4 Integration

At first glance, integration might appear straightforward; surely we can write the following?

$$\int z^2 \, \mathrm{d}z = \frac{1}{3}z^3 + c$$

Recall that such a statement *in real analysis* would reflect the equivalence of two distinct concepts:

Anti-derivatives $\frac{1}{3}z^3$ is an anti-derivative of z^2

Definite Integrals The 'area' under the curve $y = z^2$ on the interval $[0, z]$ equals $\frac{1}{3}z^3$

The equivalence of these concepts is so amazing that we call it the *fundamental* theorem of calculus. While anti-derivatives make immediate sense in complex analysis, definite integrals are more delicate. For instance, what should we mean by

$$\int_{3+i}^{4i} z^2 \, \mathrm{d}z \text{ ?}$$

A Riemann sum construction requires partitioning some *curve* joining the points $3+i$ and $4i$; but which curve? Does it matter? This question leads us to revisit the idea of a *contour* or *path integral.*

4.1 Functions of a Real Variable and Contour Integrals

We start by considering complex-valued functions of a *real* variable. Derivatives and definite integrals of such functions are built from those of their real and imaginary parts:

$$w'(t) = u'(t) + iv'(t), \qquad \int_a^b w(t) \, \mathrm{d}t = \int_a^b u(t) \, \mathrm{d}t + i \int_a^b v(t) \, \mathrm{d}t \qquad (a, b \text{ are } \textit{real} \text{ or } \pm\infty)$$

Examples 4.1. 1. If $w(t) = 5t^2 + it$, then

$$w'(t) = 10t + i, \qquad \int_1^2 w(t) \, \mathrm{d}t = \int_1^2 5t^2 \, \mathrm{d}t + i \int_1^2 t \, \mathrm{d}t = \frac{7}{3} + \frac{i}{2}$$

2. If $w(t) = t^2 + e^{it} = t^2 + \cos t + i \sin t$, then

$$w'(t) = 2t - \sin t + i \cos t, \qquad \int_0^{2\pi} w(t) \, \mathrm{d}t = \frac{1}{3}t^3 + \sin t - i \cos t \Big|_0^{2\pi} = \frac{8}{3}\pi^3$$

The natural extension of the (real) fundamental theorem is on show here; $\frac{1}{3}t^3 + \sin t - i \cos t$ is plainly an *anti-derivative* of $w(t)$.

The obvious calculus laws are easily verified by considering real and imaginary parts separately.

Lemma 4.2.

- *Linearity: if* $k \in \mathbb{C}$*, then* $\frac{\mathrm{d}}{\mathrm{d}t} kw(t) = kw'(t)$ *and* $\int_a^b kw(t) \, \mathrm{d}t = k \int_a^b w(t) \, \mathrm{d}t$
- *Product rule:* $\frac{\mathrm{d}}{\mathrm{d}t} w(t)z(t) = w'(t)z(t) + w(t)z'(t)$
- *Chain rule: If* $s(t)$ *is a* real *function then* $\frac{\mathrm{d}}{\mathrm{d}t} w\big(s(t)\big) = w'\big(s(t)\big)s'(t)$

Complex substitutions are a little more subtle, and benefit from a proof.

Lemma 4.3 (Complex Chain Rule). *Suppose $w(t) = F\big(z(t)\big)$ where*

- *$z(t) = x(t) + iy(t)$ is differentiable at t, and,*
- *$F(z) = u(x,y) + iv(x,y)$ is holomorphic at $z(t)$*

Then w is differentiable at t, and $w'(t) = F'\big(z(t)\big)z'(t)$.
If $z'(t)$ is integrable on $[a,b]$ and F is holomorphic on $z([a,b])$, we can put this in integral form

$$\int_a^b F'\big(z(t)\big)z'(t)\,\mathrm{d}t = F\big(z(b)\big) - F\big(z(a)\big)$$

Proof. Apply the multi-variable chain rule from real calculus and the Cauchy–Riemann equations:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + i\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + i\left(\frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = (u_x + iv_x)\frac{\mathrm{d}x}{\mathrm{d}t} + i(v_y - iu_y)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (u_x + iv_x)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) \qquad \text{(Cauchy–Riemann)} \\ &= F'\big(z(t)\big)z'(t)\end{aligned}$$

■

Examples 4.4.

1. Let $w(t) = e^{t-it^2} = F\big(z(t)\big)$ where $F(z) = e^z$ and $z(t) = t - it^2$. Then

$$w'(t) = e^{t-it^2}\frac{\mathrm{d}}{\mathrm{d}t}(t - it^2) = e^{t-it^2}(1 - 2it)$$

   Compare this with the method of Example 4.1 which gives the same result, if more slowly

$$w'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(e^t\cos t^2 - ie^t\sin t^2) = e^t(\cos t^2 - 2t\sin t^2) - ie^t(\sin t^2 + 2t\cos t^2)$$

2. Since $w(t) = F\big(z(t)\big) = \frac{1}{10}(1 - t^2 + it)^{10}$ has $w'(t) = (i - 2t)(1 - t^2 + it)^9$, we see that

$$\int_0^1 (i - 2t)(1 - t^2 + it)^9\,\mathrm{d}t = \frac{1}{10}(1 - t^2 + it)^{10}\bigg|_0^1 = \frac{i^{10} - 1}{10} = -\frac{1}{5}$$

   This would be horrific if we had to multiply out to work with real and imaginary parts!

3. Sometimes the real and imaginary part approach is simply not tenable;

$$\int_0^1 3it\sqrt{1 + it^2}\,\mathrm{d}t = (1 + it^2)^{3/2}\bigg|_0^1 = (1 + i)^{3/2} - 1 = 2\sqrt{2}e^{\frac{3\pi i}{8}} - 1$$

   Everything is evaluated using the principal square root since $1 + it^2$ lies in the first quadrant.

While most of the basic rules of real calculus translate to complex-valued functions of a real variable, not everything goes through. Be particularly careful of existence results such as the mean value theorem which apply perfectly well to real and imaginary parts, but not to the whole...

Contours and Contour Integrals

We now turn our attention to integrating complex functions along curves. But what sort of curves?

Definition 4.5. A *smooth arc* is an oriented curve C in the complex plane for which there exists a *regular parametrization*; a differentiable function $z : [a, b] \to \mathbb{C}$ such that,

1. $z([a, b]) = C$ where $z(a)$ is the *start* of the curve and $z(b)$ is the *end*;
2. $z'(t)$ is *continuous* on $[a, b]$ and *non-zero* on (a, b).

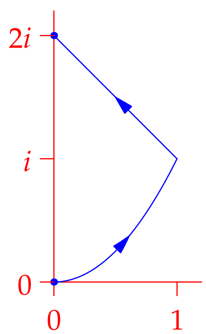
A *contour* is a piecewise smooth arc C consisting of finitely many smooth arcs joined end-to-end. A parametrization $z(t)$ of C is therefore continuous with piecewise continuous derivative.

If we *reverse the orientation* of a contour C, the resulting contour is labelled $-C$.

Additionally, we say that a contour is:

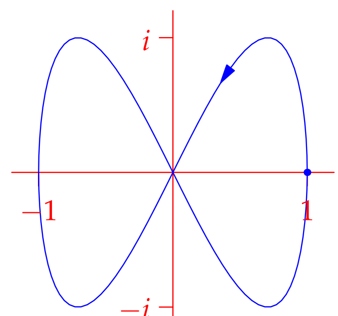
- *Closed* if it starts and ends at the same point, $z(a) = z(b)$;
- *Simple* if it does not cross itself (z is injective, $z(t) = z(s) \implies t = s$).
- *Positively oriented* if it is simple, closed and traversed counter-clockwise.

Examples 4.6. Here are three contours with explicit parametrizations:



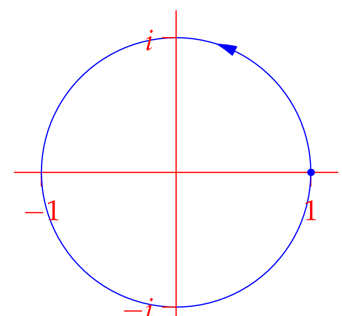
Simple, piecewise

$$z = \begin{cases} t + it^2 & t \in [0, 1] \\ 2 - t + it & t \in [1, 2] \end{cases}$$



Closed, non-simple

$$z = \cos t + i \sin 2t,\ t \in [0, 2\pi]$$



Positively oriented

$$z = e^{it},\ t \in [0, 2\pi]$$

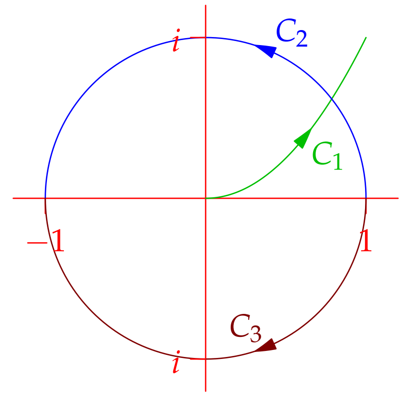
Definition 4.7. Let C be a contour parametrized by $z : [a, b] \to \mathbb{C}$ and suppose that $f(z)$ is a complex function defined on the range of z. The *contour integral* of $f(z)$ along C is

$$\int_C f = \int_C f(z)\,\mathrm{d}z := \int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t$$

This is often written $\oint_C f$ if C is positively oriented (simple and closed).

We plainly require the integrability of the function $f\big(z(t)\big)z'(t)$ on the *real interval* $[a, b]$; typically we assume that this expression is piecewise continuous. As we'll see shortly, the choice of parametrization is irrelevant.

Examples 4.8. We evaluate several contour integrals.



1. For the contour C_1 parametrized by $z(t) = t + it^2$, $t \in [0, 1]$, we compute

$$\int_{C_1} z\,dz = \int_0^1 z(t)z'(t)\,dt = \int_0^1 (t + it^2)(1 + 2it)\,dt$$
$$= \int_0^1 t - 2t^3 + 3it^2\,dt = i$$

2. For the contour C_2 with $z(t) = e^{it}$ with $t \in [0, \pi]$,

$$\int_{C_2} \frac{1}{z}\,dz = \int_0^\pi \frac{ie^{it}}{e^{it}}\,dt = \pi i$$

$$\int_{C_2} z^2 + 1\,dz = \int_0^\pi (e^{2it} + 1)ie^{it}\,dt = \frac{i}{3i}e^{3it} + \frac{i}{i}e^{it}\Big|_0^\pi = \frac{1}{3}(e^{3\pi i} - 1) + e^{\pi i} - 1 = -\frac{8}{3}$$

3. We compute the same integrals as the previous example, but over the lower semi-circle C_3 parametrized by $z(t) = e^{-it}$ with $t \in [0, \pi]$. This time

$$\int_{C_3} \frac{1}{z}\,dz = \int_0^\pi \frac{-ie^{-it}}{e^{-it}}\,dt = -\pi i$$
$$\int_{C_3} z^2 + 1\,dz = \int_0^\pi (e^{-2it} + 1)(-ie^{-it})\,dt = \frac{-i}{-3i}e^{-3it} + \frac{-i}{-i}e^{-it}\Big|_0^\pi = -\frac{8}{3}$$

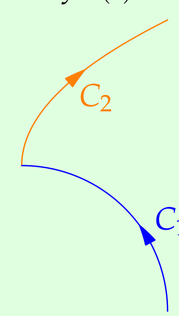
The *sign* of one integral changed but the other did not! We'll return to this problem shortly…

Before considering more examples we develop some of the basic properties of contour integrals. Several are immediate from our earlier discussion, for instance linearity:

$$\int_C \big(af(z) + bg(z)\big)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz$$

Of more importance are the following:

Theorem 4.9 (Basic rules for contour integrals). *Suppose C is a contour parametrized by $z(t)$.*



1. *If $C = C_1 \cup C_2$, where C_1 and C_2 are contours such that the end of the first is the start of the second, then*

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

2. *$\int_C f$ is independent of (orientation-preserving) parametrization.*
3. *Reversing orientation changes the sign of the integral: $\int_{-C} f = -\int_C f$.*

Example 4.10. By the previous example and part 3 of the Theorem, if C is the unit circle centered at the origin, then $\oint_C \frac{1}{z}\,dz = 2\pi i$. This can also be verified directly.

Proof. Part 1 follows from the well-known property $\int_a^b = \int_a^c + \int_c^b$ of real integrals. Armed with this, it is enough to check the other parts for a single smooth arc C.

2. Suppose $z : [a, b] \to \mathbb{C}$ and $w : [\alpha, \beta] \to \mathbb{C}$ are parametrizations of C. Then $w(s(t)) = z(t)$ for some continuously differentiable s with *positive* derivative. Now compute,

$$z'(t) = w(s(t))s'(t), \qquad s(a) = \alpha, \quad s(b) = \beta, \quad \text{from which,}$$
$$\int_a^b f(z(t))z'(t)\,dt = \int_a^b f(w(s(t)))w'(s(t))\frac{ds}{dt}\,dt = \int_{s(a)}^{s(b)} f(w(s))w'(s)\,ds$$
$$= \int_\alpha^\beta f(w(s))w'(s)\,ds$$

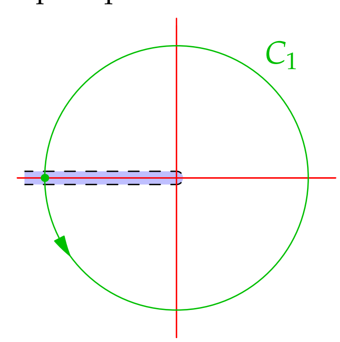
3. This is almost identical, except that $-C$ requires a reparametrization $w(s)$ with $s' < 0$, $s(a) = \beta$ and $s(b) = \alpha$. The upshot is that the limits flip on the final integral:

$$\int_C f = \int_a^b f(z(t))z'(t)\,dt = \int_\beta^\alpha f(w(s))w'(s)\,ds = -\int_\alpha^\beta f(w(s))w'(s)\,ds = -\int_{-C} f \qquad \blacksquare$$

Contour Integrals of multi-valued functions If a function is multi-valued, we must specify a *branch* before integrating. It is acceptable to have the contour start and/or finish on the branch cut.[1]

Examples 4.11. 1. Compute $\oint_{C_1} z^{1/2}\,dz$ over the unit circle using the principal square root.

Since the question specifies the principal branch, we must must start and finish the contour at $z = -1$. Parametrize via $z(t) = e^{it}$ where $t \in (-\pi, \pi)$. Since



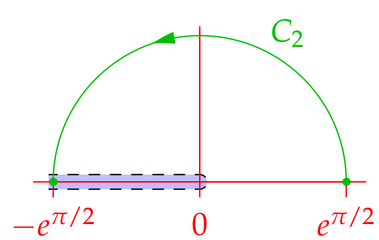
$$\sqrt{z(t)}\,z'(t) = e^{\frac{it}{2}} ie^{it} = ie^{\frac{3it}{2}}$$

is continuous on $[-\pi, \pi]$, we compute

$$\int_{C_1} z^{1/2}\,dz = \int_{-\pi}^{\pi} ie^{\frac{3it}{2}}\,dt = \frac{2}{3}e^{\frac{3it}{2}}\Big|_{-\pi}^{\pi} = \frac{2}{3}(e^{\frac{3\pi i}{2}} - e^{-\frac{3\pi i}{2}}) = -\frac{4i}{3}$$

2. Find $\int_{C_2} z^i\,dz$ over the semi-circle shown using P. V. z^i.

Let $z(t) = e^{\frac{\pi}{2}}e^{it}$ where $t \in [0, \pi]$. Since



$$z^i z' = e^{i\,\mathrm{Log}\,z} z' = \exp\left(i(\tfrac{\pi}{2} + it)\right) e^{\frac{\pi}{2}} ie^{it} = -e^{\frac{\pi}{2}} e^{(-1+i)t}$$

is continuous when $0 \le t \le \pi$, we have

$$\int_{C_2} z^i\,dz = \int_0^\pi -e^{\frac{\pi}{2}} e^{(-1+i)t}\,dt = \frac{e^{\frac{\pi}{2}}}{1-i}\left(e^{(-1+i)\pi} - 1\right) = -\frac{1+i}{2}(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}})$$

Warning! Things are more difficult if we permit a contour to *cross* a branch cut since we must then work with multiple branches simultaneously. We won't do this.

[1] We extend the integrand continuously: recall that if $g(t)$ is *uniformly continuous* on a bounded open interval (a, b), then it has a continuous extension to the closed interval $[a, b]$ and we can therefore define $\int_a^b g(t)\,dt$.

Exercises 4.1 1. Evaluate the derivatives and integrals:

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin t + i\sqrt{t}\right]$ (b) $\frac{\mathrm{d}}{\mathrm{d}t}(i+t^3)^2$ (c) $\int_0^1 e^{\pi i t}\,\mathrm{d}t$

(d) $\int_0^{\pi/2} e^{it}(1+e^{it})^2\,\mathrm{d}t$ (e) $\int_0^1 (i-t)^6\,\mathrm{d}t$ (f) $\int_0^{\pi} (i-1)\cos\big((1+i)t\big)\,\mathrm{d}t$

2. Show that if $m,n \in \mathbb{Z}$, then

$$\int_0^{2\pi} e^{imt}e^{-int}\,\mathrm{d}t = 2\pi\delta_{mn}$$

where $\delta_{mn} = 1$ when $m = n$ and 0 otherwise. Do this two ways:

(a) Using the exponential law and the chain rule/substitution.

(b) By multiplying out and working with real and imaginary parts.

3. Use the chain rule to evaluate the integral $\int_0^x e^{(1+i)t}\,\mathrm{d}t$.
Hence find both $\int_0^x e^t \cos t\,\mathrm{d}t$ and $\int_0^x e^t \sin t\,\mathrm{d}t$ *without* using integration by parts.

4. Prove the product rule for functions of a real variable:

$$\frac{\mathrm{d}}{\mathrm{d}t}(wz) = w'z + wz'$$

(*Hint: let* $w(t) = u(t) + iv(t),\ z(t) = x(t) + iy(t)$ *and multiply out…*)

5. Check that the mean value theorem fails for the function $w(t) = \sqrt{t} + it^2$ on the interval $[0,1]$. That is, there is no $\xi \in (0,1)$ for which $w'(\xi) = \frac{w(1)-w(0)}{1-0}$.

6. Justify the integral form of the complex chain rule by considering the real and imaginary parts of F. What facts from real calculus are you using?

7. Evaluate each contour integral $\int_C f(z)\,\mathrm{d}z$ by explicitly parametrizing C:

(a) $f(z) = z^2$; C is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = z$; C consists of the straight lines joining $z = 1$ to $1+i$ to $-1+i$ to -1.

(c) $f(z) = \operatorname{Log} z$; C is the circular arc of radius 3 centered at the origin, oriented counter-clockwise from $-3i$ to $3i$.

8. Explicitly check that $\int_C z\,\mathrm{d}z = \frac{1}{2}(B^2 - A^2)$ along the straight line joining A and B.
(*Hint: the line can be parametrized by* $z(t) = (1-t)A + tB$ *where* $t \in [0,1]$)

9. Let $n \in \mathbb{Z}$ and let C_0 be the positively oriented circle centered at z_0 with radius $R > 0$. Explicitly parametrize this circle to show that

$$\oint_{C_0} (z-z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

10. Suppose $z : [a,b] \to \mathbb{C}$ is a regular parametrization of a smooth arc C. Then the *arc-length* of the curve is the integral of the *speed* of the parametrization:

$$L = \int_a^b \left|z'(t)\right| \mathrm{d}t$$

(a) Compute the arc-length of the circle of radius R centered at the origin.

(b) Compute the arc-length of the simple piecewise curve in Example 4.6.
(*This requires a tough substitution; perhaps look up a table of integrals...*)

(c) By commenting on the proof of Theorem 4.9, explain why a reparametrization of C does not change the arc-length.

(d) Let $s(t) = \int_a^t |z'(\tau)| \,\mathrm{d}\tau$ be the arc-length as a *function* of $t \in [a,b]$. Consider a new parametrization $w(s) = z(t(s))$, where $t(s)$ is the inverse function of $s(t)$.
Prove that $\left|\frac{\mathrm{d}w}{\mathrm{d}s}\right| = 1$.
(*This proves that every smooth arc has a unit-speed parametrization*)

The last three questions elaborate a little on the approach in Examples 4.11.

11. (a) Compute the integral of the principal value of $z^{1/3}$ around the positively oriented unit circle starting and finishing at $z = -1$.

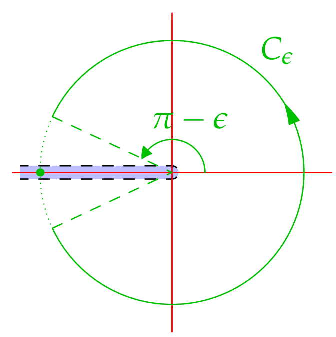
(b) Now consider the branch $z^{1/3} = \exp(\frac{1}{3}\log z)$ where $\arg z \in (0, 2\pi)$. Integrate this around the positively oriented unit circle starting and finishing at $z = 1$. What do you observe?

12. Compute $\oint_C z^{1/2} \,\mathrm{d}z$ where we take the α-branch $z^{1/2} = \exp(\frac{1}{2}\log z)$ with $\arg z \in (\alpha, \alpha + 2\pi)$ and the unit circle C traced from angle α to $\alpha + 2\pi$.

13. Let $\epsilon > 0$ be small and suppose that C_ϵ is the circular arc of radius 1 centered at the origin, traversed counter-clockwise from angle $-\pi + \epsilon$ to $\pi - \epsilon$.
By parametrizing C_ϵ, explicitly evaluate $\int_{C_\epsilon} \sqrt{z} \,\mathrm{d}z$ and verify that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \sqrt{z} \,\mathrm{d}z = -\frac{4i}{3}$$

is the value obtained previously for $\oint_C \sqrt{z} \,\mathrm{d}z$.



4.2 Path-independence, the Fundamental Theorem & Integral Estimation

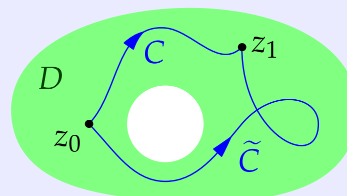
We start by revisiting an earlier example.

Example (4.8, parts 2 & 3). Let $F(z) = \frac{1}{3}z^3 + z$ and observe that $F'(z) = z^2 + 1$. If $z : [a, b] \to \mathbb{C}$ parametrizes a smooth arc C such that $z(a) = 1$ and $z(b) = -1$, then Lemma 4.3 shows that

$$\begin{aligned}\int_C z^2 + 1\,\mathrm{d}z &= \int_a^b \big(z(t)^2 + 1\big)z'(t)\,\mathrm{d}t = \int_a^b F'\big(z(t)\big)z'(t)\,\mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F\big(z(t)\big)\,\mathrm{d}t \\ &= F\big(z(b)\big) - F\big(z(b)\big) = -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3}\end{aligned}$$

The contour integral is independent of the choice of arc C!

Definition 4.12. Suppose $f : D \to \mathbb{C}$ where D is path-connected. We say that a contour integral $\int_C f$ is *path-independent* if its value depends only on the *endpoints* of C and not on the how the contour travels between these points.
Otherwise said, if $\widetilde{C}$ is any other contour with the same endpoints, then $\int_C f = \int_{\widetilde{C}} f$.
In such a situation it is permissible to write $\int_C f = \int_{z_0}^{z_1} f$.

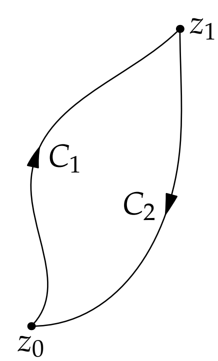


Before tackling the main result, we tidy up a connection between path independence and closed curves.

Lemma 4.13. *Every contour integral $\int_C f(z)\,\mathrm{d}z$ is path-independent if and only if $\int_C f(z)\,\mathrm{d}z = 0$ around every closed contour.*

Proof. ($\Rightarrow$) Assume all integrals are path independent and let C be a closed contour. Choose any points $z_0, z_1 \in C$ and decompose $C = C_1 \cup C_2$ into two contours from z_0 to z_1 and back again. Then,

$$\begin{aligned}\int_C f(z)\,\mathrm{d}z &= \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z && \text{(Theorem 4.9, part 1)}\\ &= \int_{C_1} f(z)\,\mathrm{d}z - \int_{-C_2} f(z)\,\mathrm{d}z && \text{(Theorem 4.9, part 3)}\\ &= 0\end{aligned}$$



since C_1 and $-C_2$ share the same endpoints.
($\Leftarrow$) Conversely, suppose $\int_C f(z)\,\mathrm{d}z = 0$ round any closed contour. Suppose $C_1, -C_2$ are any contours with the same endpoints z_0, z_1, then $C = C_1 \cup C_2$ is a closed contour and the above calculation shows that $\int_{C_1} f(z)\,\mathrm{d}z = \int_{-C_2} f(z)\,\mathrm{d}z$. ∎

The critical observation in the example was that the integrand $f(z) = z^2 + 1$ had an *anti-derivative* $F(z) = \frac{1}{3}z^2 + z$; this demonstrated path-independence and facilitated the easy computation of the integral. This should seem very familiar…

Theorem 4.14 (Fundamental Theorem). *Let f be continuous on an open domain D. Then (on D),*

$$f \text{ has an anti-derivative} \iff \text{all contour integrals of } f \text{ are path-independent}$$

In such situations, $\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0)$ where $F(z)$ is any anti-derivative of $f(z)$.

The assumptions of openness and continuity are necessary both because f has an anti-derivative, and because they'll be used explicitly in the proof. The ($\Rightarrow$) direction is sometimes known as the Fundamental Theorem of Line Integrals, especially in multi-variable calculus.
Since the proof requires a little work, we postpone it until after some examples.

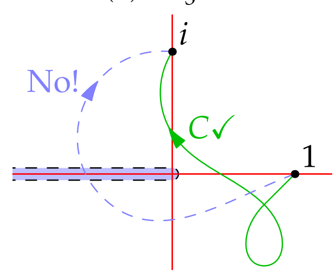
Examples 4.15.

1. $f(z) = (z+i)^3$ has anti-derivative $F(z) = \frac{1}{4}(z+i)^4$. It follows that

$$\int_0^{1-2i} (z+i)^4 = \frac{1}{4}(z+i)^4\Big|_0^{1-2i} = \frac{1}{4}\left[(1-i)^4 - i^4\right] = \frac{1}{4}(-4-1) = -\frac{5}{4}$$

regardless of the contour used to travel from $z = 0$ to $1-2i$.

2. Let $f(z) = z^{1/2}$ where we take the principal value. This has anti-derivative $F(z) = \frac{2}{3}z^{3/2}$.
If C is any contour joining $z_0 = 1$ and $z_1 = i$, which does not cross the branch cut, then



$$\int_C z^{1/2}\,\mathrm{d}z = \frac{2}{3}z^{3/2}\Big|_1^i = \frac{2}{3}(i^{3/2} - 1^{3/2}) = \frac{2}{3}\left(e^{\frac{3\pi i}{4}} - 1\right)$$

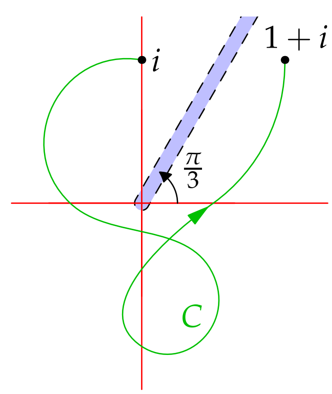
It is important to use the *same (principal) branch* to evaluate the anti-derivative: since $z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}$, we also have $z^{3/2} = e^{\frac{3}{2}\operatorname{Log} z}$.

3. Since $f(z) = \sin z$ has an anti-derivative $F(z) = -\cos z$ valid at all $z \in \mathbb{C}$, we see that $\int_C \sin z\,\mathrm{d}z = 0$ round any closed curve C.

4. By Example 4.10, $\oint_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$ round the unit circle. By the fundamental theorem, $f(z) = \frac{1}{z}$ cannot have an anti-derivative on any domain containing the circle. But this is obvious from our earlier discussion of the logarithm, any branch of which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z}$$

Since the logarithm cannot be made single-valued on any path encircling the origin, no anti-derivative of $f(z) = \frac{1}{z}$ exists on any domain containing the circle C.
We can, however, use the fundamental theorem to evaluate $\int_C \frac{1}{z}\,\mathrm{d}z$ over any contour staying within a single branch of the logarithm. In the picture, given the contour C, we choose the branch cut shown and evaluate



$$\begin{aligned}\int_C \frac{1}{z}\,\mathrm{d}z &= \log(1+i) - \log i = \log\sqrt{2}e^{\frac{\pi i}{4}} - \log e^{-\frac{3\pi i}{2}} \\ &= \ln\sqrt{2} + \frac{\pi i}{4} - \frac{3\pi i}{2} = \frac{1}{4}(\ln 4 + 7\pi i)\end{aligned}$$

Note how the arguments were chosen so that $-\frac{5\pi}{3} < \theta < \frac{\pi}{3}$.

Proving the Fundamental Theorem

The forward direction is relatively straightforward. Before reading the converse however, you may find it helpful to review part I of the fundamental theorem from real analysis, namely

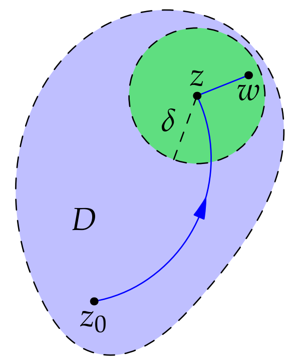
$$f \text{ continuous} \implies \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

We proceed by mimicking the original argument though there are some extra subtleties.

Proof. ($\Rightarrow$) The result is already true for every smooth arc by Lemma 4.3. Now let $C = C_1 \cup \cdots \cup C_n$ be a contour where the smooth arcs C_k are arranged in order with start/end points z_{k-1}, z_k. By Theorem 4.9, part 1, we see that

$$\int_C f = \sum_{k=1}^n \int_{C_k} f = \sum_{k=1}^n [F(z_k) - F(z_{k-1})] = F(z_n) - F(z_0)$$

($\Leftarrow$) Fix $z_0 \in D$ and *define*



$$F(z) = \int_{z_0}^z f(\zeta)\,\mathrm{d}\zeta$$

where the integral is taken along *any* curve joining z_0 to z. This is well-defined by the assumption of path-independence. To complete the proof, we need only show that $\lim\limits_{w\to z} \frac{F(w)-F(z)}{w-z} = f(z)$ on D.
Fix $z \in D$ and let $\epsilon > 0$ be given. Since f is continuous and D open,

$$\exists \delta > 0 \text{ such that } |\zeta - z| < \delta \implies \zeta \in D \text{ and } |f(\zeta) - f(z)| < \frac{\epsilon}{2}$$

Let $w \in D$ be such that $|w - z| < \delta$. Evaluating along any curve joining z, w we obtain

$$F(w) - F(z) = \int_z^w f(\zeta)\,\mathrm{d}\zeta$$

We may therefore *choose* the straight line segment. The proof is almost complete:

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \left|\frac{1}{w-z}\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right| = \frac{1}{|w-z|}\left|\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right|$$
$$\leq \frac{1}{|w-z|}|w-z|\frac{\epsilon}{2} < \epsilon$$

■

The second last inequality should give you pause. We are tempted to argue that

$$\left|\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right| \leq \int_z^w |f(\zeta) - f(z)|\,\mathrm{d}\zeta \leq \int_z^w \frac{\epsilon}{2}\,\mathrm{d}\zeta = |w-z|\frac{\epsilon}{2}$$

While this makes perfect sense in *real* analysis, it is utter nonsense in *complex* analysis, since

$$\int_z^w |f(\zeta) - f(z)|\,\mathrm{d}\zeta = \int_a^b \big|f\big(\zeta(t)\big) - f(z)\big|\,\zeta'(t)\,\mathrm{d}t$$

need not be a real number! We had better tidy this up.

Theorem 4.16 (Integral Estimation). 1. *Suppose $w : [a,b] \to \mathbb{C}$ is piecewise continuous. Then*

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| \le \int_a^b |w(t)|\,\mathrm{d}t$$

2. *Suppose C is a contour with length L, and let f be piecewise continuous on C. Then $|f(z)|$ is bounded by some $M \in \mathbb{R}_0^+$ on C, and*

$$\left| \int_C f(z)\,\mathrm{d}z \right| \le ML$$

Part 2 justifies the suspect inequality in the proof of the Fundamental Theorem: $f(\zeta) - f(z)$ is bounded by $M = \frac{\epsilon}{2}$ and the path of integration is the straight line of length $L = |w - z|$.

Proof. 1. Let $\int_a^b w(t)\,\mathrm{d}t = re^{i\theta}$. Since θ is constant, observe that $r = \int_a^b e^{-i\theta} w(t)\,\mathrm{d}t$ is *real*. In particular,

$$r = \int_a^b \mathrm{Re}\big(e^{-i\theta} w(t)\big) + i\,\mathrm{Im}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t = \int_a^b \mathrm{Re}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t$$

Appealing to $\mathrm{Re}\, z \le |z|$, we see that

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| = r = \int_a^b \mathrm{Re}(e^{-i\theta} w(t))\,\mathrm{d}t \le \int_a^b \left| e^{-i\theta} w(t) \right|\,\mathrm{d}t = \int_a^b |w(t)|\,\mathrm{d}t$$

2. Parametrize the contour integral by $z : [a,b] \to \mathbb{C}$. Since $f(z(t))$ is piecewise continuous on a closed bounded interval $[a,b]$, it is bounded and thus M exists. But now,

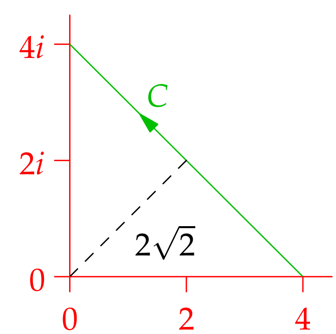
$$\left| \int_C f(z)\,\mathrm{d}z \right| = \left| \int_a^b f\big(z(t)\big) z'(t)\,\mathrm{d}t \right| \le \int_a^b \left| f\big(z(t)\big) z'(t) \right| \mathrm{d}t \le \int_a^b M \left| z'(t) \right| \mathrm{d}t = ML$$

■

While the computation of arc-length is usually impractical, for circular and straight arcs it is straightforward. Indeed the estimation of integrals around both large and small circles will prove crucial for the rest of the course.

Examples 4.17. 1. On the straight line C joining $z = 4$ to $4i$, we see that

$$2\sqrt{2} = |2(1+i)| \le |z| \le 4 \implies |z+1| \le |z| + 1 \le 5$$



By the extended triangle inequality,

$$\left| z^4 + 4 \right| \ge \left| |z|^4 - 4 \right| \ge 60$$

Since C has length $4\sqrt{2}$, it follows that

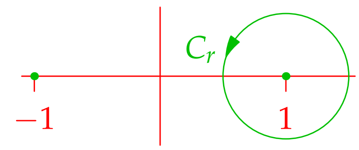
$$\left| \int_C \frac{z+1}{z^4+4}\,\mathrm{d}z \right| \le \frac{20\sqrt{2}}{60} = \frac{\sqrt{2}}{3}$$

2. For the function in the previous example, consider the circle C_R with radius $R > \sqrt[4]{4}$ centered at the origin. On C_R, we have $|z|^4 > 4$ whence the triangle inequality tells us that

$$\left|\frac{z+1}{z^4+4}\right| \le \frac{|z|+1}{|z^4|-4} = \frac{R+1}{R^4-4} \implies \left|\oint_{C_R} \frac{z+1}{z^4+4}\,dz\right| \le \frac{2\pi R(R+1)}{R^4-4}$$

In particular, this approaches zero as $R \to \infty$.

3. Let C_r be the circle with radius $r < 2$ centered at $z = 1$. Then



$$|z-1| = r \text{ and } |z+1| = |2-(1-z)| \ge 2-r$$

from which

$$\left|\oint_{C_r} \frac{1}{1-z^2}\,dz\right| \le \frac{2\pi r}{(2-r)r} = \frac{2\pi}{2-r} \xrightarrow[r\to 0]{} \pi$$

It will shortly be verified that $\oint_{C_r} \frac{1}{1-z^2}\,dz = -\pi i$ for *any* $r < 2$.

Exercises 4.2 1. Evaluate each contour integral $\int_C f(z)\,dz$ using the fundamental theorem:

(a) $f(z) = z^5$ where C is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = \frac{1}{z}$ where C is the pair of straight lines from $z = 1$ to $-1-i$ to $-i$.

(c) $f(z) = iz\sin z^2$, where C is the straight line from the origin to $z = i\sqrt{\pi}$.

(d) $f(z) = \frac{1}{1+z^2}$ where C is the straight line from $z = 1$ to $2+i$.

(e) $f(z) = \frac{1}{\sqrt{z}}$ where C is any path $z(t)$ with $\operatorname{Re} z > 0$ joining $z = 1+i$ and $z = 4$.

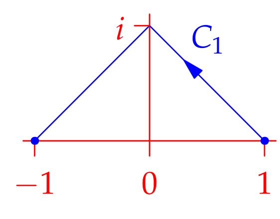
(f) $f(z) = \text{P.V.}\, z^{-1-2i}$ along the quarter circle $z(t) = e^{it}$ where $0 \le t \le \frac{\pi}{2}$.

2. Let $n \in \mathbb{N}_0$. Prove that for every contour C from z_0 to z_1

$$\int_C z^n\,dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

3. If C is a closed curve not containing z_0, and $n \in \mathbb{Z} \setminus \{0\}$, prove that $\int_C (z-z_0)^{n-1}\,dz = 0$.

4. Let $f(z) = z^{1/3}$ be the branch where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Evaluate the integral $\int_{C_1} f(z)\,dz$ where C_1 is the curve shown in the picture.



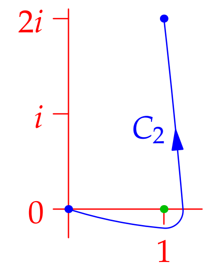
5. Evaluate $\int_{2i}^{1+i} \operatorname{Log} z\,dz$ where the curve C lies in the upper half-plane. (*Hint: use integration by parts*)

6. Evaluate $\int_{-1}^{1} z^{2-i}\,dz$ where z^{2-i} is the principal branch, and the integral is over any contour which, apart from its endpoints, lies above the real axis.

7. (a) Find an anti-derivative of $\frac{1}{1-z^2}$ on the domain $D = \mathbb{C}$ except for the real axis where $|z| \geq 1$. Evaluate $\int_0^{1+2i} \frac{1}{1-z^2}\,\mathrm{d}z$ along any curve in D.

   (b) Prove directly that $\oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z = -i\pi$ where C_r is the circle of radius $r < 2$ centered at $z = 1$.
   (*Hint: use the anti-derivative and the circle starting and finishing on the branch cut at* $z = 1 + r$)

   (c) Evaluate $\int_0^{1+2i} \frac{1}{1-z^2}\,\mathrm{d}z$ along a curve C_2 looping to the *right* of $z = 1$ as shown in the picture. Compare your answer with parts (a) and (b).



8. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $2i$. Without evaluating the integral, show that

   (a) $\left|\int_C \frac{z+4}{z^3-1}\,\mathrm{d}z\right| \leq \frac{6\pi}{7}$ (b) $\left|\int_C \frac{\mathrm{d}z}{z^2-1}\right| \leq \frac{\pi}{3}$

9. If C is the straight line joining the origin to $1+i$, show that $\left|\int_C z^3 e^{2iz}\,\mathrm{d}z\right| \leq 4$

10. If C is the boundary of the triangle with vertices $0, 3i$ and -4, prove that $\left|\oint_C (e^z - \overline{z})\,\mathrm{d}z\right| \leq 60$
    (*Hint: show that* $|e^z - \overline{z}| \leq e^x + \sqrt{x^2+y^2} \ldots$)

11. Let C_R be the circle of radius $R > 1$ centered at the origin. Prove that

$$\left|\oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

and thus prove that $\lim_{R\to\infty} \oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z = 0$.

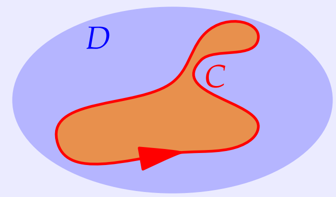
4.3 The Cauchy–Goursat Theorem and Cauchy's Integral Formula

We begin to extend the fundamental theorem with the goal of more fully understanding and evaluating holomorphic functions. We first require another piece of topology.

Definition 4.18. Suppose D is a connected region of the plane.

We say that D is *simply-connected* if every closed contour in D can by shrunk smoothly to a point without any part leaving D.

Otherwise said, if C is a simple closed contour in D then everything *inside* C also lies in D.



Theorem 4.19 (Cauchy–Goursat, version 1). *Suppose C is a closed curve in a simply-connected region D. If f is holomorphic on D, then $\int_C f(z)\,\mathrm{d}z = 0$.*

We prove a basic version where the real and imaginary parts of f are assumed to have continuous partial derivatives; we then invoke *Green's Theorem*, which you should have seen in a multi-variable calculus course. A proof without the restriction is significantly longer and more challenging.

Lemma 4.20 (Green's Theorem). *Suppose D is a closed bounded simply-connected domain with boundary C and that $P, Q : D \to \mathbb{R}$ have continuous partial derivatives. Then*

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

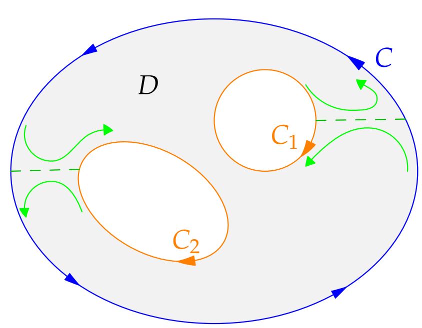
Sketch Proof of Cauchy–Goursat. Suppose $f(z) = u + iv$ has continuous partial derivatives and parametrize C: assume WLOG that this is positively oriented. Then

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \int_a^b f\left(z(t)\right) z'(t)\,\mathrm{d}t = \int_a^b \left(u(z(t)) + iv(z(t))\right)\left(x'(t) + iy'(t)\right)\mathrm{d}t \\ &= \int_a^b ux' - vy'\,\mathrm{d}t + i\int_a^b vx' + uy'\,\mathrm{d}t \\ &= \oint_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_C v\,\mathrm{d}x + u\,\mathrm{d}y \qquad \text{(definition of line integral in } \mathbb{R}^2\text{)} \\ &= \iint_D -v_x - u_y\,\mathrm{d}A + i\iint_D u_y - v_x\,\mathrm{d}A = 0\end{aligned}$$

by Green's Theorem and the Cauchy–Riemann equations. ∎

We now perform a sneaky trick. Starting with a simple closed curve C, remove from its interior the regions within simple closed non-intersecting contours $C_1, \ldots, C_k$. We orient these clockwise so that the interior region D lies on the curves' *left*.

By cutting, we can join the boundary curves into a single curve at the cost of traversing each cut twice in opposite directions. Stretching credulity slightly, we have a new simple closed curve to which we can apply Cauchy–Goursat…



Corollary 4.21 (Cauchy–Goursat, version 2). *Suppose C is a simple closed contour, oriented counter-clockwise. Let $C_1, \ldots, C_k$ be non-intersecting simple closed curves in the interior of C, oriented clockwise. If $f(z)$ is holomorphic on the region between and including C and the interior boundaries $C_1, \ldots, C_k$, then*

$$\int_C f(z)\,dz + \sum_{j=1}^{k} \int_{C_j} f(z)\,dz = 0$$

The power of this result lies in how it allows us to *compare* integrals around different contours.

Corollary 4.22. *Suppose C_1, C_2 are non-intersecting positively oriented simple closed contours. If f is holomorphic on the region between and including the curves, then*

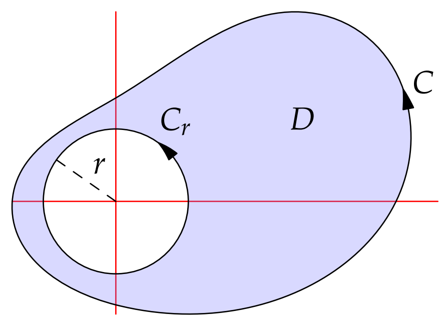
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Examples 4.23. 1. Recall Example 4.17.3. Since $f(z) = \frac{1}{1-z^2}$ is holomorphic on and between any two circles C_r centered at $z = 1$ and with radius $r < 2$, we see that the value of $\int_{C_r} \frac{1}{1-z^2}\,dz$ is independent of the radius r. At the moment, we still need to evaluate on at least one such circle to obtain the explicit value $-i\pi$.

2. We compute the integral of $f(z) = \frac{1}{z}$ round *any* simple closed contour C staying away from the origin.

   - If the origin is *exterior* to C, then $f(z)$ is holomorphic on and inside C, whence $\oint_C \frac{1}{z}\,dz = 0$.
   - If the origin is *interior* to C, then there is some minimum distance d of C to the origin. Choose any circle C_r with radius $r < d$ centered at the origin. Since $f(z)$ is holomorphic on the region D between and including C and C_r, we conclude that

$$\oint_C \frac{1}{z}\,dz = \oint_{C_r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} rie^{i\theta}\,d\theta = 2\pi i$$



More generally, if C is a positively-oriented simple closed contour not containing z_0, then

$$\oint_C \frac{1}{z - z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ lies inside } C \\ 0 & \text{if } z_0 \text{ lies outside } C \end{cases}$$

This example generalizes to one of the central results of complex analysis...

Theorem 4.24 (Cauchy's Integral Formula). *Suppose $f(z)$ is holomorphic everywhere on and inside a simple closed contour C. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

Moreover, f is infinitely differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

Example 4.25. If $f(z)$ is a polynomial, we can check the integral formula explicitly by appealing to the Cauchy–Goursat Theorem and Exercise 4.1.9: if C is a simple closed contour encircling z_0, then

$$\oint_C (z-z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

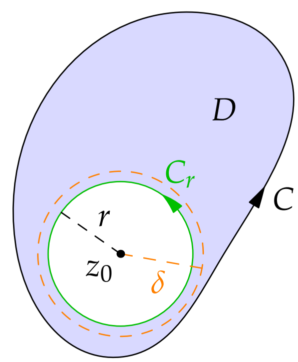
For instance, if $f(z) = 3z^2+2$ and $z_0 = 0$, then

$$\frac{1}{2\pi i}\oint_C \frac{3z^2+2}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_C 3z\,\mathrm{d}z + \frac{1}{2\pi i}\oint_C \frac{2}{z}\,\mathrm{d}z = 0+2 = f(0)$$

$$\frac{2!}{2\pi i}\oint_C \frac{3z^2+2}{z^3}\,\mathrm{d}z = \frac{2}{2\pi i}\oint_C \frac{3}{z}\,\mathrm{d}z + \frac{2}{2\pi i}\oint_C \frac{2}{z^3}\,\mathrm{d}z = 6+0 = f''(0)$$

Proof of the basic Integral formula. Denote by D the open region interior to C. Let $\epsilon > 0$ be given. Since f is holomorphic, it is also continuous. Thus

$$\exists\delta > 0 \text{ such that } |z-z_0| < \delta \implies z \in D \text{ and } |f(z)-f(z_0)| < \frac{\epsilon}{2}$$



Choose a positive $r < \delta$ and draw the circle C_r of radius r centered at z_0. This lies entirely in D. Since $\frac{f(z)}{z-z_0}$ is holomorphic between and on C and C_r, Corollary 4.22 tells us that

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

We need only bound an integral to obtain the basic formula:

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z - f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\right| \le \frac{1}{2\pi}2\pi r\frac{\epsilon}{2r} < \epsilon$$

■

We postpone a sketch proof of the derivative formula[2] to the exercises.

[2] Informally, it appears as if the formula follows from repeated differentiation:

$$f^{(n+1)}(z_0) = \frac{\mathrm{d}}{\mathrm{d}z_0}f^{(n)}(z_0) \overset{???}{=} \frac{n!}{2\pi i}\oint_C \frac{\partial}{\partial z_0}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z = \frac{(n+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+2}}\mathrm{d}z$$

Of course, one cannot blindly bring a derivative inside an integral like this, so a formal proof is required.

Examples 4.26. We can use the integral formula to evaluate certain integrals that would be difficult if not impossible to evaluate by parametrization.

1. If C is the circle centered at $z = i$ with radius 1, then $f(z) = \frac{3z \sin z}{z+i}$ is holomorphic on an inside C, whence

$$\oint_C \frac{3z \sin z}{z^2+1}\,\mathrm{d}z = \oint_C \frac{3z\sin z}{(z+i)(z-i)}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i \frac{3i \sin i}{2i} = 3\pi i \sin i = \frac{3}{2}(e^{-1} - e^{1})$$

Contrast this with parametrizing $z(t) = i + e^{it}$ and attempting to evaluate directly!

$$\int_0^{2\pi} \frac{3(i+e^{it})\sin(i+e^{it})ie^{it}}{(i+e^{it})^2+1}\,\mathrm{d}t \ldots$$

2. Let C be a simple closed contour staying away from $z_0 = 4$. Since $f(z) = \frac{3z^2+7}{e^z}$ is entire,

$$\oint_C \frac{3z^2+7}{e^z(z-4)}\,\mathrm{d}z = \begin{cases} 0 & \text{if } z_0 = 4 \text{ is outside } C \\ 2\pi i f(4) = 110\pi i e^{-4} & \text{if } z_0 = 4 \text{ is inside } C \end{cases}$$

3. Let C be the circle of radius 2 centered at $z = 1+i$. Then $g(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ is holomorphic on and inside C, except at $z = i$. We conclude that

$$\oint_C g(z)\,\mathrm{d}z = \oint_C \frac{1}{(z+i)^3(z-i)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right|_i (z+i)^{-3} = 12\pi i(2i)^{-5} = \frac{3\pi}{8}$$

We finish this section with an easy yet powerful corollary of the integral formula that we mentioned in Chapter 2.

Corollary 4.27. *If f is holomorphic, then it is infinitely differentiable and all derivatives are holomorphic. In particular, the real and imaginary parts of f have continuous partial derivatives of all orders.*

Proof. If f is holomorphic at w_0, then it is holomorphic on an open set D containing w_0. Draw a circle C_r centered at w_0 lying inside D. By the Cauchy integral formula, $f''(z_0)$ exists at every z_0 inside C_r. Otherwise said, f' is holomorphic at w_0. Now induct... ■

Finally, note that if f has an anti-derivative F, then F is necessarily holomorphic and so, by the Corollary, is f itself. Combining our results yields the following.

Theorem (Summary). *Suppose f is continuous on an open domain D and that curves C lie in D.*

all $\oint_C f(z)\,\mathrm{d}z = 0$	$\Longleftarrow$ (If D simply-connected; Cauchy–Goursat)	f holomorphic on D
$\Updownarrow$ Lemma 4.13		$\Uparrow$ Cauchy Integral Formula
all $\int_C f(z)\,\mathrm{d}z$ path independent	$\Longleftrightarrow$ (Fundamental Thm)	f has an anti-derivative on D

Exercises 4.3 1. Apply the Cauchy–Goursat Theorem to show that $\oint_C f(z)\,dz = 0$ when the contour C is the unit circle $|z| = 1$.

(a) $f(z) = \dfrac{z^2}{z+3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \operatorname{Log}(z+2)$

2. Let C_1 denote the square with sides along the lines $x = \pm 1, y = \pm 1$, and C_2 be the circle $|z| = 4$: explain why

$$\oint_{C_1} \frac{1}{3z^2+1}\,dz = \oint_{C_2} \frac{1}{3z^2+1}\,dz$$

3. Let C be the square with sides $x = 0, 1$ and $y = 0, 1$. Evaluate the integral $\oint_C \frac{1}{z-a}\,dz$ when:

(a) a is *exterior* to the square;

(b) a is *interior* to the square.

4. Let C be the positively-oriented boundary of the half-disk $0 \le r \le 1$, $0 \le \theta \le \pi$ and define $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ and $f(0) = 0$ using the branch of $z^{1/2}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Prove that

$$\oint_C f(z)\,dz = 0$$

by evaluating three contour integrals: over the semicircle, and over two segments of the real axis joining 0 to ± 1. Why does Cauchy–Goursat not apply here?

5. If C is a positively-oriented simple closed contour, prove that the area enclosed by C is

$$\frac{1}{2i}\oint_C \overline{z}\,dz$$

(*Hint: Mirror the sketch proof of Cauchy–Goursat, even though $\overline{z}$ isn't holomorphic...*)

6. Let C denote the boundary of the square with sides $x = \pm 2, y = \pm 2$. Evaluate the following:

(a) $\displaystyle\oint_C \frac{e^{-\frac{\pi z}{2}}\,dz}{z-i}$ (b) $\displaystyle\oint_C \frac{e^z + e^{-z}}{z(z^2+10)}\,dz$ (c) $\displaystyle\oint_C \frac{z\,dz}{3z+i}$ (d) $\displaystyle\oint_C \frac{\sec(z/2)}{(z-1-i)^2}\,dz$

7. Evaluate the integral $\oint_C g(z)\,dz$ around the circle of radius 3 centered at $z = i$ when:

(a) $g(z) = \dfrac{1}{z^2+9}$ (b) $g(z) = \dfrac{1}{(z^2+9)^2}$

8. Prove that if f is holomorphic on and inside a simple closed contour C and z_0 is not on C, then

$$\oint_C \frac{f'(z)}{z-z_0}\,dz = \oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

9. For the final time, recall Example 4.17.3. Use Cauchy's integral formula on a circle C_r of radius $r < 2$ centered at $z = 1$ to prove that

$$\oint_{C_r} \frac{1}{1-z^2}\,dz = -i\pi$$

10. Suppose we have a polynomial $p(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$ centered at z_0. Use Cauchy's integral formula to prove that $a_k = \frac{p^{(k)}(z_0)}{k!}$ is the usual Taylor coefficient.

11. Let C be the unit circle $z = e^{i\theta}$ where $-\pi < \theta \leq \pi$ and suppose $a \in \mathbb{R}$ is constant. By first evaluating $\oint_C z^{-1}e^{az}\,dz$, prove that

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$$

12. (a) Suppose that $f(z)$ is *continuous* on and inside a simple closed contour C. Prove that the function $g(z)$ defined by

$$g(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

is holomorphic at every point z_0 inside C and that

$$g'(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

(*Hint: consider the formula in part (b) of the next question, after replacing the first two f's by g*)

(b) If $f(z) = x(1-x)(1-y)$ and C is the square with vertices $0, 1, 1+i, i$ compute $g(z_0)$.

13. We prove the 1st derivative version of Cauchy's integral formula. As previously, let $\delta > 0$ be such that $|w - z_0| < \delta \implies w \in D$.

(a) If $|\Delta z| < \delta$ and $z \in C$ explain why

i. $|z - z_0| \geq \delta$

ii. $|z - z_0 - \Delta z| > 0$

(b) Use the basic integral formula on C to evaluate $f(z_0 + \Delta z) - f(z_0)$ and thus prove that

$$\left|\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz\right| \leq \frac{ML}{2\pi(\delta - \Delta z)\delta^2}|\Delta z|$$

where M is an upper bound for $|f(z)|$ on C, and L is the length of C. Hence conclude that $f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$.

If you want a real challenge, do the same thing for higher derivatives!

4.4 Liouville's Theorem and The Maximum Modulus Principle

In this section we derive some powerful corollaries of the Cauchy integral formula. The first is easy.

Lemma 4.28 (Cauchy's Inequality). *If f is holomorphic on and inside the circle C of radius R centered at z_0, and $|f(z)| \leq M$ on C, then*

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\right| \leq \frac{n!M}{R^n}$$

This seems innocuous, but it has surprising applications. If f is entire and bounded ($|f(z)| \leq M$ for all z) then Cauchy's inequality applies for *every* circle centered at *every* z_0:

$$\forall z_0, R,\ \left|f'(z_0)\right| \leq \frac{M}{R} \implies \forall z_0,\ f'(z_0) = 0$$

We've therefore proved:

Theorem 4.29 (Liouville). *The only bounded entire functions are constants.*

We now come to perhaps the simplest proof of one of the most famous results in mathematics.

Theorem 4.30 (Fundamental Thm. of Algebra). *Every non-constant polynomial has a root in $\mathbb{C}$.*

Proof. It costs nothing to assume that the polynomial is monic. Suppose $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ has no roots, then $\frac{1}{p(z)}$ is entire. Our goal is to prove that $\frac{1}{p(z)}$ is also *bounded* on $\mathbb{C}$: since $p(z)$ is non-constant, this contradicts Liouville's Theorem.

In fact it is enough to bound $\frac{1}{p(z)}$ for all *large* z: that is we want to find $R > 0$ such that

$$|z| > R \implies \frac{1}{p(z)} \text{ bounded}$$

This is since $\frac{1}{p(z)}$, being continuous, is already bounded on the compact disk $|z| \leq R$. So how do we find R? The idea is to use the triangle inequality to write

$$|z| > R \implies |p(z)| = |z|^n \left|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \geq |z|^n \left|1 - \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right|\right|$$

We need only choose R such that $\left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \leq \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \leq \frac{1}{2}$ to complete the proof:

$$|z| > R \implies |p(z)| \geq \frac{1}{2}R^n \implies \frac{1}{|p(z)|} \leq \frac{2}{R^n}$$

Forcing each term in the sum to be $\leq \frac{1}{2n}$ is enough: it is easy to check that this is accomplished by

$$R := \max\left\{(2n\,|a_k|)^{\frac{1}{n-k}} : k = 0, \ldots, n-1\right\}$$

∎

In the proof, we used the fact that a continuous function $f(z)$ on a compact (closed bounded) domain K is bounded. As you should recall from real analysis, the least upper bound is achieved

$$\exists z_0 \in K \text{ such that } |f(z_0)| = \sup\{|f(z)| : z \in K\}$$

For *holomorphic* functions, we can say something more restrictive and surprising. The maximum modulus of an holomorphic function on a compact domain is always and only achieved at an *edge point*.

Theorem 4.31 (Maximum Modulus Principle). *Suppose $f(z)$ is holomorphic and non-constant on an* bounded, open, connected *domain D. Then $|f(z)|$ has no maximum value on D.*

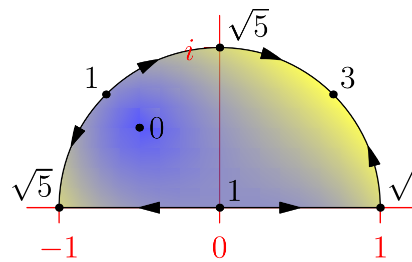
Examples 4.32. 1. Let $f(z) = e^z$ on the unit disk $|z| \le 1$. Then $|f(z)| = e^x$ which has its maximum at $z = 1$, on the edge of the disk.

2. Consider $f(z) = 2z^2 + i$ on the upper semi-disk with radius 1. There are two boundaries:
$y = 0$: plainly $|f(z)| = \sqrt{4x^4 + 1}$ is maximal at $z = \pm 1$.
$r = 1$: write $f(z) = 2e^{2i\theta} + i$ from which

$$|f(z)| = \sqrt{(2\cos 2\theta)^2 + (2\sin 2\theta + 1)^2} = \sqrt{5 + 4\sin 2\theta}$$

which is maximal when $\theta = \frac{\pi}{4}$ with $\left|f(e^{\frac{i\pi}{4}})\right| = 3$.

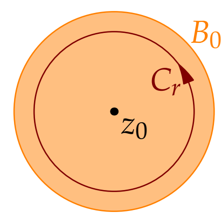
The color indicates that value of $|f(z)|$, and the arrows its direction of increase on the boundary.



Proof. First we prove a special case.

Fix $\delta > 0$, assume $f(z)$ is holomorphic on and inside $B_0 = \{z : |z - z_0| \le \delta\}$ and suppose $|f(z)|$ attains its maximum at z_0.

Let $r \le \delta$ and apply the Cauchy integral formula around the circle C_r of radius r centered at z_0:



$$f(z_0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\,dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$$\implies |f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} \left|f(z_0 + re^{it})\right| dt \qquad \text{(Theorem 4.16)}$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\,dt = |f(z_0)| \qquad (|f(z_0)| = \max\{|f(z)| : z \in B_0\})$$

The inequalities are therefore *equalities*, the middle of which now becomes

$$\int_0^{2\pi} |f(z_0)| - \left|f(z_0 + re^{it})\right| dt = 0$$

Since the integrand is non-negative and continuous, it must be zero. But then $|f(z)| = |f(z_0)|$ on *all circles* C_r, whence $|f(z)|$ is *constant* on B_0. Since $f(z)$ is holomorphic, it is also constant[3] on B_0.

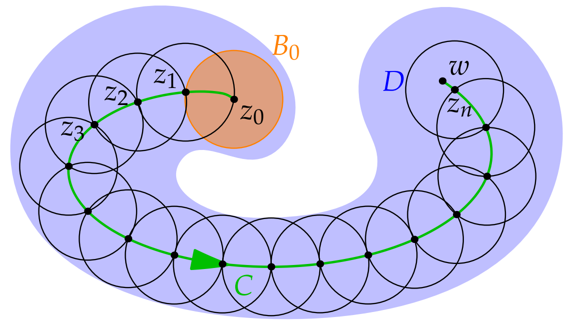
[3] Recall from chapter 2: $|f(z)| = k \ne 0 \implies \overline{f(z)} = \frac{k^2}{f(z)}$ is holomorphic, whence $u - iv$ satisfies the Cauchy–Riemann equations. . .

Now suppose that f is holomorphic on some open connected domain D and that it attains its maximum at $z_0 \in D$.

Let $w \in D$ be any other point and join z_0 to w by a simple contour C.

Since D is open, $\exists \delta > 0$ such that all points $\leq \delta$ distance of C lie within D.

Take points $z_1, \ldots, z_n$ separated by δ along C and disks $B_k = \{z : |z - z_k| \leq \delta\}$ centered at each z_k such that $w \in B_n$: this is possible since C has *finite length.*



By the simple case, $f(z)$ is constant on B_0. Since $z_1 \in B_0$, this shows that $|f(z_1)| = |f(z_0)|$ is also the maximum modulus on D. The special case now says that $f(z)$ is constant in $B_0 \cup B_1$. By induction, f is constant on all $\bigcup B_k$ and so $f(w) = f(z_0)$.

■

Exercises 4.4

1. (a) Suppose f is entire and that $|f(z)| \leq c|z|$ for some constant $c \in \mathbb{R}^+$. Prove that $f(z) = kz$ where $k \in \mathbb{C}$ satisfies $|k| \leq c$.
   (b) What can you say about f if it is entire and there exists some linear polynomial $cz + d$ with $c \neq 0$ such that $|f(z)| \leq |cz + d|$ for all $z \in \mathbb{C}$?
2. If $f(z) = u + iv$ is entire and $u(x, y)$ is bounded above, apply Liouville's Theorem to $\exp(f(z))$ to prove that u is constant.
3. Let $f(z)$ be a non-zero holomorphic function on a closed bounded domain. By considering $g(z) = \frac{1}{f(z)}$, show that the minimum value of $|f(z)|$ also occurs on the boundary.
4. Find the maximum and minimum values of $\left|z^2 + 4i\right|$ on the unit disk $|z| \leq 1$.
5. On the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$, show that $|\sin z|$ is maximal at the point $z = \frac{\pi}{2} + i$.
   (*Hint: first show that* $|\sin z|^2 = \sin^2 x + \sinh^2 y$)
6. Revisit the standard method from multivariable calculus (compute $\nabla |f(z)|^2$) to check that the maximum value of $\left|2z^2 + i\right|$ in Exercise 4.32 really does occur on the boundary.
7. Complete the proof of the fundamental theorem of algebra:
   (a) Verify that $R := \max\{(2n|a_k|)^{\frac{1}{n-k}}\}$ is *positive.*
   (b) Prove that $|z| > R \implies \forall k,\ \frac{|a_k|}{|z|^{n-k}} < \frac{1}{2n}$.
8. (a) (Hard and long) Prove the factor theorem: if $p(z_1) = 0$, then $p(z) = (z - z_1)q(z)$ for some polynomial $q(z)$.
   For a challenge, prove the full division algorithm: if $\deg f \geq \deg g$, *then there exist unique polynomials* $q(z), r(z)$ *for which*
   $$f(z) = g(z)q(z) + r(z) \quad \textit{and} \quad \deg r < \deg g$$
   (b) Prove that a degree $n \geq 1$ polynomial $p(z)$ factors uniquely over $\mathbb{C}$: up to order, exist unique $z_1, \ldots, z_n \in \mathbb{C}$ such that
   $$p(z) = a(z - z_1) \cdots (z - z_n)$$