# 5 Hyperbolic Geometry

## 5.1 Saccheri, Lambert and Absolute Geometry

As evidenced by its absence from his first 28 theorems, Euclid clearly found the parallel postulate awkward; indeed many later mathematicians believed it could not be an independent axiom. Two of the earliest to work rigorously on this were Giovanni Saccheri (1667–1733) & Johann Lambert (1728–1777), who attempted to force contradictions by assuming the negation of the parallel postulate. While they failed at their primary purpose, their insights provided the foundation of a new non-Euclidean geometry. Before considering their work, we define some terms and recall our earlier discussion regarding parallels.

**Definition 5.1.** *Absolute* or *neutral* geometry is the axiomatic system comprising all of Hilbert's axioms except Playfair's axiom. Euclidean geometry is a special case of neutral geometry.

A *non-Euclidean geometry* is typically a model satisfying some or all of Euclid's/Hilbert's axioms and for which parallels are non-unique:

There exists a line $\ell$ and a point $P \notin \ell$ through which there are either *no parallels* or *at least two.*

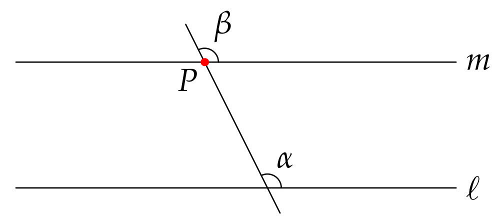
For example, spherical geometry is non-Euclidean since there are no parallel lines (Hilbert's axioms I-2 and O-3 are also false, as is the exterior angle theorem).

**Results in absolute geometry** Everything in the first 28 theorems of Euclid, including:

- Basic constructions: bisectors, perpendiculars, etc.
- The Exterior Angle Theorem.
- Triangle congruence theorems: SAS, ASA, SAA, SSS.
- Congruent/equal angles imply parallels: i.e.

  $$\alpha \cong \beta \implies \ell \parallel m$$

  This is equivalent to the existence of a parallel $m$ to a given line $\ell$ through a point $P \notin \ell$.

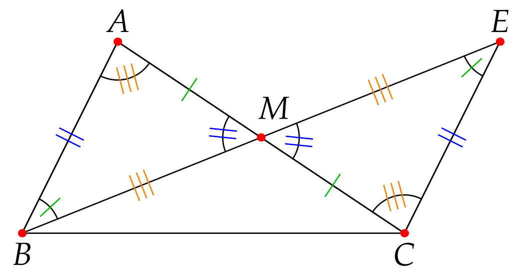



**Arguments requiring unique parallels** We have previously discussed the following results whose proofs relied on Playfair's axiom: the arguments are therefore *false* in absolute geometry:

- A line crossing parallel lines makes equal angles: in the picture, $\ell \parallel m \implies \alpha \cong \beta$. This is the uniqueness in Playfair: the parallel $m$ to $\ell$ through $P$ is unique.
- Angles in a triangle sum to a straight edge (180°).
- Constructions of squares/rectangles.
- Pythagoras' Theorem.

While our *arguments* for the above are certainly false, we cannot instantly claim that the *results* are false in absolute geometry: there might be alternative proofs! To show that the results really require unique parallels, we must exhibit a *model*: we shall describe such in the next section. The existence of this model explains why Saccheri and Lambert failed in their endeavors: the parallel postulate (Playfair) is indeed independent of Euclid's (Hilbert's) other axioms.

**The Saccheri–Legendre Theorem**




We start with an extension of the Exterior Angle Theorem based on Euclid's proof.

Given $\triangle ABC$, take $M$ to be the midpoint of $\overline{AC}$ and extend $\overline{BM}$ to $E$ such that $\overline{BM} \cong \overline{ME}$ to obtain congruent triangles. Observe:

1. $\measuredangle ACB + \measuredangle CAB = \measuredangle ACB + \measuredangle ACE < 180°$: this is essentially Euclid's Exterior Angle Theorem.
2. The sum of the angles in $\triangle ABC$ and $\triangle EBC$ are equal.[1]
3. One (or both!) of $\measuredangle EBC$ or $\measuredangle BEC$ is $\leq \frac{1}{2}\measuredangle ABC$.

We may iterate this construction to produce a sequence $\triangle_0 = \triangle ABC$, $\triangle_1 = \triangle BEC$, $\triangle_2, \triangle_3, \ldots$ each of which has the same angle sum *and* such that at least one angle in $\triangle_n$ has measure

$$\alpha_n \leq \frac{1}{2^n}\measuredangle ABC$$

If the sum $\Sigma$ of the angles in $\triangle ABC$ were *greater* than 180°, then

$$\Sigma = 180° + \epsilon$$

for some $\epsilon > 0$. Since $\frac{1}{2^n} \to 0$, we may choose $n$ large enough so that $\alpha_n < \epsilon$. But then the sum of the other two angles in $\triangle_n$ would be *greater than 180°*, contradicting the Exterior Angle Theorem!

We have therefore proved:

**Theorem 5.2 (Saccheri–Legendre).** *The angle sum in a triangle is at most 180° in absolute geometry.*

Saccheri's failed hope was to prove equality *without* invoking the parallel postulate. That he got half way there is still remarkable!

**Saccheri and Lambert Quadrilaterals** Saccheri and Lambert both considered *quadrilaterals* in the absence of the parallel postulate. Two families of such are named in their honor.

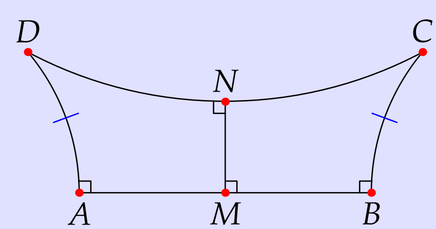
**Definition 5.3.** A *Saccheri quadrilateral* $ABCD$ satisfies

$$\overline{AD} \cong \overline{BC} \quad \text{and} \quad \measuredangle DAB = \measuredangle CBA = 90°$$

$\overline{AB}$ is the *base* and $\overline{CD}$ the *summit*.

The interior angles at $C$ and $D$ are the *summit angles*.

A *Lambert quadrilateral* has three right-angles; for instance $AMND$.




We draw these with curved sides to indicate that the summit angles need not be right-angles, though, as yet, we have no model which shows that they could be anything else. Regardless of how they are drawn, $\overline{AD}$, $\overline{BC}$ and $\overline{CD}$ are all *segmemts*!

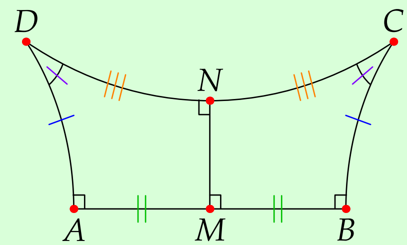
[1]With the parallel postulate, we could use congruence of angles $\measuredangle BAC \cong \measuredangle ECA$ to conclude that $\overline{CE} \parallel \overline{BA}$, from which the sum of the angles in a triangle is 180° and the observation is trivial. We cannot do this in absolute geometry!

The seeming symmetry of a Saccheri quadrilateral is not an illusion.

**Lemma 5.4.** 1. *If we bisect the base and summit of a Saccheri quadrilateral, we obtain congruent Lambert quadrilaterals.*

2. *The summit angles of a Saccheri quadrilateral are congruent.*

3. *In Euclidean geometry, Saccheri and Lambert quadrilaterals are rectangles (four right-angles).*




We leave parts 1 and 2 as an exercise.

*Proof of 3.* By part 1 we need only prove this for a Saccheri quadrilateral. Following the exterior angle theorem, $\overleftrightarrow{AB}$ is a crossing line making equal (right-) angles, whence $\overline{AD} \parallel \overline{BC}$.

However $\overleftrightarrow{CD}$ also crosses the same parallel lines; by the parallel postulate, the summit angles sum to a straight edge. Since these are congruent, they must both be right-angles. ∎

We can now prove another of the conclusions of Saccheri & Lambert.

**Theorem 5.5.** *In absolute geometry, the summit angles of a Saccheri quadrilateral measure $\leq 90°$.*

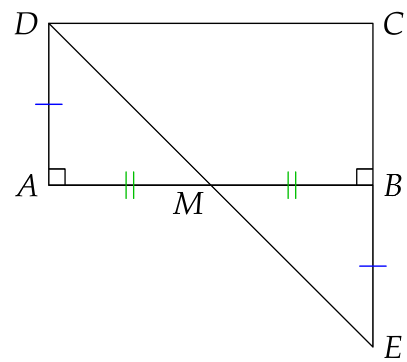
*Proof.* Extend $\overline{CB}$ to $E$ (on the opposite side of $\overline{AB}$ to $C$) such that $\overline{BE} \cong \overline{DA}$. Let $M$ be the midpoint of $\overline{AB}$.

SAS says that $\triangle DAM \cong \triangle EBM$, whence $M$ lies on $\overline{DE}$.

The summit angles at $C$ and $D$ therefore sum to

$$\begin{aligned} \measuredangle ADC + \measuredangle BCD &= \measuredangle ADM + \measuredangle EDC + \measuredangle DCE \\ &= \measuredangle CED + \measuredangle EDC + \measuredangle DCE \\ &\leq 180° \end{aligned}$$

by the Saccheri–Legendre Theorem.



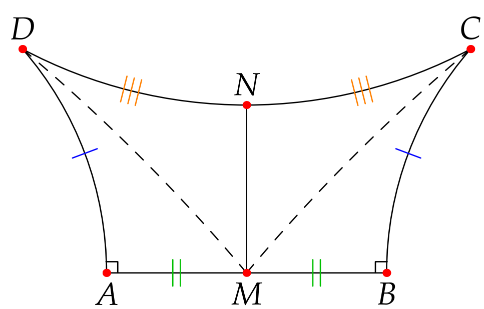

∎

**Exercises** Complete these exercises in absolute geometry; you cannot use Playfair or Euclid's parallel postulate!

1. Prove parts 1 and 2 of Lemma 5.4.

   *(Hint: use the picture: all you need are the triangle congruence theorems...)*

2. Use the same picture to give a quick alternative proof of Theorem 5.5.



## 5.2 Models of Hyperbolic Geometry

In the early 1800's James Boylai, Carl Friedrich Gauss and Nikolai Lobashevsky independently took the next step. Rather than attempting to establish the parallel postulate as a theorem within Euclidean geometry, they defined a new geometry based on the first four of Euclid's postulates plus an alternative to the parallel postulate:

**Axiom 5.6 (Boylai–Lobashevshky/Hyperbolic Postulate).** Given a line and a point not on the line, there exist *at least two* parallel lines through the point.

The resulting axiomatic system[2] is known as *hyperbolic geometry*. Consistency was proved in the late 1800's by Beltrami, Klein and Poincaré, each of whom created models of hyperbolic geometry by defining point, line, etc., in novel ways. The simplest model is arguably the *Poincaré disk*, named for Henri Poincaré though first proposed by Beltrami.

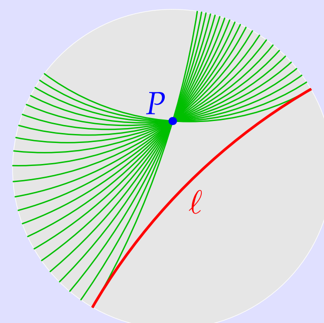
**Definition 5.7.** The *Poincaré disk* is the interior of the unit circle

$$\{(x, y) : x^2 + y^2 < 1\}$$

A *hyperbolic line* is a diameter or a circular arc meeting the unit circle at right angles.

In the picture we have a hyperbolic line $\ell$ and a point $P$: also drawn are several parallel hyperbolic lines to $\ell$ passing through $P$.

Points on the boundary circle are termed *omega points*: these are *not* in the Poincaré disk and are essentially 'points at infinity.'




It is easy to describe hyperbolic lines using equations in analytic geometry.

**Lemma 5.8.** *Every hyperbolic line in the Poincaré disk model is one of the following:*

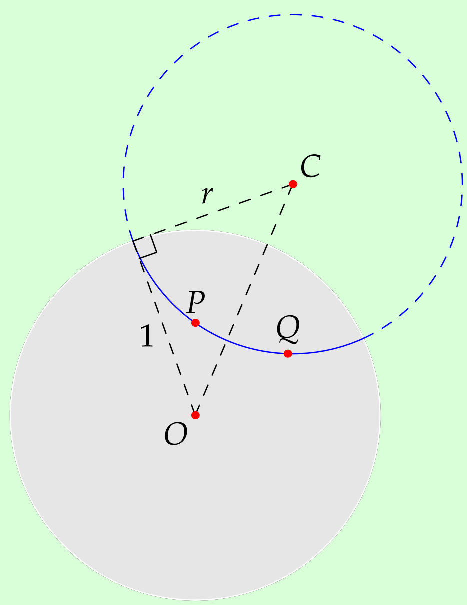
- *A diameter passing through $(a, b) \neq (0, 0)$ with Euclidean equation $bx = ay$.*
- *The arc of a (Euclidean) circle with equation*

$$x^2 + y^2 - 2ax - 2by + 1 = 0 \quad \text{where} \quad a^2 + b^2 > 1$$

*and (Euclidean) center and radius*

$$C = (a, b) \quad \text{and} \quad r = \sqrt{a^2 + b^2 - 1}$$

*Moreover, there exists a unique hyperbolic line joining any two points in the Poincaré disk.*




[2]In the modern approach we assume all of Hilbert's axioms, replacing Playfair's axiom with the hyperbolic postulate.

**Example 5.9.** Find the equation of the hyperbolic line through the points $P = (0, \frac{1}{2})$ and $Q = (\frac{1}{2}, \frac{1}{3})$ in the Poincaré disk: this is the picture shown in Lemma 5.8.
Substitute into $x^2 + y^2 - 2ax - 2by + 1 = 0$ to obtain a system of equations for $a, b$:

$$\begin{cases} \frac{1}{4} - b + 1 = 0 \\ \frac{1}{4} + \frac{1}{9} - a - \frac{2}{3}b + 1 = 0 \end{cases} \implies (a, b) = \left(\frac{19}{36}, \frac{5}{4}\right)$$

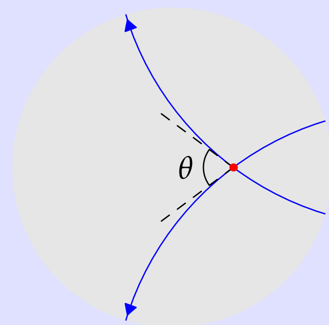
The required hyperbolic line $\overleftrightarrow{PQ}$ therefore has equation

$$x^2 + y^2 - \frac{19}{18}x - \frac{5}{2}y + 1 = 0 \quad \text{or} \quad \left(x - \frac{19}{36}\right)^2 + \left(y - \frac{5}{4}\right)^2 = \frac{545}{648}$$

To complete the model, we need to define congruence of hyperbolic segments and angles.

**Definition (5.7 continued).** The *hyperbolic distance* between points $P, Q$ in the Poincaré disk is

$$d(P, Q) := \cosh^{-1}\left(1 + \frac{2\,|PQ|^2}{(1 - |P|^2)(1 - |Q|^2)}\right)$$




where $|PQ|$ is the Euclidean distance and $|P|, |Q|$ are the Euclidean distances of $P, Q$ from the origin.
Hyperbolic line segments are *congruent* if they have the same length.
The *angle* between hyperbolic rays is that between their (Euclidean) tangent lines: angles are congruent if they have the same measure.

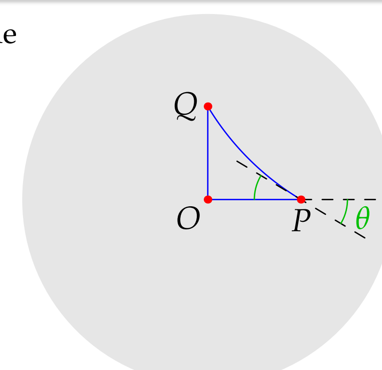
**Lemma 5.10.** *The hyperbolic distance*[a] *of a point $P$ from the origin is*

$$d(O, P) = \cosh^{-1}\frac{1 + |P|^2}{1 - |P|^2} = \ln\frac{1 + |P|}{1 - |P|}$$

[a] It should seem reasonable for hyperbolic functions to play some role in hyperbolic geometry! As a primer:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{and} \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

**Example 5.11.** We calculate the sides and angles in the right-triangle with vertices $O = (0, 0)$, $P = (\frac{1}{2}, 0)$ and $Q = (0, \frac{1}{2})$.




$$|P| = \tfrac{1}{2} = |Q|, \qquad |PQ|^2 = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

$$d(O, P) = d(O, Q) = \ln\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \ln 3 = \cosh^{-1}\frac{5}{3} \approx 1.099$$

$$d(P, Q) = \cosh^{-1}\left(1 + \frac{2 \cdot \frac{1}{2}}{(1 - \frac{1}{4})^2}\right) = \cosh^{-1}\frac{25}{9} \approx 1.681$$

Now observe that the hyperbolic line $\overleftrightarrow{PQ}$ has equation

$$x^2 + y^2 - \frac{5}{2}x - \frac{5}{2}y + 1 = 0$$

Implicit differentiation yields

$$2x - \frac{5}{2} + \left[2y - \frac{5}{2}\right]\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x-5}{5-4y} \implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_P = -\frac{3}{5}$$

Since the side $\overline{OP}$ is horizontal, we conclude that the interior angle to the triangle at $P$ is

$$\theta = \tan^{-1}\frac{3}{5} \approx 30.96°$$

By symmetry, we have the same angle at $Q$. With a right-angle at $O$, we conclude that the sum of the angles in the triangle is approximately $151.93° < 180°$!

As a sanity check, we compare some data for the hyperbolic triangle $\triangle OPQ$ and the *Euclidean* triangle with the same vertices

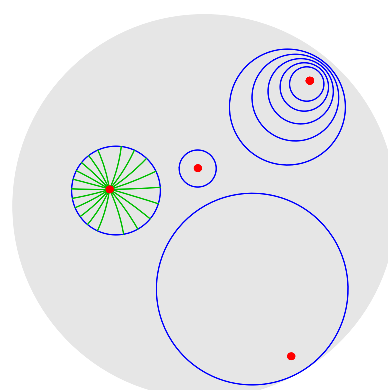
| Property | Hyperbolic Triangle | Euclidean Triangle |
|---|---|---|
| Edge lengths | 1.099 : 1.099 : 1.681 | 0.5 : 0.5 : 0.707 |
| Relative edge ratios | 1 : 1 : 1.530 | 1 : 1 : 1.414 |
| Angles | 30.06°, 30.96°, 90° | 45°, 45°, 90° |

Observe that the hyperbolic side lengths are longer, and that the hypotenuse is *relatively* longer in the hyperbolic case. It should also be obvious that the hyperbolic side lengths also do not satisfy the usual Pythagorean relation $a^2 + b^2 = c^2$.

We leave the next result as an exercise: it says that distance increases smoothly as one moves along a hyperbolic line.

**Lemma 5.12.** *Fix $P$ and a hyperbolic line through $P$. Then the distance function $Q \mapsto d(P,Q)$ maps the set of points on one side of $P$ differentiably and bijectively onto the interval $(0,\infty)$.*

The Lemma means that hyperbolic circles are well-defined and look like one expects: the circle of hyperbolic radius $\delta$ centered at $P$ is the set of points $Q$ such that $d(P,Q) = \delta$.



In the picture we've drawn several hyperbolic circles and their centers. One of the circles has several of its radii drawn. Notice how the centers are closer (in a Euclidean sense) to the boundary circle than one might expect: this is since hyperbolic distances measure greater the further one is from the origin.

You might be suspicious (and you'd be correct—see Exercise 5.2.5) that hyperbolic circles in the Poincaré disk model are also Euclidean circles! Moreover, their hyperbolic radii intersect the circles at right-angles as we'd expect.

**Theorem 5.13.** *The Poincaré disk is a model of hyperbolic geometry.*

*Sketch Proof.* A rigorous proof would require us to check the hyperbolic postulate and all Hilbert's axioms except Playfair. Instead we check Euclid's postulates 1–4 and the hyperbolic postulate 5.

1. Lemma 5.8 says we can join any given points in the Poincaré disk by a unique segment.

2. A hyperbolic segment joins two points *inside* the (open) Poincaré disk. The distance formula increases (Lemma 5.12) unboundedly as $P$ moves towards the boundary circle, so we can always make a hyperbolic line longer.

3. Hyperbolic circles are defined above.

4. All right-angles are equal since the notion of angle is unchanged from Euclidean geometry.

5. The first picture on page 4 shows multiple parallels...

■

**Other Models of Hyperbolic Space: non-examinable**

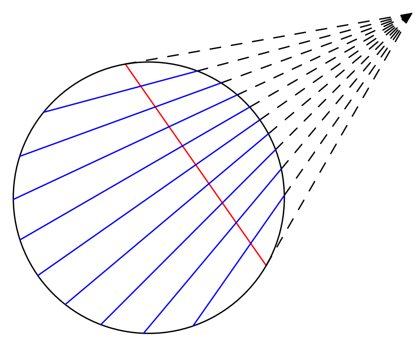
There are several other models of hyperbolic space. Here are three of the most common.

**The Klein Disk Model** The approach is similar to the Poincaré disk except that lines are taken to be chords of the unit circle and the distance function is defined differently:

$$d_K(P,Q) = \frac{1}{2}\left|\ln \frac{|P\Theta|\,|Q\Omega|}{|P\Omega|\,|Q\Theta|}\right|$$

where $\Omega, \Theta$ are the points where the chord meets the circle. It is easier to compute distances in this model since hyperbolic lines are the same as Euclidean lines.

The cost is that the notion of *angle* is different. Perpendicularity comes from the following idea: Take a hyperbolic line and find the tangents to the unit circle where it meets. Any chord whose extension passes through the intersection of these tangents is orthogonal to the original line. Measuring other angles is difficult!



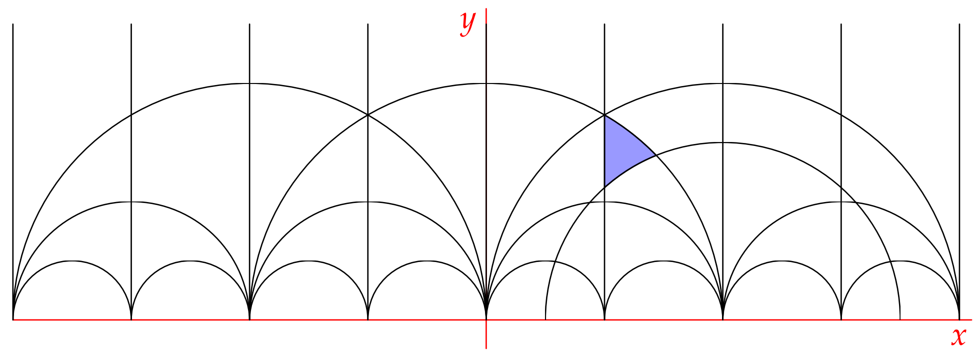
A famous result from differential geometry (Gauss' Theorem Egregium) says that this problem is unavoidable. If $H$ is a model of hyperbolic geometry, then its intrinsic curvature says that it is impossible to find an embedding $f : H \to \mathbb{R}^2$ which preserves *both* the concepts of straight line and angle. Poincaré's model preserves angles but results in 'bendy' lines: Klein's hyperbolic lines are 'Euclidean straight' but his angles are ugly. The best we can do is to have one concept or the other: we cannot have both.[3]

[3] The same problem arises when trying to make a map of part of the Earth (another curved geometry). One can have maps which preserve distance or angle, but not both.

**The Poincaré Half-plane Model** This is equivalent to the Poincaré disk via a modified stereographic projection and is widely used in complex analysis. The set of points comprises the upper half-plane ($y > 0$) in $\mathbb{R}^2$. Hyperbolic lines are vertical lines or semicircles centered on the $x$-axis:

$$x = \text{constant}, \quad \text{or} \quad (x-a)^2 + y^2 = r^2$$

Two advantages are the simple expressions for lines and that angles are measured as in Euclidean space. The expression for hyperbolic distance remains horrific!




The Poincaré half-plane: a hyperbolic triangle is shaded

**The Hyperboloid Model** Unlike the other models, this one is embedded in three dimensions. Points comprise the upper sheet ($z \geq 1$) of the hyperboloid

$$x^2 + y^2 = z^2 - 1$$

A hyperbolic line is the intersection of the hyperbolid with a plane through the origin. Isometries (congruence) can be described using matrix-multiplication and the formula for hyperbolic distance is particularly easy: between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in this model, the hyperbolic distance is

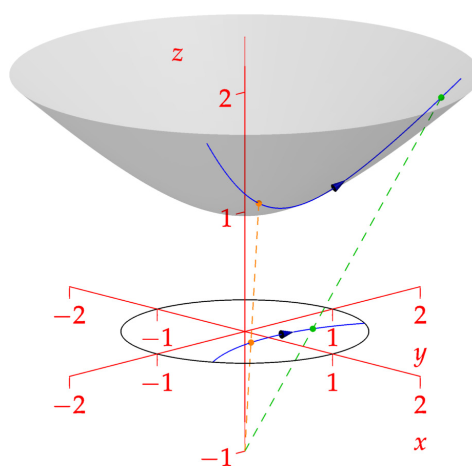
$$d(P, Q) = \cosh^{-1}(cz - ax - by)$$

Difficulties include working in three dimensions and the fact that angles are awkward.

The relationship between the Hyperboloid and Poincaré disk models is via projection. Place the disk in the $x, y$-plane centered at the origin and draw a line through a point in the disk and the point $(0, 0, -1)$. The intersection of this line with the hyperboloid gives the correspondence.

In the picture, the orange and green points on the blue hyperbolic lines correspond in the two models.

Under this correspondence, it is easy to check that the inverse-cosh formulæ for hyperbolic distance are in accord.



**Exercises** *All questions are within the Poincaré disk model*

1. (a) Find the equation of the hyperbolic line joining $P = (\frac{1}{4}, 0)$ and $Q = (0, \frac{1}{2})$.
   (b) Find the side lengths of the hyperbolic triangle $\triangle OPQ$ where $O = (0, 0)$ is the origin.
   (c) The triangle in part (b) is right-angled at $O$. If $o, p, q$ represent the hyperbolic lengths of the sides opposite $O, P, Q$ respectively, check that the Pythagorean theorem $p^2 + q^2 = o^2$ is *false*. Now compute $\cosh p \cosh q$: what do you observe?

2. Let $P = \left(\frac{1}{2}, \sqrt{\frac{5}{12}}\right)$ and $Q = \left(\frac{1}{2}, -\sqrt{\frac{5}{12}}\right)$

   (a) Compute the hyperbolic distances $d(O, P)$, $d(O, Q)$ and $d(P, Q)$, where $O$ is the origin.
   (b) Compute the angle $\measuredangle POQ$.
   (c) Show that the hyperbolic line $\ell = \overleftrightarrow{PQ}$ has equation

   $$x^2 - \frac{10}{3}x + y^2 + 1 = 0$$

   (d) Calculate $\frac{dy}{dx}$ and hence show that a tangent vector to $\ell$ at $P$ is $\sqrt{15}\mathbf{i} + 7\mathbf{j}$. Use this to compute $\measuredangle OPQ$.

3. We extend Example 5.11. Let $c \in (0, 1)$ and label $O = (0, 0)$, $P = (c, 0)$ and $Q = (0, c)$.

   (a) Compute the hyperbolic side lengths of $\triangle OPQ$.
   (b) Find the equation of the hyperbolic line joining $P = (c, 0)$ and $Q = (0, c)$.
   (c) Use implicit differentiation to prove that the interior angles at $P$ and $Q$ measure $\tan^{-1} \frac{1-c^2}{1+c^2}$. What happens as $c \to 0^+$ and as $c \to 1^-$?

4. Let $0 < r < 1$ and find the hyperbolic side lengths and interior angles of the equilateral triangle with vertices $(r, 0)$, $(-\frac{r}{2}, \frac{\sqrt{3}r}{2})$ and $(-\frac{r}{2}, -\frac{\sqrt{3}r}{2})$.What do you observe as $r \to 0^+$ and $r \to 1^-$?

5. (a) Use the cosh distance formula to prove that the hyperbolic circle of hyperbolic radius $\rho = \ln 3$ and center $C = (\frac{1}{2}, 0)$ in the Poincaré disk has *Euclidean* equation

   $$\left(x - \frac{2}{5}\right)^2 + y^2 = \frac{4}{25}$$

   (b) Prove that every hyperbolic circle in the Poincaré disk is in fact a Euclidean circle.

6. We sketch a proof of Lemma 5.12.

   (a) Prove that $f(x) = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ is strictly increasing on the interval $(1, \infty)$.
   (b) By part (a), it is enough to show that $\frac{|PQ|^2}{1-|Q|^2}$ increases as $Q$ moves away from $P$ along a hyperbolic line. Appealing to symmetry, let $P = (0, c)$ lie on the hyperbolic line with equation $x^2 + y^2 - 2by + 1 = 0$. Prove that

   $$\frac{|PQ|^2}{1 - |Q|^2} = \frac{(b - c)y + bc - 1}{1 - by}$$

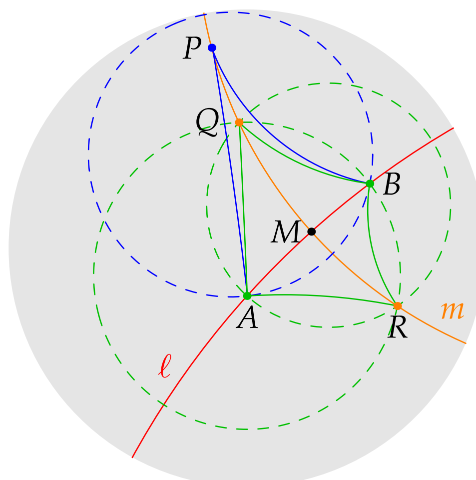
   and hence show that this is an increasing function of $y$ when $c < y < \frac{1}{b}$.

## 5.3 Parallels and Perpendiculars in Hyperbolic Geometry

From now on, all pictures and examples will be illustrated within the Poincaré disk model. Recall (page 1) that we may use anything from absolute geometry: in case you need to be convinced, here is such a result, proved in the style of Euclid but illustrated in the Poincaré disk.

**Lemma 5.14.** *Through a point $P$ not on a line $\ell$ there exists a unique perpendicular to $\ell$.*

*Proof.* Choose a point $A \in \ell$ and join $\overline{AP}$. If $\overline{AP}$ is perpendicular to $\ell$, we only need uniqueness.




Otherwise, $\ell$ is *not tangent* to the circle centered at $P$ with radius $|AP|$. It follows that there exists a second intersection point $B \in \ell$.

Construct the circles with radius $|AB|$ centered at $A$ and $B$ respectively: these have two intersections $Q, R$. Let $m = \overleftrightarrow{QR}$.

Checking the following should be an easy exercise:

- $m$ intersects $\ell$ at right-angles ($M$ in the picture)
- $P \in m$
- $M$ is the midpoint of $\overline{AB}$

To help, note that the blue and green arcs are radii of their respective circles, so we have several isosceles triangles...

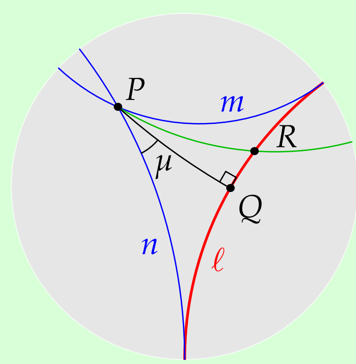
For uniqueness, suppose we have two perpendiculars to $\ell$ through $P$ intersecting $\ell$ at distinct points $M, N$. Then $\triangle PMN$ has two right-angles which contradicts Saccheri–Legendre (Theorem 5.2). ∎

### The Fundamental Theorem of Parallels in Hyperbolic Geometry

We now consider a major departure from Euclidean geometry.

**Theorem 5.15 (Fundamental Theorem of Parallels).** *Given a hyperbolic line $\ell$ and a point $P$ not on $\ell$, drop the perpendicular $\overline{PQ}$ to $\ell$.*

*There exist precisely two parallel lines $m, n$ to $\ell$ through $P$ with the following properties:*




1. *A ray based at $P$ intersects $\ell$ if and only if it lies between $m$ and $n$ in the same fashion as $\overrightarrow{PQ}$.*
2. *$m$ and $n$ make congruent acute angles $\mu$ with $\overrightarrow{PQ}$.*

**Definition 5.16.** The *limiting*, or *asymptotic*, *parallels* to $\ell$ through $P$ are the lines $m, n$. Every other parallel is termed *ultraparallel*.

The *angle of parallelism* at $P$ relative to $\ell$ is the acute angle $\mu$.

The proof depends crucially on ideas from analysis, particularly continuity & suprema.

*Proof.* The lines through $P$ are in bijective correspondence with angles in the interval $(-90°, 90°]$ measured with respect to $\overline{PQ}$. Points $R \in \ell$ are in bijective continuous correspondence with the real numbers. We therefore have a *continuous* function

$$f : \mathbb{R} \to (-90°, 90°] \quad \text{where} \quad f(r) = \measuredangle QPR$$

By the exterior angle theorem, $90° \notin \operatorname{range} f$.

Since $\operatorname{dom} f = \mathbb{R}$ is an interval, the intermediate value theorem forces $\operatorname{range} f$ to be a *subinterval* of $(-90°, 90°)$.

Transfer $\overline{QR}$ to the other side of $Q$ to produce $S \in \ell$. Applying SAS we see that $\measuredangle QPS = -\measuredangle QPR$, whence the interval $I = \operatorname{range} f$ is *symmetric*:

$$\theta \in I \iff -\theta \in I$$

Let $\mu = \sup I \leq 90°$ be the least upper bound; by symmetry, $-\mu = \inf I$ is the greatest lower bound. Let $m$ and $n$ be the lines making angles $\pm\mu$ respectively. Plainly every ray making angle $\theta \in (-\mu, \mu)$ intersects $\ell$.

Suppose $m$ intersected $\ell$ at $M$. Let $N \in \ell$ lie on the other side of $M$ from $Q$. Then $\measuredangle \overline{QPN} > \mu$ is a contradiction. It follows that $m$ is parallel to $\ell$. Similarly $n \parallel \ell$ and we have part 1.

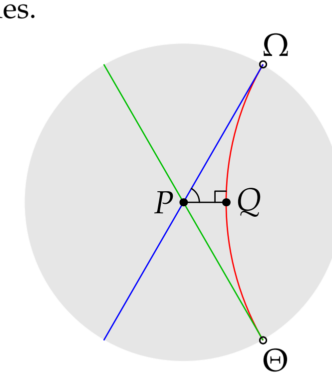
Finally $m = n \iff \mu = 90°$. In such a case there would exist only one parallel to $\ell$ through $P$, contradicting the hyperbolic postulate. ∎

Except for the last line, the proof works perfectly in absolute geometry: if we restrict to Euclidean geometry, then the 'angle of parallelism' is always 90°!

**Corollary 5.17.** *The perpendicular distance $\delta = d(P, Q)$ and the angle of parallelism are related via*

$$\cosh\delta = \csc\mu \quad \textit{or equivalently} \quad \tan\frac{\mu}{2} = e^{-\delta}$$

We postpone the proof to Exercise 5.3.3 and a discussion of omega-triangles.




**Examples 5.18.** 1. Let $\ell$ be the hyperbolic line with equation

$$x^2 + y^2 - 4x + 1 = 0 \qquad \text{(Euclidean center } (2,0) \text{ radius } \sqrt{3})$$

Its omega points are $\Omega = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\Theta = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

By symmetry, the perpendicular from $P = (0,0)$ to $\ell$ has equation $y = 0$ and results in $Q = (2 - \sqrt{3}, 0)$.

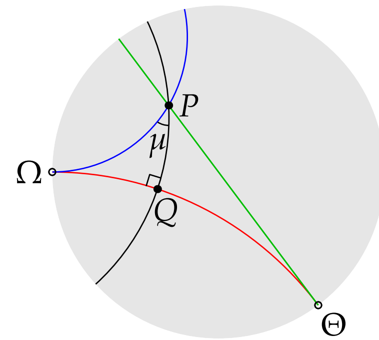
The limiting parallels clearly have equations $y = \pm\sqrt{3}x$, from which the angle of parallelism is $\mu = \tan^{-1}\sqrt{3} = 60°$.

In accordance with Corollary 5.17, we easily verify that

$$\delta = d(P, Q) = \ln\frac{1 + (2 - \sqrt{3})}{1 - (2 - \sqrt{3})} = \ln\sqrt{3} \rightsquigarrow e^{-\delta} = \frac{1}{\sqrt{3}} = \tan\frac{60°}{2}$$

2. We find the limiting parallels and the angle of parallelism when

$$P = \left(-\frac{3}{10}, \frac{4}{10}\right) \quad \text{and} \quad x^2 + y^2 + 2x + 4y + 1 = 0$$




First find the omega-points by intersecting the line with the boundary circle $x^2 + y^2 = 1$:

$$\Omega = (-1, 0), \quad \Theta = \left(\frac{3}{5}, -\frac{4}{5}\right)$$

$P$ lies on the diameter containing $\Theta$, whence $\overleftrightarrow{P\Theta}$ immediately has equation $y = -\frac{4}{3}x$.

For $\overleftrightarrow{P\Omega}$, substitute into the usual expression $x^2 + y^2 - 2ax - 2by + 1 = 0$ to obtain

$$x^2 + y^2 + 2x - \frac{13}{8}y + 1 = 0 \qquad \text{(Euclidean center } (a, b) = (-1, \tfrac{13}{16}), \text{ radius } \tfrac{13}{16})$$

Clearly $\overleftrightarrow{P\Theta}$ has slope $-\frac{4}{3}$. For $\overleftrightarrow{P\Omega}$, implicit differentiation yields a slope of

$$\frac{dy}{dx} = \frac{a - x}{y - b} = \frac{16(1 + x)}{13 - 16y} \implies \left.\frac{dy}{dx}\right|_P = \frac{16 \cdot \frac{7}{10}}{13 - \frac{64}{10}} = \frac{56}{33}$$

whence the angle of parallelism is *half* that between the vectors $\binom{-33}{-56}$ and $\binom{3}{-4}$:

$$\mu = \frac{1}{2}\cos^{-1}\frac{\binom{-33}{-56}\cdot\binom{3}{-4}}{\left|\binom{-33}{-56}\right|\left|\binom{3}{-4}\right|} = \frac{1}{2}\cos^{-1}\frac{5}{13} \approx 33.69°$$

Corollary 5.17 can now be used to find the perpendicular distance $d(P, Q) = \ln \frac{3+\sqrt{13}}{2}$.

By contrast, without the development of later machinery, it is very tricky to find the co-ordinates of $Q$. If you want a serious challenge, see if you can convince yourself that $Q = \left(\frac{93(-29+2\sqrt{117})}{1865}, \frac{26(-29+2\sqrt{117})}{1865}\right)$.

**Angles in Triangles, Rectangles and the AAA congruence**

We finish this section three important observations that follow from the Hyperbolic Postulate.

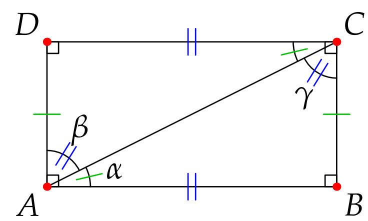
**Theorem 5.19.** *In hyperbolic geometry:*

1. *There are no rectangles (quadrilaterals with four right-angles): in particular, the summit angles of a Saccheri quadrilateral are acute.*
2. *The angles in a triangle always sum to less than 180°.*
3. *(AAA Congruence) If $\triangle ABC$ and $\triangle DEF$ have angles congruent in pairs, then their sides are congruent in pairs and so $\triangle ABC \cong \triangle DEF$.*

We leave parts 2 and 3 to the homework: both follow easily from part 1. Note particularly that AAA is a triangle *congruence* theorem in hyperbolic geometry, not a *similarity* theorem!

**Lemma 5.20.** *In absolute geometry, a diagonal splits a rectangle into congruent triangles. In particular, the opposite sides of a rectangle are congruent.*

*Proof.* Given a rectangle $ABCD$, draw a diagonal to obtain two sub-triangles as shown.




Each triangle has angle sum $\leq 180°$, yet the sum of these equals that of the rectangle, $360°$. Both triangles therefore have angle sum $180°$.

Since both triangles are right-angled, their remaining angles sum to $90°$. But the angles meeting at $A$ also sum to $90°$, whence we have congruent angles: in the language of the picture,

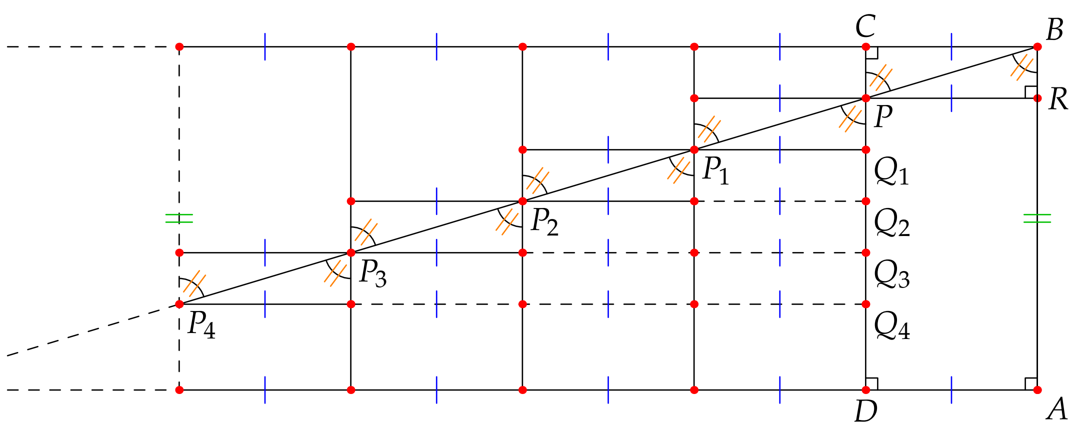
$$\alpha + \beta = 90° = \alpha + \gamma \implies \beta = \gamma$$

ASA using the diagonal as the common side proves that the triangles, and thus opposite sides of the rectangle, are congruent. ∎

We can now prove that rectangles are impossible in hyperbolic geometry. To be more precise, we prove that if a rectangle exists *within absolute geometry*, then the hyperbolic postulate is false.

*Proof of Theorem 5.19, part 1.* Given a rectangle $ABCD$, let $P \in \overline{CD}$ and drop the perpendicular from $P$ to $R \in \overline{AB}$. Clearly $PRBC$ is a rectangle, since otherwise one of $ARPD$ and $RBCP$ would have angle sum exceeding $360°$.

By Lemma 5.20, $\overrightarrow{BP}$ splits $PRBC$ so that the orange marked angles are congruent. In particular, $\overrightarrow{BP}$ crosses $\overline{CD}$ at the same angle as it leaves $B$!




Now repeat the construction to obtain a sequence of points $P, P_1, P_2, P_3, \ldots$ and congruent rectangles as indicated. The equidistant sequence of points $P, Q_1, Q_2, Q_3, \ldots$ must eventually pass $D$ since $\overline{CD}$ is finite: clearly $\overrightarrow{BP}$ intersects $\overleftrightarrow{AD}$.

Since $P$ was generic, we conclude that the angle of parallelism of $B$ with respect to $\overleftrightarrow{AD}$ is $90°$ and that the hyperbolic postulate is therefore false. There are no rectangles in hyperbolic geometry. ∎

**Exercises** 1. Prove the following in hyperbolic geometry (*use Theorem 5.19*).

(a) Two hyperbolic lines cannot have more than one common perpendicular.

(b) Saccheri quadrilaterals with congruent summits and summit angles are congruent.

2. Let $\ell$ be the line $x^2 + y^2 - 4x + 2y + 1 = 0$ and drop a perpendicular from $O$ to $Q \in \ell$.

(a) Explain why $Q$ has co-ordinates $(\frac{2}{\sqrt{5}}t, -\frac{1}{\sqrt{5}}t)$ for some $t \in (0, 1)$.

(b) Show that the hyperbolic distance $\delta = d(O, Q)$ of $\ell$ from the origin is $\ln \frac{1+\sqrt{5}}{2}$.

(c) By observing that $\Omega = (0, -1)$ is an omega-point for $\ell$, compute the angle of parallelism $\mu = \measuredangle QO\Omega$ explicitly and check that $\cosh \delta = \csc \mu$.

3. We prove a simplified version of Corollary 5.17. Let $P = (0, 0)$ be the origin, let $0 < r < 1$ and consider the hyperbolic line $\ell$ passing through $Q = (r, 0)$ at right-angles to $\overline{PQ}$.

(a) Find the equation of $\ell$ and prove that the limiting parallels of $\ell$ through $P$ have equations

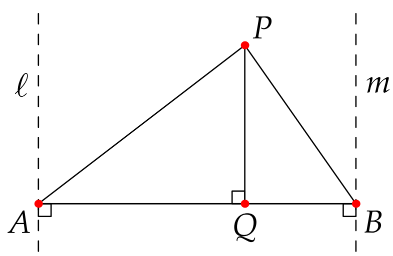
$$y = \pm \frac{1 - r^2}{2r} x$$

(*Hint: what does symmetry tell you about the location of the Euclidean center of* $\ell$*?*)

(b) Let $\mu$ be the angle of parallelism of $P$ relative to $\ell$ and $\delta = d(P, Q)$ the hyperbolic distance. Prove that $\cosh \delta = \csc \mu$.
(*Hint:* $\csc^2 \mu = 1 + \cot^2 \mu = 1 + \frac{1}{\tan^2 \mu} = \ldots$)

4. We work in *absolute geometry*.

(a) Suppose $A$, $B$ and $P$ are non-collinear and drop the perpendicular from $P$ to $Q \in \overleftrightarrow{AB}$.
If $P$ lies between the perpendiculars $\ell, m$ to $\overleftrightarrow{AB}$ through $A$ and $B$, prove that $Q$ is interior to $\overline{AB}$.
(*Hint: show that the other cases are impossible*)




(b) Suppose there exists a triangle with angle sum 180°. Show that there exists a *right-triangle* with angle sum 180° and therefore a rectangle.
*Since rectangles are impossible in hyperbolic geometry, this proves part 2 of Theorem 5.19.*

5. We prove the AAA congruence theorem (Theorem 5.19, part 3).

Suppose $\triangle ABC$ and $\triangle DEF$ are *non-congruent* but have angles congruent in pairs. WLOG assume $\overline{DE} < \overline{AB}$. By uniqueness of angle/segment transfer, there exist unique points $G \in \overline{AB}$ and $H \in \overrightarrow{AC}$ such that (SAS) $\triangle DEF \cong \triangle AGH$.

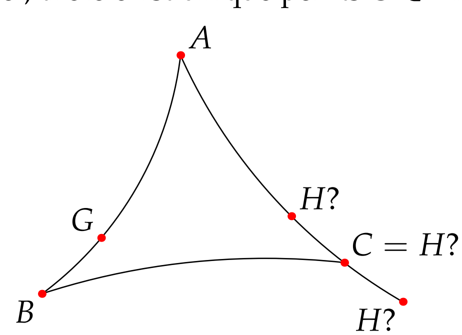
The picture shows the three possible arrangements.

(a) $H$ is interior to $\overline{AC}$.

(b) $H = C$.

(c) $C$ lies between $A$ and $H$.

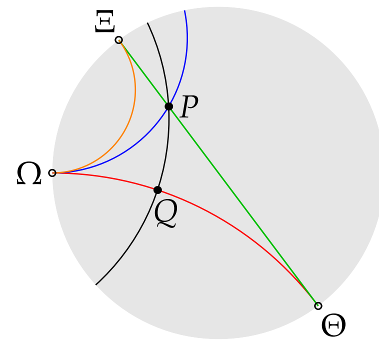
In each case, explain why we have a contradiction.



### 5.4 Omega-triangles

The concept of limiting parallels allows us to extend the notion of *triangle.*

**Definition 5.21.** An *omega-triangle* or *ideal-triangle* is a 'triangle' one or more of whose vertices is an omega-point. At least two of the sides of an omega-triangle form a pair of limiting parallels.




There are three types of omega-triangle depending on how many omega-points they contain. In the picture, $\triangle PQ\Omega$ has one omega-point, $\triangle P\Omega\Theta$ has two and $\triangle\Omega\Theta\Xi$ three!

It is perhaps surprising that many of the standard results (exterior angle and congruence theorems) also apply to omega-triangles!

The first can be thought of as the AAA congruence theorem where one 'angle' is zero.

**Theorem 5.22 (Angle-Angle Congruence for Omega-triangles).** *Suppose $\triangle PQ\Omega$ and $\triangle RS\Theta$ are omega-triangles with a single omega-point. If the the angles are congruent in pairs*

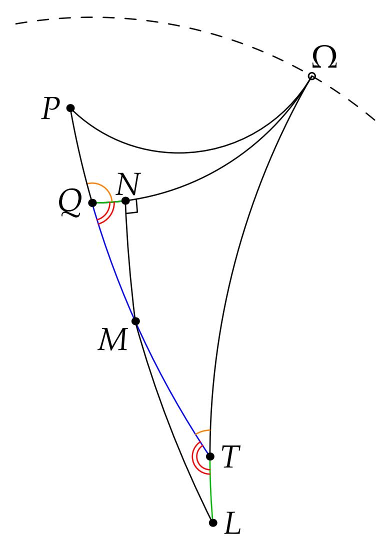
$$\angle PQ\Omega \cong \angle RS\Theta \qquad \angle QP\Omega \cong \angle SR\Theta$$

*then the finite sides of each triangle are also congruent: $\overline{PQ} \cong \overline{RS}$.*

It doesn't really make sense to speak of the 'infinite' sides, or the 'angles' at omega points, being congruent. If one defines congruence in terms of isometries (later), then the claim is more reasonable.

*Proof.* Transfer $\angle SR\Theta$ to $P$ and choose $T \in \overrightarrow{PQ}$ such that $\overline{PT} \cong \overline{RS}$. If $T = Q$ we are done.

Otherwise, first assume $\overline{PQ} < \overline{PT}$ as in the picture. The hypothesis states that the marked orange angles at $Q$ and $T$ are congruent.




Let $M$ be the midpoint of $\overline{QT}$ and drop the perpendicular to $\overleftrightarrow{Q\Omega}$ at $N$.

Choose $L \in \overleftrightarrow{\Omega T}$ on the opposite side of $\overleftrightarrow{QT}$ to $N$ such that $\overline{TL} \cong \overline{NQ}$.

The red angles are congruent, as are the pairs of green and blue lines: SAS says $\triangle MQN \cong \triangle MTL$, whence $M$ lies on $\overline{LN}$ and we have a right-angle(!) at $L$.

The angle of parallelism of $L$ relative to $\overleftrightarrow{QN}$ is now 90°: contradiction.

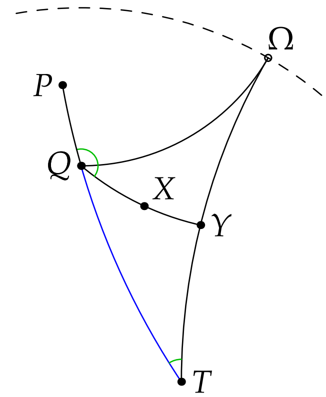
There are several other possible orientations:

- $T$ could lie on the same side of $Q$ as $P$ but the resulting argument is the same after reversing the roles of $Q$ and $T$.
- $N$ could lie on the opposite side of $Q$ from $\Omega$. In this case SAS is applied to the same triangles but with respect to the congruent *orange* angles.
- In the special case that $N = Q$, the orange angles are right-angles and the same contradiction appears.

■

**Theorem 5.23 (Exterior Angle Theorem for Omega-Triangles).** *Suppose $\triangle QT\Omega$ has a single omega-point. Extend $\overline{TQ}$ to $P$. Then $\angle PQ\Omega > \angle QT\Omega$.*

*Proof.* We show that the other cases are impossible.




To see that $\angle PQ\Omega$ and $\angle QT\Omega$ cannot be congruent, consider the picture in the proof of the AA congruence theorem. The orange angles cannot be congruent since the entire picture is a contradiction!

If $\angle PQ\Omega < \angle QT\Omega$, then we have the picture on the right. The goal is to create a triangle contradicting the usual exterior angle theorem.

Transfer $\angle QT\Omega$ to $Q$ to obtain $\overrightarrow{QX}$ *interior* to $\angle TQ\Omega$.

Since $\overleftrightarrow{Q\Omega}$ is a limiting parallel to $\overleftrightarrow{T\Omega}$, the Fundamental Theorem says that $\overrightarrow{QX}$ intersects $\overline{T\Omega}$ at a point $Y$.

But now $\triangle QTY$ contradicts the standard exterior angle theorem. ■

The final congruence theorem is an exercise based on the previous picture.

**Corollary 5.24 (Side-Angle Congruence for Omega-triangles).** *Suppose $\triangle QT\Omega$ and $\triangle RS\Theta$ have a single omega-point. If $\angle QT\Omega \cong \angle RS\Theta$ and $\overline{QT} \cong \overline{RS}$ then $\angle TQ\Omega \cong \angle SR\Theta$.*

A triangle with one omega-point only has three pieces of data: two finite angles and one finite edge. The AA and SA congruence theorems say that two of these determine the third.

### Other observations

*Pasch's Axiom*: Versions of this are *theorems* for omega-triangles.

- If a line crosses a side of an omega-triangle and does not pass through any vertex (including $\Omega$), then it must pass through exactly one of the other sides.
- If a line passes through an interior point and exactly one vertex (including $\Omega$) of an omega-triangle, then it passes through the opposite side. This is partly embedded in the proof of Theorem 5.23.

*Perpendicular Distance and the Angle of Parallelism*: Applied to right-angled omega-triangles, the AA and SA congruence theorems prove that the angle of parallelism is a bijective function of the perpendicular distance. Moreover, by transferring the right-angle to the positive $x$-axis and the other vertex to the origin, we obtain the arrangment in Exercise 5.3.3: this calculation therefore completes the proof of Corollary 5.17.

**Exercises** 1. Let $\triangle PQ\Omega$ be an omega-triangle. Prove that $\angle PQ\Omega + \angle QP\Omega < 180°$

2. Suppose that $\ell$ and $m$ are limiting parallels. Explain why they cannot have a common perpendicular.

3. Prove the Side-Angle congruence theorem for omega-triangles with one omega-point.

### 5.5 Area and Angle-defect

We now discuss one of the triumphs of Johann Lambert, the astonishing fact that the sum of the angles in a hyperbolic triangle determines its area. We start with a loose axiomatization of area as a relative measure in *absolute geometry.*

*Axiom I* Two geometric figures have the same area if and only if they can be sub-divided into finitely many pairs of mutually congruent triangles.[4]

*Axiom II* The area of a triangle is positive.

*Axiom III* The area of the union of disjoint figures is the sum of the areas of the figures.

**Area determines angle-sum in *absolute* geometry**

For this we require a useful quantity.

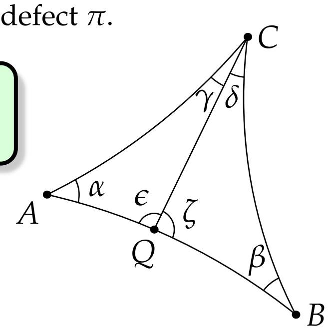
**Definition 5.25.** Let $\Sigma_\triangle$ be the sum of the angles in a triangle. Measured in radians, the *angle-defect* of $\triangle$ is $\pi - \Sigma_\triangle$.

Since triangles have angle-sum $\leq \pi$ (Saccheri-Legendre), it follows that

$$0 \leq \pi - \Sigma_\triangle \leq \pi$$

In Euclidean geometry the defect is always zero, while in hyperbolic geometry the defect is strictly positive (Theorem 5.19). A 'triangle' with three omega-points would have defect $\pi$.

**Lemma 5.26.** *Angle-defect is additive: If a triangle is split into two sub-triangles, then the defect of the whole is the sum of the defects of the parts.*




The proof should be clear from the picture:

$$[\pi - (\alpha + \gamma + \epsilon)] + [\pi - (\beta + \delta + \zeta)] = \pi - (\alpha + \beta + \gamma + \delta)$$

since $\epsilon + \zeta = \pi$. Notice that angle-*sum* is not additive!

**Theorem 5.27.** *If two triangles have the same area, then their angle-sums are identical.*

Of course this is utterly trivial in Euclidean geometry!

*Proof.* The Lemma provides the induction step: if $\triangle_1$ and $\triangle_2$ have the same area, then their interiors are disjoint unions of a finite collection of mutually congruent triangles:

$$\triangle_1 = \bigcup_{k=1}^{n} \triangle_{1,k} \quad \text{and} \quad \triangle_2 = \bigcup_{k=1}^{n} \triangle_{2,k} \quad \text{where} \quad \triangle_{1,k} \cong \triangle_{2,k}$$

Each pair $\triangle_{1,k}, \triangle_{2,k}$ has the same angle-defect, whence the angle-defects of $\triangle_1$ and $\triangle_2$ are equal:

$$\text{defect}(\triangle_1) = \sum_{k=1}^{n} \text{defect}(\triangle_{1,k}) = \sum_{k=1}^{n} \text{defect}(\triangle_{2,k}) = \text{defect}(\triangle_2)$$

■

[4] To allow infinitely many infinitessimal sub-triangles would require ideas from calculus and complexify our discussion.

### Angle-sum determines area in *hyperbolic* geometry

The converse relies on a reversible construction relating triangles and Saccheri quadrilaterals. The construction itself is valid in absolute geometry, even though the ultimate conclusion that angle-sum determines area is not.

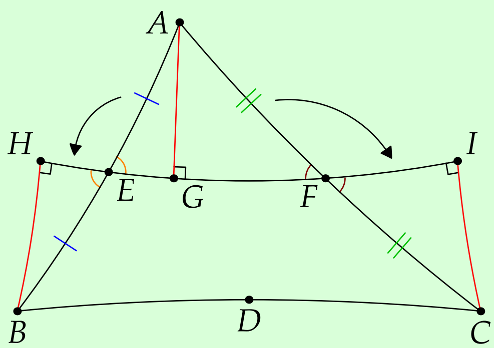
**Lemma 5.28.** 1. *Given $\triangle ABC$, choose a side $\overline{BC}$. Bisect the remaining sides at $E, F$ and drop perpendiculars from $A$, $B$ ,$C$ to $\overleftrightarrow{EF}$. Then $HICB$ is a Saccheri quadrilateral with base $\overline{HI}$.*

2. *Conversely, given a Saccheri quadrilateral $HICB$ with summit $\overline{BC}$, let $A$ be any point such that $\overleftrightarrow{HI}$ bisects $\overline{AB}$ at $E$. Then the intersection $F = \overleftrightarrow{HI} \cap \overline{AC}$ is the midpoint of $\overline{AC}$.*

*Both constructions yield the same picture and the following conclusions:*

- *The triangle and quadrilateral have equal area.*
- *The sum of the summit angles of the quadrilateral equals the angle sum of the triangle.*

*We've chosen $\overline{BC}$ to be the longest side of $\triangle ABC$; this isn't required, though it helpfully forces $E, F$ to lie between $H, I$.*



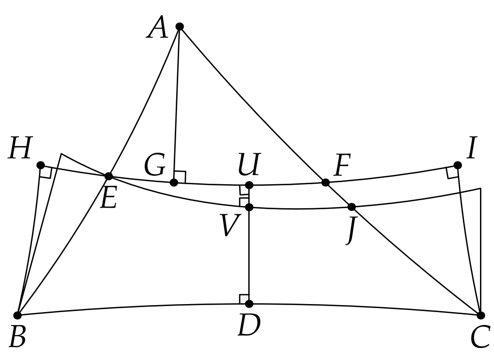

*Proof.* 1. By two applications of the SAA congruence theorem (follow the arrows...)

$$\triangle BEH \cong \triangle AEG \quad \text{and} \quad \triangle CFI \cong \triangle AFG$$

We conclude that $\overline{BH} \cong \overline{AG} \cong \overline{CI}$ whence $HICB$ is a Saccheri quadrilateral. The area and angle-sum correspondences are immediate from the picture.

2. Suppose the midpoint were at $J \neq F$. By part 1, we may create a new Saccheri quadrilateral with base $\overline{BC}$ using the midpoints $E, J$.

   The perpendicular bisector of $\overline{BC}$ (at $D$) bisects the bases of both Saccheri quadrilaterals perpendicularly, creating $\triangle EUV$ with two right-angles: contradiction.




∎

We now prove a special case of the main result.

**Lemma 5.29.** *Suppose* hyperbolic *triangles $\triangle ABC$ and $\triangle PQR$ have congruent sides $\overline{BC} \cong \overline{QR}$ and the same angle-sum. Then the triangles have the same area.*

*Proof.* Construct the quadrilaterals corresponding to $\triangle ABC$ and $\triangle PQR$ with summits $\overline{BC} \cong \overline{QR}$. These have congruent summits *and* summit angles: by Exercise 5.3.1b they are congruent. ∎

The final observation is what makes this special to *hyperbolic* geometry. In the Euclidean case, Saccheri quadrilaterals are *rectangles*: congruent summits do not force congruence of the remaining sides.

**Theorem 5.30.** *In hyperbolic geometry, if $\triangle ABC$ and $\triangle PQR$ have the same angle-sum then they have the same area.*

*Proof.* If the triangles have a congruent pair of edges, we are done by the previous result. Otherwise, we create a new triangle $\triangle LBC$ which matches the same Saccheri quadrilateral as $\triangle ABC$.

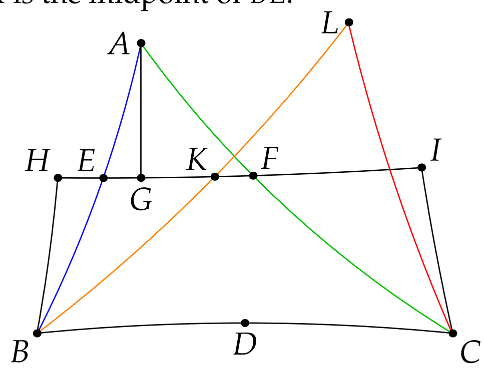
Otherwise, WLOG suppose $|AB| < |PQ|$ and construct the Saccheri quadrilateral with summit $\overline{BC}$. Select $K$ on $\overleftrightarrow{EF}$ such that $|BK| = \frac{1}{2}|PQ|$ and extend such that $K$ is the midpoint of $\overline{BL}$.

- By Lemma 5.28,

  $$\text{Area}(\triangle LBC) = \text{Area}(HICB) = \text{Area}(\triangle ABC)$$

- By Theorem 5.27, $\triangle LBC$ has the same angle-sum as $\triangle ABC$ and thus $\triangle PQR$.

- $\triangle LBC$ and $\triangle PQR$ share a congruent side ($\overline{LB} \cong \overline{PQ}$) and have the same angle-sum. Lemma 5.29 says their areas are equal.




∎

Since both area and angle-*defect* are additive, we immediately conclude:

**Corollary 5.31.** *The angle-defect of a hyperbolic triangle is an additive function of its area: by normalizing the definition of area,[a] we conclude that*

$$\pi - \Sigma_\triangle = \text{Area}(\triangle)$$

[a] We have strictly only proved that $\pi - \Sigma_\triangle = k \cdot \text{Area}(\triangle)$ for some positive constant $k$. However, it can be shown that $k = 1$ if we choose the area measure arising naturally from the hyperbolic distance function. In fact this is a special case of the famous Gauss–Bonnet theorem: for any triangle on a surface with Gauss curvature $K$, we have

$$\Sigma_\triangle - \pi = \iint_\triangle K \, \mathrm{d}A$$

The three special constant-curvature examples of this result are:

*Euclidean space* This is *flat* ($K = 0$): the angle-defect is always zero.

*Hyperbolic space* This has *constant negative curvature* $K = -1$: the area $\iint_\triangle \mathrm{d}A$ is precisely the angle-defect $\pi - \Sigma_\triangle$.

*Spherical geometry* A sphere of radius 1 has *constant positive curvature* $K = 1$: the area of a triangle is its angle-*excess* $\Sigma_\triangle - \pi$.

A full discussion of this result, of Gauss curvature and what is meant by 'arising naturally' are properly subjects for a (long!) course in differential geometry.

Note finally how the Angle-Angle-Angle congruence (Theorem 5.19, part 3) is related to the corollary:

$$\begin{array}{ccc} \triangle ABC \cong \triangle DEF & \overset{\text{AAA}}{\Longleftrightarrow} & \text{angles congruent in pairs} \\ \Downarrow & & \Downarrow \\ \text{equal area} & \overset{\text{Cor}}{\Longleftrightarrow} & \text{same angle-defect} \end{array}$$

**Example 5.32.** Recall the isosceles right-triangle from Example 5.11 with vertices $O$, $P = (\frac{1}{2}, 0)$ and $Q = (0, \frac{1}{2})$. Its angle-sum and area are

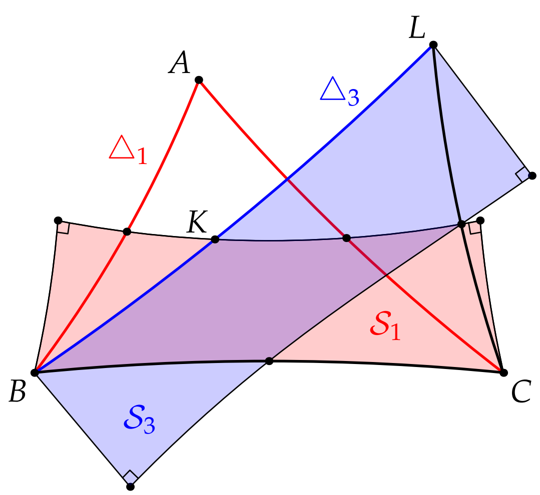
$$\frac{\pi}{2} + 2\tan^{-1}\frac{3}{5} \approx 151.93^\circ \implies \text{area} = \pi - \left(\frac{\pi}{2} + 2\tan^{-1}\frac{3}{5}\right) = \frac{\pi}{2} - 2\tan^{-1}\frac{3}{5} \approx 0.490$$

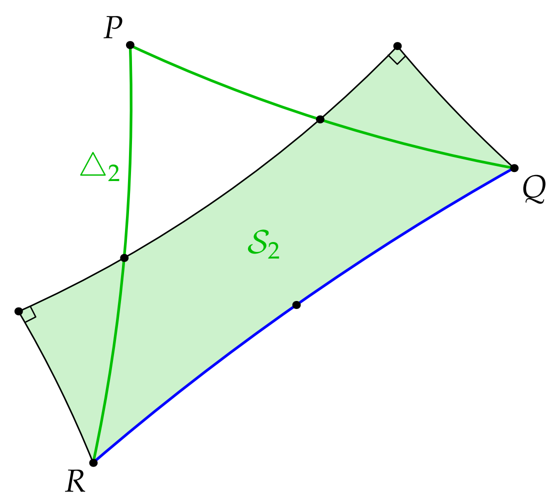
Note that a Euclidean triangle with the same vertices has area $\frac{1}{8} = 0.125$.
Generalizing this (see Exercise 5.2.3), the triangle with vertices $O, P = (c, 0)$ and $Q = (0, c)$ has area

$$\pi - \left(\frac{\pi}{2} + 2\tan^{-1}\frac{1-c^2}{1+c^2}\right) = \frac{\pi}{2} - 2\tan^{-1}\frac{1-c^2}{1+c^2}$$

As expected, $\lim\limits_{c\to 0^+} \text{area}(c) = 0$. In the other limit, the triangle becomes an omega-triangle with two omega-points and we have $\lim\limits_{c\to 1^-} \text{area}(c) = \frac{\pi}{2}$: an infinitely large triangle with finite 'area'!

Our discussion in fact provides an explicit method for cutting a triangle into subtriangles and rearranging its pieces to create a triangle with equal area.




Suppose $\triangle_1$ and $\triangle_2$ have equal area. Let $L, K$ be chosen as in the proof of Theorem 5.30 so that $\overline{QR} \cong \overline{BL}$. We now have:

- $\triangle_1, \triangle_2, \triangle_3, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ have the same area.
- The summit angles of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are congruent (half the angle sum of each triangle).
- $\mathcal{S}_2, \mathcal{S}_3$ are *congruent* since they share a summit and have congruent summit angles.

We can now follow the steps in Lemma 5.28 to transform $\triangle_1$ to $\triangle_2$:

$$\triangle_1 \to \mathcal{S}_1 \to \triangle_3 \to \mathcal{S}_3 \cong \mathcal{S}_2 \to \triangle_2$$

where each arrow represents cutting off two triangles and moving them. Indeed this works even for triangles in Euclidean geometry: try it!

**Exercises** 1. Use Corollary 5.31 to find the area of the hyperbolic triangle with given vertices.

   (a) $O = (0,0)$, $P = (\frac{1}{2}, \sqrt{\frac{5}{12}})$ and $Q = (\frac{1}{2}, -\sqrt{\frac{5}{12}})$.
   (*Hint: you should already know the angles from previous exercises!*)
   (b) $O = (0,0)$, $P = (\frac{1}{4}, 0)$, $Q = (0, \frac{1}{2})$.
   (c) $P = (r, 0)$, $Q = \left(-\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$, $R = \left(-\frac{r}{2}, -\frac{\sqrt{3}r}{2}\right)$ where $0 < r < 1$.

2. In the proof of Theorem 5.30, explain why we can find $K$ such that $|BK| = \frac{1}{2}|PQ|$.

3. Show that there is no finite triangle in hyperbolic geometry that achieves the maximum area bound $\pi$.

   (Hard!) For a challenge, try to prove that all omega-triangles also satisfy the angle-defect formula: Area $= \pi - \Sigma$, so that only triangles with three omega-points have maximum area.

4. Let $\Omega_1, \ldots, \Omega_n$ be $n$ distinct omega-points arranged counter-clockwise around the boundary circle of the Poincaré disk. A region is bounded by the $n$ hyperbolic lines

   $$\overleftrightarrow{\Omega_1\Omega_2}, \quad \overleftrightarrow{\Omega_2\Omega_3}, \quad \ldots, \quad \overleftrightarrow{\Omega_n\Omega_1}$$

   What is the area of the region? Hence argue that the 'area' of hyperbolic space is infinite.

5. Suppose that an omega-triangle is drawn with vertices at $O = (0,0)$, $\Omega = (1,0)$ and $P = (0,h)$ where $h > 0$.

   (a) Prove that the hyperbolic segment $\overline{P\Omega}$ is an arc of a circle with equation

   $$(x-1)^2 + (y-k)^2 = k^2$$

   for some $k > 0$.

   (b) Prove that the area of $\triangle OP\Omega$ is given by

   $$A(h) = \sin^{-1}\frac{2h}{1+h^2}$$

### 5.6 Isometries and Calculation

There are (at least!) two major issues in our approach to hyperbolic geometry.

*Calculations are difficult* In analytic (Euclidean) geometry we typically choose the origin and orientation of axes to ease calculation. We'd like to do the same in hyperbolic geometry.

*We assumed too much* We defined *distance, angle* and *line* separately, yet these concepts are *not independent*! In Euclidean geometry, the distance function, or *metric,* defines angle measure via the dot product,[5] and (with some calculus) the arc-length of any curve. One *proves* that the paths of shortest length (*geodesics*) are straight lines: the metric therefore *defines* the notion of line!

There is a related remedy for these issues: *isometries*, the rigid motions of the Poincaré disk. To work with these we make an alternative definition within the complex plane.

**Definition 5.33.** The *Poincaré disk* is the set $D := \{z \in \mathbb{C} : |z| < 1\}$ equipped with the distance function

$$d(z,w) := \left| \ln \frac{|z-\Omega|\,|w-\Theta|}{|z-\Theta|\,|w-\Omega|} \right|$$

where $\Omega, \Theta$ are the omega-points for the hyperbolic line through $z, w$ (defined via circles).

We'll see later that this is the same distance given by the usual cosh formula: it is already easy to see that $d(z,0) = \ln \frac{1+|z|}{1-|z|}$ as in Lemma 5.10. For the present, we state some facts from complex analysis.

**Definition 5.34.** A *Möbius (fractional-linear) transformation* is a function of the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

**Theorem 5.35.** *A Möbius transformation $f(z) = \frac{az+b}{cz+d}$ has the following properties:*

1. *(Invertibility)* $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ *is bijective, with inverse* $f^{-1}(z) = \frac{dz-b}{-cz+a}$.
2. *(Conformality)* *If two curves in $\mathbb{C}$ intersect at an angle, then the images of these curves under $f$ intersect at the same angle.*
3. *(Line/circle preservation)* *Every line/circle[a] is mapped to another line/circle.*
4. *(Cross-ratio)* *For any four distinct points $z_1, z_2, z_3, z_4$, we have*

$$\frac{\big(f(z_1)-f(z_2)\big)\big(f(z_3)-f(z_4)\big)}{\big(f(z_2)-f(z_3)\big)\big(f(z_4)-f(z_1)\big)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

[a] In $\mathbb{C} \cup \{\infty\}$ a line is just a circle containing $\infty$...

[5] Writing $|\mathbf{u}| = |PQ|$ for the length of a line segment, we see that for any $\mathbf{u}, \mathbf{v}$,

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\left(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2\right)$$

so that the metric defines the dot product. Now define angle measure via $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$.

The isometries of the Poincaré disk are built from a subset of the Möbius transformations.

**Theorem 5.36.** *The* orientation-preserving[a] isometries *of the Poincaré disk have the form*

$$f(z) = e^{i\theta}\frac{\alpha - z}{\overline{\alpha}z - 1} \quad \text{where } |\alpha| < 1 \text{ and } \theta \in [0, 2\pi)$$

*All isometries can be found by composing $f$ with complex conjugation (reflection in the real axis).*

[a] If $C$ is to the left of $\overrightarrow{AB}$, then $f(C)$ is to the left of $\overrightarrow{f(A)f(B)}$. This is the usual 'right-hand rule.'

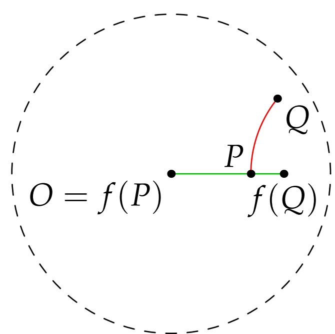
Referring to the properties in Theorem 5.35:

1. The isometries are precisely the set of Möbius transformations which map $D$ bijectively to itself.
2. Preservation of angles is required for a rigid motion.
3. The unit circle is preserved, so that omega-points are mapped to omega-points. Moreover, the class of hyperbolic lines is preserved: any circle or line intersecting the unit circle at right-angles is mapped to another such (angle-preservation is used here).
4. Let $\Omega$ and $\Theta$ be the omega-points on the hyperbolic line joining $z, w \in D$. By properties 2 and 3, we see that $f(\Omega)$ and $f(\Theta)$ are the omega-points on the hyperbolic line through $f(z), f(w)$. Preservation of the cross-ratio shows that $f$ is an isometry:

$$d(f(z), f(w)) = \left| \ln \frac{|f(z) - f(\Omega)|\,|f(w) - f(\Theta)|}{|f(z) - f(\Theta)|\,|f(w) - f(\Omega)|} \right| = \left| \ln \frac{|z - \Omega|\,|w - \Theta|}{|z - \Theta|\,|w - \Omega|} \right| = d(z, w)$$

How does this help us compute? The isometry $f$ moves $\alpha$ to the origin; one can then choose $\theta$ to orient whichever direction you like along the positive $x$-axis.

**Example 5.37.** Let $P = \frac{1}{2}$ and $Q = \frac{2}{3} + \frac{\sqrt{2}}{3}i$. Move $P$ to the origin using an isometry with $\alpha = P$:




$$f(z) = e^{i\theta}\frac{\alpha - z}{\overline{\alpha}z - 1} = e^{i\theta}\frac{1 - 2z}{z - 2} \implies f(P) = O$$

$$f(Q) = e^{i\theta}\frac{1 - \frac{4}{3} - \frac{2\sqrt{2}}{3}i}{\frac{2}{3} - 2 + \frac{\sqrt{2}}{3}i} = -\frac{1 + 2\sqrt{2}i}{-4 + \sqrt{2}i}e^{i\theta} = \frac{i}{\sqrt{2}}e^{i\theta}$$

Choosing $e^{i\theta} = -i$ places $f(Q) = \frac{1}{\sqrt{2}}$ on the positive $x$-axis. Now check:

$$d(P, Q) = \cosh^{-1}\left(1 + \frac{2\,|PQ|^2}{(1 - |P|^2)(1 - |Q|^2)}\right) = \cosh^{-1}\left(1 + \frac{\frac{2}{4}}{(1 - \frac{1}{4})(1 - \frac{2}{3})}\right)$$
$$= \cosh^{-1} 3 = \ln(3 + 2\sqrt{2})$$
$$d(f(P), f(Q)) = \ln\frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \ln\frac{\sqrt{2} + 1}{\sqrt{2} - 1} = \ln(3 + 2\sqrt{2})$$

The points really are the same distance apart! Indeed the hyperbolic segment $\overline{PQ}$ (with equation $x^2 + y^2 - \frac{5}{2}x + 1 = 0$) is transformed by $f$ to a segment $\overline{f(P)f(Q)}$ of the $x$-axis.

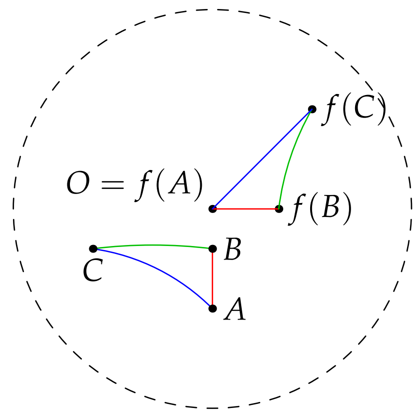
Recall (e.g. Example 5.11) how we previously computed angles. Isometries make this *much* easier.

**Example 5.38.** Given $A = -\frac{i}{2}$, $B = -\frac{i}{5}$ and $C = -\frac{1}{5}(3+i)$, we find $d(A,B)$, $d(A,C)$ and $\measuredangle BAC$. Start by moving $A$ to the origin and consider $f(B)$:

$$f(z) = e^{i\theta}\frac{-\frac{i}{2}-z}{\frac{i}{2}z-1} = \frac{2z+i}{2-iz}e^{i\theta}, \qquad f(B) = \frac{-\frac{2i}{5}+i}{2-\frac{1}{5}}e^{i\theta} = \frac{i}{3}e^{i\theta} = \frac{1}{3}$$

It is unnecessary, but $e^{i\theta} = -i$ forces $f(B)$ onto the positive $x$-axis. Our isometry is therefore

$$f(z) = \frac{2z+i}{z+2i} \implies f(C) = \frac{-\frac{2}{5}(3+i)+i}{-\frac{1}{5}(3+i)+2i} = \frac{1+i}{2}$$




By mapping $A$ to the origin, two sides of the triangle are now Euclidean straight lines and the computations are easy:

$$d(A,B) = d(O,f(B)) = \ln\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \ln 2$$

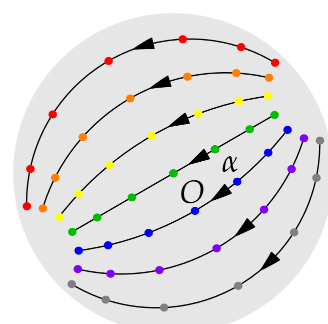
$$d(A,C) = d(O,f(C)) = \ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = 2\ln(\sqrt{2}+1)$$

$$\measuredangle BAC = \arg\left(\frac{1+i}{2}\right) = \frac{\pi}{4}$$

The picture hopefully makes clear the meaning of *orientation-preserving*: $\triangle ABC$ is rotated and translated to obtain $\triangle f(A)f(B)f(C)$, but not reflected.

**Interpretation of Isometries (non-examinable)** Following Euclidean geometry, we want to interpret isometries as rotations, reflections and translations. Here is the dictionary in hyperbolic space.

*Translation* of $\alpha$ to the origin is accomplished by $T_\alpha(z) = \frac{\alpha-z}{\overline{\alpha}z-1}$
The picture shows repeated applications of $T_\alpha$ to seven colored points.
To translate $\alpha$ to $\beta$, do a composition!

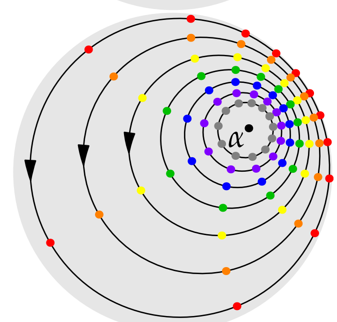



$$T_{-\beta}\circ T_\alpha(z) = \frac{(\overline{\alpha}\beta-1)z-\alpha+\beta}{(\overline{\alpha}-\overline{\beta})z+\alpha\overline{\beta}-1}$$

*Rotations* $R_\theta(z) = e^{i\theta}z$ rotates counter-clockwise around the origin. To rotate around $\alpha$, one computes the composition

$$T_{-\alpha}\circ R_\theta \circ T_\alpha$$

The picture shows repeated rotation by $30° = \frac{\pi}{6}$ around $\alpha$.




*Reflection* $P_\theta(z) = e^{2i\theta}\overline{z}$ reflects across the line making angle $\theta$ with the real axis. Composition permits more general reflections

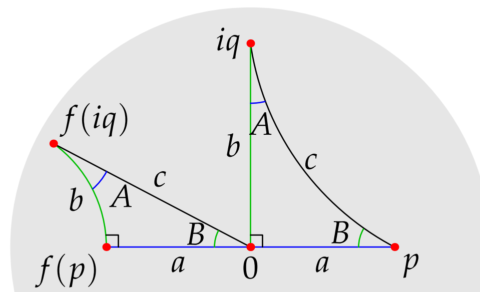
$$T_{-\alpha}\circ P_\theta \circ T_\alpha$$

**Hyperbolic Trigonometry**

The goal of trigonometry is to 'solve' triangles: given minimal numerical data, we compute the remaining side lengths and angle measures. In hyperbolic geometry, the triangle congruence theorems (SAS, ASA, SSS, SAA *and* AAA) provide suitable minimal data.

Given a right triangle, we may apply an isometry to move the right-angle to the origin and the non-hypotenuse sides to the positive axes. The non-hypotenuse side lengths are

$$a = \cosh^{-1}\left(\frac{1+p^2}{1-p^2}\right), \qquad b = \cosh^{-1}\left(\frac{1+q^2}{1-q^2}\right)$$




To measure the hypotenuse, first translate $p$ to the origin

$$f(z) = \frac{p-z}{pz-1}$$

But then

$$f(iq) = \frac{p-iq}{ipq-1} = \frac{-p(1+q^2)+iq(1-p^2)}{p^2q^2+1} \implies \tan B = \frac{q(1-p^2)}{p(1+q^2)} \text{ and } |f(iq)|^2 = \frac{p^2+q^2}{p^2q^2+1}$$

We therefore see that

$$\cosh c = \frac{1+|f(iq)|^2}{1-|f(iq)|^2} = \frac{1+p^2+q^2+p^2q^2}{1-p^2-q^2+p^2q^2} = \frac{1+p^2}{1-p^2}\cdot\frac{1+q^2}{1-q^2} = \cosh a\cosh b$$

Moreover, applying the hyperbolic identity $\sinh^2 b = \cosh^2 b - 1$, we obtain

$$\sinh b = \frac{2q}{1-q^2} \implies \tanh b = \frac{\sinh b}{\cosh b} = \frac{2q}{1+q^2}$$

Applying standard trig identities such as $\sec^2 B = 1+\tan^2 B$, we finally conclude:

**Theorem 5.39.** *In a hyperbolic right-triangle with adjacent $a$, opposite $b$, and hypotenuse $c$,*

$$\cos B = \frac{\tanh a}{\tanh c} \qquad \sin B = \frac{\sinh b}{\sinh c} \qquad \tan B = \frac{\tanh b}{\sinh a} \qquad \cosh c = \cosh a\cosh b$$

The final expression is Pythagoras' Theorem for hyperbolic triangles.

**Corollary 5.40 (Hyperbolic Cosine Rule).** *Apply the above argument to a triangle with vertices $0, p, qe^{i\theta}$ to obtain*

$$\cosh c = \cosh a\cosh b - \sinh a\sinh b\cos C$$

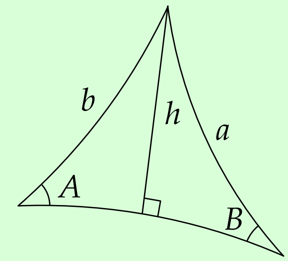
*where* $\cosh a = \frac{1+p^2}{1-p^2}$, $\sinh a = \sqrt{\cosh^2 a - 1} = \frac{2p}{1-p^2}$, *etc.*

*Expressing the right-hand side of this in terms of $p, q$ and applying the Euclidean cosine rule yields our original cosh-formula for distance (page 5).*

**Corollary 5.41 (Hyperbolic Sine Rule).** *In any hyperbolic triangle,*

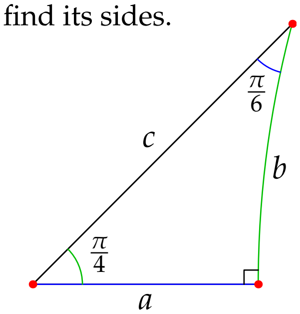
$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$$

*The picture shows the generic situation: simply apply Theorem 5.39 and equate the* $\sinh h$*-terms...*




Armed with these results, one can solve any hyperbolic triangle numerically given the information in one of the triangle congruence theorems. Admittedly some (particularly ASA and the general AAA) are very messy to compute with, but others are straightforward.

**Examples 5.42.** 1. (right-angled AAA) A triangle has angles $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$: find its sides.




Using the tan-formula,

$$\frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} = \frac{\tanh b}{\sinh a} = \frac{\sinh b}{\sinh a \cosh b}$$
$$1 = \tan\frac{\pi}{4} = \frac{\tanh a}{\sinh b} = \frac{\sinh a}{\cosh a \sinh b}$$

Now multiply together and use hyperbolic Pythagoras,

$$\frac{1}{\sqrt{3}} = \frac{1}{\cosh a \cosh b} = \frac{1}{\cosh c} \implies c = \cosh^{-1}\sqrt{3} = \ln(\sqrt{3}+\sqrt{2}) \approx 1.1462$$

We quickly see that $\sinh c = \sqrt{\cosh^2 c - 1} = \sqrt{2}$, whence the sine-rule yields the other sides:

$$\sinh b = \sin\frac{\pi}{4} \cdot \frac{\sinh c}{\sin \pi} = 1 \implies b = \sinh^{-1} 1 = \cosh^{-1}\sqrt{2} \approx 0.8814$$
$$\implies \cosh a = \frac{\cosh c}{\cosh b} = \sqrt{\frac{3}{2}} \implies a \approx 0.6565$$

2. (SAS) A triangle has angle $C = \frac{\pi}{3}$ between sides $a = b = \cosh^{-1} 2$. Find the remaining data.

We have $\sinh a = \sinh b = \sqrt{\cosh^2 a - 1} = \sqrt{3}$. By the cosine rule,

$$\cosh c = 2 \cdot 2 - \sqrt{3}\sqrt{3} \cdot \frac{1}{2} = \frac{5}{2} \implies c = \cosh^{-1}\frac{5}{2}$$

Finally, apply the sine rule:

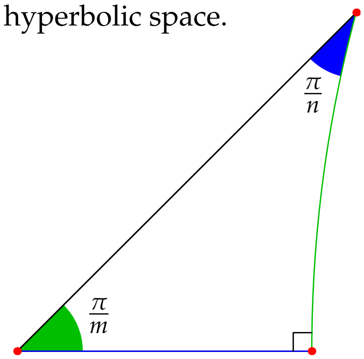
$$\sin B = \sin A = \frac{\sin C \sinh a}{\sinh c} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{\sqrt{21}/2} = \frac{3}{\sqrt{21}} = \sqrt{\frac{3}{7}}$$

The area of this triangle is therefore

$$\pi - \frac{\pi}{3} - 2\sin^{-1}\sqrt{\frac{3}{7}} \approx 0.6669$$

### Hyperbolic Tilings

The first example above can be used to make a regular tiling of hyperbolic space.

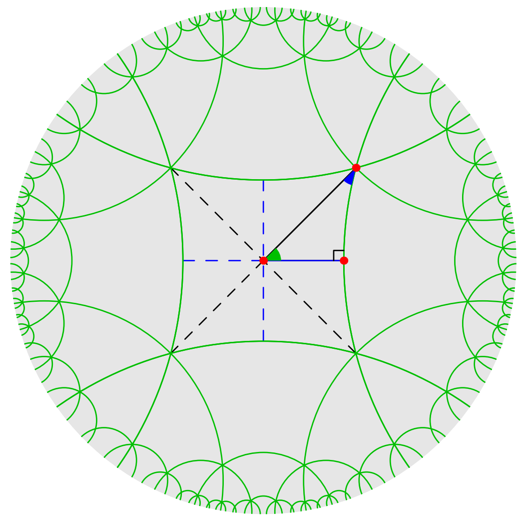



The fundamental triangle

Take eight congruent copies of the triangle and arrange them around the origin as in the picture. Now reflect the quadrilateral over each of its edges and repeat the process in all directions. We obtain a regular tiling of hyperbolic space comprising *four-sided* figures with *six* meeting at every vertex!

In hyperbolic space, many different regular tilings are possible. Suppose such is to be made using regular $m$-sided polygons, $n$ of which are to meet at each vertex: each polygon comprises $2m$ copies of the fundamental right-triangle, whose angles are therefore $\frac{\pi}{2}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$. Since the angles sum to less than $\pi$ radians, we see that there exists a regular tiling of hyperbolic space whenever $m, n$ satisfy

$$\frac{\pi}{2} + \frac{\pi}{m} + \frac{\pi}{n} < \pi \iff (m-2)(n-2) > 4$$

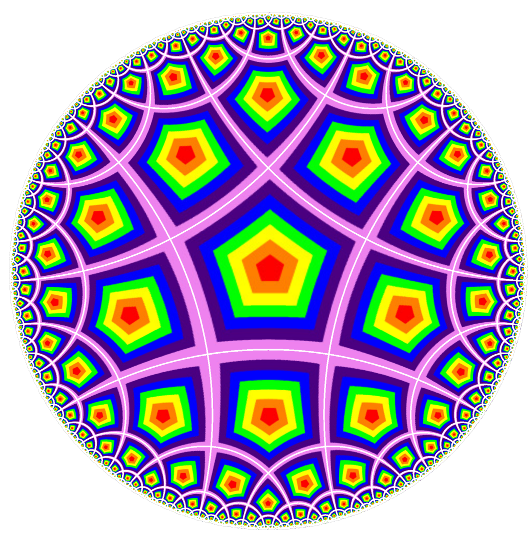


$(m, n) = (4, 6)$

The first example is $m = 4$ and $n = 6$, where the fundamental triangle is clear. In the second example four pentagons meet at each vertex and the interiors of the polygons have been colored. This was produced using the tools found here and here: have a play!

The multitude of possible tilings in hyperbolic geometry is in contrast to Euclidean geometry, where a regular tiling requires *equality*

$$(m-2)(n-2) = 4$$



$(m, n) = (5, 4)$

The three solutions $(m, n) = (3, 6), (4, 4), (6, 3)$ correspond to the only tilings of Euclidean geometry by regular polygons (equilateral triangles, squares and hexagons). However, all can be scaled to arbitrary side-lengths. In hyperbolic geometry, there are infinitely many distinct tilings, but each has a unique side-length.

For related fun, look up M.C. Escher's *Circle Limit* artworks, some of which are based on hyperbolic tilings. If you want an excuse to play video games while pretending to study geometry, have a look at Hyper Rogue, which relies on (sometimes irregular) tilings.

**Exercises** 1. Use Definition 5.33 to prove that $d(z,0) = \ln \frac{1+|z|}{1-|z|}$.

(*Hint: what are the omega-points for the line through* 0 *and* $z$*?*)

2. Use an isometry to find angle $\measuredangle ABC$ when

$$A = 0, \quad B = \frac{i}{2}, \quad C = \frac{1+i}{2}$$

3. Associate a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ with the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in an obvious way.

   If $g$ is another Möbius transformation, prove that the composition $f \circ g$ is associated to the product $AB$ of the matrices associated to $f, g$. Hence verify that $f^{-1}(z) = \frac{dz-b}{a-cz}$.

   (*Since multiplying $a, b, c, d$ by a non-zero scalar doesn't change $f$, we see that the group of Möbius transformations is isomorphic to the* projective special linear group $\mathrm{PSL}_2(\mathbb{R})$. *The isometries of hyperbolic space form a proper subgroup.*)

4. (a) A triangle has vertices $A = \frac{1}{3}$, $B = \frac{1}{2}$ and $C$, where $C$ lies in the upper half-plane (positive imaginary part) such that $\measuredangle BAC = 45°$ and $b = d(A,C) = \cosh^{-1} 3$
   Compute $a = d(B,C)$ using the hyperbolic cosine rule.

   (b) The isometry

   $$f(z) = \frac{\frac{1}{3} - z}{\frac{1}{3}z - 1} = \frac{1-3z}{z-3}$$

   moves $A$ to the origin. What is $f(B)$ and therefore $f(C)$?
   (*Hint: remember that $f$ is orientation preserving*)

   (c) Use the *inverse* of the isometry $f$ to compute the co-ordinates of $C$. As a sanity-check, use the cosh distance formula to recover your answer to part (a).

5. Use the Maclaurin series $\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots$ to multiply out the terms of the hyperbolic Pythagorean theorem $\cosh c = \cosh a \cosh b$ to order 4 (i.e. $a^4$, $a^2b^2$, etc.). What is the relationship to the Euclidean Pythagorean theorem?

6. A right hyperbolic triangle has non-hypotenuse sides $a = \cosh^{-1} 2$ and $b = \cosh^{-1} 3$. Find the hypotenuse, the angles and the area of the triangle.

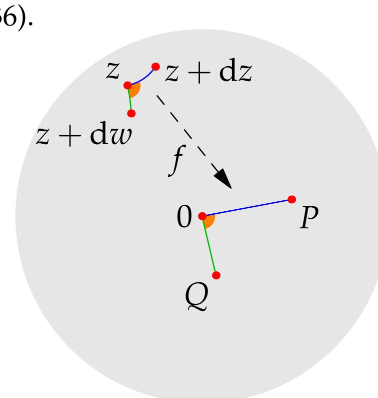
7. Use the hyperbolic cosine rule to prove that the cosh distance formula is valid.

8. An equilateral hyperbolic triangle has side-length $a$ and angle $A$. Prove that $\cos A = \frac{\cosh a}{\cosh a + 1}$. If an equilateral triangle has each angle 45°, what is its side-length?

   (*Hint: Apply the cosine rule*)

**The Poincaré Disk for Differential Geometers (non-examinable)**

This last optional section should be accessible to anyone who's taken a vector-calculus course covering line-integrals and surface area. All we really need is the Poincaré Disk model with its distance function $d(z,w)$ and description of the isometeries (Theorems 5.35, 5.36).




We first need a way to measure infinitessimal change. Consider the infinitessimally separated points $z$ and $z+\mathrm{d}z$. Use an isometry

$$f:\xi\mapsto\frac{z-\xi}{\overline{z}\xi-1}$$

to map $z$ to the origin. Then $z+\mathrm{d}z$ is mapped to

$$P:=f(z+\mathrm{d}z)=\frac{-\mathrm{d}z}{\overline{z}(z+\mathrm{d}z)-1}=\frac{\mathrm{d}z}{1-|z|^2}$$

where we deleted the $\overline{z}\,\mathrm{d}z$ term since it is infinitessimally small compared to $1-|z|^2$.

Since isometries must preserve length and angle, this construction has several consequences:

*Angle measure* If we repeat the exercise for a second infinitessimal segment $z\to z+\mathrm{d}w$, we see that the angle between the original segments is precisely that between the infinitessimal vectors $\mathrm{d}z$ and $\mathrm{d}w$. This is precisely the conformality observation in Theorem 5.35, and moreover shows how the distance function determines the angle measure.

*Infinitessimal distance and arc-length* The hyperbolic distance from $z$ to $z+\mathrm{d}z$ is

$$d(z,z+\mathrm{d}z)=d(O,P)=\ln\frac{1+|P|}{1-|P|}=\ln(1+|P|)-\ln(1-|P|)=2\,|P|=\frac{2\,|\mathrm{d}z|}{1-|z(t)|^2}$$

where the approximation $\ln(1\pm|P|)=\pm|P|$ is used since $|P|$ is infinitessimal.

If $z(t)$ parametrizes a curve in the disk, then the infinitessimal distance formula allows us to compute the arc-length

$$\int_{t_0}^{t_1}\frac{2\,|z'(t)|}{1-|z(t)|^2}\,\mathrm{d}t$$

*Area* If $\mathrm{d}x$ and $i\mathrm{d}y$ are infinitessimal horizontal and vertical changes in $z=x+iy$, then the area of the infinitessimal rectangle spanned by $z\to z+\mathrm{d}x$ and $z\to z+i\mathrm{d}y$ is the area element

$$\mathrm{d}A=\frac{2\,\mathrm{d}x}{1-|z|^2}\frac{2\,\mathrm{d}y}{1-|z|^2}=\frac{4\,\mathrm{d}x\mathrm{d}y}{(1-x^2-y^2)^2}$$

The area of a region $R$ in the Poincaré disk is therefore given by the double integrals

$$\iint_R\frac{4\,\mathrm{d}x\mathrm{d}y}{(1-x^2-y^2)^2}=\iint_R\frac{4r\,\mathrm{d}r\,\mathrm{d}\theta}{(1-r^2)^2}=\iint_R\sinh\delta\,\mathrm{d}\delta\,\mathrm{d}\theta$$

where the last expression is written in polar co-ordinates using the hyperbolic distance $\delta$. In this way the measure of area also depends on the distance function.

**Example 5.43 (Circles and 'hyperbolic $\pi$').** Suppose that a circle has hyperbolic radius $\delta$. By moving its center to the origin via an isometry, we can parameterize it in the usual manner:

$$z(t) = r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad \theta\in[0,2\pi) \quad \text{where} \quad \delta = \ln\frac{1+r}{1-r} \rightsquigarrow r = \frac{e^\delta-1}{e^\delta+1}$$

Its circumference (hyperbolic arc-length) is then

$$\int_0^{2\pi}\frac{2r}{1-r^2}\,\mathrm{d}\theta = \frac{4\pi r}{1-r^2} = 2\pi\sinh\delta = 2\pi\left(\delta+\frac{1}{3!}\delta^3+\frac{1}{4!}\delta^5+\cdots\right) > 2\pi\delta$$

where we used the Maclaurin series to compare. Its area is

$$\int_0^{2\pi}\int_0^\delta \sinh\delta\,\mathrm{d}\delta\,\mathrm{d}\theta = 2\pi(\cosh\delta-1) = \pi\left(\delta^2+\frac{2}{4!}\delta^4+\frac{2}{6!}\delta^6+\cdots\right) > \pi\delta^2$$

A hyperbolic circle therefore has larger ratios of circumference : radius and area : radius$^2$ than for a Euclidean circle. Moreover, these ratios are *not constant*: one might say that the hyperbolic version of $\pi$ is a function!

**Hyperbolic Lines as Geodesics** Finally, we perform a calculus of variations argument to see that the distance function in fact *defines* our notion of a hyperbolic line.

**Definition 5.44.** A *geodesic* is a path of shortest length between two points.

**Theorem 5.45.** *The geodesics in the Poincaré disk model are precisely the hyperbolic lines.*

*Proof.* First suppose that $b$ lies on the positive $x$-axis. Parametrize a curve from 0 to $b$ via

$$z(t) = x(t)+iy(t) \quad \text{where} \quad 0\le t\le 1, \quad z(0)=0,\ z(1)=b$$

Now compute its arc-length:

$$\int_0^1\frac{2\,|z'(t)|}{1-|z(t)|^2}\,\mathrm{d}t = \int_0^1\frac{2\sqrt{x'^2+y'^2}}{1-x^2-y^2}\,\mathrm{d}t \ge \int_0^1\frac{2\,|x'|}{1-x^2}\,\mathrm{d}t \ge \int_0^1\frac{2x'(t)}{1-x(t)^2}\,\mathrm{d}t = \int_0^b\frac{2\,\mathrm{d}x}{1-x^2}$$
$$= \ln\frac{1+b}{1-b} = d(0,b)$$

where we have equality if and only if $y(t)\equiv 0$ and $x(t)$ is increasing. The length-minimizing path is therefore along the $x$-axis.

More generally, given points $A, B$, apply an isometry $f$ such that $f(A)=0$ and $f(B)=b$ lies on the positive $x$-axis. The geodesic from $A$ to $B$ is therefore the image of the segment $\overline{0b}$ under the inverse isometry $f^{-1}$. By the properties of Möbius transforms, this is an arc of a Euclidean circle through $A, B$ intersecting the unit circle at right-angles: our original definition of a hyperbolic line. ∎