a_2\, b_2 \ldots k_2 \\ \ldots\ldots\ldots\ldots \\ a_n\, b_n \ldots k_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

gives a linear transformation in n dimensions.

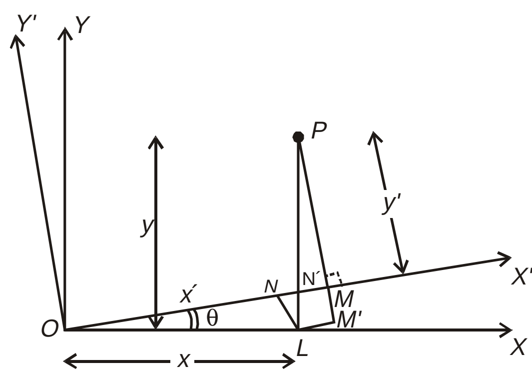

Fig. 1.1

This transformation is linear because the relations $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$, hold for transformation.

If the determinant value of the transformation matrix is zero, i.e. $|A| = 0$, the transformation is termed as **'Singular-transformation'**, otherwise, **'non-singular'**.

Non-singular transformation is also called **'regular-transformation'.**

Corollary: If $Y = AX$ denotes the transformation of (x_1, x_2, x_3) to (y_1, y_2, y_3) and $Z = BY$ denotes the transformation from (y_1, y_2, y_3) to (z_1, z_2, z_3), thus follows:

$$Z = BY = B(AX) = BAX.$$

If $$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

then the transformation of (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by $Z = (BA)X$, where

$$BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -7 & 12 & -1 \end{bmatrix}$$

Observations: It is seen that every square matrix defines a linear transformation. Further more, it is possible to write the inverse transformation $X = A^{-1}Y$ for only non-singular matrix A.

II. Orthogonal Transformations

A transformation from one set of rectangular coordinates to another set of rectangular coordinates is called an 'orthogonal transformation' or in other words, the linear transformation $Y = AX$ is said to be orthogonal, if matrix A is orthogonal, i.e. $AA' = I = A'A$.

Thus, an important property of this transformation is carried out only if transformation matrix is orthogonal or *vice versa.*

We have $$X'X = \begin{bmatrix} x_1 \, x_2 \ldots x_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots + x_n^2$$

Similarly, $$Y'Y = \begin{bmatrix} y_1 \; y_2 \; \ldots \; y_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y_1^2 + y_2^2 + \ldots + y_n^2$$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = Y'Y = (AX)'\,AX = X'A'AX = X'(A'A)X$$

which is possible only if $A'A = I = AA'$ and $A^{-1} = A'$.

Hence a square matrix 'A' is said to be orthogonal if $AA' = A'A$ and $A^{-1} = A'$.

Observations:

(*i*) A linear transformation preserves length if and only if its matrix is orthogonal.

(*ii*) The column vectors (row vectors) of an orthogonal matrix are mutually orthogonal unit vectors.

(*iii*) The product of two or more orthogonal matrices is orthogonal.

(*iv*) The determinant of an orthogonal matrix is ± 1.

(*v*) If the real n-square matrix A is orthogonal, its column vector (row-vectors) are an orthogonal basis of $V_n R$ (n-dimensional vector space in field of real) and conversely.

Example 35: If $\xi = x\cos\alpha - y\sin\alpha$, $\eta = x\sin\alpha + y\cos\alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.

Solution: Given $$\left.\begin{aligned} \xi &= x\cos\alpha - y\sin\alpha \\ \eta &= x\sin\alpha + y\cos\alpha \end{aligned}\right\} \qquad \ldots(1)$$

We can write the above system of equations in matrix notation as:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \ldots(2)$$

or more precisely, $Y = AX$, where $Y = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$, representing linear transformation with A as the matrix of transformation.

Now, $$A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad \ldots(3)$$

Find, $$AA' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

Also $A'A = I$. Hence A is an orthogonal matrix.
But if A is an orthogonal, then $A' = A^{-1}$.
Thus, for the transformation $Y = AX$, we can write the inverse transformation

$$X = A^{-1}Y, \text{ where } A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A'.$$

Example 36: Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ orthogonal? If not, can it be converted into an orthogonal matrix? [KUK, 2005]

Solution: Let the given matrix be A. Then to check its orthogonality, find AA'
Thus

$$AA' = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix}$$

As $AA' \neq I$, hence A is not an orthogonal matrix.

However, it can be made an orthogonal by nromalization, i.e. on dividing every element of a row by the square root of the sum of squares of each element of the respective row so that product of resultant matrix (normalization) with its transpose would be a unit matrix.

Hence, the orthogonal form of the matrix A is $\begin{bmatrix} \frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ -\frac{3}{\sqrt{91}} & \frac{1}{\sqrt{91}} & \frac{9}{\sqrt{91}} \end{bmatrix}$.

Example 37: Prove that $\begin{bmatrix} \mathbf{l} & \boldsymbol{m} & \boldsymbol{n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \boldsymbol{n} & \mathbf{l} & -\boldsymbol{m} & \mathbf{0} \\ -\boldsymbol{m} & \boldsymbol{n} & -\mathbf{l} & \mathbf{0} \end{bmatrix}$ **is orthogonal, when** $\boldsymbol{l} = \frac{\mathbf{2}}{\mathbf{7}}, \ \boldsymbol{m} = \frac{\mathbf{3}}{\mathbf{7}}, \ \boldsymbol{n} = \frac{\mathbf{6}}{\mathbf{7}}$.

Solution: If we denote the given matrix by 'A' then it implies that (l, m, n) must have $\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$ is their one of the values that makes A as an orthogonal matrix. In other words, deduce that $AA' = I$ is possible with $l = \frac{2}{7}, m = \frac{3}{7}, n = \frac{6}{7}$.

Now
$$AA' = \begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix} \begin{bmatrix} l & 0 & n & -m \\ m & 0 & l & n \\ n & 0 & -m & -l \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \qquad AA' = \begin{bmatrix} l^2+m^2+n^2 & 0 & nl+ml-mn & -lm+mn-nl \\ 0 & 1 & 0 & 0 \\ nl+ml-nm & 0 & n^2+m^2+l^2 & -nm+ln+lm \\ -ml+nm-ln & 0 & -mn+nl+ml & m^2+n^2+l^2 \end{bmatrix}$$

For matrix A to be rothogonal, $AA' = 1$

i.e. $\quad nl + ml - nm = 0 \qquad \ldots(1)$

and $\quad l^2 + m^2 + n^2 = 1 \qquad \ldots(2)$

From (1), we have, $\left(\frac{l}{m}\right) + \frac{l}{n} = 1$

Let $\quad \frac{l}{n} = k$, then $\frac{l}{m} = (1-k) \qquad \ldots(3)$

Again suppose $k = \frac{1}{3}$, then $\left.\begin{array}{l} \frac{l}{m} = \frac{2}{3} \ \text{ or } \ m = \frac{3l}{2} \\ \frac{l}{n} = \frac{1}{3} \ \text{ or } \ n = 3l \end{array}\right\} \qquad \ldots(4)$

$\therefore$ Then using (4) in (2), we get

$$l^2 + m^2 + n^2 = \left(l^2 + \frac{9}{4}l^2 + 9l^2\right) = \frac{49}{4}l^2 = 1$$

$\Rightarrow \quad$ Either $l = \frac{2}{7}$ or $l = -\frac{2}{7} \qquad \ldots(5)$

Taking $l = \frac{2}{7}$, we get $m = \frac{3}{7}$ and $n = \frac{6}{7}$

Hence with $(l, m, n) = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$, A is orthogonal.

Theorem 1: Prove that both 'the inverse and transpose' of an orthogonal matrix are also orthogonal.

Solution: As we know that for an orthogonal matrix say A,

$$AA' = I = A'A \text{ and } A' = A^{-1}$$

Let $\quad A^{-1} = B$

Case I: Then for B to be an orthogonal, we are to prove that

$$BB' = B'B = I$$

$\therefore \quad BB' = (A^{-1})(A^{-1})' = A^{-1}(A')^{-1} = A^{-1}(A^{-1})^{-1} = A^{-1}A = I$

Similarly, $\quad B'B = (A^{-1})' A^{-1} = (A')^{-1}A^{-1} = (A^{-1})^{-1}A^{-1} = AA^{-1} = I$

Hence inverse of an orthogonal matrix is also an orthogonal.

Case II: Let $A' = B$. For B to be orthogonal, we need to prove that

$$BB' = I = B'B$$

$\therefore \quad BB' = A'(A')' = A'A = I;$

Also $\quad B'B = (A')'A' = AA' = I$

Hence transpose of an orthogonal matrix is also orthogonal.

Theorem 2: A linear transformation preserves length if and only if its matrix is orthogonal.

Solution: Let Y_1, Y_2 be the respective images of X_1, X_2 under the linear transformation

$$Y = AX$$

Suppose A is orthogonal, then $AA' = I = A'A$

Now,

$$Y_1 \cdot Y_2 = Y_1'Y_2 = (AX_1)'(AX_2) = X_1'(A'A)X_2 = X_1 \cdot X_2 \text{ inner product.}$$

Hence the transformation preserves length.

For *vice versa*, suppose lengths (i.e., inner products) are preserved.

Then, $\quad Y_1 \cdot Y_2 = Y_1' Y_2 = (AX_1)'(AX_2) = X_1'(A'A)X_2$

But, $Y_1 \cdot Y_2 = X_1 \cdot X_2$ (given) i.e., $X_1'(A'A)X_2$ must be equal to $X_1 \cdot X_2$ which is only possible when $\quad A'A = I$

Hence A is orthogonal.

For example, the linear transformation $Y = AX = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} X$

is orthogonal.

The image of $X = [a\ b\ c]'$ is $Y = \begin{bmatrix} \frac{a}{3} + \frac{2b}{3} + \frac{2c}{3} & \frac{2a}{3} + \frac{b}{3} - \frac{2c}{3} & \frac{2a}{3} - \frac{2b}{3} + \frac{c}{3} \end{bmatrix}$

and both vectors are of length $\sqrt{a^2 + b^2 + c^2}$.

Example 38: Given that $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$**, where** a, b, c **are roots of** $x^3 + x^2 + k = 0$ **(**k **is a constant). Prove that** A **is orthogonal.**

Solution: a, b, c are the roots of the cubic $x^3 + x^2 + k = 0$ implies

S_1 = Sum of the roots taken one at a time

$$a + b + c = (-1)\frac{\text{co–eff. of } x^2}{\text{co–eff. of } x^3} = -1 \quad \ldots(1)$$

S_2 = Sum of the roots taken two at a time

$$ab + bc + ca = (-1)^2 \frac{\text{co–eff. of } x}{\text{co–eff. of } x^3} = 0 \quad \ldots(2)$$

S_3 = Sum of the roots taken three at a time ...(3)

$$abc = (-1)^3 \frac{\text{constant term}}{\text{co–efficient of } x^3} = -k$$

Now, to check whether A is orthogonal, find the product AA'

Here

$$AA' = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ca + ab + bc \\ ab + bc + ca & b^2 + c^2 + a^2 & bc + ca + ab \\ ca + ab + bc & bc + ca + ab & c^2 + a^2 + b^2 \end{bmatrix} \quad \ldots(4)$$

On using the values of S_1 and S_2, i.e. $a + b + c = -1$ and $ab + bc + ca = 0$

we see that $(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$ results in $a^2 + b^2 + c^2 = 1$. ...(5)

On using (1), (2), (3) and (5) $AA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

Hence with a, b, c as the roots of the given cubic, the matrix A is an orthogonal.

Example 39: If $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ **defines an orthogonal transformation, then show that** $l_i l_j + m_i m_j + n_i n_j = 0 (i \neq j); = 1 (i = j); i, j = 1, 2, 3.$

Solution: We know that for an orthogonal matrix A, $AA' = I = A'A$ and $A' = A^{-1}$

$$\therefore \quad AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \text{ for given } A.$$

$$= \begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_2 l_1 + m_2 m_1 + n_2 n_1 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_3 l_1 + m_3 m_1 + n_3 n_1 & l_3 l_2 + m_3 m_2 + n_3 n_2 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix}$$

For A to be an orthogonal, $AA' = I$ which is possible only if,

$$(l_1^2 + m_1^2 + n_1^2) = (l_2^2 + m_2^2 + n_2^2) = (l_3^2 + m_3^2 + n_3^2) = 1$$

and $\quad (l_1 l_2 + m_1 m_2 + n_1 n_2) = (l_2 l_3 + m_2 m_3 + n_2 n_3) = (l_3 l_1 + m_3 m_1 + n_3 n_1) = 0.$

ASSIGNMENT 3

1. Prove that the product of two orthogonal matrix is orthogonal.
2. Prove that the matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal matrix.
3. Given that $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where a, b, c are the roots of $x^3 + x^2 + k = 0$

   (where k is a constant). Prove that 'A' is orthogonal.
4. Show that the modulus of an orthogonal transformation is either 1 or –1.

   [Hint: Since $AA' = I$, then $|A|\,|A'| = |1|$]

1.10 DIAGONALISATION OF MATRICES, THEIR QUADRATIC AND CANONICAL FORMS

1. **Diagonalization:** If a square matrix A of order n has n linearly independent eigen values, then a matrix P can be found such that $P^{-1}AP$, called a **matrix of transformation**.

   We prove this theorem for a square matrix of order $n = 3$ as follows:

   Let $\lambda_1, \lambda_2, \lambda_3$ be the three eigen values of the square matrix A. Let X_1, X_2, X_3 be the corresponding eigen vectors, where $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

   Let a square matrix whose elements are three column matrices X_1, X_2, X_3 be denoted by P or more precisely,

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

then $\quad AP = A[X_1 \;\; X_2 \;\; X_3] = [AX_1 \;\; AX_2 \;\; AX_3] = [\lambda_1 X_1 \;\; \lambda_2 X_2 \;\; \lambda_3 X_3]$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$= PD$, where D is the **diagonal matrix** such that $P^{-1} AP = D$.

The resulting diagonal matrix D, contains the eigen values on its diagonal.

This transformation of a square matrix A by a non-singular matrix P to $P^{-1}AP$ is termed as **Similarity Transformation**. The **matrix *P*** which diagonalizes the transformation matrix A is called the **Modal Matrix** and the **matrix *D***, so obtained by the process of diagonalization is termed as **Spectral Matrix**.

Observations: The diagonalizing matrix for matrix $A_{n\times n}$ may contain complex elements because the zeros of the characteristics equation of $A_{n\times n}$ will be either real or in conjugate pairs. Further, diagonalizing matrix is not unique because its form depends on the order in which the eigen values of $A_{n\times n}$ are taken.

2. Quadratic Forms: A homogeneous expression of second degree in several variables is called a quadratic form.

e.g. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X' = [x \;\; y \;\; z]$

then $\quad X'AX = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{31}zx,$ ...(1)

(for $a_{12} = a_{21}$, $a_{23} = a_{32}$, $a_{13} = a_{31}$) is a quadratic form in three variable x, y, z where the given matrix A is symmetric.

3. Transformation to Cannoncial Form: Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the three eigen vectors in their normalized form (i.e. each element is divided by the square root of the sum of the squares of all the three elements in the respective eigen vector corresponding to the eigen values λ_1, λ_2, λ_3 of a square matrix A).

Then through the non-singular linear transformation, $X = PY$

We get $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Hence the quadratic form (1) is reduced to a sum of squeres, i.e. cononical form:

$F = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$ ...(2)

P is the matrix of transformation which is an orthogonal matrix. That is why the above method of reduction is called the orthogonal transformation.

Observations:

(*i*) Here in this case, D and A are congruent matrices and the transformation $X = PY$ is known as congruent transformation.

(*ii*) The number of positive terms in cononical form of the quadratic is the index (s) of the form.

(*iii*) Rank r of matrix D (or A) is called the rank of the form.

(*iv*) The difference to the number of positive terms and negative terms to the quadratic form is the signature of the quadratic form.

4. Nature of Quadratic Forms: Let $Q = X'AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$. **Index** of a quadratic form is the number of positive terms in its canonical form and signalize of the quadratic form is the difference of positive and negative number of terms in its canonical form.

A real quadratic form $X'AX$ is said to be

(*i*) **positive definite** if all the eigen values of A are > 0 (in this case, the rank r and index, s of the square matrix A are equal to the number of variables, i.e. $r = s = n$);

(*ii*) **negative definite** if all the eigen values of A are < 0 (here $r = n$ and $s = 0$);

(*iii*) **positive semi-definite** if all the eigen values of $A \geq 0$, with atleast one eigen value is zero (in this case, $r = s < n$);

(*iv*) **negative semi-definite** if all the eigen values of A are ≤ 0 with at least one eigen value is zero (it is the case, when $r < n$, $s = 0$);

(*v*) **indefinite** if the eigen values occur with mixed signs.

5. Determination of the Nature of quadratic Form without Reduction To Canonical Form: Let the quadratic form

$$X'AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let
$$A_1 = a_{11},\ A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the quadratic form $X'AX$ is said to be

(*i*) **positive definite** if $A_i > 0$ for $i = 1, 2, 3$;

(*ii*) **negative definite** if $A_2 > 0$ and $A_1 < 0$, $A_3 < 0$;

(*iii*) **positive semi-definite** if $A_i > 0$ and atleast one $A_i = 0$;

(*iv*) **negative semi-definite** if some of A_i are zero in case (*ii*);

(*v*) **indefinite** in all other cases;

Example 40: Obtain eigen values, eigen vectors and diagonalize the matrix,

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}.$$ **[NIT Jalandhar, 2005]**

Solution: The corresponding characteristic equation is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \; -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

Clearly, it is a qubic in λ and has roots 0, 3, 15.

If x_1, x_2, x_3 be the three components of an eigen vector say 'X' corresponding to the eigen values λ, then

We have $[A-\lambda]X = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

For $\lambda = 0$, $8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_1 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

These equations determine a single linearly independent solution.

On solving them, $\dfrac{x_1}{21-16} = \dfrac{x_2}{-8+18} = \dfrac{x_3}{24-14} = k(\text{say})$

$\Rightarrow$ $(x_1, x_2, x_3) = (k, 2k, 2k)$

$\therefore$ Let the linearly independent solution be (1, 2, 2), as every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 0$.

Likewise, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 15$ are the arbitrary non-zero multiple of vectors (2, 1, –2) and (2, –2, 1).

Hence the three eigen vectors may be considered as (1, 2, 2), (2, 1, –2), (2, –2, 1).

$\therefore$ The diagonalizing matrix 'P' $= [X_1 \;\; X_2 \;\; X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Example 41: Find the Latent roots, Eigen vectors, the modal matrix (i.e., diagonalizing matrix ('*P*'), sepectral matrix of the given matrix $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$ and hence reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to canonical form.

Solution: The corresponding characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} \Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Clearly, it is a qubic in 'λ' and has three values, viz. 1, 2, 4.

Hence the latent roots of 'A' are 1, 2 and 4.

If x, y, z be the three components of eigen vector corresponding to these eigen values, $\lambda = 1, 2, 4$, then

$$\text{for} \quad \lambda = 1, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} [X_1] = 0 \quad \text{with} \quad X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$\Rightarrow$ $\left.\begin{aligned} 2y_1 - z_1 &= 0 \\ -y_1 + 2z_1 &= 0 \end{aligned}\right\}$ having one of the possible set of values, say, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Likewise,

$$\text{for} \quad \lambda = 2, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0 \quad \Rightarrow \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{for } \lambda = 4, \quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} [X_3] = 0 \quad \text{or} \quad y_3 + z_3 = 0$$

$$\therefore \qquad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence, we have Modal Matrix, $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

and Spectral Matrix $'D' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Canonical form as: $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$, i.e. $x^2 + 2y^2 + 4z^2$

Example 42: Reduce the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form and hence reduce it to canonical form. **[UP Tech, 2006; Raipur, 2004]**

Solution: The characteristic equation is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_3 - \lambda_2 - 5\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = 1 \pm \sqrt{5}$$

Thus, the eigen values for matrix 'A' are $1, \pm\sqrt{5}$

$$\therefore \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigen vector, so that $\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

For $\lambda = 1, \sqrt{5}, -\sqrt{5}$, we get vectors in the form

$$\text{'}P\text{'} = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \text{ the diagonalizing matrix.}$$

Its canonical form is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = x^2 + \sqrt{5}y^2 - \sqrt{5}z^2$.

Example 43: Show that the transformation matrix

$$\boldsymbol{H} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{with} \quad \theta = \frac{1}{2}\tan^{-1}\frac{2h}{(a-b)}$$

changes the matrix $\boldsymbol{C} = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ **to the diagonal form** $\boldsymbol{D = HCH'}$.

Solution: $$HCH' = \begin{bmatrix} (a\cos\theta + h\sin\theta) & (h\cos\theta + b\sin\theta) \\ (-a\sin\theta + h\cos\theta) & (-h\sin\theta + b\cos\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} (a\cos\theta + h\sin\theta) & (h\cos\theta + b\sin\theta) \\ (-a\sin\theta + h\cos\theta) & (-h\sin\theta + b\cos\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta(a\cos\theta + h\sin\theta) + \sin\theta(h\cos\theta + b\sin\theta) & -\sin\theta(a\cos\theta + h\sin\theta) + \cos\theta(h\cos\theta + b\sin\theta) \\ \cos\theta(-a\sin\theta + h\cos\theta) + \sin\theta(-h\sin\theta + b\cos\theta) & -\sin\theta(-a\sin\theta + h\cos\theta) + \cos\theta(-h\sin\theta + b\cos\theta) \end{bmatrix}$$

$$= \begin{bmatrix} a^2\cos^2\theta + b^2\sin^2\theta + 2h\sin\theta\cos\theta & -(a-b)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) \\ (a-b)\sin\theta\cos\theta - h(\cos^2\theta - \sin^2\theta) & a^2\sin^2\theta + b^2\cos^2\theta - 2h\sin\theta\cos\theta \end{bmatrix}$$

$$HCH' = \begin{bmatrix} a^2\cos^2\theta + b^2\sin^2\theta + 2h\sin\theta\cos\theta & 0 \\ 0 & a^2\sin^2\theta + b^2\cos^2\theta - 2h\sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

as $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{(a-b)}$, i.e. $(a-b)\sin\theta\cos\theta - h(\cos^2\theta - \sin^2\theta) = 0$

Hence the result.

Example 44: Find the eigen vector of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **and hence reduce**

$\boldsymbol{6x^2 + 3y^2 + 3x^2 - 2yz + 4zx - 4xy}$ **to a sum of squares.** **[KUK, 2006, 04, 01]**

Solution: The characteristic equation is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \qquad \ldots(1)$$

$\Rightarrow$ $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ giving values $\lambda = 2, 2, 8$

Corresponding to $\lambda = 2$, the eigen vectors are given by

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \ldots(2)$$

Clearly, we have only one set of linearly independent values of x_1, x_2, x_3.

Since form above, we get only one independent equation viz.

$$2x_1 - x_2 + x_3 = 0 \qquad \ldots(3)$$

If we take $x_3 = 0$ in (3), we get $2x_1 = x_2$ i.e. $x_1 = \dfrac{x}{2}$

$\therefore$ $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{0}$ $\Rightarrow$ $X = [1, 2, 0]$

Now, choosing $x_2 = 0$ in (3), we get $2x_1 = -x_3$, giving eigen vector (1, 0, –2)

Any other Eigen vector corresponding to $\lambda = 2$ will be a linear combination of these two.

Corresponding to $\lambda = 8$, we have

$$[A - \lambda I]X = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

giving equations, $\left.\begin{array}{r} -2x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 - 5x_2 - x_3 = 0 \end{array}\right\}$

Solving them, we get $\dfrac{x_1}{2} = \dfrac{x_2}{-1} = \dfrac{x_3}{1}$

$\therefore$ $X = [2, -1, 1]$.

Hence $$P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ -0 & -2 & 1 \end{bmatrix}$$

The 'sum of squares' viz. the canonical form of the given quadratic is

$$8x^2 + 2y^2 + 2z^2 = 4x^2 + y^2 + z^2$$

Example 45: Reduce the quadratic form $2xy + 2yz + 2zx$ to the canonical form by an orthogonal reduction and state its nature.

[Kurukshetra, 2006; Bombay, 2003; Madras, 2002]

Solution: The given quadratic form in matrix notations is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The eigen values for this matrix are 2, –1, –1 and the corresponding eigen vectors for

$$\left.\begin{aligned} \lambda = 2, \quad & x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \\ \lambda = -1, \quad & x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \\ \lambda = -1, \quad & x_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}\right\}$$

(Eigen vector corresponding to the repeated eigen value –1, is obtained by assigning arbitrary values to the variable as usual.)

Here we observe that x_2 and x_3 are not orthogonal vectors as the inner product,

$$x_2 \cdot x_3 = -1(0) + 1(1) + 0(-1) \neq 0.$$

Therefore, take $x_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ so that x_1, x_2 and x_3 are mutually orthogonal.

Now, the normalized modal matrix $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

Consider the orthogonal transformation $X = PY$, i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$

Using this orthogonal transformation, the quadratic form reduces to canonical form, $Q = 2x'^2 - y'^2 - z'^2$. The quadratic form is an indefinite in nature as the eigen values are with mixed sign and rank $r = 3$; index $s = 1$.

Example 46: Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into 'a sum of squares' by an orthogonal transformation and give the matrix of transformation.
[KUK, 2008; NIT Kurukshetra, 2002]

Solution: On comparing the given quadratic with the general quadratic $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, the matrix is given by

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The desired characteristic equation becomes

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0,$$

which is a cubic in λ and has three values viz., 1, 4, 4.

Hence the desired canonical form i.e., 'a sum of squares' is $x^2 + 4y^2 + 4z^2$.

Solving $[A - \lambda I][X] = 0$ for three values of λ

For $\lambda = 1$, we have $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$

or $\left.\begin{matrix} 2x_1 + y_1 + z_1 = 0 \\ x_1 + 2y_1 - z_1 = 0 \end{matrix}\right\}$, i.e. $\dfrac{x_1}{-1-2} = \dfrac{y_1}{1+2} = \dfrac{z_1}{4-1} = k$

$$\therefore \qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Similarly for $\lambda = 4$, $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$

We have two linearly independent vectors $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

As the transformation has to be an orthogonal one, therefore to obtain 'P', first divide each elements of a corresponding eigen vector by the square root of sum of the squares of its respective elements and then express as $[X \; Y \; Z]$

Hence the matrix of transformation, $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Example 47: Discuss the nature of the quadratic $2xy + 2yz + 2zx$ without reduction to canonical form.

Solution: The given quadratic in matrix form is, $A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$

Here $A_1 = 0$; $A_2=\begin{vmatrix}0&1\\1&0\end{vmatrix}=-1<0$; $A_3=\begin{vmatrix}0&1&1\\1&0&1\\1&1&0\end{vmatrix}=2>0$

$\therefore$ The quadratic is indefinite in nature.

1.11 CAYLEY-HAMILTON THEOREM [PTU, 2009; NIT Kurukshetra, 2002]

Statement: *Every square matrix satisfies its own characteristic equation.*

Proof: Let A be any n-square matrix such that its characteristic equation is given by

$$|A-\lambda I| = (-1)^n\lambda^n + k_1\lambda^{n-1} + \ldots + k_n = 0 \qquad \ldots(1)$$

We need to prove that $|A-\lambda I| = (-1)^n A^n + k_1 A^{n-1} + \ldots + k_n = 0$

The elements of the n-square matrix $[A - \lambda I]$ are at the most first degree in λ and, therefore, the adjoint of the matrix $[A - \lambda I]$, say B, which consists of the co-factors of the elements in $|A - \lambda I|$ must represent a polynomial of degree $(n - 1)$ in λ. Further the adjoint B can be broken up into a number of matrices such that

$$B = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \ldots\ldots + B_n \qquad \ldots(2)$$

where all B_i's are the square matrices whose elements are the functions of the elements of the given matrix A.

We also known that $A \cdot \text{adj} \cdot A = |A|\, I$

$$\Rightarrow \quad [A-\lambda I] \text{ adjoint } [A-\lambda I] = |A-\lambda I| I \qquad \ldots(3)$$

By (1), (2) and (3), we have

$$[A-\lambda I]\,[B_1\lambda^{n-1} + B_2\lambda^{n-2} + \ldots + B_{n-1}\lambda + B_n]$$
$$= [(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n]I \qquad \ldots(4)$$

Equating the co-efficients of equal powers of λ on both sides, we get

$$\left.\begin{aligned} -B_1 &= (-1)^n I \\ AB_1 - B_2 &= k_1 I \\ AB_2 - B_3 &= k_2 I \\ &\ldots\ldots\ldots\ldots \\ AB_{n-1} - B_n &= k_{n-1} I \\ AB_n &= k_n I \end{aligned}\right\} \qquad \ldots(5)$$

Pre-multiplying the equations by $A^n, A^{n-1}, \ldots, A, I$ respectively and adding, we obtain

$$0 = (-1)^n A^n + k_1 A^{n-1} + \ldots\ldots + k_{n-1} A + k_n I$$

or $$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_n = 0 \qquad \ldots(6)$$

Observation: In equation (6) on transferring $k_n I$ to the left hand side and then multiplying throughout by A^{-1}, we can obtain the inverse of the matrix A

$$-A^{-1}k_n = [(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots]A^{-1}$$

or $$A^{-1} = -\frac{1}{k_n}\left[(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1}\right]$$

Example 48: Verify Cayley-Hamilton theorem for the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$**. Hence compute** $\mathbf{A^{-1}}$**.** **[KUK, 2005, 2008; Madras, 2006; UP Tech, 2005]**

Solution: The characteristic equation, is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \qquad \ldots(1)$$

To prove that 'Cayley-Hamilton' theorem, we have to prove that

$$A^3 - 6A^2 + 9A - 4I = 0$$

Obtain
$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \qquad \ldots(2)$$

Similarly,
$$A^3 = A^2 \times A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} \qquad \ldots(3)$$

Now
$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 \qquad \ldots(4)$$

To compute A^{-1}, multiply both side of by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

or
$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$
$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Example 49: Find the characteristic equation of the matrix $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix}$ **and hence, find the matrix represented by** $\mathbf{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I}$**.**

[Rajasthan, 2005; UP Tech, 2003]

Solution: The characteristic equation of the given matrix,

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \qquad \ldots(1)$$

Further, as we know that every matrix satisfies its own characteristic equation

Hence $\quad A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(2)$

Rewrite, $\quad A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

as $\quad (A^8 - 5A^7 + 7A^6 - 3A^5) + (A^4 - 5A^3 + 7A^2 - 3A) + A + I$

or $\quad A^5 (A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$

On using (2), it nearly becomes $(A^2 + A + I)$

Hence, the given expression (A^8 – 5A7 + 7A6 – 3A5 + A4 – 5A3 + 8A2 – 2A + I)

represents the matrix, $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$.

ASSIGNMENT 4

1. Find the eigen values, eigen vectors, modal matrix and the spectral matrix of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ and hence reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to *a* canonical form. **[NIT Kurukshetra, 2004; Andhara, 2000]**

2. Write down the quadratic form corresponding to the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

[**HINT:** Quadratic Form = $X'AX$]

3. Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$ into a 'sum of squares' by an orthogonal transformation. State the nature of the quadratic. Also find the set of values of x, y, z which will make the form vanish. **[NIT Kurukshetra, 2009]**

4. Verify Cayley Hamilton theorem for the matrix A and find ifs inverse if $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

1.12 SOME SPECIAL MATRICES

Complex Matrices: If a matrix 'A' = $[a_{rs}]$, whose elements are $a_{rs} = \alpha_{rs} + i\beta_{rs}$ where α_{rs}, β_{rs} being real is called a complex matrix. The matrix $\bar{A} = [\bar{a}_{rs}] = [\alpha_{rs} - i\beta_{rs}]$ is known as the conjugate matrix. The transpose conjugate of A, i.e. $\bar{A}'$ is oftenly denoted by A^θ.

Further, if $A = \begin{bmatrix} a_1 + i\,b_1 \\ a_2 + i\,b_2 \\ \cdots\cdots\cdots \\ a_n + i\,b_n \end{bmatrix}$, then

$$\bar{A}' A = A^{\theta} A = (a_1 - i\,b_1)(a_1 + ib_1) + \ldots + (a_n - i\,b_n)(a_n + i\,b_n)$$
$$= (a_1^2 + b_1^2) + \ldots + (a_n^2 + b_n^2)$$

Orhtogonal Matrix (Rotational Matrix): If for a square matrix $A = [a_{ij}]$ of order n, we have $AA' = I = A'A$, then A is said to be an 'orthogonal' or 'rotational matrix'.

e.g. (*i*) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, (*ii*) $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

Unitary Matrix: If a square matrix A in a complex field is such that $\bar{A}' = A^{-1}$, then A is called a unitary matrix. The determinant of a unitary matrix is of unit modulus and thus is non-singular.

e.g. Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ so that $\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$

and $\bar{A}' = \overset{\Theta}{A} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$\therefore$ $A\overset{\Theta}{A} = \frac{1}{4}\begin{bmatrix} 1+i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} (1-i)^2 + (1-i^2) & (1-i^2) - (1-i^2) \\ (1-i^2) - (1-i^2) & (1-i^2) + (1-i^2) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 1.$$

Hermitian Matrix: A square matrix A is said to be Hermitian if $\bar{A}' = A$ where $\bar{A}$ denotes the matrix whose elements are the complex conjugates of the elements of A. **[PTU, 2007, 2008]**

In terms of general elements, the above assertion implies $A' = \bar{A}$ ($a_{ji} = \bar{a}_{ij}$ or $a_{ii} = \bar{a}_{ii}$) which shows that all the diagonal elements are real.

A square matrix A is said to be skew-Hermitian if $\bar{A}' = -A$. Whence, the leading diagonal elements of a **skew-Hermitian** matrix are either all purely imaginary or zero.

Thus, Hermitian and skew-Hermitian matrices are the generalization in the complex field of symmetric and skew-symmetric matrices respectively.

e.g. (*i*) $\begin{bmatrix} 1 & 5+4i \\ 5-4i & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & 3+4i \\ 2-3i & 3-4i & 3 \end{bmatrix}$ (*iii*) $\begin{bmatrix} i & 1+i & 2+3i \\ -1+i & 2i & 3+4i \\ -2+3i & -3+4i & 3i \end{bmatrix}$

Clearly (*i*) and (*ii*) are the examples of two Hermitian matrices of which all the diagonal elements are real numbers while (*iii*) is an example of skew-Hermitian as all of its diagonal element are purely imaginary.

Example 50: Show that $\boldsymbol{A = \begin{bmatrix} i & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}}$ **is a Hermitian.**

Solution: Let the transpose A' of square matrix [A] is equal to its conjugate complex, i.e. $A' = \bar{A}$, then A is said to be the Hermitian matrix.

Clearly, $$A' = \begin{bmatrix} 1i & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

each $a_{rs} = (\alpha_{rs} + i\beta_{rs})$ elements of A' is equal to the elements $a_{rs} = (\alpha_{rs} - i\beta_{rs})$ of $\bar{A}$.

Hence the matrix A is Hermitian Matrix.

Normal Matrices: A square matrix A is called normal if $A\bar{A}' = \bar{A}'A$; where $\bar{A}'$ or $\tilde{A}$, stands for conjugate transpose of A. Normal matrices include Diagonal, Real, Symmetric, Real-Skew symmetric, Orthogonal, Hermitian, Skew-Hermitian or Unitary matrices.

***Note*:** If A is any normal matrix and U is a unitary matrix then $\bar{U}'AU$ is normal as:

Let $\bar{U}'AU = X$ then $\bar{X}' = \left(\bar{U}'AU\right)'$

$$= \bar{U}'\bar{A}'\bar{\bar{U}} \qquad \because (\bar{U}')' = \bar{U}$$

$$= \bar{U}'\bar{A}'U, \qquad \because \bar{\bar{U}} = U$$

Here we need to prove $\bar{X}'X = X\bar{X}'$

$$\bar{X}'X = \left(\bar{U}'\bar{A}'U\right).\left(\bar{U}'AU\right) \text{ (Taking } U\bar{U}' = I)$$

$$= \bar{U}'\bar{A}'AU = \bar{U}'A\bar{A}'U \qquad \text{(Rewrite } \bar{A}'A = A\bar{A}')$$

$$= \bar{U}'AU\bar{U}'\bar{A}'U = X\bar{X}' \qquad \text{(As } I = U\bar{U}')$$

Theorem 1: Any square matrix can be expressed uniquely as a sum of Hermitian and Skew-Hermitian Matrix.

Proof: Let A be a square matrix (complex or real) such that

$A = H + S$, where $H = \frac{1}{2}(A + A')$ is a symmetric matrix

$S = \frac{1}{2}(A - A')$ is a skew-symmetric matrix

Now, we need to prove that H is Hermitian and S is skew-Hermitian.

$$H' = \frac{1}{2}(A + A')' = \frac{1}{2}\left(A' + (A')\right)'$$

$$= \frac{1}{2}(A' + A) = H.$$

[$\because$ Transpose of the transpose of a matrix is the matrix itself]

Hence H is Hermitian,

and $$S' = \frac{1}{2}(A - A')' = \frac{1}{2}\left(A' - (A')'\right)$$

$$= \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -S$$

Hence S is a skew-Hermitian.

Uniqueness: Suppose $A = (K + T)$, where K is Hermitian and T skew-Hermitian

then $A' = K' + T'$ or $A' = K - T$ [$\because K' = K$ and $T' = -T$ by supposition]

Adding the two, $(A + A') = 2K$ or $K = \frac{1}{2}(A + A')$

$K = H$ from defintion of A above.

On substsacting $(A - A') = 2T$ or $T = \frac{1}{2}(A - A')$

$T = S$ from definition of 'A' above.

Hence H and S are unique.

Theorem 2: Show that the inverse of a unitary matrix is unitary.

Proof: Let U is an unitary matrix i.e., $U' = U^{-1}$ ...(1)

Thus, $(U^{-1})(U^{-1})' = (U^{-1})(U')^{-1}$

$= (U)^{-1}(U')^{-1}$

$= (U'U)^{-1}$ $\quad |\because \; B^{-1}A^{-1} = (AB)^{-1}|$

$= (U^{-1}U)^{-1}$ $\quad \because$ By (1)

$= (I)^{-1} = I$ ...(2)

Similarly, $(U^{-1})'(U^{-1}) = (U')^{-1}U^{-1}$

$= (UU')^{-1}$ $\quad [\because \; B^{-1}A^{-1} = (AB)^{-1}]$

$= (UU^{-1})^{-1}$

$= (I)^{-1} = I$ ...(3)

Hence the result.

Theorem 3: Show that the product of two *n*-rowed unitary matrix is unitary.

Proof: A square matrix X will be unitary if $XX' = I_n$,
then suppose the U and V are two unitary $n \times n$ matrices

i.e., $UU' = I_n = VV'$

Thus, $(UV)(UV)' = UV \cdot V'U' = U(VV')U' = UI_n U' = UU' = I_n$

Similary, $(UV)'(UV) = V'U'UV = V'(U'U)V = V'I_n V = V'V = I_n$

Hence $(UV)(UV)' = I_n = (UV)'(UV)$ and thus the product is unitary.

Theorem 4: Modulus of each characteristic roots of a unitary matrix is unity.

OR

Show that the eigen values of a unitary matrix have absolute values.

Proof: Let 'A' is an unitary matrix and $AX = \lambda X$ …(1)

Then taking conjugate transpose of each side

$$\left(\overline{AX}\right)' = \overline{\lambda X}'$$

or $$X^{\Theta}A^{\Theta} = \bar{\lambda}X^{\Theta} \qquad \ldots(2)$$

with A^{Θ} and X^{Θ} as conjugate transpose of A and X respectively.

Multiplying (1) and (2),

$$\left(X^{\Theta}A^{\Theta}\right)\left(AX\right) = \bar{\lambda}X^{\Theta}\lambda X$$

$$X^{\Theta}\left(A^{\Theta}A\right)X = \bar{\lambda}\lambda X^{\Theta}X$$

$$X^{\Theta}X = \bar{\lambda}\lambda X^{\Theta}X$$

$$\left(1-\bar{\lambda}\lambda\right)X^{\Theta}X = 0$$

Hence, either $(1 - \lambda\lambda) = 0$ or $X^{\Theta}X = 0$

But $X^{\Theta}X \neq 0$. $\therefore$ $\left(1-\bar{\lambda}\lambda\right) = 0$ implying $\bar{\lambda}\lambda = 1$

So that modulus of λ is unity.

(**Cor:** Modulus of each characteristic root of an orthogonal matrix is unity. In particular, theorem also applies to orthogonal matrices).

Theorme 5. Eigen values or characteristic roots of a Skew-Hermitian (and thus of a Skew-Symmetric) are purely imaginary or zero. **[KUK, 2006]**

Proof: Let A be a skew-Hermitian Matrix and $AX = \lambda X$

then $$(iA)X = (i\lambda)X$$

But 'iA' is Hermitian and as such '$i\lambda$', a characteristic root of 'iA' is real.

Thus for $i\lambda$ to be real either $\lambda = 0$ or λ is a purely imaginary number.

Theorem 6: Characteristic roots of a Hermitian Matrix and thus of a Symmetric Matrix are all real.

Proof: Let λ be any characteristic root of a Hermitian Matrix 'A'. Means there exists a vector $X \neq 0$, such that

$$AX = \lambda X \qquad \ldots(1)$$

Pre-multiplying with X^{Θ}, we obtain

$$X^{\Theta}(AX) = X^{\Theta}\lambda X$$

or $$= \lambda X^{\Theta}X = \lambda X^{\Theta}X \qquad \ldots(2)$$

Being the values of Hermitian forms, $X^{\Theta}AX$ and $X^{\Theta}X$ are both real.

Also $$X^{\Theta}X \neq 0 \text{ for } X \neq 0 \qquad \ldots(3)$$

Thus from (2) and (3), we have

$$\lambda = \frac{X^{\Theta}(AX)}{X^{\Theta}X} \text{ is real.}$$

Alternately: If λ is a latent root of a Hermitian matrix H, and X the corresponding eigen vector, then

$$HX = \lambda X \qquad \ldots(1)$$

$$\overline{HX} = \overline{\lambda X}$$

$$\left(\overline{HX}\right)' = \left(\overline{\lambda X}\right)'$$

$$\bar{X}'\bar{H}' = \overline{\lambda X}'$$

Hence $$X^{\Theta}H = \bar{\lambda}X^{\Theta} \qquad \ldots(2)$$

with $\bar{X}' = X^{\Theta}$ as transpose of the conjugate complex of X and $H^{\Theta} = H$, since H is Hermitian.

Also From (1), $\bar{X}'HX = \bar{X}'\lambda X$

or $$X^{\Theta}HX = \lambda X^{\Theta}X \qquad \ldots(3)$$

$$\bar{\lambda}X^{\Theta}X = \lambda X^{\Theta}X \qquad \text{using (2)}$$

Since $X^{\Theta}X \neq 0$, it follows that $\bar{\lambda} = \lambda$

Hence λ is real (all λ_i's are real).

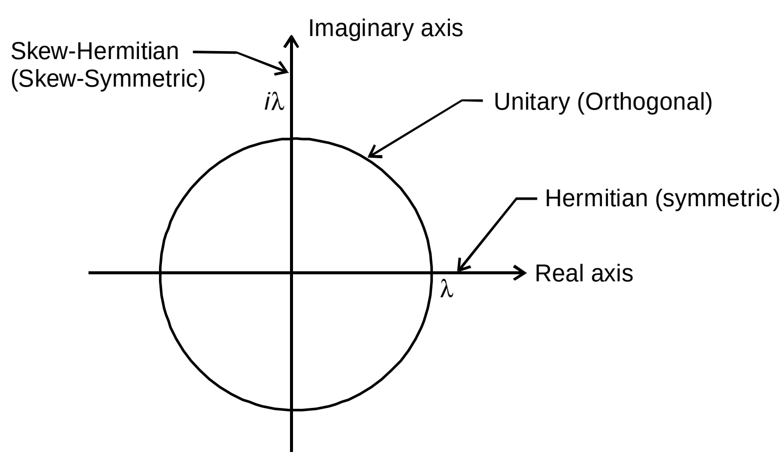

Fig. 1.2

Theorem 7: Show that for any square matrix A; $(A + A^{\Theta})$, $A^{\Theta}A$ are Hermitian and $(A - A^{\Theta})$ is Skew-Hermitian, where A^{Θ} stands for transpose conjugate of A.

Proof: By definition a square matrix A is said to be Hermitian, if $\bar{A}' = A$, i.e., $A^{\Theta} = A$.

Here, $(A + A^{\Theta})^{\Theta} = A^{\Theta} + (A^{\Theta})^{\Theta} = A^{\Theta} + A$

which shows that conjugate transpose of $(A + A^{\Theta})$ is equal to itself. Hence $(A + A^{\Theta})$ is Hermitian.

Likewise, $(AA^{\Theta})^{\Theta} = (A^{\Theta})^{\Theta} A^{\Theta} = AA^{\Theta}$. Hence AA^{Θ} is Hermitian.

Again, $(A - A^{\Theta})^{\Theta} = -(A^{\Theta})^{\Theta} + A^{\Theta} = -A + A^{\Theta} = -(A - A^{\Theta})$. Hence $(A - A^{\Theta})$ is skew-Hermitian.

Theorem 8: Prove that any matrix *A* which is similar to a diagonal matrix, *D* has *n* linearly independent invariant vectors.

Proof: Let P be a non-singular matrix such that

$$P^{-1}AP = D = \text{dig.}\ (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Per-multiplying by P on both sides, we get

$$AP = PD \qquad (\because\ PP^{-1} = I)$$

Let $P[X_1, X_2, \ldots, X_n]$, the above relation becomes

$$A\left[X_1, X_2, \ldots, X_n\right] = \left[X_1, X_2 \ldots X_n\right]\begin{bmatrix} \lambda_1 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & \lambda_n \end{bmatrix}$$

or $\quad [AX_1, AX_2, \ldots AX_n] = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$

which clearly shows that $X_1, X_2, \ldots X_n$ are n eigen vectors of the matrix A corrseponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$

Since these vectors constitutes the columns of a non-singular matrix, hence there exists a linearly independent set of eigen values.

Theorem 9: If *X* is a characteristic vector corresponding to a characteristic root λ of a normal matrix *A*, then *X* is a characteristic vector of $\bar{A}'$ (conjugate transpose) corresponding to the characteristic root λ.

Proof. As matrix A is given normal i.e., $\bar{A}'A = A\bar{A}'$ …(1)

Then, $\quad (A - \lambda I)\overline{(A - \lambda I)'} = (A - \lambda I)\left(\bar{A}' - \bar{\lambda} I\right)$

$$= A\bar{A}' - A\bar{\lambda}I - \lambda I\bar{A}' + \lambda\bar{\lambda}I$$

$$= \left(\bar{A}'A - A\bar{\lambda}I\right) + \left(-\lambda I\,\bar{A}' + \lambda\bar{\lambda}I\right)$$

$$= \left(\bar{A}' - \bar{\lambda}I\right)A - \lambda I \cdot \left(\bar{A}' - \bar{\lambda}I\right)$$

$$= \left(\bar{A}' - \bar{\lambda}I\right)(A - \lambda I)$$

$$= \overline{(A - \lambda I)'}(A - \lambda I) \qquad \ldots(2)$$

Thus $(A - \lambda I)$ is normal

Now, let $(A - \lambda I) = B$ and by hypothesis $BX = 0$ …(3)

So that $\overline{(BX)'}(BX) = 0$ …(4)

Further $\quad \overline{(B'X')'} = \overline{X'(B')'}$

$$= \overline{X'B} \qquad \because\ (B')' = B$$

$$= \bar{X}'\bar{B}$$

$$= \bar{X}'B \qquad \because \quad \left(\bar{B}\right) = B$$

$$\overline{(B'X')}'(BX) = \left(\bar{X}'B\right)\left(\bar{B}'X\right) = \overline{(BX)}'\left(\bar{B}'X\right) \qquad \ldots(5)$$

By (3) and (5), we have

$$\bar{B}'X = 0 \quad \text{or} \quad \left(\bar{A}' - \lambda\bar{I}\right)X = 0 \qquad \ldots(6)$$

Thus, X is a characteristic vector of $\bar{A}'$ corresponding to the characteristic value λ.

Example 51: If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{2i\pi/3}$, show that $S^{-1} = \frac{1}{3}\bar{S}$.

Solution: Let $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ...(1)

Now $$a = e^{2i\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right); \qquad \ldots(2)$$

$$a^2 = e^{4i\pi/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right); \qquad \ldots(3)$$

$$a^3 = e^{2i\pi} = (\cos 2\pi + i\sin 2\pi) = 1; \qquad \ldots(4)$$

and $$\bar{a} = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \ \overline{a^2} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \qquad \ldots(5)$$

Thus from equations (2) to (5), we see that

$$\bar{a} = a^2, \ \overline{a^2} = a \ \text{ and } \ a^4 = a^3 \cdot a = a \qquad \ldots(6)$$

Now write, $$\bar{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{a^2} & \bar{a} \\ 1 & \bar{a} & \overline{a^2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \ \text{(Using 6)} \qquad \ldots(7)$$

Find co-factors a_{ij}' s:

Co-factor of $a_{11} = (a - a^2)$ = Co-factors of a_{12}, a_{21}, a_{13}, a_{31}

Co-factor of $a_{22} = (a^2 - 1)$ = Co-factors of a_{33}

Co-factor of $a_{23} = (-a + 1)$ = Co-factor fo a_{32}

Also $|S| = 1(a - a^2) + 1(a - a^2) + 1(a - a^2) = 3(a - a^2)$

$$\therefore \quad S^{-1} = \frac{1}{3(a-a^2)}\begin{bmatrix}(a-a^2) & (a-a^2) & (a-a^2)\\ (a-a^2) & (a^2-1) & (-a+1)\\ (a-a^2) & (-a+1) & (a^2-1)\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \dfrac{(a^2-a^3)}{(a-a^2)} & \dfrac{(-a^4+a^3)}{(a-a^2)}\\ 1 & \dfrac{(-a^4+a^3)}{(a-a^2)} & \dfrac{(a^2-a^3)}{(a-a^2)}\end{bmatrix} \quad \text{(On replacing 1 by } a^3\text{)}$$

or $$S^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & a & a^2\\ 1 & a^2 & a\end{bmatrix} = \frac{1}{3}\bar{S}$$

Hence the result.

Example 52: If $N = \begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$, obtain the matrix $(1-N)(1+N)^{-1}$, and show that it is unitary. **[KUK, 2008]**

Solution: Let $$N = \begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \ldots(1)$$

Then $$(I-N) = \begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix} \qquad \ldots(2)$$

and $$(I+N) = \begin{bmatrix}1 & 1+2i\\ -1+2i & 1\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \qquad \ldots(3)$$

Find co-factors of a_{ij}'s:

Co-factors of $a_{11} = 1$

Co-factors of $a_{12} = -(-1+2i) = (1-2i)$

Co-factors of $a_{21} = -(1+2i) = (-1-2i)$

Co-factors of $a_{22} = 1$ …(4)

Also $|I+N| = 1-(2i+1)(2i-1) = 1-(-4-1) = 6$ …(5)

whence $$(I+N)^{-1} = \frac{1}{6}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix} \qquad \ldots(6)$$

Take product of $(I-N)(I+N)^{-1}$ with the help of equations (2) and (6)

$$\therefore \quad (I-N)(I+N)^{-1} = \frac{1}{6}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 4i^1 & -2-4i \\ 2-4i & 4i^2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \qquad \ldots(7)$$

Let $(I - N)(1 + \mathrm{N})^{-1} = U,$

then for U to be unitary, we must have $\bar{U}'U = I$

Thus, from equation (7) obtain $\bar{U} = \frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$

which implies $\bar{U}' = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

Now $$\bar{U}'U = \frac{1}{6\times 6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 4\times 4+(2+4i)(2-4i) & -4(-2-4i)+(2+4i)(-4) \\ (-2+4i)(-4)-4(2-4i) & (-2+4i)(-2-4i)+16 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 16+4-16i^2 & 8+16i-8-16i \\ 8-16i-8+16i & 4-16i^2+16 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$$

Hence $U = (1 - N)(1 + N)^{-1}$ is unitary.

Brief about special types of matrices

To any matrix $[a_{ij}]$, we call

(i) Symmetric if $[a_{ij}] = [a_{ij}]'$ (ii) Skew-symmetric if $[a_{ij}] = -[a_{ij}]'$

(iii) Involutary if $[a_{ij}] = [a_{ij}]^{-1}$ (iv) Orthogonal if $[a_{ij}]' = [a_{ij}]^{-1}$

(v) Real if $\left[a_{ij}\right] = \overline{\left[a_{ij}\right]}$ (vi) Hermitian if $\left[a_{ij}\right] = \left[\bar{a}_{ij}\right]'$

(vii) Skew-Hermitian if $\left[a_{ij}\right] = -\overline{\left[aij\right]}'$ ($viii$) Unitary if $\left[a_{ij}\right] = \left(\overline{\left[aij\right]}'\right)^{-1}$

(ix) Pure Imaginary if $\left[a_{ij}\right] = -\overline{\left[a_{ij}\right]}$

1.13 DIFFERENTIATION AND INTEGRATION OF MATRICES

Suppose we have a matrix $[a_{ij}(t)]$, where enteries $a_{ij}(t)$ of the matrix are functions of a certain argument t:

$$\left[a_{ij}(t)\right] = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}(t) & a_{m2}(t) & \ldots & a_{mn}(t) \end{bmatrix} \qquad \ldots(1)$$

We can write this more precisely

$$[A(t)] = [a_{ij}(t)];\ (i = 1, 2 \ldots\ m;\ j = 1, 2, \ldots, n) \qquad \ldots(2)$$

Let the elements of the matrix have derivatives $\frac{d}{dt}a_{11}(t), \ldots, \frac{d}{dt}a_{mn}(t)$

Definition 1: The derivative of a matrix $[A(t)]$ is a matrix denoted by $\frac{d}{dt}[A(t)]$, whose enteries are the elements of the matrix $[A(t)]$; i.e.

$$\frac{d}{dt}[A(t)] = \begin{bmatrix} \frac{d}{dt}a_{11} & \frac{d}{dt}a_{12} \ldots \frac{d}{dt}a_{1n} \\ \frac{d}{dt}a_{21} & \frac{d}{dt}a_{22} \ldots \frac{d}{dt}a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ \frac{d}{dt}a_{m1} & \frac{d}{dt}a_{m2} \ldots \frac{d}{dt}a_{mn} \end{bmatrix} \qquad \ldots(3)$$

Remarks: This definition of the derivatives of a matrix comes quite naturally if to the operations of substraction of matrices and multiplication by a scalar, we adjoin the operation of passage to limit:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{[A(t+\Delta t)] - [A(t)]\right\}$$
$$= \lim_{\Delta t \to 0}\left[\frac{a_{ij}(t+\Delta t) - a_{ij}(t)}{\Delta t}\right]$$
$$= \left[\lim_{\Delta t \to 0}\frac{a_{ij}(t+\Delta t) - a_{ij}}{\Delta t}\right]$$

We can write equation (3) more precisely in the symbolic form as below:

$$\frac{d}{dt}[A(t)] = \left[\frac{d}{dt}a_{ij}(t)\right] \text{ or } \frac{d}{dt}[A(t)] = \left[\frac{d}{dt}A(t)\right] \qquad \ldots(4)$$

More commonly 'D' is used in place of $\frac{d}{dt}$,

Hence $\qquad D[A(t)] = [D(A(t))] \qquad \ldots(5)$

Definition 2: The integral of the matrix $[A(t)]$ is a matrix to which we denote as $\int [A(t)]dt$ whose elements are equal to the integrals of the elements of the given matrix:

$$\int A(t)\,dt = \begin{bmatrix} \int a_{11}(t)\,dt & \int a_{12}(t)\,dt & \ldots\int a_{1n}(t)\,dt \\ \int a_{21}(t)\,dt & \int a_{22}(t)\,dt & \ldots\int a_{2n}(t)\,dt \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ \int a_{m}(t)\,dt & \int a_{m2}(t)\,dt & \ldots\int a_{mn}(t)\,dt \end{bmatrix} \qquad \ldots(6)$$

More precisely,

$$\int A(t)dt = [\int a_{ij}(t)dt] = [\int A(t)dt] \qquad \ldots(7)$$

The symbol $\int (\,)\,dt$ is sometimes denoted by a single letter, say S, and then we can write equation (7), like, we did in (5)

$$S[A] = [SA]$$

A. Solutions of System of Differential Equations with Constant coefficients

We consider a system of linear differential equations with n unknowns $x_1(t)$, $x_2(t)$, ... $x_n(t)$:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad \ldots(1)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n$$

The coefficients a_{ij} are constants. We introduce the notations:

$$[x] = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad \ldots(2)$$

This is solution matrix or the vector solution of the system (1). Writing the matrix of derivatives of the solutions:

$$\left[\frac{dx}{dt}\right] = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} \qquad \ldots(3)$$

Let us write down the matrix of coefficients of the system of differential equations:

$$[a] = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{bmatrix} \qquad \ldots(4)$$

Using the rule for matrix multiplication, we can write the system of differential equations (1) in matrix form:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_3(t) \end{bmatrix}. \qquad \ldots(5)$$

or, more precisely on the basis of the rule for differentiation,

$$\frac{d}{dt}\left[x(t)\right] = [a][x] \qquad \ldots(6)$$

The equation can also be written as:

$$\frac{dx}{dt} = a\,x \qquad \ldots(7)$$

where x is also called the vector solution; a is short notation for the matrix $[a_{ij}]$.

If we have
$$[\alpha] = \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \ldots(8)$$

where α_i are certain scalars, then the set of solutions of a system of differential equations will be sought in the form

$$[x] = e^{\lambda t}[\alpha] \quad \text{or} \quad x = e^{\lambda t}\alpha \qquad \ldots(9)$$

The solution of a Leibnitz linear differential equation $\frac{dx}{dt} - kx = 0$ will be $x = e^{-kt}C$, where C is an arbitrary constant. Again if x is a vector quantity then for different scalars k_i and constants C_i, we can write

$$x = Ce^{kt} \text{ with } C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \qquad \ldots(10)$$

Substituting (9) into (7), viz. the rule for multiplication of matrix by a scalar and the rule for differentiating matrices, we get both sides as

$$\frac{d}{dt}\left(e^{\lambda t}\alpha\right) = a\,e^{\lambda t}\alpha \qquad \ldots(11)$$

Whnce we have $\lambda\alpha = a\lambda$

or
$$a\alpha - \lambda\alpha = 0 \qquad \ldots(12)$$

The matrix equation (12) can also be written as:

$$(a - \lambda I)\alpha = 0, \qquad \ldots(13)$$

where I is an identity matrix of order n.

In expanded form, equation (13) is thus:

$$\begin{bmatrix} a_{11-\lambda} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22-\lambda} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn-\lambda} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = 0 \qquad \ldots(14)$$

Equation (12) shows that the vector ‘α’, can be transformed by the matrix ‘a’ into a parallel vector ‘$\lambda\alpha$’. Hence, the vector ‘α’ is an ‘eigen vector’ of the matrix ‘a’ corresponding to the ‘eigen value’ λ. In scalar form, equation (12) as a system of algebraic equations is thus:

$$\left.\begin{aligned} &\left(a_{11} - \lambda_1\right)\alpha_1 + a_{12}\,\alpha_2 + \;\ldots\ldots a_{1n}\,\alpha_n = 0 \\ &a_{21}\alpha_1 + \left(a_{22} - \lambda_2\right)\alpha_2 + \;\ldots\ldots a_{2n}\,\alpha_n = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{n1}\,\alpha_1 + a_{n2}\,\alpha_2 + \ldots + \left(a_{nn} - \lambda_n\right)\alpha_n = 0 \end{aligned}\right\} \qquad \ldots(15)$$

The scalar λ must be determined from (15).

If λ is such that the determinant value Δ of the coefficient matrix, $[a - \lambda I]$ is different from zero, then the system (15) has only trivial solutions, $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ and, hence formulates only trivial solutions

$$x_1(t) = x_2(t) = \dots = x_n(t) = 0 \qquad \dots(16)$$

If λ is such that the determinant Δ of the coefficient matrix $[a - \lambda I]$ vanishes, we arrive at resulting an equation of order n for determining λ:

$$\begin{vmatrix} a_{11-\lambda} & a_{12} \dots a_{1n} \\ a_{21} & a_{22-\lambda} \dots a_{2n} \\ \dots\dots\dots\dots \\ a_{n1} & a_{n2} \dots a_{nn-\lambda} \end{vmatrix} = 0 \qquad \dots(17)$$

This equation is called the auxiliary equation or characteristic equation and its roots are called the roots of the auxiliary characteristic equation.

Case I: The roots of the auxiliary equation are real and distinct.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of the auxiliary equation. For each root λ_i, write the system of equations (15) and determine the coefficients $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_n^{(i)}$. It may be shown that one of them is arbitrary and be considered equal to unity. Thus, we obtain:

For the root λ_1, the solution of the system (10)

$$x_1^{(1)} = \alpha_1^1 e^{\lambda_1 t}, \quad x_2^{(1)} = \alpha_2^1 e^{\lambda_1 t}, \dots\dots \quad x_n^{(1)} = \alpha_n^{(1)} e^{\lambda_1 t}$$

For the root λ_2, solution of the system (10)

$$x_1^{(2)} = \alpha_1^{(2)} e^{\lambda_2 t}, \quad x_2^{(2)} = \alpha_2^{(2)} e^{\lambda_2 t}, \dots\dots, \quad x_n^{(2)} = \alpha_n^{(2)} e^{\lambda_2 t}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

For the root λ_n, the solution of the system (10)

$$x_1^{(n)} = \alpha_1^{(n)} e^{\lambda_n t}, \quad x_2^{(n)} = \alpha_2^{(n)} e^{\lambda_n t}, \dots\dots, \quad x_n^{(n)} = \alpha_n^{(n)} e^{\lambda_n t}$$

Thus on substitution of values of $x_i^{(n)}$, the system of functions becomes

$$\left.\begin{aligned} x_1 &= C_1 \alpha_1^{(1)} e^{\lambda_1 t} + C_2 \alpha_1^{(2)} e^{\lambda_2 t} + \dots + C_n \alpha_1^{(n)} e^{\lambda_n t} \\ x_2 &= C_1 \alpha_2^{(1)} e^{\lambda_1 t} + C_2 \alpha_2^{(2)} e^{\lambda_2 t} + \dots + C_n \alpha_2^{(n)} e^{\lambda_n t} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= C_1 \alpha_n^{(1)} e^{\lambda_1 t} + C_2 \alpha_n^{(2)} e^{\lambda_2 t} + \dots + C_n \alpha_n^{(n)} e^{\lambda_n t} \end{aligned}\right\} \qquad \dots(18)$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants. This is the general solution of system (1). A particular solution can be obtained by giving particular values to the arbitrary constants.

In matrix form, the solution (18) of the system can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} \dots \alpha_1^{(n)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} \dots \alpha_2^{(n)} \\ \dots\dots\dots\dots \\ \alpha_n^{(1)} & \alpha_n^{(2)} \dots \alpha_n^{(n)} \end{bmatrix} \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{bmatrix} \qquad \dots(19)$$

where C_i are arbitrary constants.

Precisely, $[x] = [a]\,[Ce^{\lambda t}]$ …(20)

Case II: The roots of the auxiliary equations are distinct, but imaginary.

Among the roots of the auxiliary equation, let there be two complex conjugate roots:

$$\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta \qquad \text{…(21)}$$

To these roots will correspond the solutions:

$$\left.\begin{aligned} x_j^{(1)} &= \alpha_j^{(1)}\, e^{(\alpha+i\beta)t},\ (j = 1, 2, \ldots, n) \\ x_j^{(2)} &= \alpha_j^{(2)}\, e^{(\alpha+i\beta)t},\ (j = 1, 2, \ldots, n) \end{aligned}\right\} \qquad \text{…(22)}$$

The coefficients $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ are determined from the system of equation (14). Since the real and imaginary parts of the complex solution are also solutions.

We, thus, obtain two particular solutions:

$$\left.\begin{aligned} x_j^{-(1)} &= e^{\alpha t}\left(\lambda_j^{(1)} \cos \beta t + \lambda_j^{(2)} \sin \beta t\right) \\ x_j^{-(2)} &= e^{\alpha t}\left(\lambda_j^{-(1)} \sin \beta t + \lambda_j^{-(2)} \cos \beta t\right) \end{aligned}\right\} \qquad \text{…(23)}$$

where $\lambda_j^{(1)}, \lambda_j^{(2)}, \lambda_j^{-(1)}, \lambda_j^{-(2)}$ are real numbers determined in terms of $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$.

Appropriate combinations of functions (23) will enter into general solution of the system.

Example 53: Write down in the matrix form of the system and the solution of the system of linear differential equations:

$$\frac{dx_1}{dt} = 2x_1 + 2x_2,\ \frac{dx_2}{dt} = x_1 + 3x_2.$$

Solution: In the matrix form, the system of equations is written as

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{…(1)}$$

Now the corresponding characteristic equation is

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0 \quad \text{i.e.,} \quad \lambda^2 - 5\lambda + 4 = 0$$

whence $\lambda_1 = 1,\ \lambda_2 = 4$ …(2)

Now, formulate matrix equation $[A - \lambda I]\,[\alpha] = 0$ with column matrix $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

i.e.,
$$\left.\begin{aligned} (a_{11} - \lambda)\alpha_1 + a_{12}\alpha_2 &= 0 \\ a_{21}\alpha_1 + (a_{22} - \lambda)\alpha_2 &= 0 \end{aligned}\right\} \qquad \text{…(3)}$$

For $\lambda = 1$,
$$\left.\begin{aligned} (2-1)\alpha_1^{(1)} + a_{12}\alpha_2^{(1)} &= 0 \\ \alpha_1^{(1)} + (3-1)\alpha_2^{(1)} &= 0 \end{aligned}\right\}$$

i.e. simply one equation, $\alpha_1^{(1)} + 2\alpha_2^{(1)} = 0$ ...(4)

Setting $\alpha_1^{(1)} = 1$, we get $\alpha_2^{(1)} = -\frac{1}{2}$ ...(5)

In the same fashion, corresponding to the root $\lambda = 4$.

Now we can write the solution of the system in matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \end{bmatrix}$$

i.e.
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \end{bmatrix}$$

Therefore, we have
$$\left.\begin{aligned} x_1 &= C_1 e^{t} + C_2 e^{4t} \\ x_2 &= -\frac{1}{2} C_1 e^{t} + C_2 e^{4t} \end{aligned}\right\}.$$

Example 54: Write in matrix form the system and the solution of the system of differential equations

$$\boldsymbol{\frac{dx_1}{dt} = x_1, \quad \frac{dx_2}{dt} = x_1 + 2x_2, \quad \frac{dx_3}{dt} = x_1 + x_2 + 3x_3}.$$

Solution: In matrix form, the system of equations is written as:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let us form the characteristic equation and find its roots,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0, \text{ i.e. } (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

whence $\lambda = 1, 2, 3.$

Corresponding to $\lambda = 1$, finding $\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}$ from the system of equations as below:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

i.e.,
$$\left.\begin{aligned} \alpha_1^{(1)} + \alpha_2^{(1)} &= 0 \\ \alpha_1^{(1)} + \alpha_2^{(1)} + 2\alpha_3^{(1)} &= 0 \end{aligned}\right\}$$

From above, we have $\alpha_3^{(1)} = 0$ with $\alpha_1^{(1)} = 1$, $\alpha_2^{(1)}, = -1$

Similarly, corresponding to $\lambda = 2$, determine $\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}$.

i.e.,
$$\left.\begin{aligned} -\alpha_1^{(2)} &= 0 \\ \alpha_1^{(2)} &= 0 \\ \alpha_1^{(2)} + \alpha_2^{(2)} + \alpha_3^{(2)} &= 0 \end{aligned}\right\}$$

From above, we find $\alpha_1^{(2)} = 0$, $\alpha_2^{(2)} = 1$, $\alpha_3^{(2)} = -1$

Likewise, corresponding to $\lambda = 3$, we determine $\alpha_1^{(3)}$, $\alpha_2^{(3)}$, $\alpha_3^{(3)}$.

We obtain
$$\left.\begin{aligned} -2\alpha_1^{(3)} &= 0 \\ \alpha_1^{(3)} - \alpha_2^{(3)} &= 0 \\ \alpha_1^{(3)} + \alpha_2^{(3)} &= 0 \end{aligned}\right\}$$

or $\alpha_1^{(3)} = 0$, $\alpha_2^{(3)} = 0$, $\alpha_3^{(3)} = 1$

Consequently, in the matrix form, the solution of the given system of equations can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^t \\ C_2 e^{2t} \\ C_3 e^{3t} \end{bmatrix}$$

or
$$\left.\begin{aligned} x_1 &= C_1 e^t \\ x_2 &= -C_1 e^t + C_2 e^{2t} \\ x_3 &= -C_2 e^{2t} + C_3 e^{3t} \end{aligned}\right\}$$

ASSIGNMENT 5

Solve the following system of linear differential equations by the matrix method:

$$\frac{dx_1}{dt} + x_2 = 0, \quad \frac{dx_2}{dt} + 4x_1 = 0$$

B. Matrix Notation for a Linear Equations of Order *n*

Suppose we have an nth order linear differential equation with constant coefficients:

$$\frac{d^n x}{dt^n} = a_n \frac{d^{n-1} x}{dt^{n-1}} + a_{n-1} \frac{d^{n-2} x}{dt^{n-2}} + \ldots + a_1 x \qquad \ldots(1)$$

Later we will observe that this way of numbering the coefficients is convenient.

Take $x = x_1$

and
$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_{n-1}}{dt} &= x_n \\ \frac{dx_n}{dt} &= a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \end{aligned}\right\} \qquad \ldots(2)$$

Let us write down coefficient matrix of the system:

$$[a*] = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & a_4 & \dots & a_n \end{bmatrix} \qquad \dots(3)$$

***Note*:** Here we do not discuss the question of passage to a limit for operations performed on matrices. Then, the system (91) can be written as follows:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_{n-1}}{dt} \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad \dots(4)$$

or, briefly

$$\frac{d}{dt}[x] = [a^*] \cdot [x] \qquad \dots(5)$$

Example 55: Write the equation $\frac{d^2x}{dt^2} = p\frac{dx}{dt} + qx$ in matrix-form.

Solution: Put $x = x_1$, then $\frac{dx_1}{dt} = x_2$ and $\frac{dx_2}{dt} = px_2 + qx_1$

The system of equation in matrix form looks like this:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q & p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

C. Solving System of Linear Differential Equations with Variable Coefficients by the Method of Successive Approximations

Let it required to find the solution of the system of linear differential equations with variable coefficients.

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n \end{aligned}\right\} \qquad \dots(1)$$

that satisfy the initial conditions.

$$x_1 = x_{10},\ x_2 = x_{20},\ \ldots,\ x_n = x_{no},\ \text{for } t = t_0 \qquad \ldots(2)$$

If, besides the matrix of coefficient of the system and the matrix of solution, we introduce the matrix of initial conditions

$$[x_0] = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{bmatrix} \qquad \ldots(3)$$

then the system of equations (1) with initial conditions (2), can be written as:

$$\frac{d}{dt}[x] = [a(t)] \cdot [x] \qquad \ldots(4)$$

Here, $[a(t)]$ is again coefficients matrix of the system. We will solve the problem by the method of successive approximations.

To get a better grasp of the material that follows, let us apply to the method of successive approximations first to a single linear equation of the first order.

It is required to find the solution of the single equation

$$\frac{dx}{dt} = a(t)x \qquad \ldots(5)$$

for the initial conditions, $x = x_0$ for $t = t_0$

On assumption that $a(t)$ is a continuous function, the solution of the differential equation (5) with initial conditions, reduces to the integral equation

$$x = x_0 + \int_{t_0}^{t} a(z)x(z)\,dz \qquad \ldots(6)$$

We will solve this equation by the method of successive approximations:

$$\left.\begin{aligned} x_1 &= x_0 + \int_{t_0}^{t} a(z)x_0\,dz \\ x_2 &= x_0 + \int_{t_0}^{t} a(z)x_1(z)\,dz \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m &= x_0 + \int_{t_0}^{t} a(z)x_{m-1}(z)\,dz \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad \ldots(7)$$

We introduce the operator S, (the integration operator)

$$S(\ ) = \int_{t_0}^{t} (\ )\,dz \qquad \ldots(8)$$

Using the operator S, we can write the equations (101) as follows:

$$\left.\begin{aligned} x_1 &= x_0 + S(ax_0) \\ x_2 &= x_0 + S(ax_1) = x_0 + S\Big(a\big(x_0 + S(ax_0)\big)\Big) \\ x_3 &= x_0 + S\bigg(a\Big(x_0 + S\Big(a\big(x_0 + S(ax_0)\big)\Big)\Big)\bigg) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m &= x_0 + S\bigg(a\Big(x_0 + S\Big(x_0 + S\big(x_0 + S(ax_0)\big)\Big)\Big)\bigg) \end{aligned}\right\} \qquad \ldots(9)$$

Expanding, we get

$$x_m = x_0 + \underbrace{Sax_0 + SaSax_0 + SaSaSax_0 + \ldots + SaSaSa\ldots Sax_0}_{m \text{ times}}$$

Taking x_0 outside the brackets (x_0 constatn), we obtain

$$x_m = \{1 + \underbrace{Sa + SaSa + \ldots + SaSa\ldots Sa}_{m \text{ times}}\}x_0 \qquad \ldots(10)$$

It has been proved that if $a(t)$ is a continuous function, then the sequence $[x_m]$ converges. The limit of this sequence is a convergent series:

$$x = [1 + Sa + SaSa + \ldots]x_0 \qquad \ldots(11)$$

***Note*:** If $a(t)$ = const., then formula (11) assumes a simple form. Indeed, by (8) we can write

$$\left.\begin{array}{l} Sa = aSI = a(t - t_0) \\ SaSa = a^2S(t - t_0) = a^2\dfrac{(t - t_0)^2}{2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underbrace{SaSa\ldots Sa}_{m \text{ times}} = a^m\dfrac{(t - t_0)^m}{m!} \end{array}\right\} \qquad \ldots(12)$$

In this case, (11) takes the form

$$x = \left[1 + a\frac{(t - t_0)}{1!} + a^2\frac{(t - t_0)^2}{2!} + \ldots + a^m\frac{(t - t_0)^m}{m!}\right]x_0$$

$$x = x_0e^{a(t - t_0)} \qquad \ldots(13)$$

The method of solving the single equation (5) that we have just reviewed is carried over in its entirety to the solution of system (1) for the initial conditions (2).

In matrix form, system (1) with initial conditions (2) can be written as:

$$\frac{d}{dx}[x] = [a(t_0)][x] \qquad \ldots(14)$$

For the final conditions, $[x] = [x_0]$ for $t = t_0$, if we use the rule of matrix multiplication and matrix integration, the solution of system (14), under the given conditions, can be reduced to the solution of the matrix integral equation.

$$[x(t)] = [x_0] + \int_{t_0}^{t}[a(z)]\cdot[x(z)]dz \qquad \ldots(15)$$

We find the successive approximations

$$[x_m(t)] = [x_0] + \int_{t_0}^{t}[a(z)]\cdot[x_{m-1}(z)]dz \qquad \ldots(16)$$

By successive substitution of the successive approximations under the integral, the solution of the system comes out like this in matrix form:

$$[x(t)] = [x_0] + \int_{t_0}^{t}[a(z_1)]\left\{[x_0] + \int_{t_0}^{z_1}[a(z_2)]\left([x_0] + \int_{t_0}^{z_2}[a(z_3)](\ldots)dz_3\right)dz_2\right\}dz_1 + \ldots$$

or

$$[x(t)] = [x_0] + \int_{t_0}^{t}[a(z_1)]\cdot[x_0]dz_1 + \int_{t_0}^{t}[a(z_1)]\int_{t_0}^{z_1}[a(z_2)]\cdot[x_0]dz_2\,dz_1 + \ldots \qquad \ldots(17)$$

Using the integration operator S, we can write (17) as

$$[x(t)] = \{[E] + S[a]\, S[a]\, S[a] + \ldots\}\, [x_0] \quad \ldots(18)$$

The operator in brackets { } can be denoted by a single letter. We denote it by $\eta^{(t_0\, t)}[a(t)]$. Then equation (18) is precisely written as

$$[x(t)] = \eta^{(t_0\, t)}[a(t)][x_0] \quad \ldots(19)$$

It is interesting to note that if the coefficients of system (1) are constants, then using the rule for taking a common factor all entries of the matrix outside the matrix symbol, * we can write

$$S[a] = \frac{(t-t_0)}{1}[a]$$

$$S[a]\, S[a] = \frac{(t-t_0)^2}{2!}[a]^2$$

$$S[a]\, S[a]\, S[a] = \frac{(t-t_0)^3}{3!}[a]^3 \text{ and so on.}$$

In the case of constant coefficient, formula (18) assumes the form

$$[x(t)] = \left[[E] + \frac{t-t_0}{1!}[a] + \frac{(t-t_0)^2}{2!}[a]^2 + \ldots + \frac{(t-t_0)^m}{m!}[a]^m + \ldots\right][x_0] \quad \ldots(20)$$

This equation can be symbolized in compact form as

$$[x(t)] = e^{(t-t_0)}\,[a]\,[x_0] \quad \ldots(21)$$

ANSWERS

Assignment 4

1. 1,2,4; (1, 0, 0), (0, 1, 1), (0, 1, – 1,); $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$; $x_1^2 + 2x_2^2 + 4x_3^2$

2. $x_1^2 + 4x_3^2 + 4x_1x_2 + 10x_1x_3 + 6x_2x_3$

Assignment 5

$x_1 = c_1e^{-2t} + c_2e^{2t}, \quad x_2 = 2c_1e^{-2t} - 2c_2e^{2t}$