5 Continuity

This chapter defines continuity and develops its basic properties, again without recourse to limits. We shall discuss limits in Chapter 13.

The Definition of Continuity

Naively, we think of a curve as being continuous if we can draw it "without removing the pencil from the paper." Let (x_0, y_0) be a point on the curve, and draw the lines $y = c_1$ and $y = c_2$ with $c_1 < y_0 < c_2$. If the curve is continuous, at least a "piece" of the curve on each side of (x_0, y_0) should be between these lines, as in Fig. 5-1 (left). Compare this with the behavior of the discontinuous curve in Fig. 5-1 (right). The following definition is a precise formulation, for functions, of this idea.

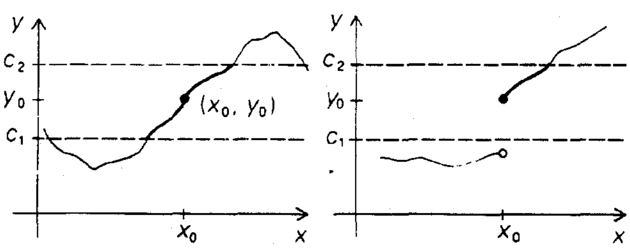

Fig. 5-1 A continuous curve (left) and a discontinuous curve (right).

Definition If x_0 is an element of the domain D of a function f, we say that f is *continuous at* x_0 if:

1. For each $c_1 < f(x_0)$ there is an open interval I about x_0 such that, for those x in I *which also lie in* D, $c_1 < f(x)$.

2. For each $c_2 > f(x_0)$ there is an open interval J about x_0 such that, for those x in J *which also lie in* D, $f(x) < c_2$.

If f is continuous at every point of its domain, we simply say that f is continuous or f is *continuous on* D.

Warning It is tempting to define a continuous motion $f(t)$ as one which never passes from $f(t_0)$ to $f(t_1)$ without going through every point between these two. This is a desirable property, but for technical reasons it is not suitable as a definition of continuity; (See Figure 5-4).

The property by which continuity is defined might be called the "principle of persistence of inequalities": f is continuous at x_0 when every strict inequality which is satisfied by $f(x_0)$ continues to be satisfied by $f(x)$ for x in some open interval about x_0. The intervals I and J in the definition may depend upon the value of c_1 and c_2. The definition of continuity may also be phrased in terms of transitions using the idea of Solved Exercise 2, Chapter 13.

Another way to paraphrase the definition of continuity is to say that $f(x)$ is close to $f(x_0)$ when x is close to x_0. The lines $y = c_1$ and $y = c_2$ in Fig. 5-1 provide a measure of closeness. The following example illustrates this idea.

Worked Example 1 The mass y (in grams) of a silver plate which is deposited on a wire during a plating process is given by a function $f(x)$, where x is the length of time (in seconds) during which the plating apparatus is allowed to operate. Suppose that you wish to deposit 2 grams of silver on the wire and that $f(3) = 2$. Being realistic, you know that you cannot control the time *precisely*, but you are willing to accept the result if the mass is less than 0.003 gram in error. Show that if f is continuous, there is a certain tolerance τ such that, if the time is within τ of 3 seconds, the resulting mass of silver plate will be acceptable.

Solution We wish to restrict x so that $f(x)$ will satisfy the inequalities $1.997 < f(x) < 2.003$. We apply the definition of continuity, with $x_0 = 3$, $c_1 = 1.997$, and $c_2 = 2.003$. From condition 1 of the definition, there is an open interval I containing 3 such that $1.997 < f(x)$ for all $x \in I$. From condition 2, there is J such that $f(x) < 2.003$ for all $x \in J$. For τ less than the distance from 3 to either endpoint of I or J, the interval $[3 - \tau, 3 + \tau]$ is contained in both I and J; for x in this interval, we have, therefore, $1.997 < f(x) < 2.003$.

Of course, to get a specific value of τ which works, we must know more about the function f. Continuity tells us only that such a tolerance τ exists.

Theorem 1, which appears later in this chapter, gives an easy way to verify that many functions are continuous. First, though, we try out the definition on a few simple cases in the following exercises.

Solved Exercises*

1. Let $g(x)$ be the *step function* defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Show that g is not continuous at $x_0 = 0$.

*Solutions appear in the Appendix.

2. Let $f(x)$ be the *absolute value function*, $f(x) = |x|$. Show that f is continuous at $x_0 = 0$.

3. Let f be continuous at x_0 and suppose that $f(x_0) \neq 0$. Show that $1/f(x)$ is defined on an open interval about x_0.

4. Decide whether each of the functions whose graphs appear in Fig. 5-2 is continuous. Explain your answers.

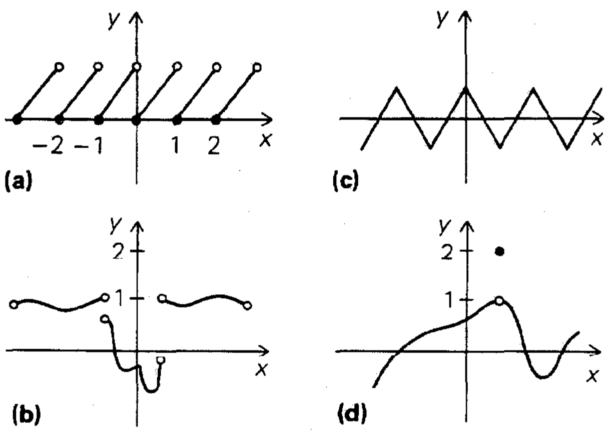

Fig. 5-2 Which functions are continuous?

Exercises

1. Let $f(x)$ be the step function defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 2 & \text{if } x \geqslant 0 \end{cases}$$

Show that f is discontinuous at 0.

2. Show that, for any constants a and b, the linear function $f(x) = ax + b$ is continuous at $x_0 = 2$.

3. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ ? & \text{if } 1 \leqslant x \leqslant 3 \\ x - 6 & \text{if } 3 < x \end{cases}$$

How can you define $f(x)$ on the interval $[1,3]$ in order to make f continuous on $(-\infty, \infty)$? (A geometric argument will suffice.)

4. Let $f(x)$ be defined by $f(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$. How should you define $f(1)$ to make the resulting function continuous? [*Hint*: Plot a graph of $f(x)$ for x near 1 by factoring the numerator.]
5. Let $f(x)$ be defined by $f(x) = 1/x$ for $x \neq 0$. Is there any way to define $f(0)$ so that the resulting function will be continuous?
6. Prove from the definition that the function $s(x) = x^2 + 1$ is continuous at 0.

Differentiability and Continuity

If a function $f(x)$ is differentiable at $x = x_0$, then the graph of f has a tangent line at $(x_0, f(x_0))$. Our intuition suggests that if a curve is smooth enough to have a tangent line then the curve should have no breaks—that is, a differentiable function is continuous. The following theorem says just that.

Theorem 1 *If the function f is differentiable at x_0, then f is continuous at x_0.*

Proof We need to verify that conditions 1 and 2 of the definition of continuity hold, under the assumption that the definition of differentiability is met.

We begin by verifying condition 2, so let c_2 be any number such that $f(x_0) < c_2$. We shall produce an open interval I about x_0 such that $f(x) < c_2$ for all x in I.

Choose a positive number M such that $-M < f'(x_0) < M$, and let l_- and l_+ be the lines through $(x_0, f(x_0))$ with slopes $-M$ and M. Referring to Fig. 5-3, we see that l_+ lies below the horizontal line $y = c_2$ for a certain

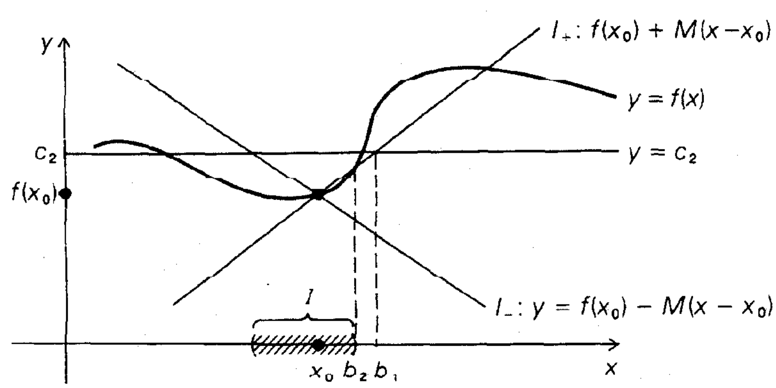

Fig. 5-3 The geometry needed for the proof of Theorem 1.

distance to the right of x_0, and that the graph of f lies below l_+ for a certain distance to the right of x_0 because l_+ overtakes the graph of f at x_0. More precisely, the line $l_+: y = f(x_0) + M(x - x_0)$ intersects $y = c_2$ at

$$b_1 = \frac{c_2 - f(x_0)}{M} + x_0 > x_0$$

and $f(x_0) + M(x - x_0) < c_2$ if $x < b_1$. (The reader should verify this.) Let (a_2, b_2) be an interval which works for l_+ overtaking the graph of f at x_0, so that $f(x) < f(x_0) + M(x - x_0)$ for $x \in (x_0, b_2)$.

If b is the smaller of b_1 and b_2, then

$$f(x) < f(x_0) + M(x - x_0) < c_2 \quad \text{for } x_0 < x < b \tag{1}$$

Similarly, by using the line l_- to the left of x_0, we may find $a < x_0$ such that

$$f(x) < f(x_0) - M(x - x_0) < c_2 \quad \text{for } a < x < x_0 \tag{2}$$

(The reader may wish to add the appropriate lines to Fig. 5-3.) Let $I = (a, b)$. Then inequalities (1) and (2), together with the assumption $f(x_0) < c_2$, imply that

$$f(x) < c_2 \quad \text{for } x \in I,$$

so condition 2 of the definition of continuity is verified.

Condition 1 is verified in an analogous manner. One begins with $c_1 < f(x_0)$ and uses the line l_+ to the left of x_0 and l_- to the right of x_0. We leave the details to the reader.

Worked Example 2 Show that the function $f(x) = (x - 1)/3x^2$ is continuous at $x_0 = 4$.

Solution We know from Chapter 3 that x, $x - 1$, x^2, $3x^2$, and hence $(x - 1)/3x^2$ are differentiable (when $x \neq 0$). Since $4 \neq 0$, Theorem 1 implies that f is continuous at 4.

This method is certainly much easier than attempting to verify directly the conditions in the definition of continuity.

The argument used in this example leads to the following general result.

Corollary

1. *Any polynomial $P(x)$ is continuous.*
2. *Let $P(x)$ and $Q(x)$ be polynomials, with $Q(x)$ not identically zero. Then the rational function $R(x) = P(x)/Q(x)$ is continuous at all points of its domain; i.e., R is continuous at all x_0 such that $Q(x_0) \neq 0$.*

In Chapter 3, we proved theorems concerning sums, products, and quotients of differentiable functions. One can do the same for continuous functions: the sum product and quotient (where the denominator is nonzero) of continuous functions is continuous. Using such theorems one can proceed directly, without recourse to differentiability, to prove that polynomials are continuous. Interested readers can try to work these theorems out for themselves (see Exercise 11 below) or else wait until Chapter 13, where they will be discussed in connection with the theory of limits.

Solved Exercises

5. Prove that $(x^2 - 1)/(x^3 + 3x)$ is continuous at $x = 1$.
6. Is the converse of Theorem 1 true; i.e., is a function which is continuous at x_0 necessarily differentiable there? Prove or give an example.
7. Prove that there is a number $\delta > 0$ such that $x^3 + 8x^2 + x < 1/1000$ if $0 \leqslant x < \delta$.
8. Let f be continuous at x_0 and A a constant. Prove that $f(x) + A$ is continuous at x_0.

Exercises

7. Why can't we ask whether the function $(x^3 - 1)/(x^2 - 1)$ is continuous at 1?
8. Let $f(x) = \dfrac{1}{x} + \dfrac{x^2 - 1}{x}$.

   Can you define $f(0)$ so that the resulting function is continuous at all x?
9. Find a function which is continuous on the whole real line, and which is differentiable for all x except 1, 2, and 3. (A sketch will do.)

10. In Solved Exercise 7, show that $\delta = 1/2000$ works; i.e., $x^3 + 8x^2 + x < 1/1000$ if $0 \leqslant x < 1/2000$.

11. (a) Prove that if $f(x) < c_1$ for all x in I and $g(x) < c_2$ for all x in I, then $(f+g)(x) < c_1 + c_2$ for all x in I.

 (b) Prove that, if f and g are continuous at x_0, so is $f+g$.

12. $$\text{Let } f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

 (a) At which points is f continuous?

 (b) At which points is f differentiable?

13. Let f be defined in an open interval about x_0. Suppose that $f(x) = f(x_0) + (x - x_0)g(x)$, where g is continuous at a. Prove that f is differentiable at x_0 and that $f'(x_0) = g(x_0)$. [*Hint*: Prove that $(x - x_0)(g(x) - g(a))$ vanishes rapidly at x_0.]

The Intermediate Value Theorem

A function f is said to have the *intermediate value property* if, whenever f is defined on $[x_1, x_2]$, then $f(x)$ takes every value between $f(x_1)$ and $f(x_2)$ as x runs from x_1 to x_2. Our intuitive notions of continuity suggest that every continuous function has the intermediate value property, and indeed we will prove that this is true. Unfortunately, the intermediate value property is not suitable as a *definition* of continuity; in Fig. 5-4 we have sketched the graph of a function which has the intermediate value property but which is not continuous at 0.

Before, proving the main intermediate value theorem, it is convenient to begin by proving an alternative version.

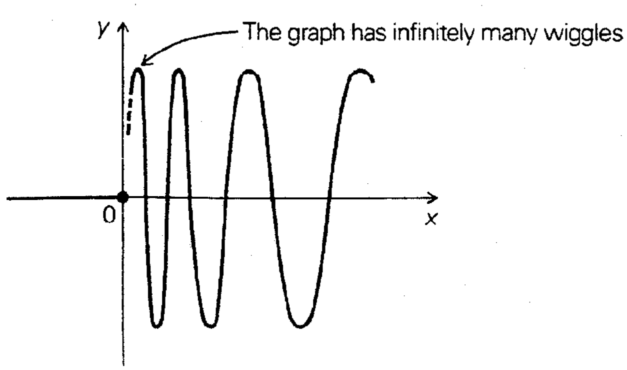

Fig. 5-4 A discontinuous function which has the intermediate value property.

Lemma Intermediate Value Theorem (alternative version). *Suppose that f is continuous on $[a, b]$ and that $f(a)$ is less [greater] than some number d. If $f(x) \neq d$ for all $x \in [a, b]$, then $f(b)$ is less [greater] than d as well.*

Proof We write out the proof for the case $f(a) < d$, leaving the case $f(a) > d$ to the reader. We look at the set S consisting of all those x in $[a, b]$ for which $f < d$ on $[a, x]$; i.e., $x \in S$ means that $x \in [a, b]$ and that $f(z) < d$ for all z in $[a, x]$. The idea of the proof is to show that S is an interval and then to prove that $S = [a, b]$. The completeness axiom states that every convex set is an interval. Thus, we begin by proving that S is convex; i.e., if $x_1 < y < x_2$, and x_1 and x_2 are in S, then y is in S as well. First of all, $x_1 \in S$ implies $a \leqslant x_1$, and $x_2 \in S$ implies $x_2 \leqslant b$, so we have $a \leqslant x_1 < y < x_2 \leqslant b$, so $y \in [a, b]$. To prove that $y \in S$, then, we must show that $f < d$ on $[a, y]$. But $f < d$ on $[a, x_2]$, since $x_2 \in S$, and $[a, y]$ is contained in $[a, x_2]$; thus, for $z \in [a, y]$, we have $z \in [a, x_2]$, so $f(z) < d$; i.e., $f < d$ on $[a, y]$. We have proven that $y \in S$, so S is convex.

By the completeness axiom, S is an interval. Since $f(a) < d$, we have $f < d$ on $[a, a]$, so $a \in S$. Nothing less than a can be in S, so a is the left-hand endpoint of S. Since S is contained in $[a, b]$, it cannot extend infinitely far to the right; we conclude that $S = [a, c)$ or $[a, c]$ for some c in $[a, b]$.

Case 1 Suppose $S = [a, c)$. Then, for every $z \in [a, c)$, we have $f < d$ on $[a, z]$, so $f(z) < d$; we have thus shown that $f < d$ on $[a, c)$. If $f(c)$ were less than d, we would have $f < d$ on $[a, c]$, so that c would belong to S, contradicting the statement that $S = [a, c)$. $f(c)$ cannot equal d, by the assumption of the theorem we are proving, so the only remaining possibility is $f(c) > d$. By the continuity of f at c, we must have $f(x) > d$ for all x in some open interval about c; but this contradicts the fact that $f < d$ on $[a, c)$. We conclude that the case $S = [a, c)$ simply cannot occur.

Case 2 Suppose that $S = [a, c]$, with $c < b$. Since $c \in S$, we have $f < d$ on $[a, c]$. By the continuity of f at c and the fact that $f(c) < d$, we conclude that $f < d$ also on some open interval (p, q) containing c. (Here we use the fact that $c < b$, so that f is defined on an open interval about c.) But if $f(z) < d$ for z in $[a, c]$ and for z in (p, q), with $p < c < q \leqslant b$, then $f < d$ on $[a, q)$. Let $y = \frac{1}{2}(c + q)$, the point halfway between c and q. Then $f < d$ on $[a, y]$, so $y \in S$. Since $y > c$, this contradicts the statement that $S = [a, c]$. Thus, case 2 cannot occur.

The only possibility which remains, and which therefore must be true, is that $S = [a, b]$. Thus, $f(z) < d$ for all $z \in [a, b]$, and in particular for $z = b$, so $f(b) < d$, which is what was to be proven.

We can now deduce the usual form of the intermediate value theorem.

Theorem 2 Intermediate Value Theorem *Let f be continuous on $[a,b]$, and assume that $f(a) < d$ and $f(b) > d$, or $f(a) > d$ and $f(b) < d$. Then there exists a number c in (a,b) such that $f(c) = d$.*

Proof If there were no such c, then the alternative version, which we have just proven, would enable us to conclude that $f(a)$ and $f(b)$ lie on the same side of d. Since we have assumed in the statement of the theorem that $f(a)$ and $f(b)$ lie on opposite sides of d, the absence of a c such that $f(c) = d$ leads to a contradiction, so the c must exist.

Solved Exercises

9. In the proof of the intermediate value theorem, why did we not use, instead of S, the set T consisting of those x in $[a,b]$ for which $f(x) < d$?
10. Find a formula for a function like that shown in Fig. 5-4. (You may use trigonometric functions.)
11. Prove that the polynomial $x^5 + x^4 - 3x^2 + 2x + 8$ has at least one real root.
12. Let T be the set of *values* of a function f which is continuous on $[a,b]$; i.e., $y \in T$ if and only if $y = f(x)$ for some $x \in [a,b]$. Prove that T is convex.

Exercises

14. Let f be a polynomial and suppose that $f'(-1) < 0$ while $f'(1) > 0$. Prove that f must have a critical point (a point where f' vanishes) somewhere on the interval $(-1,1)$.
15. Let $f(x) = x^4 - x^2 + 35x - 7$. Prove that f has at least two real roots.
16. (a) Give a direct proof of the Intermediate Value Theorem.
    (b) Use the intermediate value theorem to prove the alternative version.

Increasing and Decreasing at a Point

Intuitively, if we say $f(x)$ is increasing with x at x_0, we mean that when x is increased a little, $f(x)$ increases and when x is decreased a little, $f(x)$ decreases. Study Fig. 5-5 to see why we say "a little."

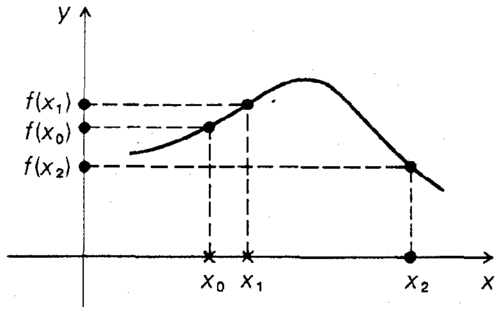

Fig. 5-5 x_1 is a little greater than x_0 and $f(x_1) > f(x_0)$; x_2 is a lot greater than x_0 and now $f(x_2) < f(x_0)$.

Definition Let f be a function whose domain contains an open interval about x_0.

1. f is said to be *increasing* at x_0 if the graph of f overtakes the horizontal line through $(x_0, f(x_0))$ at x_0.

2. f is said to be *decreasing* at x_0 if the graph is overtaken by the line at x_0. (See Fig. 5-6.)

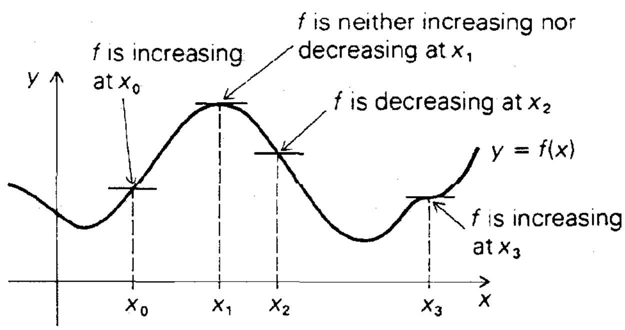

Fig. 5-6 Where is f increasing and decreasing?

Thus, f is increasing at x_0 if and only if $f(x) - f(x_0)$ changes sign from negative to positive at x_0. Similarly f is decreasing if and only if $f(x) - f(x_0)$ changes sign from positive to negative at x_0.

If we substitute the definition of "overtake" in the definition of increasing, we obtain the following equivalent reformulation.

Definition' Let f be a function whose domain contains an open interval about x_0.

1. f is said to be *increasing* at x_0 if there is an open interval I about x_0 such that:

(i) $f(x) < f(x_0)$ for $x < x_0$ in I
(ii) $f(x) > f(x_0)$ for $x > x_0$ in I

2. f is said to be *decreasing* at x_0 if there is an open interval I about x_0 such that:

(i) $f(x) > f(x_0)$ for $x < x_0$ in I
(ii) $f(x) < f(x_0)$ for $x > x_0$ in I

Pictorially speaking, a function f is increasing at x_0 when moving x a little to the left of x_0 lowers $f(x)$ while moving x a little to the right of x_0 raises $f(x)$. (The opposite happens if the function is decreasing at x_0.)

Worked Example 3 Show that $f(x) = x^2$ is increasing at $x_0 = 2$.

Solution Let I be $(1,3)$. If $x < x_0$ is in I, we have $1 < x < 2$, so $f(x) = x^2 < 4 = x_0^2$. If $x > x_0$ is in I, then $2 < x < 3$, and so $f(x) = x^2 > 4 = x_0^2$. We have verified (i) and (ii) of part 1 of Definition$'$, so f is increasing at 2.

The transition definition of the derivative of f at x_0 tells us which lines overtake and are overtaken by the graph of f at x_0. This leads to the next theorem.

Theorem 3 *Let f be differentiable at x_0.*

1. *If $f'(x_0) > 0$, then f is increasing at x_0.*
2. *If $f'(x_0) < 0$, then f is decreasing at x_0.*
3. *If $f'(x_0) = 0$, then f may be increasing at x_0, decreasing at x_0, or neither.*

Proof We shall prove parts 1 and 3; the proof of part 2 is similar to 1.

The definition of the derivative, as formulated in Theorem 4, Chapter 2, includes the statement that any line through $(x_0, f(x_0))$ whose slope is less than $f'(x_0)$ is overtaken by the graph of f at x_0. If $f'(x_0) > 0$, then the horizontal line through $(x_0, f(x_0))$, whose slope is 0, must be overtaken by the graph of f at x_0; thus, f is increasing at x_0, by definition.

The functions x^3, $-x^3$, and x^2, all of which have derivative 0 at $x_0 = 0$, establish part 3; see Solved Exercise 15.

The following is another proof of Theorem 3 directly using the original definition of the derivative in Chapter 1.

Alternative Proof From condition 1 of the definition of the derivative (p. 6), we know that if $m < f'(x_0)$, then the function $f(x) - [f(x_0) + m(x - x_0)]$ changes sign from negative to positive at x_0. Thus if $0 < f'(x_0)$, then we may choose $m = 0$ and conclude that $f(x) - f(x_0)$ changes sign from negative to positive at x_0; that is, f is increasing at x_0. This establishes part 1 of the theorem. Part 2 is similar, and the functions x^3, $-x^3$, and x^2 establish part 3 as in the first proof.

Worked Example 4 Is $x^5 - x^3 - 2x^2$ increasing or decreasing at -2?

Solution Letting $f(x) = x^5 - x^3 - 2x^2$, we have $f'(x) = 5x^4 - 3x^2 - 4x$ and $f'(-2) = 5(-2)^4 - 3(-2)^2 - 4(-2) = 80 - 12 + 8 = 76$, which is positive. By Theorem 3 part 3, $x^5 - x^3 - 2x^2$ is increasing at -2.

Theorem 3 can be interpreted geometrically: if the linear approximation to f at x_0 (that is, the tangent line) is an increasing or decreasing function, then f itself is increasing or decreasing at x_0. If the tangent line is horizontal, the behavior of f at x_0 is not determined by the tangent line. (See Fig. 5-7.)

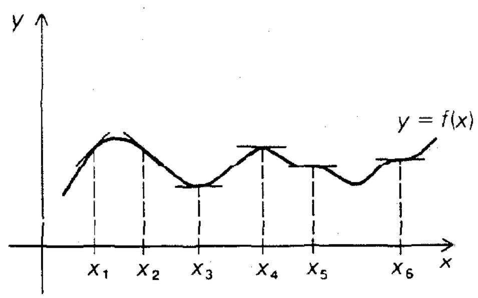

Fig. 5-7
$f'(x_1) > 0$; f is increasing at x_1
$f'(x_2) < 0$; f is decreasing at x_2
$f'(x_3) = f'(x_4) = 0$; f is neither increasing nor decreasing at x_3 and x_4
$f'(x_5) = 0$; f is decreasing at x_5
$f'(x_6) = 0$; f is increasing at x_6

We can also interpret Theorem 3 in terms of velocities. If $f(t)$ is the position of a particle on the real-number line at time t, and $f'(t_0) > 0$, then the particle is moving to the right at time t_0; if $f'(t_0) < 0$, the particle is moving to the left.

Combined with the techniques for differentiation in Chapter 3, Theorem 3 provides an effective means for deciding where a function is increasing or decreasing.

Solved Exercises

13. The temperature at time x is given by $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$. Is it getting warmer or colder at $x = 0$?

14. Using Theorem 3, find the points at which $f(x) = 2x^3 - 9x^2 + 12x + 5$ is increasing or decreasing.

15. Decide whether each of the functions x^3, $-x^3$, and x^2 is increasing, decreasing, or neither at $x = 0$.

Exercises

17. If $f(t) = t^5 - t^4 + 2t^2$ is the position of a particle on the real-number line at time t, is it moving to the left or right at $t = 1$?

18. Find the points at which $f(x) = x^2 - 1$ is increasing or decreasing.

19. Find the points at which $x^3 - 3x^2 + 2x$ is increasing or decreasing.

20. Is $f(x) = 1/(x^2 + 1)$ increasing or decreasing at $x = 1, -3, \frac{3}{4}, 25, -36$?

21. A ball is thrown upward with an initial velocity of 30 meters per second. The ball's height above the ground at time t is $h(t) = 30t - 4.9t^2$. When is the ball rising? When is it falling?

22. (a) Prove that, if $f(x_0) = 0$, then f is increasing [decreasing] at x_0 if and only if $f(x)$ changes sign from negative to positive [positive to negative] at x_0.

    (b) Prove that f is increasing [decreasing] at x_0 if and only if $f(x) - f(x_0)$ changes sign from negative to positive [positive to negative] at x_0.

Increasing or Decreasing on an Interval

Suppose that f is increasing at every point of an interval $[a, b]$. We would expect $f(b)$ to be larger than $f(a)$. In fact, we have the following useful result.

Theorem 4 *Let f be continuous on $[a, b]$, where $a < b$, and suppose that f is increasing [decreasing] at all points of (a, b). Then $f(b) > f(a)$ $[f(b) < f(a)]$.*

The statement of Theorem 4 may appear to be tautological—that is, "trivially true"—but in fact it requires a proof (which will be given shortly). Like the intermediate value theorem, Theorem 4 connects a *local* property of functions (increasing at each point of an interval) with a *global* property (relation between values of the function at endpoints). We do not insist that f be increasing or decreasing at a or b because we wish the theorem to apply in cases of the type illustrated in Fig. 5-8. Also, we note that if f is not continuous, the result is not valid (see Fig. 5-9).

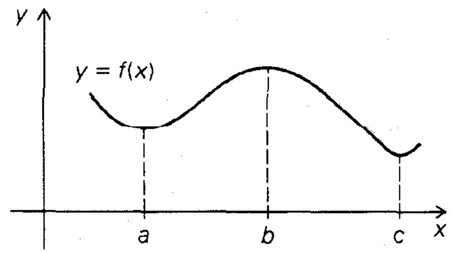

Fig. 5-8 f is increasing at each point of (a, b); $f(b) > f(a)$
f is decreasing at each point of (b, c); $f(c) < f(b)$
f is neither increasing nor decreasing at a, b, c

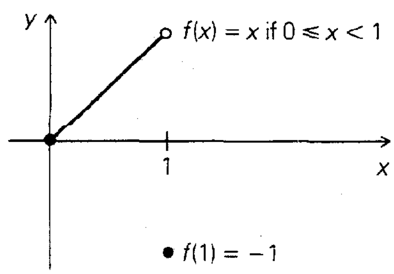

Fig. 5-9 f is increasing at all points of $(0, 1)$, but $f(1)$ is not greater than $f(0)$.

Proof of Theorem 4 We proceed in several steps.

Step 1 If $a < x < y < b$, then $f(x) < f(y)$.

To prove this, we choose any x in (a, b) and let S be the set consisting of those y in (x, b) for which $f(x) < f(z)$, for all z in $(x, y]$. If we can show that $S = (x, b)$, then for any y such that $x < y < b$ we will have $y \in S$, so that $f(x) < f(z)$ for all z in $(x, y]$; in particular, we will have $f(x) < f(y)$.

We proceed to show that $S = (x, b)$. By the same kind of argument as we used in the proof of the intermediate value theorem (alternative version) it is easy to show that S is convex. By the completeness axiom, S is an interval. Since f is increasing at x, S contains all the points sufficiently near to x and to the right of x, so x is the left-hand endpoint of S. Thus $S = (x, c)$ or $(x, c]$ for some c, $c \leqslant b$.

Suppose that $c < b$. Then f is increasing at c, so we can find points p and q such that:

$$x < p < c < q < b \qquad (1)$$

$$f(y) < f(c) \qquad \text{for all } y \text{ in } (p, c) \qquad (2)$$

and

$$f(c) < f(y) \qquad \text{for all } y \text{ in } (c, q) \qquad (3)$$

Since $S = (x, c)$, we have $f(x) < f(y)$ for all y in (p, c). By (2), we have $f(x) < f(c)$; by (3), we then have $f(x) < f(y)$ for all y in (c, q). Thus, we have $f(x) < f(y)$ for all y in (x, c) and $[c, q)$, so $f(x) < f$ on (x, q), i.e., $f(x) < f(y)$ for all $y \in (x, q)$. Thus S contains points to the right of c, contradicting the fact that c is the righthand endpoint of S. Hence c must equal b, and so $S = (x, b)$.

Notice that we have not yet used the continuity of f at a and b. (See the corollary below.)

Step 2 If $y \in (a, b)$, then $f(a) \leqslant f(y)$ and $f(y) \leqslant f(b)$.

To prove that $f(a) \leqslant f(y)$, we assume the opposite, namely, $f(a) > f(y)$, and derive a contradiction. In fact, if $f(a) > f(y)$ for some $y \in (a, b)$, the continuity of f at a implies that $f > f(y)$ on some interval $[a, r)$. Choosing x in (a, r) such that $x < y$, we have $f(x) > f(y)$, contradicting step 1. Using the continuity of f at b, we can prove in a similar manner that $f(y) \leqslant f(b)$.

Step 3 If $y \in (a, b)$, then $f(a) < f(y)$ and $f(y) < f(b)$.

To prove that $f(a) < f(y)$, we choose any x between a and y (i.e., let $x = \frac{1}{2}(a + y)$). By step 1, $f(x) < f(y)$; by step 2, $f(a) \leqslant f(x)$. So $f(a) \leqslant f(x) < f(y)$, and $f(a) < f(y)$. Similarly, we prove $f(y) < f(b)$.

Step 4 $f(a) < f(b)$.

Choose any y in (a, b). By step 3, $f(a) < f(y)$ and $f(y) < f(b)$, so $f(a) < f(b)$.

We shall now rephrase Theorem 4. The following terminology will be convenient.

Definition Let f be a function defined on an interval I. If $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I, we say that f is *increasing* on I. If $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I, we say that f is *decreasing* on I. If f is either increasing on I or decreasing on I, we say that f is *monotonic* on I.

Theorem 4′ *Let f be continuous on $[a, b]$ and increasing [decreasing] at all points of (a, b). Then f is increasing [decreasing] on $[a, b]$.*

For example, the function in Fig. 5-8 is monotonic on $[a, b]$ and monotonic on $[b, c]$, but it is not monotonic on $[a, c]$. The function $f(x) = x^2$ is monotonic on $(-\infty, 0)$ and $(0, \infty)$, but not on $(-\infty, \infty)$. (Draw a sketch to convince yourself.)

Combining Theorems 3 and 4′ with the intermediate value theorem gives a result which is useful for graphing.

Theorem 5 *Suppose that:*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b) and f' is continuous on (a, b).*
3. *f' is never zero on (a, b).*

Then f is monotonic on $[a, b]$. To check whether f is increasing or decreasing on $[a, b]$, it suffices to compute the value of f' at any one point of (a, b) and see whether it is positive or negative.

Proof* By the intermediate value theorem, f' must either be positive everywhere on (a, b) or negative everywhere on (a, b). (If f' took values with both signs, it would have to be zero somewhere in between.) If f' is positive everywhere, f is increasing at each point of (a, b) by Theorem 3. By Theorem 4', f is increasing on $[a, b]$. If f' is negative everywhere, f is decreasing on $[a, b]$.

We can also apply Theorem 5 to intervals which are not closed.

Corollary *Suppose that:*

1. *f is continuous on an open interval (a, b).*
2. *f is differentiable and f' is continuous at each point of (a, b).*
3. *f' is not zero at any point of (a, b).*

Then f is monotonic on (a, b) (increasing if $f' > 0$, decreasing if $f' < 0$).

Proof By the intermediate value theorem, f' is everywhere positive or everywhere negative on (a, b). Suppose it to be everywhere positive. Let $x_1 < x_2$ be in (a, b). We may apply Theorems 3 and 4 to f on $[x_1, x_2]$ and conclude that $f(x_1) < f(x_2)$. Thus f is increasing on (a, b). Similarly, if f' is everywhere negative on (a, b), f is decreasing on (a, b).

Similar statements hold for half-open intervals $[a, b)$ or $(a, b]$.

Applications of these results to the shape of graphs will be given in the next chapter.

*Theorem 5 is still true if f' is not continuous; see Problem 10 of Chapter 7.

Solved Exercises

16. The function $f(x) = -2/x$ has $f'(x) = 2/x^2 > 0$ at all points of its domain, but $f(1) = -2$ is not greater than $f(-1) = 2$. What is wrong here?

17. Let $a < b < c$, and suppose that f, defined on (a, c), is increasing at each point of (a, b) and (b, c) and is continuous at b. Prove that f is increasing at b.

Exercises

23. Prove the analogue of Theorem 4 for decreasing functions.

24. Show by example that the conclusion in Solved Exercise 17 may not be valid if f is discontinuous at b.

25. (a) Suppose that the functions f and g defined on $[a, b]$ are continuous at a and b, f' and g' are continuous on (a, b) and that $f'(x) > g'(x)$ for all x in (a, b). If $f(a) = g(a)$, prove that $f(b) > g(b)$. [*Hint*: You may assume that $f - g$ is continuous at a and b.]

    (b) Give a physical interpretation of the result of part (a), letting x be *time*.

26. Let f be a polynomial such that $f(0) = f(1)$ and $f'(0) > 0$. Prove that $f'(x) = 0$ for some x in $(0, 1)$.

The Extreme Value Theorem

The last theorem in this chapter asserts that a continuous function on a closed interval has maximum and minimum values. Again, the proof uses the completeness axiom.

We begin with a lemma which gets us part way to the theorem.

Boundedness Lemma *Let f be continuous on $[a, b]$. Then there is a number B such that $f(x) \leqslant B$ for all x in $[a, b]$. We say that f is bounded above by B on $[a, b]$ (see Fig. 5-10).*

Proof If P is any subset of $[a, b]$, we will say that f is *bounded above* on P if there is some number m (which may depend on P) such that $f(x) \leqslant m$ for all x in P. Let S be the set consisting of those y in $[a, b]$ such that f is bounded above on $[a, y]$. If $y_1 < y < y_2$, where y_1 and y_2 are in S, then f

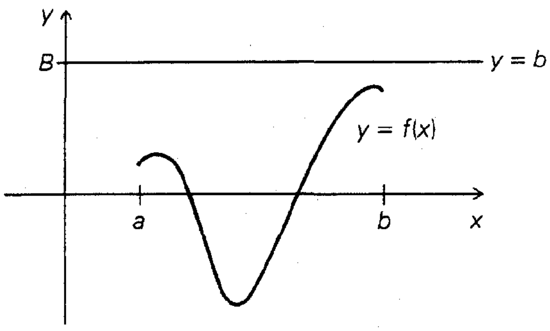

Fig. 5-10 The line $y = B$ lies above the graph of f on $[a, b]$.

is bounded above on $[a, y_2]$, so it must be bounded above on $[a, y]$. Also, y must belong to $[a, b]$ if y_1 and y_2 do, so S is convex.

By the completeness axiom, S is an interval which is contained in $[a, b]$. f is obviously bounded above on $[a, a]$ (let $m = f(a)$), so $a \in S$, and S is of the form $[a, c)$ or $[a, c]$ for some c in $[a, b]$.

Let c be the right-hand endpoint of S. Since f is continuous at c, f is bounded above on some interval about c; let e be the left-hand endpoint of that interval. Since $e \in [a, c)$, f is bounded above on $[a, e]$ as well as on $[e, c]$, so f is bounded above on $[a, c]$, and $c \in S$. Thus, S is of the form $[a, c]$. If $c < b$, an argument like the one we just used shows that S must contain points to the right of c, which is a contradiction. So we must have $c = b$, i.e., $S = [a, b]$, and f is bounded above on $[a, b]$.

By taking the "best" bound for f on $[a, b]$ (imagine lowering the horizontal line in Fig. 5-10 until it just touches the graph of f), we will obtain the maximum value.

Theorem 6 Extreme Value Theorem *Let f be continuous on $[a, b]$. Then f has both a maximum and minimum value on $[a, b]$; i.e., there are points x_M and x_m in $(a, b]$ such that $f(x_m) \leqslant f(x) \leqslant f(x_M)$ for all x in $[a, b]$.*

Proof We prove that there is a maximum value, leaving the case of a minimum to the reader. Consider the set T of values of f, i.e., $q \in T$ if and only if $q = f(x)$ for some x in $[a, b]$. We saw in Solved Exercise 12 that T is convex. By the completeness axiom, T is an interval. By the boundedness lemma, T cannot extend infinitely far in the positive direction, so it has an upper endpoint, which we call M. (See Fig. 5-11.) We wish to show that the graph of f actually touches the line $y = M$ at some point, so that M will be a value of $f(x)$ for some x in $[a, b]$, and thus M will be the maximum value.

If there is no x in $[a, b]$ for which $f(x) = M$, then $f(x) < M$ for all x in $[a, b]$, and an argument identical to that used in the proof of the

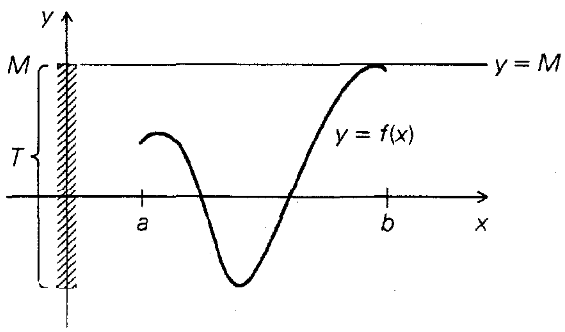

Fig. 5-11 $y = M$ is the maximum value of f on $[a, b]$.

boundedness lemma shows that there is some $M_1 < M$ such that $f(x) \leqslant M_1$ for all x in $[a, b]$. But this contradicts the assumption that M is the upper endpoint of the set T of values of f, so there must be some x_M in $[a, b]$ for which $f(x_M) = M$. For any y in $[a, b]$, we now have $f(y) \in T$, so $f(y) \leqslant M = f(x_M)$, and so M is the maximum value of f on $[a, b]$.

Solved Exercises

18. Prove that there exists a number M such that $x^8 + x^4 + 8x^9 - x < M$ if $0 \leqslant x \leqslant 10{,}000$.

19. Prove that, if f is continuous on $[a, b]$, then the set T of values of f (see Solved Exercise 12) is a closed interval. Must $f(a)$ and $f(b)$ be the endpoints of this interval?

Exercises

27. (a) Prove that, if f is continuous on $[a, b]$, so is the function $-f$ defined by $(-f)(x) = -[f(x)]$.
    (b) Prove the "minimum" part of the extreme value theorem by applying the "maximum" part to $-f$.

28. Find a specific number M which works in Solved Exercise 18.

29. Give an example of a continuous function f on $[0, 1]$ such that neither $f(0)$ nor $f(1)$ is an endpoint of the set of values of f on $[0, 1]$.

30. Is the boundedness lemma true if the closed interval $[a, b]$ is replaced by an open interval (a, b)?

Problems for Chapter 5

1. We can define all the notions of this section, including continuity, differentiability, maximum and minimum values, etc., for functions of a *rational* variable; i.e., we may replace real numbers by rationals everywhere in the definitions. In particular, $[a, b]$ then means the set of rational x for which $a \leqslant x \leqslant b$.
   (a) Prove that the function
   $$f(x) = \frac{1}{x^2 - 2}$$
   is defined everywhere on the rational interval $[0, 2]$. It is possible to prove that f is continuous on $[0, 2]$; you may assume this now.
   (b) Show that the function f in part (a) does not satisfy the conclusions of the intermediate value theorem, the boundedness lemma, or the extreme value theorem. Thus, it is really necessary to work with the real numbers.

2. Prove that, if f is a continuous function on an interval I (not necessarily closed), then the set of values of f on I is an interval. Could the set of values be a closed interval even if I is not?

3. Prove that the volume of a cube is a continuous function of the length of its edges.

4. "Prove" that you were once exactly 1 meter tall. Did you ever weigh 15 kilograms?

5. Write a direct proof of the "minimum" part of the extreme value theorem.

6. Prove that, if f and g are continuous on $[a, b]$, then so is $f - g$.

7. Assuming the result of Problem 6, prove that, if $f(a) < a$, $f(b) > b$, and f is continuous on $[a, b]$, then there is some x in $[a, b]$ such that $f(x) = x$.

8. Give an example of (discontinuous) functions f and g on $[0, 1]$ such that f and g both have maximum values on $[0, 1]$, but $f + g$ does not.

9. Let $f(x) = x^{17} + 3x^4 - 2$ and $g(x) = 5x^6 - 10x + 3$. Prove that there is a number x_0 such that $f(x_0) = g(x_0)$.

10. (a) Prove that, if f is increasing at each point of an open interval I, then the set S consisting of all those x in I for which $f(x) \in (0, 1)$ is convex.
    (b) Show that S might not be convex if f is not increasing on I.

11. Show by example that the sum of two functions with the intermediate value property need not have this property.

12. Prove that the intermediate value theorem implies the completeness axiom. [*Hint*: If there were a convex set which was not an interval, show that you could construct a continuous function which takes on only the values 0 and 1.]