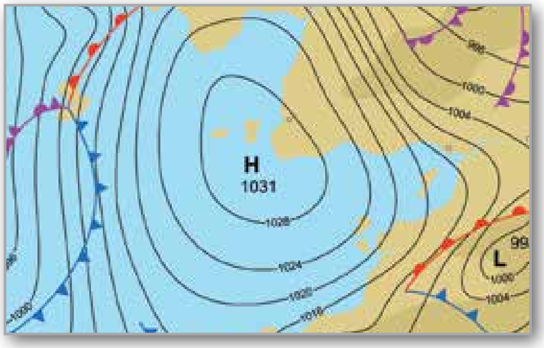
5 Logarithmic, Exponential, and Other Transcendental Functions

Breaking Strength *(Exercise 102, p. 360)*

Radioactive Half-Life Model *(Example 1, p. 352)*



Atmospheric Pressure *(Exercise 85, p. 349)*

Heat Transfer *(Exercise 93, p. 332)*

Sound Intensity *(Exercise 104, p. 323)*

5.1 The Natural Logarithmic Function: Differentiation

- **Develop and use properties of the natural logarithmic function.**
- **Understand the definition of the number *e*.**
- **Find derivatives of functions involving the natural logarithmic function.**

JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Napier coined the term *logarithm*, from the two Greek words *logos* (or ratio) and *arithmos* (or number), to describe the theory that he spent 20 years developing and that first appeared in the book *Mirifici Logarithmorum canonis descriptio* (A Description of the Marvelous Rule of Logarithms). Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

See LarsonCalculus.com to read more of this biography.

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it does not apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, you will use the Second Fundamental Theorem of Calculus to *define* such a function. This antiderivative is a function that you have not encountered previously in the text. It is neither algebraic nor trigonometric but falls into a new class of functions called *logarithmic functions.* This particular function is the **natural logarithmic function.**

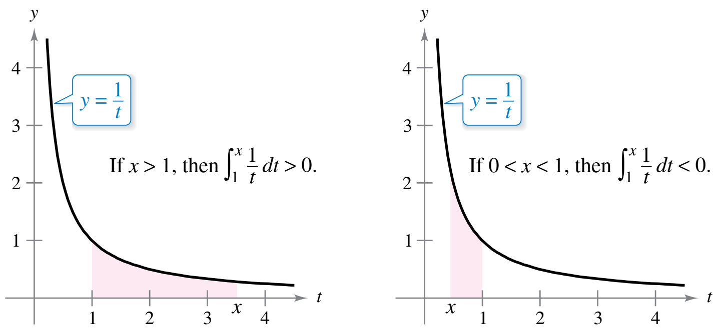
Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln 1 = 0$, because the upper and lower limits of integration are equal when $x = 1$.



If $x > 1$, then $\ln x > 0$. If $0 < x < 1$, then $\ln x < 0$.
Figure 5.1

Exploration

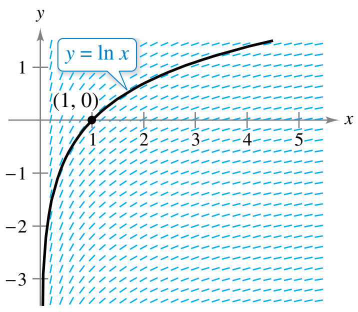
Graphing the Natural Logarithmic Function Using *only* the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.

The Granger Collection, NYC

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

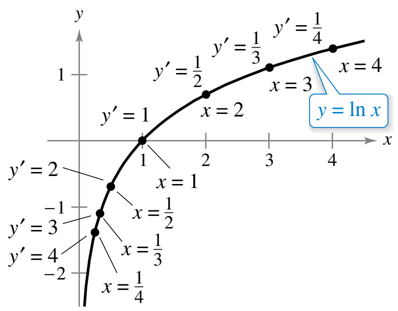
Figure 5.2 is a computer-generated graph, called a *slope field (or direction field)*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$. (You will study slope fields in Section 6.1.)



Each small line segment has a slope of $\frac{1}{x}$.

Figure 5.2

> **THEOREM 5.1 Properties of the Natural Logarithmic Function**
>
> The natural logarithmic function has the following properties.
>
> **1.** The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
>
> **2.** The function is continuous, increasing, and one-to-one.
>
> **3.** The graph is concave downward.



The natural logarithmic function is increasing, and its graph is concave downward.

Figure 5.3

Proof The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

$$f'(x) = \frac{1}{x} \qquad \text{First derivative}$$

is positive for $x > 0$, as shown in Figure 5.3. It is concave downward because its second derivative

$$f''(x) = -\frac{1}{x^2} \qquad \text{Second derivative}$$

is negative for $x > 0$. The proof that f is one-to-one is given in Appendix A. The following limits imply that its range is the entire real number line.

$$\lim_{x\to 0^+} \ln x = -\infty$$

and

$$\lim_{x\to\infty} \ln x = \infty$$

Verification of these two limits is given in Appendix A. ∎

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that the properties listed on the next page are characteristic of all logarithms.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln 1 = 0$

2. $\ln(ab) = \ln a + \ln b$

3. $\ln(a^n) = n \ln a$

4. $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$

Proof The first property has already been discussed. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

So, consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant, $\ln(ax) = \ln a + \ln x + C$. By letting $x = 1$, you can see that $C = 0$. The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and third properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b$$

EXAMPLE 1 Expanding Logarithmic Expressions

a. $\ln \dfrac{10}{9} = \ln 10 - \ln 9$ — Property 4

b. $\ln\sqrt{3x + 2} = \ln(3x + 2)^{1/2}$ — Rewrite with rational exponent.

$= \dfrac{1}{2}\ln(3x + 2)$ — Property 3

c. $\ln\dfrac{6x}{5} = \ln(6x) - \ln 5$ — Property 4

$= \ln 6 + \ln x - \ln 5$ — Property 2

d. $\ln\dfrac{(x^2 + 3)^2}{x\sqrt[3]{x^2 + 1}} = \ln(x^2 + 3)^2 - \ln\left(x\sqrt[3]{x^2 + 1}\right)$

$= 2\ln(x^2 + 3) - [\ln x + \ln(x^2 + 1)^{1/3}]$

$= 2\ln(x^2 + 3) - \ln x - \ln(x^2 + 1)^{1/3}$

$= 2\ln(x^2 + 3) - \ln x - \dfrac{1}{3}\ln(x^2 + 1)$

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln x$ is all positive real numbers. (See Figure 5.4.)

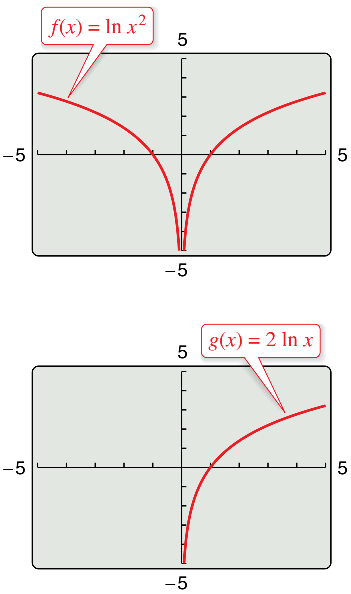


Figure 5.4

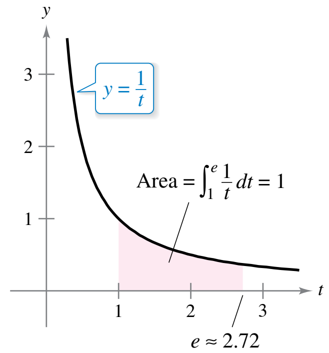
The Number e

THE NUMBER e

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$. (You will learn more about this in Section 5.5.)

The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.



e is the base for the natural logarithm because $\ln e = 1$.
Figure 5.5

$$e \approx 2.71828182846$$

Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

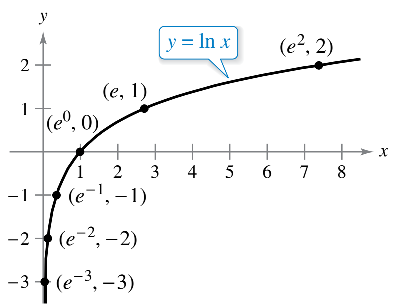
FOR FURTHER INFORMATION To learn more about the number e, see the article "Unexpected Occurrences of the Number e" by Harris S. Shultz and Bill Leonard in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e \\ &= n(1) \\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2



If $x = e^n$, then $\ln x = n$.
Figure 5.6

The logarithms shown in the table above are convenient because the x-values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

EXAMPLE 2 **Evaluating Natural Logarithmic Expressions**

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}, \quad x > 0$

2. $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

EXAMPLE 3 Differentiation of Logarithmic Functions

See LarsonCalculus.com for an interactive version of this type of example.

a. $\dfrac{d}{dx}[\ln 2x] = \dfrac{u'}{u} = \dfrac{2}{2x} = \dfrac{1}{x}$ $\qquad u = 2x$

b. $\dfrac{d}{dx}[\ln(x^2 + 1)] = \dfrac{u'}{u} = \dfrac{2x}{x^2 + 1}$ $\qquad u = x^2 + 1$

c. $\dfrac{d}{dx}[x \ln x] = x\left(\dfrac{d}{dx}[\ln x]\right) + (\ln x)\left(\dfrac{d}{dx}[x]\right)$ $\qquad$ Product Rule

$= x\left(\dfrac{1}{x}\right) + (\ln x)(1)$

$= 1 + \ln x$

d. $\dfrac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \dfrac{d}{dx}[\ln x]$ $\qquad$ Chain Rule

$= 3(\ln x)^2 \dfrac{1}{x}$

Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

EXAMPLE 4 Logarithmic Properties as Aids to Differentiation

Differentiate

$f(x) = \ln\sqrt{x + 1}.$

Solution Because

$$f(x) = \ln\sqrt{x + 1} = \ln(x + 1)^{1/2} = \frac{1}{2}\ln(x + 1) \qquad \text{Rewrite before differentiating.}$$

you can write

$$f'(x) = \frac{1}{2}\left(\frac{1}{x + 1}\right) = \frac{1}{2(x + 1)}. \qquad \text{Differentiate.}$$

EXAMPLE 5 **Logarithmic Properties as Aids to Differentiation**

Differentiate $f(x) = \ln \dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

Solution Because

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \text{Write original function.}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \qquad \text{Rewrite before differentiating.}$$

you can write

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}. \qquad \text{Simplify.}$$

In Examples 4 and 5, be sure you see the benefit of applying logarithmic properties *before* differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5.

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions. This procedure is called **logarithmic differentiation.** In general, use logarithmic differentiation when differentiating (1) a function involving many factors or (2) a function having both a variable base and a variable exponent [see Section 5.5, Example 5(d)].

EXAMPLE 6 **Logarithmic Differentiation**

Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$

Solution Note that $y > 0$ for all $x \neq 2$. So, $\ln y$ is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 \qquad \text{Write original equation.}$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take natural log of each side.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$\frac{y'}{y} = \frac{x^2+2x+2}{(x-2)(x^2+1)} \qquad \text{Simplify.}$$

$$y' = y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Solve for } y'.$$

$$y' = \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$y' = \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

REMARK You could also solve the problem in Example 6 without using logarithmic differentiation by using the Power and Quotient Rules. Use these rules to find the derivative and show that the result is equivalent to the one in Example 6. Which method do you prefer?

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The next theorem states that you can differentiate functions of the form $y = \ln|u|$ as though the absolute value notation was not present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Proof If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\frac{d}{dx}[\ln|u|] = \frac{d}{dx}[\ln(-u)] = \frac{-u'}{-u} = \frac{u'}{u}.$$

EXAMPLE 7 Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Solution Using Theorem 5.4, let $u = \cos x$ and write

$$\frac{d}{dx}[\ln|\cos x|] = \frac{u'}{u} \qquad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

$$= \frac{-\sin x}{\cos x} \qquad u = \cos x$$

$$= -\tan x. \qquad \text{Simplify.}$$

EXAMPLE 8 Finding Relative Extrema

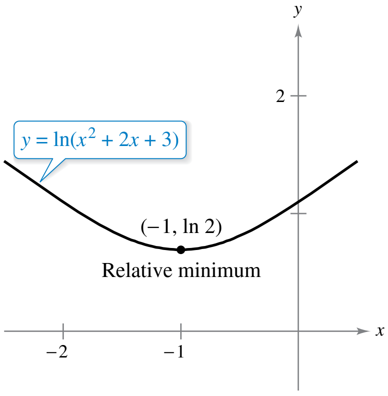
Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

Solution Differentiating y, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test and conclude that a relative minimum occurs at the point $(-1, \ln 2)$. Because there are no other critical points, it follows that this is the only relative extremum, as shown in the figure.



The derivative of y changes from negative to positive at $x = -1$.

5.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Natural Logarithmic Function** Explain why $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$.
2. **Logarithmic Properties** What is the value of n?

$$\ln 4 + \ln(n^{-1}) = \ln 4 - \ln 7$$

3. **The Number e** How is the number e defined?
4. **Differentiation of Logarithmic Functions** State the Chain Rule version of the derivative of the natural logarithmic function in your own words.

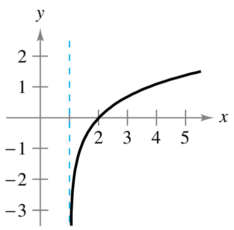
Evaluating a Logarithm Using Technology **In Exercises 5–8, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (1/t)\,dt$.**

5. $\ln 45$ **6.** $\ln 8.3$

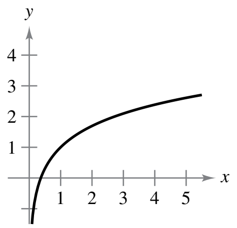
7. $\ln 0.8$ **8.** $\ln 0.6$

Matching **In Exercises 9–12, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

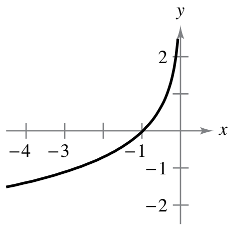
(a)



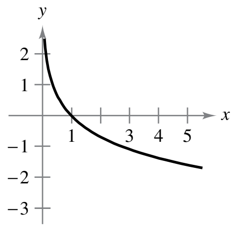
(b)



(c)



(d)



9. $f(x) = \ln x + 1$ **10.** $f(x) = -\ln x$

11. $f(x) = \ln(x - 1)$ **12.** $f(x) = -\ln(-x)$

Sketching a Graph **In Exercises 13–18, sketch the graph of the function and state its domain.**

13. $f(x) = 3 \ln x$

14. $f(x) = -2 \ln x$

15. $f(x) = \ln 2x$

16. $f(x) = \ln|x|$

17. $f(x) = \ln(x - 3)$

18. $f(x) = \ln x - 4$

Using Properties of Logarithms **In Exercises 19 and 20, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.**

19. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$

20. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

Expanding a Logarithmic Expression **In Exercises 21–30, use the properties of logarithms to expand the logarithmic expression.**

21. $\ln \dfrac{x}{4}$ **22.** $\ln \sqrt{x^5}$

23. $\ln \dfrac{xy}{z}$ **24.** $\ln(xyz)$

25. $\ln\left(x\sqrt{x^2 + 5}\right)$ **26.** $x \ln \sqrt{x - 4}$

27. $\ln \sqrt{\dfrac{x - 1}{x}}$ **28.** $\ln(3e^2)$

29. $\ln z(z - 1)^2$ **30.** $\ln \dfrac{z}{e}$

Condensing a Logarithmic Expression **In Exercises 31–36, write the expression as a logarithm of a single quantity.**

31. $\ln(x - 2) - \ln(x + 2)$ **32.** $3 \ln x + 2 \ln y - 4 \ln z$

33. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

34. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

35. $4 \ln 2 - \frac{1}{2} \ln(x^3 + 6x)$

36. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

Verifying Properties of Logarithms **In Exercises 37 and 38, (a) verify that $f = g$ by using a graphing utility to graph f and g in the same viewing window and (b) verify that $f = g$ algebraically.**

37. $f(x) = \ln \dfrac{x^2}{4}$, $x > 0$, $g(x) = 2 \ln x - \ln 4$

38. $f(x) = \ln \sqrt{x(x^2 + 1)}$, $g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

Finding a Limit **In Exercises 39–42, find the limit.**

39. $\lim\limits_{x \to 3^+} \ln(x - 3)$ **40.** $\lim\limits_{x \to 6^-} \ln(6 - x)$

41. $\lim\limits_{x \to 2^-} \ln[x^2(3 - x)]$ **42.** $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

Finding a Derivative **In Exercises 43–66, find the derivative of the function.**

43. $f(x) = \ln 3x$ **44.** $f(x) = \ln(x - 1)$

45. $f(x) = \ln(x^2 + 3)$ **46.** $h(x) = \ln(2x^2 + 1)$

47. $y = (\ln x)^4$ **48.** $y = x^2 \ln x$

49. $y = \ln(t + 1)^2$ **50.** $y = \ln \sqrt{x^2 - 4}$

51. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

52. $y = \ln[t(t^2 + 3)^3]$

53. $f(x) = \ln \dfrac{x}{x^2 + 1}$

54. $f(x) = \ln \dfrac{2x}{x + 3}$

55. $g(t) = \dfrac{\ln t}{t^2}$

56. $h(t) = \dfrac{\ln t}{t^3 + 5}$

57. $y = \ln(\ln x^2)$

58. $y = \ln(\ln x)$

59. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$

60. $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$

61. $f(x) = \ln \dfrac{\sqrt{4 + x^2}}{x}$

62. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

Finding an Equation of a Tangent Line **In Exercises 67–74, (a) find an equation of the tangent line to the graph of the function at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *tangent* feature of a graphing utility to confirm your results.**

67. $y = \ln x^4, \quad (1, 0)$

68. $y = \ln x^{2/3}, \quad (-1, 0)$

69. $f(x) = 3x^2 - \ln x, \quad (1, 3)$

70. $f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

71. $f(x) = \ln\sqrt{1 + \sin^2 x}, \quad \left(\dfrac{\pi}{4}, \ln\sqrt{\dfrac{3}{2}}\right)$

72. $f(x) = \sin 2x \ln x^2, \quad (1, 0)$

73. $y = x^3 \ln x^4, \quad (-1, 0)$

74. $f(x) = \dfrac{1}{2}x \ln x^2, \quad (-1, 0)$

Logarithmic Differentiation **In Exercises 75–80, use logarithmic differentiation to find dy/dx.**

75. $y = x\sqrt{x^2 + 1}, \quad x > 0$

76. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

77. $y = \dfrac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \dfrac{2}{3}$

78. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

79. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

80. $y = \dfrac{(x + 1)(x - 2)}{(x - 1)(x + 2)}, \quad x > 2$

Implicit Differentiation **In Exercises 81–84, use implicit differentiation to find dy/dx.**

81. $x^2 - 3\ln y + y^2 = 10$

82. $\ln xy + 5x = 30$

83. $4x^3 + \ln y^2 + 2y = 2x$

84. $4xy + \ln x^2y = 7$

Differential Equation **In Exercises 85 and 86, verify that the function is a solution of the differential equation.**

	Function	Differential Equation
85.	$y = 2\ln x + 3$	$xy'' + y' = 0$
86.	$y = x\ln x - 4x$	$x + y - xy' = 0$

Relative Extrema and Points of Inflection **In Exercises 87–92, locate any relative extrema and points of inflection. Use a graphing utility to confirm your results.**

87. $y = \dfrac{x^2}{2} - \ln x$

88. $y = 2x - \ln 2x$

89. $y = x\ln x$

90. $y = \dfrac{\ln x}{x}$

91. $y = \dfrac{x}{\ln x}$

92. $y = x^2 \ln \dfrac{x}{4}$

Using Newton's Method **In Exercises 93 and 94, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.**

93. $y = \ln x, \quad y = -x$

94. $y = \ln x, \quad y = 3 - x$

EXPLORING CONCEPTS

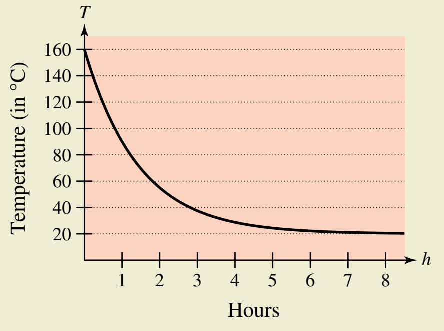
Comparing Functions **In Exercises 95 and 96, let f be a function that is positive and differentiable on the entire real number line and let $g(x) = \ln f(x)$.**

95. When g is increasing, must f be increasing? Explain.

96. When the graph of f is concave upward, must the graph of g be concave upward? Explain.

97. **Think About It** Is $\ln xy = \ln x \ln y$ a valid property of logarithms, where $x > 0$ and $y > 0$? Explain.

98. **HOW DO YOU SEE IT?** The graph shows the temperature T (in degrees Celsius) of an object h hours after it is removed from a furnace.



(a) Find $\lim_{h \to \infty} T$. What does this limit represent?

(b) When is the temperature changing most rapidly?

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. $\ln(a^{n+m}) = n\ln a + m\ln a$, where $a > 0$ and m and n are rational.

100. $\dfrac{d}{dx}[\ln(cx)] = \dfrac{d}{dx}[\ln x]$, where $c > 0$

101. If $y = \ln \pi$, then $y' = 1/\pi$.

102. If $y = \ln e$, then $y' = 1$.

103. Home Mortgage The term t (in years) of a \$200,000 home mortgage at 7.5% interest can be approximated by

$$t = 13.375 \ln\left(\frac{x}{x - 1250}\right), \quad x > 1250$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1398.43. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1611.19. What is the total amount paid?

(d) Find the instantaneous rates of change of t with respect to x when $x = \$1398.43$ and $x = \$1611.19$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

104. Sound Intensity The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = \frac{10}{\ln 10} \ln\left(\frac{I}{10^{-16}}\right).$$

(a) Use the properties of logarithms to write the formula in simpler form.

(b) Determine the number of decibels of a sound with an intensity of 10^{-5} watt per square centimeter.

105. Modeling Data The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). (*Source: Standard Handbook of Mechanical Engineers*)

p	5	10	14.696 (1 atm)	20
T	162.24	193.21	212.00	227.96

p	30	40	60	80	100
T	250.33	267.25	292.71	312.03	327.81

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rates of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find $\lim_{p\to\infty} T'(p)$ and interpret the result in the context of the problem.

106. Modeling Data The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the model from part (b).

(d) Use the model to estimate the altitude when $p = 0.75$.

(e) Use the model to estimate the pressure when $h = 13$.

(f) Use the model to find the rates of change of pressure when $h = 5$ and $h = 20$. Interpret the results.

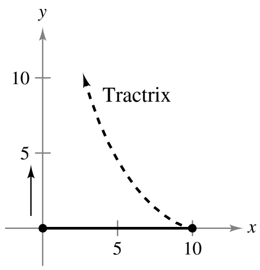
107. Tractrix A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What are the slopes of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as x approaches 10 from the left?



108. Prime Number Theorem There are 25 prime numbers less than 100. The **Prime Number Theorem** states that the number of primes less than x approaches

$$p(x) \approx \frac{x}{\ln x}.$$

Use this approximation to estimate the rate (in primes per 100 integers) at which the prime numbers occur when

(a) $x = 1000$.

(b) $x = 1{,}000{,}000$.

(c) $x = 1{,}000{,}000{,}000$.

109. Conjecture Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate for large values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$

Aceshot1/Shutterstock.com

5.2 The Natural Logarithmic Function: Integration

- **Use the Log Rule for Integration to integrate a rational function.**
- **Integrate trigonometric functions.**

Exploration

Integrating Rational Functions
Early in Chapter 4, you learned rules that allowed you to integrate *any* polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.

$\frac{2}{x}$	Example 1
$\frac{1}{4x-1}$	Example 2
$\frac{x}{x^2+1}$	Example 3
$\frac{3x^2+1}{x^3+x}$	Example 4(a)
$\frac{x+1}{x^2+2x}$	Example 4(c)
$\frac{1}{3x+2}$	Example 4(d)
$\frac{x^2+x+1}{x^2+1}$	Example 5
$\frac{2x}{(x+1)^2}$	Example 6

There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions and explain your reasoning.

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C.$$

Alternative form of Log Rule

EXAMPLE 1 Using the Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx \qquad \text{Constant Multiple Rule}$$

$$= 2\ln|x| + C \qquad \text{Log Rule for Integration}$$

$$= \ln x^2 + C \qquad \text{Property of logarithms}$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative.

EXAMPLE 2 Using the Log Rule with a Change of Variables

Find $\displaystyle\int \frac{1}{4x-1}\,dx$.

Solution If you let $u = 4x - 1$, then $du = 4\,dx$.

$$\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int \left(\frac{1}{4x-1}\right)4\,dx \qquad \text{Multiply and divide by 4.}$$

$$= \frac{1}{4}\int \frac{1}{u}\,du \qquad \text{Substitute: } u = 4x - 1.$$

$$= \frac{1}{4}\ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \frac{1}{4}\ln|4x-1| + C \qquad \text{Back-substitute.}$$

Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

EXAMPLE 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.

Solution In Figure 5.7, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2 + 1}\,dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as shown.

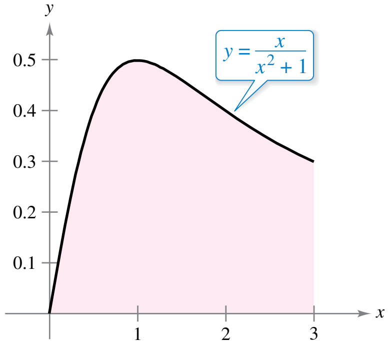
$$\int_0^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\Big[\ln(x^2+1)\Big]_0^3 \qquad \int \frac{u'}{u}\,dx = \ln|u| + C$$

$$= \frac{1}{2}(\ln 10 - \ln 1)$$

$$= \frac{1}{2}\ln 10 \qquad \ln 1 = 0$$

$$\approx 1.151$$



Area $= \int_0^3 \frac{x}{x^2+1}\,dx$

The area of the region bounded by the graph of y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.

Figure 5.7

EXAMPLE 4 Recognizing Quotient Forms of the Log Rule

a. $\displaystyle\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x^3+x| + C \qquad u = x^3 + x$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C \qquad u = \tan x$

c. $\displaystyle\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx \qquad u = x^2 + 2x$

$$= \frac{1}{2}\ln|x^2+2x| + C$$

d. $\displaystyle\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{3}{3x+2}\,dx \qquad u = 3x + 2$

$$= \frac{1}{3}\ln|3x+2| + C$$

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, both

$$\ln|(3x+2)^{1/3}| + C$$

and

$$\ln|3x+2|^{1/3} + C$$

are equivalent to the antiderivative listed in Example 4(d).

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, when a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

See LarsonCalculus.com for an interactive version of this type of example.

Find the indefinite integral.

$$\int \frac{x^2 + x + 1}{x^2 + 1}\, dx$$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2 + x + 1}{x^2 + 1} \Rightarrow \begin{array}{r} 1 \\ x^2 + 1 \overline{) x^2 + x + 1} \\ \underline{x^2 \quad\quad + 1} \\ x \end{array} \Rightarrow 1 + \frac{x}{x^2 + 1}$$

Now, you can integrate to obtain

$$\int \frac{x^2 + x + 1}{x^2 + 1}\, dx = \int \left(1 + \frac{x}{x^2 + 1}\right) dx \qquad \text{Rewrite using long division.}$$

$$= \int dx + \frac{1}{2}\int \frac{2x}{x^2 + 1}\, dx \qquad \text{Rewrite as two integrals.}$$

$$= x + \frac{1}{2}\ln(x^2 + 1) + C. \qquad \text{Integrate.}$$

Check this result by differentiating to obtain the original integrand. ■

The next example presents another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find the indefinite integral.

$$\int \frac{2x}{(x + 1)^2}\, dx$$

Solution If you let $u = x + 1$, then $du = dx$ and $x = u - 1$.

$$\int \frac{2x}{(x + 1)^2}\, dx = \int \frac{2(u - 1)}{u^2}\, du \qquad \text{Substitute.}$$

$$= 2\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du \qquad \text{Rewrite as two fractions.}$$

$$= 2\int \frac{du}{u} - 2\int u^{-2}\, du \qquad \text{Rewrite as two integrals.}$$

$$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C \qquad \text{Integrate.}$$

$$= 2\ln|u| + \frac{2}{u} + C \qquad \text{Simplify.}$$

$$= 2\ln|x + 1| + \frac{2}{x + 1} + C \qquad \text{Back-substitute.}$$

Check this result by differentiating to obtain the original integrand. ■

TECHNOLOGY If you have access to a computer algebra system, use it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

"Here is the question; what is the answer?"

Integration is more like

"Here is the answer; what is the question?"

Here are some guidelines you can use for integration.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas.
2. Find an integration formula that resembles all or part of the integrand and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.
5. Check your result by differentiating to obtain the original integrand.

EXAMPLE 7 *u*-Substitution and the Log Rule

Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x \ln x}.$$

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices

$$u = x \quad \text{and} \quad u = x \ln x$$

fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\, dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$.

REMARK Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x\, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx \quad \text{Apply trigonometric identity and multiply and divide by } -1.$$

$$= -\int \frac{u'}{u}\, dx \quad \text{Substitute: } u = \cos x.$$

$$= -\ln|u| + C \quad \text{Apply Log Rule.}$$

$$= -\ln|\cos x| + C. \quad \text{Back-substitute.}$$

Example 8 used a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x\, dx$.

Solution Consider the following procedure.

$$\int \sec x\, dx = \int (\sec x)\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx \quad \text{Multiply and divide by } \sec x + \tan x.$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x$$

and

$$u' = \sec x \tan x + \sec^2 x.$$

So, you can conclude that

$$\int \sec x\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \quad \text{Rewrite integrand.}$$

$$= \int \frac{u'}{u}\, dx \quad \text{Substitute: } u = \sec x + \tan x.$$

$$= \ln|u| + C \quad \text{Apply Log Rule.}$$

$$= \ln|\sec x + \tan x| + C. \quad \text{Back-substitute.}$$

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. The integrals of the six basic trigonometric functions are summarized below. (For proofs of $\cot u$ and $\csc u$, see Exercises 85 and 86.)

REMARK Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\, du = \ln|\csc u - \cot u| + C.$$

(See Exercises 87–90.)

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \tan u\, du = -\ln|\cos u| + C \qquad \int \cot u\, du = \ln|\sin u| + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx \\ &= \int_0^{\pi/4} \sec x\, dx && \sec x \ge 0 \text{ for } 0 \le x \le \frac{\pi}{4}. \\ &= \ln|\sec x + \tan x|\Big]_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln 1 \\ &\approx 0.881.\end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of

$f(x) = \tan x$

on the interval $[0, \pi/4]$.

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{(\pi/4) - 0}\int_0^{\pi/4} \tan x\, dx && \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\, dx \\ &= \frac{4}{\pi}\int_0^{\pi/4} \tan x\, dx && \text{Simplify.} \\ &= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} && \text{Integrate.} \\ &= -\frac{4}{\pi}\left[\ln\frac{\sqrt{2}}{2} - \ln 1\right] \\ &= -\frac{4}{\pi}\ln\frac{\sqrt{2}}{2} \\ &\approx 0.441\end{aligned}$$

The average value is about 0.441, as shown in Figure 5.8.

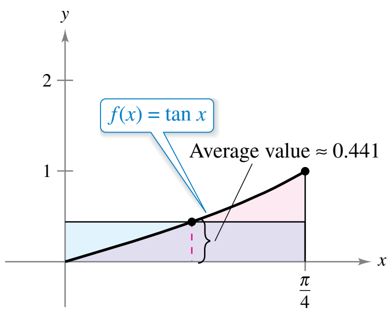


Figure 5.8

5.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Log Rule for Integration** Can you use the Log Rule to find the integral below? Explain.

$$\int \frac{x}{(x^2-4)^3}\,dx$$

2. **Long Division** Explain when to use long division before applying the Log Rule.
3. **Guidelines for Integration** Describe two ways to alter an integrand so that it fits an integration formula.
4. **Trigonometric Functions** Integrating which trigonometric function results in $\ln|\sin x| + C$?

Finding an Indefinite Integral **In Exercises 5–28, find the indefinite integral.**

5. $\int \frac{5}{x}\,dx$
6. $\int \frac{1}{x-5}\,dx$
7. $\int \frac{1}{2x+5}\,dx$
8. $\int \frac{9}{5-4x}\,dx$
9. $\int \frac{x}{x^2-3}\,dx$
10. $\int \frac{x^2}{5-x^3}\,dx$
11. $\int \frac{4x^3+3}{x^4+3x}\,dx$
12. $\int \frac{x^2-2x}{x^3-3x^2}\,dx$
13. $\int \frac{x^2-7}{7x}\,dx$
14. $\int \frac{x^3-8x}{x^2}\,dx$
15. $\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$
16. $\int \frac{x^2+4x}{x^3+6x^2+5}\,dx$
17. $\int \frac{x^2-3x+2}{x+1}\,dx$
18. $\int \frac{2x^2+7x-3}{x-2}\,dx$
19. $\int \frac{x^3-3x^2+5}{x-3}\,dx$
20. $\int \frac{x^3-6x-20}{x+5}\,dx$
21. $\int \frac{x^4+x-4}{x^2+2}\,dx$
22. $\int \frac{x^3-4x^2-4x+20}{x^2-5}\,dx$
23. $\int \frac{(\ln x)^2}{x}\,dx$
24. $\int \frac{dx}{x(\ln x^2)^3}$
25. $\int \frac{1}{\sqrt{x}(1-3\sqrt{x})}\,dx$
26. $\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$
27. $\int \frac{6x}{(x-5)^2}\,dx$
28. $\int \frac{x(x-2)}{(x-1)^3}\,dx$

Change of Variables **In Exercises 29–32, find the indefinite integral by making a change of variables (*Hint:* Let *u* be the denominator of the integrand.)**

29. $\int \frac{1}{1+\sqrt{2x}}\,dx$
30. $\int \frac{4}{1+\sqrt{5x}}\,dx$
31. $\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$
32. $\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

Finding an Indefinite Integral of a Trigonometric Function **In Exercises 33–42, find the indefinite integral.**

33. $\int \cot\frac{\theta}{3}\,d\theta$
34. $\int \theta\tan 2\theta^2\,d\theta$
35. $\int \csc 2x\,dx$
36. $\int \sec\frac{x}{2}\,dx$
37. $\int (5-\cos 3\theta)\,d\theta$
38. $\int \left(2-\tan\frac{\theta}{4}\right)d\theta$
39. $\int \frac{\cos t}{1+\sin t}\,dt$
40. $\int \frac{\csc^2 t}{\cot t}\,dt$
41. $\int \frac{\sec x\tan x}{\sec x-1}\,dx$
42. $\int (\sec 2x+\tan 2x)\,dx$

Differential Equation **In Exercises 43–46, find the general solution of the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.**

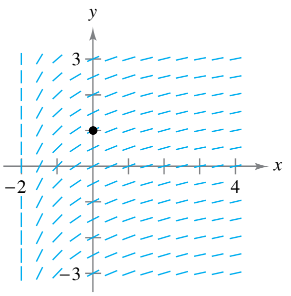
43. $\frac{dy}{dx} = \frac{3}{2-x}$, $(1, 0)$
44. $\frac{dy}{dx} = \frac{x-2}{x}$, $(-1, 0)$
45. $\frac{dy}{dx} = \frac{2x}{x^2-9}$, $(0, 4)$
46. $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t+1}$, $(\pi, 4)$

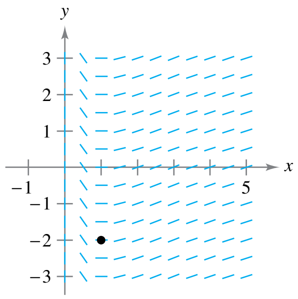
Finding a Particular Solution **In Exercises 47 and 48, find the particular solution of the differential equation that satisfies the initial conditions.**

47. $f''(x) = \frac{2}{x^2}$, $f'(1) = 1$, $f(1) = 1$, $x > 0$
48. $f''(x) = -\frac{4}{(x-1)^2} - 2$, $f'(2) = 0$, $f(2) = 3$, $x > 1$

Slope Field **In Exercises 49 and 50, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

49. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$
50. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$





Evaluating a Definite Integral In Exercises 51–58, evaluate the definite integral. Use a graphing utility to verify your result.

51. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

52. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx$

53. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

54. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

55. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

56. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

57. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

58. $\displaystyle\int_{\pi/8}^{\pi/4} (\csc 2\theta - \cot 2\theta)\,d\theta$

Finding an Integral Using Technology In Exercises 59 and 60, use a computer algebra system to find or evaluate the integral.

59. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

Finding a Derivative In Exercises 61–64, find $F'(x)$.

61. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

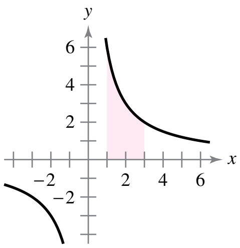
62. $\displaystyle F(x) = \int_0^x \tan t\,dt$

63. $\displaystyle F(x) = \int_1^{4x} \cot t\,dt$

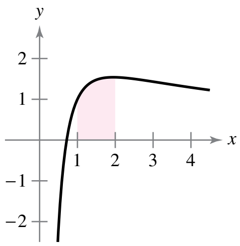
64. $\displaystyle F(x) = \int_0^{x^2} \frac{3}{t+1}\,dt$

Area In Exercises 65–68, find the area of the given region. Use a graphing utility to verify your result.

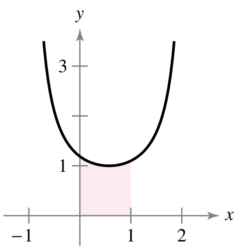
65. $y = \dfrac{6}{x}$



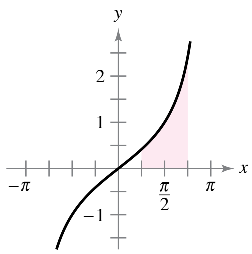
66. $y = \dfrac{1+\ln x^3}{x}$



67. $y = \csc(x+1)$



68. $y = \dfrac{\sin x}{1+\cos x}$



Area In Exercises 69–72, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.

69. $y = \dfrac{x^2+4}{x}, \quad x = 1, \quad x = 4, \quad y = 0$

70. $y = \dfrac{5x}{x^2+2}, \quad x = 1, \quad x = 5, \quad y = 0$

71. $y = 2\sec\dfrac{\pi x}{6}, \quad x = 0, \quad x = 2, \quad y = 0$

72. $y = 2x - \tan 0.3x, \quad x = 1, \quad x = 4, \quad y = 0$

Finding the Average Value of a Function In Exercises 73–76, find the average value of the function over the given interval.

73. $f(x) = \dfrac{8}{x^2}, \quad [2, 4]$

74. $f(x) = \dfrac{4(x+1)}{x^2}, \quad [2, 4]$

75. $f(x) = \dfrac{2\ln x}{x}, \quad [1, e]$

76. $f(x) = \sec\dfrac{\pi x}{6}, \quad [0, 2]$

Midpoint Rule In Exercises 77 and 78, use the Midpoint Rule with $n = 4$ to approximate the value of the definite integral. Use a graphing utility to verify your result.

77. $\displaystyle\int_1^3 \frac{12}{x}\,dx$

78. $\displaystyle\int_0^{\pi/4} \sec x\,dx$

EXPLORING CONCEPTS

Approximation In Exercises 79 and 80, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. Make your selection on the basis of a sketch of the region, not by performing calculations.

79. $f(x) = \sec x, \quad [0, 1]$

(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

80. $f(x) = \dfrac{2x}{x^2+1}, \quad [0, 4]$

(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

81. **Napier's Inequality** For $0 < x < y$, use the Mean Value Theorem to show that

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

82. **Think About It** Is the function

$$F(x) = \int_x^{2x} \frac{1}{t}\,dt$$

constant, increasing, or decreasing on the interval $(0, \infty)$? Explain.

83. **Finding a Value** Find a value of x such that

$$\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt.$$

84. **Finding a Value** Find a value of x such that

$$\int_1^x \frac{1}{t}\,dt$$

is equal to (a) $\ln 5$ and (b) 1.

85. Proof Prove that

$$\int \cot u \, du = \ln|\sin u| + C.$$

86. Proof Prove that

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C.$$

Using Properties of Logarithms and Trigonometric Identities **In Exercises 87–90, show that the two formulas are equivalent.**

87. $\int \tan x \, dx = -\ln|\cos x| + C$

$\int \tan x \, dx = \ln|\sec x| + C$

88. $\int \cot x \, dx = \ln|\sin x| + C$

$\int \cot x \, dx = -\ln|\csc x| + C$

89. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\int \sec x \, dx = -\ln|\sec x - \tan x| + C$

90. $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

$\int \csc x \, dx = \ln|\csc x - \cot x| + C$

91. Population Growth A population of bacteria P is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000.

(a) Write an equation that gives the population at any time t.

(b) Find the population when $t = 3$ days.

92. Sales The rate of change in sales S is inversely proportional to time t ($t > 1$), measured in weeks. Find S as a function of t when the sales after 2 and 4 weeks are 200 units and 300 units, respectively.

93. Heat Transfer

Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100} \, dT$$

where t is time in minutes.

94. Average Price The demand equation for a product is

$$p = \frac{90{,}000}{400 + 3x}$$

where p is the price (in dollars) and x is the number of units (in thousands). Find the average price p on the interval $40 \le x \le 50$.

95. Area and Slope Graph the function

$$f(x) = \frac{x}{1 + x^2}$$

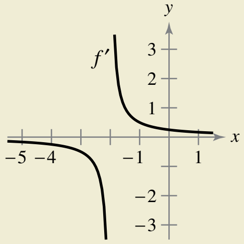
on the interval $[0, \infty)$.

(a) Find the area bounded by the graph of f and the line $y = \frac{1}{2}x$.

(b) Determine the values of the slope m such that the line $y = mx$ and the graph of f enclose a finite region.

(c) Calculate the area of this region as a function of m.

96. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Approximate the slope of f at $x = -1$. Explain.

(b) Approximate any open intervals on which the graph of f is increasing and any open intervals on which it is decreasing. Explain.

True or False? **In Exercises 97–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

97. $\ln|x^4| = \ln x^4$

98. $\ln|\cos \theta^2| = \ln(\cos \theta^2)$

99. $\int \frac{1}{x} \, dx = \ln|cx|, \quad c \neq 0$

100. $\int_{-1}^{2} \frac{1}{x} \, dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

PUTNAM EXAM CHALLENGE

101. Suppose that f is a function on the interval $[1, 3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x) \, dx = 0$. How large can $\int_1^3 \frac{f(x)}{x} \, dx$ be?

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5.3 Inverse Functions

- **Verify that one function is the inverse function of another function.**
- **Determine whether a function has an inverse function.**
- **Find the derivative of an inverse function.**

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

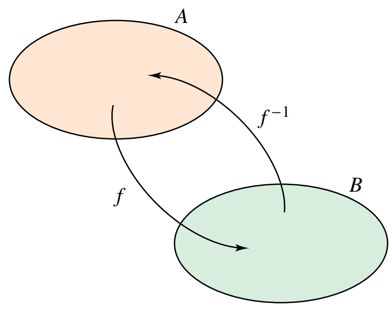
$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.9. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.9

REMARK Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Exploration

Finding Inverse Functions Explain how to "undo" each of the functions below. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$

b. $f(x) = 6x$

c. $f(x) = \dfrac{x}{2}$

d. $f(x) = 3x + 2$

e. $f(x) = x^3$

f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$$f(g(x)) = x \text{ for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but when it does, the inverse function is unique (see Exercise 94).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. So,

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c$$ Subtraction can be used to undo addition.

are inverse functions of each other and

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c}, \quad c \neq 0$$ Division can be used to undo multiplication.

are inverse functions of each other.

EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

REMARK In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

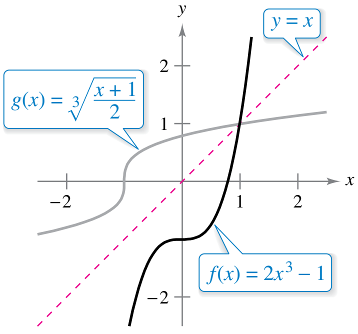
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of g with f is

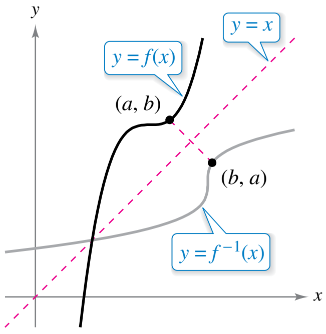
$$g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x.$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.10).



f and g are inverse functions of each other.
Figure 5.10

In Figure 5.10, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the next theorem.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 5.11

THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

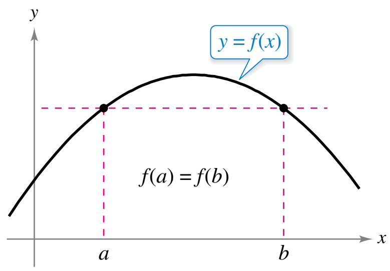
Proof If (a, b) is on the graph of f, then $f(a) = b$, and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.11. A similar argument will prove the theorem in the other direction.

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.12). The next theorem formally states why the Horizontal Line Test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* when it is either increasing on its entire domain or decreasing on its entire domain.)



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.12

THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof The proof of the first part of the theorem is left as an exercise (see Exercise 95). To prove the second part of the theorem, recall from Section P.3 that f is one-to-one when for x_1 and x_2 in its domain

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. ■

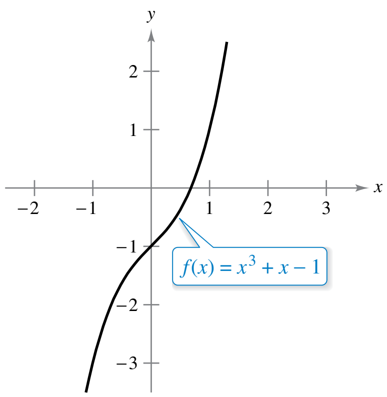
EXAMPLE 2 The Existence of an Inverse Function

a. From the graph of $f(x) = x^3 + x - 1$ shown in Figure 5.13(a), it appears that f is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. So, f is strictly monotonic, and it must have an inverse function.

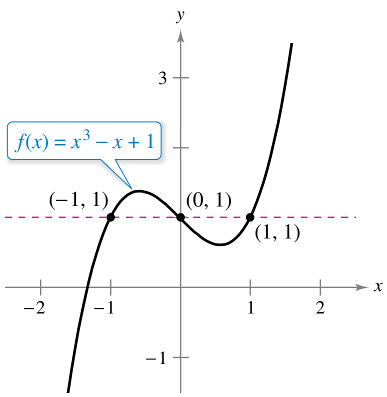
b. From the graph of $f(x) = x^3 - x + 1$ shown in Figure 5.13(b), you can see that the function does not pass the Horizontal Line Test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, f does not have an inverse function. ■



(a) Because f is increasing over its entire domain, it has an inverse function.



(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.13

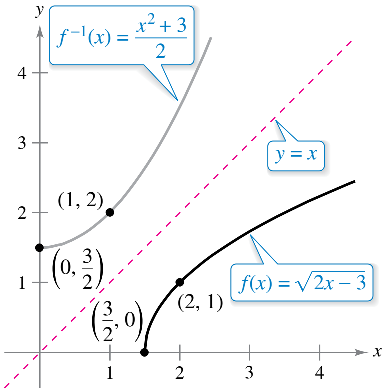
Often, it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a).

GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of $f(x) = \sqrt{2x - 3}$.



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.14

Solution From the graph of f in Figure 5.14, it appears that f is increasing over its entire domain, $[3/2, \infty)$. To verify this, note that

$$f'(x) = \frac{1}{\sqrt{2x - 3}}$$

is positive on the domain of f. So, f is strictly monotonic, and it must have an inverse function. To find an equation for the inverse function, let $y = f(x)$, and solve for x in terms of y.

$$\sqrt{2x - 3} = y \qquad \text{Let } y = f(x).$$
$$2x - 3 = y^2 \qquad \text{Square each side.}$$
$$x = \frac{y^2 + 3}{2} \qquad \text{Solve for } x.$$
$$y = \frac{x^2 + 3}{2} \qquad \text{Interchange } x \text{ and } y.$$
$$f^{-1}(x) = \frac{x^2 + 3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x - 3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

Theorem 5.7 is useful in the next type of problem. You are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

See LarsonCalculus.com for an interactive version of this type of example.

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real number line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

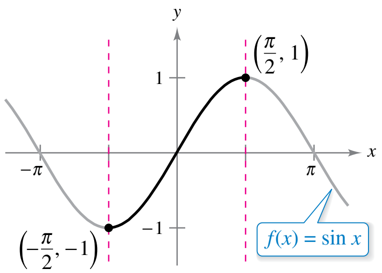
Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin 0 = 0 = \sin \pi.$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that on any larger interval the function is not strictly monotonic (see Figure 5.15).



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.15

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions, as shown in Figure 5.11.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.

Exploration

Graph the inverse functions $f(x) = x^3$ and $g(x) = x^{1/3}$. Calculate the slopes of f at $(1, 1)$, $(2, 8)$, and $(3, 27)$, and the slopes of g at $(1, 1)$, $(8, 2)$, and $(27, 3)$. What do you observe? What happens at $(0, 0)$?

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

EXAMPLE 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and therefore has an inverse function.

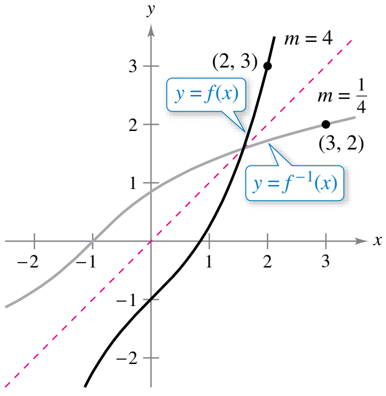
a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 5.16

In Example 5, note that at the point (2, 3), the slope of the graph of f is $m = 4$, and at the point (3, 2), the slope of the graph of f^{-1} is

$$m = \frac{1}{4}$$

as shown in Figure 5.16. In general, if $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. It follows from Theorem 5.9 that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

This reciprocal relationship is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

EXAMPLE 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$), and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2) **b.** (3, 9) and (9, 3)

Solution The derivatives of f and f^{-1} are

$$f'(x) = 2x \quad \text{and} \quad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

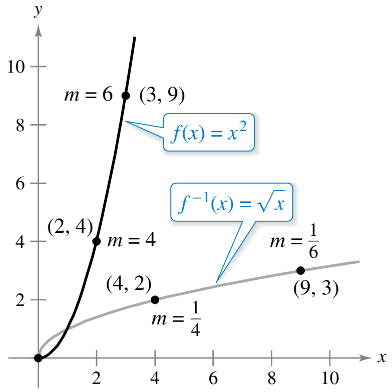
a. At (2, 4), the slope of the graph of f is $f'(2) = 2(2) = 4$. At (4, 2), the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At (3, 9), the slope of the graph of f is $f'(3) = 2(3) = 6$. At (9, 3), the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.17.



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist.
Figure 5.17

5.3 Exercises

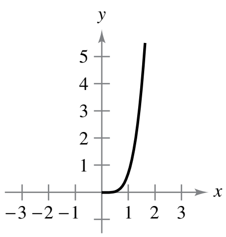
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

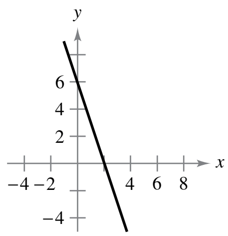
1. **Inverse Functions** In your own words, describe what it means to say that the function g is the inverse function of the function f.
2. **Reflective Property of Inverse Functions** Describe the relationship between the graph of a function and the graph of its inverse function.
3. **Domain of an Inverse Function** The function f has an inverse function, f^{-1}. Is the domain of f the same as the domain of f^{-1}? Explain.
4. **Behavior of an Inverse Function** The function f is decreasing on its domain and has an inverse function, f^{-1}. Is f^{-1} increasing, decreasing, or constant on its domain?

Matching **In Exercises 5–8, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

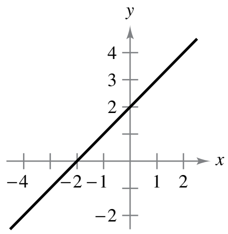
(a)



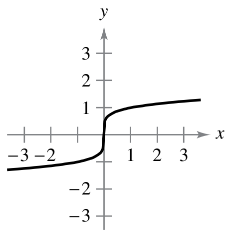
(b)



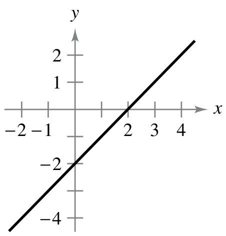
(c)



(d)



5.



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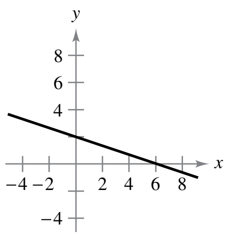
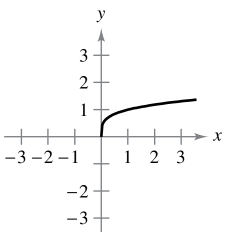


7.



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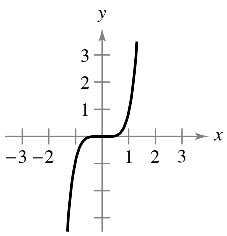


Verifying Inverse Functions **In Exercises 9–16, show that f and g are inverse functions (a) analytically and (b) graphically.**

9. $f(x) = 5x + 1$, $\quad g(x) = \dfrac{x - 1}{5}$
10. $f(x) = 3 - 4x$, $\quad g(x) = \dfrac{3 - x}{4}$
11. $f(x) = x^3$, $\quad g(x) = \sqrt[3]{x}$
12. $f(x) = 1 - x^3$, $\quad g(x) = \sqrt[3]{1 - x}$
13. $f(x) = \sqrt{x - 4}$, $\quad g(x) = x^2 + 4, \quad x \geq 0$
14. $f(x) = 16 - x^2, \quad x \geq 0$, $\quad g(x) = \sqrt{16 - x}$
15. $f(x) = \dfrac{1}{x}$, $\quad g(x) = \dfrac{1}{x}$
16. $f(x) = \dfrac{1}{1 + x}, \quad x \geq 0$, $\quad g(x) = \dfrac{1 - x}{x}, \quad 0 < x \leq 1$

Using the Horizontal Line Test **In Exercises 17–24, use a graphing utility to graph the function. Then use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function.**

17. $f(x) = \frac{3}{4}x + 6$
18. $f(x) = 1 - x^3$
19. $f(\theta) = \sin \theta$
20. $f(x) = x \cos x$
21. $h(s) = \dfrac{1}{s - 2} - 3$
22. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$
23. $f(x) = \ln x$
24. $h(x) = \ln x^2$

Determining Whether a Function Has an Inverse Function **In Exercises 25–30, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.**

25. $f(x) = 2 - x - x^3$
26. $f(x) = x^3 - 6x^2 + 12x$
27. $f(x) = 8x^3 + x^2 - 1$
28. $f(x) = 1 - x^3 - 6x^5$
29. $f(x) = \ln(x - 3)$
30. $f(x) = \cos \dfrac{3x}{2}$

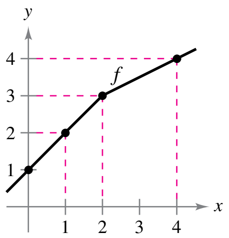
Verifying a Function Has an Inverse Function **In Exercises 31–34, show that f is strictly monotonic on the given interval and therefore has an inverse function on that interval.**

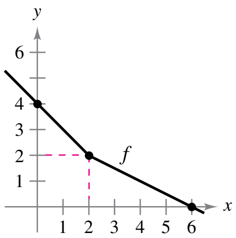
31. $f(x) = (x - 4)^2, \quad [4, \infty)$
32. $f(x) = |x + 2|, \quad [-2, \infty)$
33. $f(x) = \cot x, \quad (0, \pi)$
34. $f(x) = \sec x, \quad \left[0, \dfrac{\pi}{2}\right)$

Finding an Inverse Function **In Exercises 35–46, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domains and ranges of f and f^{-1}.**

35. $f(x) = 2x - 3$
36. $f(x) = 9 - 5x$
37. $f(x) = x^5$
38. $f(x) = x^3 - 1$
39. $f(x) = \sqrt{x}$
40. $f(x) = x^4, \quad x \geq 0$
41. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$
42. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$
43. $f(x) = \sqrt[3]{x - 1}$
44. $f(x) = x^{2/3}, \quad x \geq 0$
45. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$
46. $f(x) = \dfrac{x + 2}{x}$

Finding an Inverse Function **In Exercises 47 and 48, use the graph of the function f to make a table of values for the given points. Then make a second table that can be used to find f^{-1} and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

47.


48.


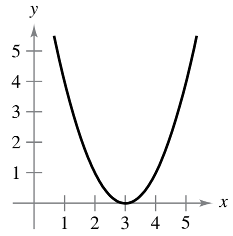
49. **Cost** You need a total of 50 pounds of two commodities costing \$1.25 and \$2.75 per pound.
(a) Verify that the total cost is $y = 1.25x + 2.75(50 - x)$, where x is the number of pounds of the less expensive commodity.
(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?
(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.
(d) Determine the number of pounds of the less expensive commodity purchased when the total cost is \$73.

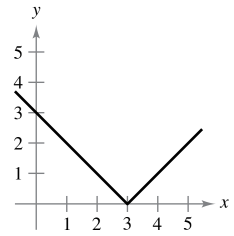
50. **Temperature** The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents Celsius temperature C as a function of Fahrenheit temperature F.
(a) Find the inverse function of C.
(b) What does the inverse function represent?
(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.
(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

Testing Whether a Function Is One-to-One **In Exercises 51–54, determine whether the function is one-to-one. If it is, find its inverse function.**

51. $f(x) = \sqrt{x - 2}$
52. $f(x) = -3$
53. $f(x) = |x - 2|, \quad x \leq 2$
54. $f(x) = ax + b, \quad a \neq 0$

Making a Function One-to-One **In Exercises 55–58, the function is not one-to-one. Delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)**

55. $f(x) = (x - 3)^2$


56. $f(x) = |x - 3|$


57. $f(x) = |x + 3|$
58. $f(x) = 16 - x^4$

Think About It **In Exercises 59–62, decide whether the function has an inverse function. If so, describe what the inverse function represents.**

59. $g(t)$ is the volume of water that has passed through a water line t minutes after a control valve is opened.
60. $h(t)$ is the height of the tide t hours after midnight, where $0 \leq t < 24$.
61. $C(t)$ is the cost of a long-distance phone call lasting t minutes.
62. $A(r)$ is the area of a circle of radius r.

Evaluating the Derivative of an Inverse Function **In Exercises 63–70, verify that f has an inverse function. Then use the function f and the given real number a to find $(f^{-1})'(a)$. (*Hint:* See Example 5.)**

63. $f(x) = 5 - 2x^3, \quad a = 7$
64. $f(x) = x^3 + 3x - 1, \quad a = -5$
65. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$
66. $f(x) = \sqrt{x - 4}, \quad a = 2$
67. $f(x) = \sin x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}, \quad a = \dfrac{1}{2}$
68. $f(x) = \cos 2x, \quad 0 \leq x \leq \dfrac{\pi}{2}, \quad a = 1$
69. $f(x) = \dfrac{x + 6}{x - 2}, \quad x > 2, \quad a = 3$
70. $f(x) = \dfrac{x + 3}{x + 1}, \quad x > -1, \quad a = 2$

Using Inverse Functions **In Exercises 71–74, (a) find the domains of f and f^{-1}, (b) find the ranges of f and f^{-1}, (c) graph f and f^{-1}, and (d) show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.**

	Functions	Point
71.	$f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
	$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
72.	$f(x) = 3 - 4x$	$(1, -1)$
	$f^{-1}(x) = \dfrac{3 - x}{4}$	$(-1, 1)$

	Functions	Point
73.	$f(x) = \sqrt{x - 4}$	$(5, 1)$
	$f^{-1}(x) = x^2 + 4, \quad x \geq 0$	$(1, 5)$
74.	$f(x) = \dfrac{4}{1 + x^2}, \quad x \geq 0$	$(1, 2)$
	$f^{-1}(x) = \sqrt{\dfrac{4 - x}{x}}$	$(2, 1)$

Using Composite and Inverse Functions **In Exercises 75–78, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the given value.**

75. $(f^{-1} \circ g^{-1})(1)$

76. $(g^{-1} \circ f^{-1})(-3)$

77. $(f^{-1} \circ f^{-1})(-2)$

78. $(g^{-1} \circ g^{-1})(8)$

Using Composite and Inverse Functions **In Exercises 79–82, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the given function.**

79. $g^{-1} \circ f^{-1}$

80. $f^{-1} \circ g^{-1}$

81. $(f \circ g)^{-1}$

82. $(g \circ f)^{-1}$

EXPLORING CONCEPTS

83. Inverse Function Consider the function $f(x) = x^n$, where n is odd. Does f^{-1} exist? Explain.

84. Think About It Does adding a constant term to a function affect the existence of an inverse function? Explain.

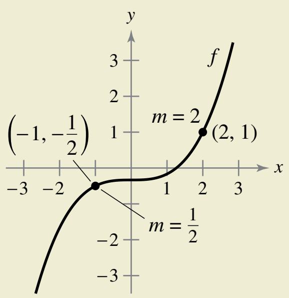
Explaining Why a Function Is Not One-to-One **In Exercises 85 and 86, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain why the function is not one-to-one.**

85. $f(x) = \tan x$

86. $f(x) = \dfrac{x}{x^2 - 4}$

87. Think About It The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find k.

88. HOW DO YOU SEE IT? Use the information in the graph of f below.



(a) What is the slope of the tangent line to the graph of f^{-1} at the point $\left(-\frac{1}{2}, -1\right)$? Explain.

(b) What is the slope of the tangent line to the graph of f^{-1} at the point $(1, 2)$? Explain.

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If f is an even function, then f^{-1} exists.

90. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

91. Making a Function One-to-One

(a) Show that $f(x) = 2x^3 + 3x^2 - 36x$ is not one-to-one on $(-\infty, \infty)$.

(b) Determine the greatest value c such that f is one-to-one on $(-c, c)$.

92. Proof Let f and g be one-to-one functions. Prove that

(a) $f \circ g$ is one-to-one.

(b) $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

93. Proof Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

94. Proof Prove that if a function has an inverse function, then the inverse function is unique.

95. Proof Prove that a function has an inverse function if and only if it is one-to-one.

96. Using Theorem 5.7 Is the converse of the second part of Theorem 5.7 true? That is, if a function is one-to-one (and therefore has an inverse function), then must the function be strictly monotonic? If so, prove it. If not, give a counterexample.

97. Derivative of an Inverse Function Show that

$$f(x) = \int_2^x \sqrt{1 + t^2}\, dt$$

is one-to-one and find $(f^{-1})'(0)$.

98. Derivative of an Inverse Function Show that

$$f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}$$

is one-to-one and find $(f^{-1})'(0)$.

99. Inverse Function Let

$$f(x) = \frac{x - 2}{x - 1}.$$

Show that f is its own inverse function. What can you conclude about the graph of f? Explain.

100. Using a Function Let $f(x) = \dfrac{ax + b}{cx + d}$.

(a) Show that f is one-to-one if and only if $bc - ad \neq 0$.

(b) Given $bc - ad \neq 0$, find f^{-1}.

(c) Determine the values of a, b, c, and d such that $f = f^{-1}$.

101. Concavity Let f be twice-differentiable and one-to-one on an open interval I. Show that its inverse function g satisfies

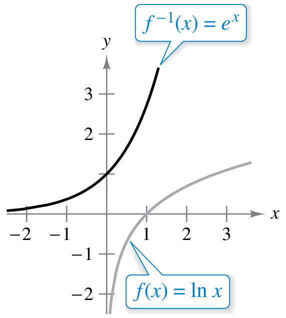
$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

When f is increasing and concave downward, what is the concavity of g?

5.4 Exponential Functions: Differentiation and Integration

- **Develop properties of the natural exponential function.**
- **Differentiate natural exponential functions.**
- **Integrate natural exponential functions.**

The Natural Exponential Function



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.18

The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all real numbers, and the range is the set of positive real numbers, as shown in Figure 5.18. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x is rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The next definition extends the meaning of e^x to include *all* real values of x.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as shown.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

EXAMPLE 1 Solving an Exponential Equation

Solve $7 = e^{x+1}$.

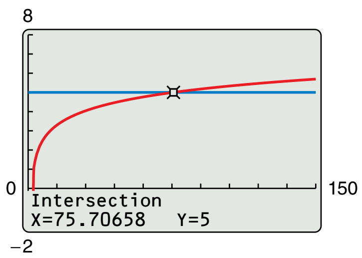
Solution You can convert from exponential form to logarithmic form by *taking the natural logarithm of each side* of the equation.

$$\begin{aligned} 7 &= e^{x+1} && \text{Write original equation.} \\ \ln 7 &= \ln(e^{x+1}) && \text{Take natural logarithm of each side.} \\ \ln 7 &= x + 1 && \text{Apply inverse property.} \\ -1 + \ln 7 &= x && \text{Solve for } x. \end{aligned}$$

So, the solution is $-1 + \ln 7 \approx 0.946$. You can check this solution as shown.

$$\begin{aligned} 7 &= e^{x+1} && \text{Write original equation.} \\ 7 &\stackrel{?}{=} e^{(-1+\ln 7)+1} && \text{Substitute } -1 + \ln 7 \text{ for } x \text{ in original equation.} \\ 7 &\stackrel{?}{=} e^{\ln 7} && \text{Simplify.} \\ 7 &= 7 \checkmark && \text{Solution checks.} \end{aligned}$$

▷ **TECHNOLOGY** You can use a graphing utility to check a solution of an equation. One way to do this is to graph the left- and right-hand sides of the equation and then use the *intersect* feature. For instance, to check the solution to Example 2, enter $y_1 = \ln(2x - 3)$ and $y_2 = 5$. The solution of the original equation is the x-value of each point of intersection (see figure). So the solution of the original equation is $x \approx 75.707$.



EXAMPLE 2 **Solving a Logarithmic Equation**

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponentiate each side* of the logarithmic equation.

$$\ln(2x - 3) = 5 \qquad \text{Write original equation.}$$

$$e^{\ln(2x-3)} = e^5 \qquad \text{Exponentiate each side.}$$

$$2x - 3 = e^5 \qquad \text{Apply inverse property.}$$

$$x = \tfrac{1}{2}(e^5 + 3) \qquad \text{Solve for } x.$$

$$x \approx 75.707 \qquad \text{Use a calculator.}$$

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the next theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$ **2.** $\dfrac{e^a}{e^b} = e^{a-b}$

Proof To prove Property 1, you can write

$$\ln(e^a e^b) = \ln(e^a) + \ln(e^b) = a + b = \ln(e^{a+b}).$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the other property is given in Appendix A.

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the properties listed below from the natural logarithmic function.

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is

   $(-\infty, \infty)$

   and the range is

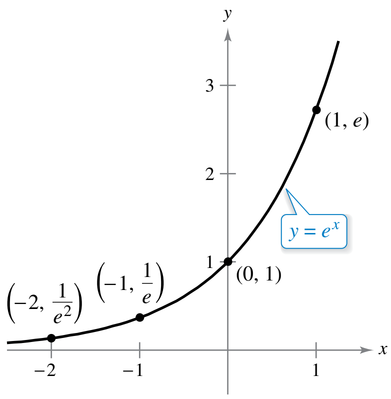
   $(0, \infty)$.

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $f(x) = e^x$ is concave upward on its entire domain.

4. $\lim\limits_{x \to -\infty} e^x = 0$

5. $\lim\limits_{x \to \infty} e^x = \infty$



The natural exponential function is increasing, and its graph is concave upward.

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution of the differential equation $y' = y$. This result is stated in the next theorem.

REMARK You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

THEOREM 5.11 Derivatives of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$
2. $\dfrac{d}{dx}[e^u] = e^u \dfrac{du}{dx}$

Proof To prove Property 1, use the fact that $\ln e^x = x$ and differentiate each side of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate each side with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x \qquad \text{Multiply each side by } e^x.$$

The derivative of e^u follows from the Chain Rule.

EXAMPLE 3 Differentiating Exponential Functions

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u \dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u \dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

c. $\dfrac{d}{dx}[x^2e^x] = x^2(e^x) + e^x(2x) = xe^x(x + 2)$ $\qquad$ Product Rule and Theorem 5.11

d. $\dfrac{d}{dx}\left[\dfrac{e^{3x}}{e^x + 1}\right] = \dfrac{(e^x + 1)(3e^{3x}) - e^{3x}(e^x)}{(e^x + 1)^2} = \dfrac{3e^{4x} + 3e^{3x} - e^{4x}}{(e^x + 1)^2} = \dfrac{e^{3x}(2e^x + 3)}{(e^x + 1)^2}$

EXAMPLE 4 Locating Relative Extrema

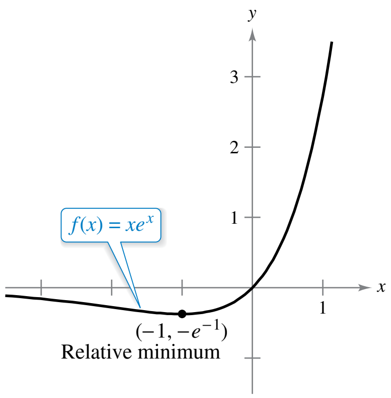
Find the relative extrema of

$$f(x) = xe^x.$$

Solution The derivative of f is

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.19. Because the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points.



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.19

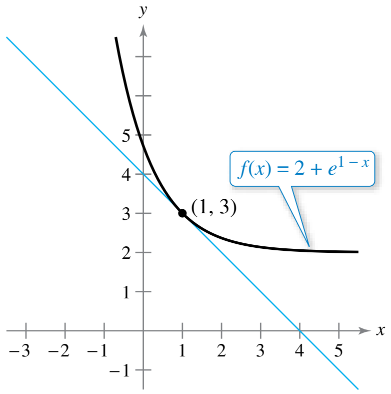


Figure 5.20

EXAMPLE 5 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = 2 + e^{1-x}$ at the point $(1, 3)$.

Solution Begin by finding $f'(x)$.

$$f(x) = 2 + e^{1-x} \quad \text{Write original function.}$$
$$f'(x) = e^{1-x}(-1) \quad u = 1 - x$$
$$= -e^{1-x} \quad \text{First derivative}$$

To find the slope of the tangent line at $(1, 3)$, evaluate $f'(1)$.

$$f'(1) = -e^{1-1} = -e^0 = -1 \quad \text{Slope of tangent line at } (1, 3)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - 3 = -1(x - 1) \quad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = -x + 4. \quad \text{Equation of tangent line at } (1, 3)$$

The graph of f and its tangent line at $(1, 3)$ are shown in Figure 5.20.

EXAMPLE 6 The Standard Normal Probability Density Function

See LarsonCalculus.com for an interactive version of this type of example.

REMARK The general form of a normal probability density function (whose mean is 0) is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-shaped curve" has points of inflection when $x = \pm\sigma$.

Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

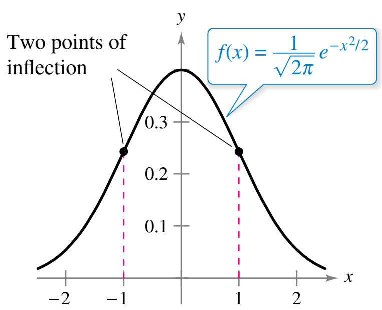
has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which the second derivative is 0.

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{Write original function.}$$
$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} \quad \text{First derivative}$$
$$f''(x) = \frac{1}{\sqrt{2\pi}}[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}] \quad \text{Product Rule}$$
$$= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) \quad \text{Second derivative}$$

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in the figure below.

FOR FURTHER INFORMATION To learn about derivatives of exponential functions of order 1/2, see the article "A Child's Garden of Fractional Derivatives" by Marcia Kleinz and Thomas J. Osler in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.



The bell-shaped curve given by a standard normal probability density function

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x \, dx = e^x + C$ **2.** $\int e^u \, du = e^u + C$

EXAMPLE 7 Integrating Exponential Functions

Find the indefinite integral.

$$\int e^{3x+1} \, dx$$

Solution If you let $u = 3x + 1$, then $du = 3 \, dx$.

$$\begin{aligned} \int e^{3x+1} \, dx &= \frac{1}{3}\int e^{3x+1}(3) \, dx && \text{Multiply and divide by 3.} \\ &= \frac{1}{3}\int e^u \, du && \text{Substitute: } u = 3x + 1. \\ &= \frac{1}{3}e^u + C && \text{Apply Exponential Rule.} \\ &= \frac{e^{3x+1}}{3} + C && \text{Back-substitute.} \end{aligned}$$

REMARK In Example 7, the missing *constant* factor 3 was introduced to create $du = 3 \, dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2} \, dx \neq \frac{1}{x}\int e^{-x^2}(x \, dx).$$

EXAMPLE 8 Integrating Exponential Functions

Find the indefinite integral.

$$\int 5xe^{-x^2} \, dx$$

Solution If you let $u = -x^2$, then $du = -2x \, dx$ or $x \, dx = -du/2$.

$$\begin{aligned} \int 5xe^{-x^2} \, dx &= \int 5e^{-x^2}(x \, dx) && \text{Regroup integrand.} \\ &= \int 5e^u\left(-\frac{du}{2}\right) && \text{Substitute: } u = -x^2. \\ &= -\frac{5}{2}\int e^u \, du && \text{Constant Multiple Rule} \\ &= -\frac{5}{2}e^u + C && \text{Apply Exponential Rule.} \\ &= -\frac{5}{2}e^{-x^2} + C && \text{Back-substitute.} \end{aligned}$$

EXAMPLE 9 Integrating Exponential Functions

Find each indefinite integral.

a. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$ **b.** $\displaystyle\int \sin x\, e^{\cos x}\,dx$

Solution

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \underbrace{e^{1/x}}_{e^u}\underbrace{\left(-\frac{1}{x^2}\right)dx}_{du} = -e^{1/x} + C$$ $u = \dfrac{1}{x}$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \underbrace{e^{\cos x}}_{e^u}\underbrace{(-\sin x)\,dx}_{du} = -e^{\cos x} + C$$ $u = \cos x$

EXAMPLE 10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

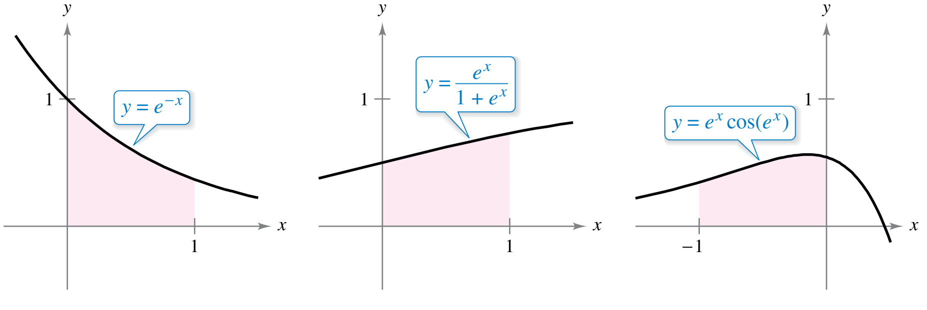
a. $\displaystyle\int_0^1 e^{-x}\,dx$ **b.** $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\displaystyle\int_{-1}^0 e^x \cos(e^x)\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 = -e^{-1} - (-1) = 1 - \frac{1}{e} \approx 0.632$$ See Figure 5.21(a).

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 = \ln(1+e) - \ln 2 \approx 0.620$$ See Figure 5.21(b).

c. $$\int_{-1}^0 e^x\cos(e^x)\,dx = \sin(e^x)\Big]_{-1}^0 = \sin 1 - \sin(e^{-1}) \approx 0.482$$ See Figure 5.21(c).



(a) (b) (c)

Figure 5.21

5.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. Natural Exponential Function Describe the graph of $f(x) = e^x$.

2. A Function and Its Derivative Which of the following functions are their own derivative?

$y = e^x + 4 \qquad y = e^x \qquad y = e^{4x} \qquad y = 4e^x$

Solving an Exponential or Logarithmic Equation **In Exercises 3–18, solve for x accurate to three decimal places.**

3. $e^{\ln x} = 4$

4. $e^{\ln 3x} = 24$

5. $e^x = 12$

6. $5e^x = 36$

7. $9 - 2e^x = 7$

8. $8e^x - 12 = 7$

9. $50e^{-x} = 30$

10. $100e^{-2x} = 35$

11. $\dfrac{800}{100 - e^{x/2}} = 50$

12. $\dfrac{5000}{1 + e^{2x}} = 2$

13. $\ln x = 2$

14. $\ln x^2 = -8$

15. $\ln(x - 3) = 2$

16. $\ln 4x = 1$

17. $\ln\sqrt{x + 2} = 1$

18. $\ln(x - 2)^2 = 12$

Sketching a Graph **In Exercises 19–24, sketch the graph of the function.**

19. $y = e^{-x}$

20. $y = \frac{1}{3}e^x$

21. $y = e^x + 1$

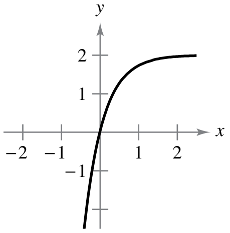
22. $y = -e^{x-1}$

23. $y = e^{-x^2}$

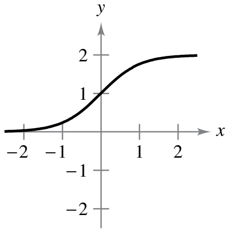
24. $y = e^{-x/2}$

Matching **In Exercises 25–28, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]**

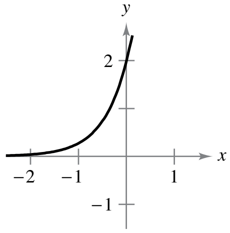
(a)



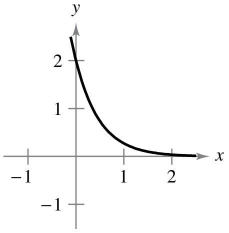
(b)



(c)



(d)



25. $y = Ce^{ax}$

26. $y = Ce^{-ax}$

27. $y = C(1 - e^{-ax})$

28. $y = \dfrac{C}{1 + e^{-ax}}$

Inverse Functions **In Exercises 29–32, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.**

29. $f(x) = e^{2x}$, $g(x) = \ln\sqrt{x}$

30. $f(x) = e^{x/3}$, $g(x) = \ln x^3$

31. $f(x) = e^x - 1$, $g(x) = \ln(x + 1)$

32. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

Finding a Derivative **In Exercises 33–54, find the derivative of the function.**

33. $y = e^{5x}$

34. $y = e^{-8x}$

35. $y = e^{\sqrt{x}}$

36. $y = e^{-2x^3}$

37. $y = e^{x-4}$

38. $y = 5e^{x^2+5}$

39. $y = e^x \ln x$

40. $y = xe^{4x}$

41. $y = (x + 1)^2 e^x$

42. $y = x^2 e^{-x}$

43. $g(t) = (e^{-t} + e^t)^3$

44. $g(t) = e^{-3/t^2}$

45. $y = \ln(2 - e^{5x})$

46. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

47. $y = \dfrac{2}{e^x + e^{-x}}$

48. $y = \dfrac{e^x - e^{-x}}{2}$

49. $y = \dfrac{e^x + 1}{e^x - 1}$

50. $y = \dfrac{e^{2x}}{e^{2x} + 1}$

51. $y = e^x(\sin x + \cos x)$

52. $y = e^{2x} \tan 2x$

53. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t \, dt$

54. $F(x) = \displaystyle\int_{0}^{e^{2x}} \ln(t + 1) \, dt$

Finding an Equation of a Tangent Line **In Exercises 55–62, find an equation of the tangent line to the graph of the function at the given point.**

55. $f(x) = e^{3x}, \quad (0, 1)$

56. $f(x) = e^{-x} - 6, \quad (0, -5)$

57. $y = e^{3x - x^2}, \quad (3, 1)$

58. $y = e^{-2x + x^2}, \quad (2, 1)$

59. $f(x) = e^{-x} \ln x, \quad (1, 0)$

60. $y = \ln \dfrac{e^x + e^{-x}}{2}, \quad (0, 0)$

61. $y = x^2e^x - 2xe^x + 2e^x, \quad (1, e)$

62. $y = xe^x - e^x, \quad (1, 0)$

Implicit Differentiation **In Exercises 63 and 64, use implicit differentiation to find dy/dx.**

63. $xe^y - 10x + 3y = 0$

64. $e^{xy} + x^2 - y^2 = 10$

Finding the Equation of a Tangent Line **In Exercises 65 and 66, use implicit differentiation to find an equation of the tangent line to the graph of the equation at the given point.**

65. $xe^y + ye^x = 1, \quad (0, 1)$

66. $1 + \ln xy = e^{x-y}, \quad (1, 1)$

Finding a Second Derivative **In Exercises 67 and 68, find the second derivative of the function.**

67. $f(x) = (3 + 2x)e^{-3x}$

68. $g(x) = \sqrt{x} + e^x \ln x$

Differential Equation **In Exercises 69 and 70, show that the function $y = f(x)$ is a solution of the differential equation.**

69. $y = 4e^{-x}$

$y'' - y = 0$

70. $y = e^{3x} + e^{-3x}$

$y'' - 9y = 0$

Relative Extrema and Points of Inflection **In Exercises 71–78, find the relative extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.**

71. $f(x) = \dfrac{e^x + e^{-x}}{2}$

72. $f(x) = \dfrac{e^x - e^{-x}}{2}$

73. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

74. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

75. $f(x) = (2 - x)e^x$

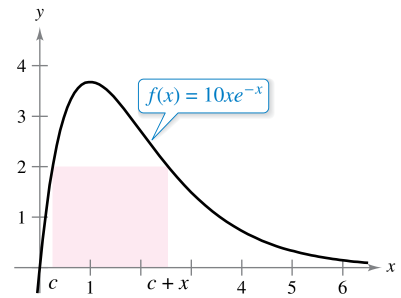
76. $f(x) = xe^{-x}$

77. $g(t) = 1 + (2 + t)e^{-t}$

78. $f(x) = -2 + e^{3x}(4 - 2x)$

79. **Area** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

80. **Area** Perform the following steps to find the maximum area of the rectangle shown in the figure.



(a) Solve for c in the equation $f(c) = f(c + x)$.

(b) Use the result in part (a) to write the area A as a function of x. [*Hint:* $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.

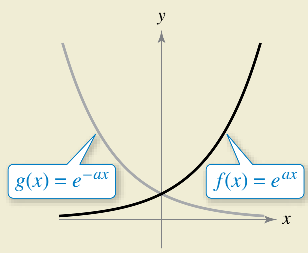
(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.

81. **Finding an Equation of a Tangent Line** Find the point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

82. **HOW DO YOU SEE IT?** The figure shows the graphs of f and g, where a is a positive real number. Identify the open interval(s) on which the graphs of f and g are (a) increasing or decreasing and (b) concave upward or concave downward.



83. **Depreciation** The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 5$.

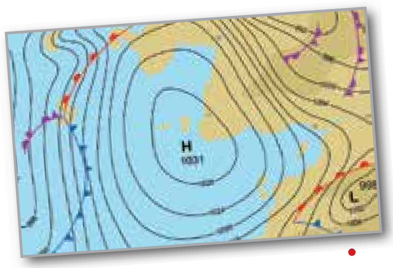
(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 5$.

84. **Harmonic Motion** The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is $y = 1.56e^{-0.22t} \cos 4.9t$, where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

85. **Atmospheric Pressure**

A meteorologist measures the atmospheric pressure P (in millibars) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	1013.2	547.5	233.0	121.6	50.7



(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rates of change of the pressure when $h = 5$ and $h = 18$.

Robert Adrian Hillman/Shutterstock.com

86. Modeling Data The table lists the approximate values V of a mid-sized sedan for the years 2010 through 2016. The variable t represents the time (in years), with $t = 10$ corresponding to 2010.

t	10	11	12	13	14	15	16
V	\$23,046	\$20,596	\$18,851	\$17,001	\$15,226	\$14,101	\$12,841

(a) Use the regression capabilities of a graphing utility to fit linear and quadratic models to the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Use the exponential model to find the rates of decrease in the value of the sedan when $t = 12$ and $t = 15$.

Linear and Quadratic Approximation **In Exercises 87 and 88, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0) \quad \text{and}$$

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, P_2, and their first derivatives at $x = 0$.

87. $f(x) = e^x$

88. $f(x) = e^{x/2}$

Stirling's Formula **For large values of n,**

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n$$

can be approximated by Stirling's Formula,

$$n! \approx \left(\frac{n}{3}\right)^n \sqrt{2\pi n}.$$

In Exercises 89 and 90, find the exact value of $n!$ and then approximate $n!$ using Stirling's Formula.

89. $n = 12$

90. $n = 15$

Finding an Indefinite Integral **In Exercises 91–108, find the indefinite integral.**

91. $\int e^{5x}(5)\,dx$

92. $\int e^{-x^4}(-4x^3)\,dx$

93. $\int e^{5x-3}\,dx$

94. $\int e^{1-3x}\,dx$

95. $\int (2x + 1)e^{x^2+x}\,dx$

96. $\int e^x(e^x + 1)^2\,dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

98. $\int \frac{e^{1/x^2}}{x^3}\,dx$

99. $\int \frac{e^{-x}}{1 + e^{-x}}\,dx$

100. $\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

101. $\int e^x\sqrt{1 - e^x}\,dx$

102. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

103. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

104. $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

105. $\int \frac{5 - e^x}{e^{2x}}\,dx$

106. $\int \frac{e^{-3x} + 2e^{2x} + 3}{e^x}\,dx$

107. $\int e^{-x}\tan(e^{-x})\,dx$

108. $\int e^{2x}\csc(e^{2x})\,dx$

Evaluating a Definite Integral **In Exercises 109–118, evaluate the definite integral. Use a graphing utility to verify your result.**

109. $\int_0^1 e^{-2x}\,dx$

110. $\int_{-1}^1 e^{1+4x}\,dx$

111. $\int_0^1 xe^{-x^2}\,dx$

112. $\int_{-2}^0 x^2e^{x^3/2}\,dx$

113. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

114. $\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx$

115. $\int_0^2 \frac{e^{4x}}{1 + e^{4x}}\,dx$

116. $\int_{-2}^0 \frac{e^{x+1}}{7 - e^{x+1}}\,dx$

117. $\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx$

118. $\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x\tan 2x\,dx$

Differential Equation **In Exercises 119 and 120, find the general solution of the differential equation.**

119. $\frac{dy}{dx} = xe^{9x^2}$

120. $\frac{dy}{dx} = (e^x - e^{-x})^2$

Differential Equation **In Exercises 121 and 122, find the particular solution of the differential equation that satisfies the initial conditions.**

121. $f''(x) = \frac{1}{2}(e^x + e^{-x}),\ f(0) = 1,\ f'(0) = 0$

122. $f''(x) = \sin x + e^{2x},\ f(0) = \frac{1}{4},\ f'(0) = \frac{1}{2}$

Area **In Exercises 123–126, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

123. $y = e^x,\ y = 0,\ x = 0,\ x = 6$

124. $y = e^{-2x},\ y = 0,\ x = -1,\ x = 3$

125. $y = xe^{-x^2/4},\ y = 0,\ x = 0,\ x = \sqrt{6}$

126. $y = e^{-2x} + 2,\ y = 0,\ x = 0,\ x = 2$

Midpoint Rule **In Exercises 127 and 128, use the Midpoint Rule with $n = 12$ to approximate the value of the definite integral. Use a graphing utility to verify your result.**

127. $\int_0^4 \sqrt{x}\,e^x\,dx$

128. $\int_0^2 2xe^{-x}\,dx$

EXPLORING CONCEPTS

129. Asymptotes Compare the asymptotes of the natural exponential function with those of the natural logarithmic function.

130. Comparing Graphs Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

True or False? **In Exercises 131–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

131. If $f(x) = g(x)e^x$, then $f'(x) = g'(x)e^x$.

132. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

133. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

134. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

135. Probability A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0065\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

136. Probability The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation

$$\int_0^x 0.3e^{-0.3t}\,dt = \frac{1}{2}.$$

What is the median waiting time?

137. Modeling Data A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use a definite integral to approximate the number of liters of chemical released during the 4 hours.

138. Using the Area of a Region Find the value of a such that the area bounded by $y = e^{-x}$, the x-axis, $x = -a$, and $x = a$ is $\frac{8}{3}$.

139. Analyzing a Graph Consider the function

$$f(x) = \frac{2}{1 + e^{1/x}}.$$

(a) Use a graphing utility to graph f.

(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

140. Analyzing a Function Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \le A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

141. Deriving an Inequality Given $e^x \ge 1$ for $x \ge 0$, it follows that

$$\int_0^x e^t\,dt \ge \int_0^x 1\,dt.$$

Perform this integration to derive the inequality

$$e^x \ge 1 + x$$

for $x \ge 0$.

142. Solving an Equation Find, to three decimal places, the value of x such that $e^{-x} = x$. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

143. Analyzing a Graph Consider

$$f(x) = xe^{-kx}$$

for $k > 0$. Find the relative extrema and the points of inflection of the function.

144. Finding the Maximum Rate of Change Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, \quad b > 0, \quad L > 0$$

increases at a maximum rate when $y = \dfrac{L}{2}$.

PUTNAM EXAM CHALLENGE

145. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \ge g(x)$ for all $x \ge 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

5.5 Bases Other than *e* and Applications

- **Define exponential functions that have bases other than e.**
- **Differentiate and integrate exponential functions that have bases other than e.**
- **Use exponential functions to model compound interest and exponential growth.**

Bases Other than e

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

> **Definition of Exponential Function to Base *a***
>
> If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by
>
> $$a^x = e^{(\ln a)x}.$$
>
> If $a = 1$, then $y = 1^x = 1$ is a constant function.

Exponential functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $\dfrac{a^x}{a^y} = a^{x-y}$ **4.** $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (*Half-life* is the number of years required for half of the atoms in a sample of radioactive material to decay.)

Carbon dating uses the radioactive isotope carbon-14 to estimate the age of dead organic materials. The method is based on the decay rate of carbon-14 (see Example 1), a compound organisms take in when they are alive.

EXAMPLE 1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

Solution Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample. Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5715}.$$

Notice that when $t = 5715$, the amount is reduced to half of the original amount.

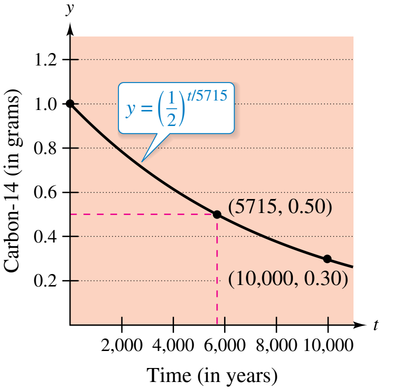
$$y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2} \text{ gram}$$

When $t = 11{,}430$, the amount is reduced to a quarter of the original amount and so on. To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5715}$$

$$\approx 0.30 \text{ gram}$$

The graph of y is shown at the right.



The half-life of carbon-14 is about 5715 years.

Michal Ninger/Shutterstock.com

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

REMARK In precalculus, you learned that $\log_a x$ is the value to which a must be raised to produce x. This agrees with the definition at the right because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

Definition of Logarithmic Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 5.2. (Assume x and y are positive numbers and n is rational.)

1. $\log_a 1 = 0$ — Log of 1
2. $\log_a xy = \log_a x + \log_a y$ — Log of a product
3. $\log_a x^n = n \log_a x$ — Log of a power
4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ — Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarithmic function.** So, for common logarithms,

$$y = 10^x \quad \text{if and only if} \quad x = \log_{10} y. \qquad \text{Property of inverse functions}$$

EXAMPLE 2 Bases Other than e

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$ **b.** $\log_2 x = -4$

Solution

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3 3^x &= \log_3 \frac{1}{81} \\ x &= \log_3 3^{-4} \\ x &= -4 \end{aligned}$$

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\begin{aligned} \log_2 x &= -4 \\ 2^{\log_2 x} &= 2^{-4} \\ x &= \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the differentiation rules for bases other than e given in the next theorem.

REMARK These differentiation rules are similar to those for the natural exponential function and the natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base.

THEOREM 5.13 Derivatives for Bases Other than *e*

Let a be a positive real number $(a \neq 1)$, and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$
2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u \dfrac{du}{dx}$
3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$
4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u}\dfrac{du}{dx}$

Proof By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u\frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules. ■

EXAMPLE 3 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$ **b.** $y = 2^{3x}$ **c.** $y = \log_{10} \cos x$ **d.** $y = \log_3 \dfrac{\sqrt{x}}{x+5}$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

REMARK Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10}\tan x$

d. Before differentiating, rewrite the function using logarithmic properties.

$$y = \log_3 \frac{\sqrt{x}}{x+5} = \frac{1}{2}\log_3 x - \log_3(x+5)$$

Next, apply Theorem 5.13 to differentiate the function.

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{1}{2}\log_3 x - \log_3(x+5)\right] \\ &= \frac{1}{2(\ln 3)x} - \frac{1}{(\ln 3)(x+5)} \\ &= \frac{5-x}{2(\ln 3)x(x+5)} \end{aligned}$$

■

Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

which follows from Theorem 5.13.

EXAMPLE 4 **Integrating an Exponential Function to Another Base**

Find $\int 2^x\,dx$.

Solution

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$$

When the Power Rule, $D_x[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent n was required to be a rational number. Now the rule is extended to cover any real value of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 **The Power Rule for Real Exponents**

Let n be any real number, and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$ **2.** $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

The next example compares the derivatives of four types of functions. Each function uses a different differentiation formula, depending on whether the base and the exponent are constants or variables.

EXAMPLE 5 **Comparing Variables and Constants**

a. $\dfrac{d}{dx}[e^e] = 0$ Constant Rule

b. $\dfrac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\dfrac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Use logarithmic differentiation.

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1)$$
$$\frac{y'}{y} = 1 + \ln x$$
$$y' = y(1 + \ln x)$$
$$y' = x^x(1 + \ln x)$$

REMARK Be sure you see that there is no simple differentiation rule for calculating the derivative of $y = x^x$. In general, when $y = u(x)^{v(x)}$, you need to use logarithmic differentiation.

Applications of Exponential Functions

An amount of P dollars is deposited in an account at an annual interest rate r (in decimal form). What is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

For instance, the result for a deposit of $1000 at 8% interest compounded n times a year is shown in the table at the right.

n	A
1	$1080.00
2	$1081.60
4	$1082.43
12	$1083.00
365	$1083.28

As n increases, the balance A approaches a limit. To develop this limit, use the next theorem. To test the reasonableness of this theorem, try evaluating

$$\left(\frac{x+1}{x}\right)^x$$

for several values of x, as shown in the table at the left.

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

THEOREM 5.15 A Limit Involving *e*

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

A proof of this theorem is given in Appendix A.

Given Theorem 5.15, take another look at the formula for the balance A in an account in which the interest is compounded n times per year. By taking the limit as n approaches infinity, you obtain

$$\begin{aligned} A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^n && \text{Take limit as } n \to \infty. \\ &= P \lim_{n\to\infty}\left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^r && \text{Rewrite.} \\ &= P\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right]^r && \text{Let } x = n/r. \text{ Then } x \to \infty \text{ as } n \to \infty. \\ &= Pe^r. && \text{Apply Theorem 5.15.} \end{aligned}$$

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of $1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08} \approx \$1083.29.$$

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (in decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

EXAMPLE 6 Continuous, Quarterly, and Monthly Compounding

See LarsonCalculus.com for an interactive version of this type of example.

A deposit of \$2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years when the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

Solution

a. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ — Compounded quarterly

$= 2500\left(1 + \frac{0.05}{4}\right)^{4(5)}$

$= 2500(1.0125)^{20}$

$= \$3205.09$

b. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ — Compounded monthly

$= 2500\left(1 + \frac{0.05}{12}\right)^{12(5)}$

$\approx 2500(1.0041667)^{60}$

$= \$3208.40$

c. $A = Pe^{rt}$ — Compounded continuously

$= 2500\left[e^{0.05(5)}\right]$

$= 2500e^{0.25}$

$= \$3210.06$

EXAMPLE 7 Bacterial Culture Growth

A bacterial culture is growing according to the *logistic growth function*

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}}, \quad t \geq 0$$

where y is the weight of the culture in grams and t is the time in hours. Find the weight of the culture after (a) 0 hours, (b) 1 hour, and (c) 10 hours. (d) What is the limit as t approaches infinity?

Solution

a. When $t = 0$, $y = \frac{1.25}{1 + 0.25e^{-0.4(0)}}$

$= 1$ gram.

b. When $t = 1$, $y = \frac{1.25}{1 + 0.25e^{-0.4(1)}}$

≈ 1.071 grams.

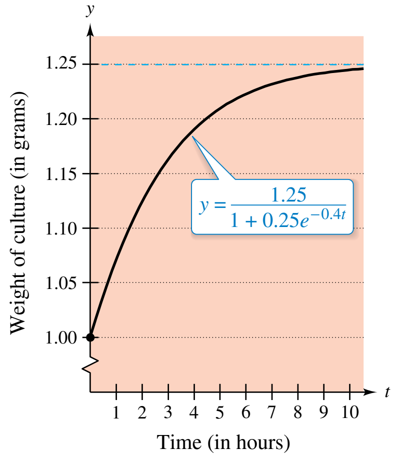
c. When $t = 10$, $y = \frac{1.25}{1 + 0.25e^{-0.4(10)}}$

≈ 1.244 grams.

d. Taking the limit as t approaches infinity, you obtain

$$\lim_{t \to \infty} \frac{1.25}{1 + 0.25e^{-0.4t}} = \frac{1.25}{1 + 0} = 1.25 \text{ grams.}$$

The graph of the function is shown in Figure 5.22.



The limit of the weight of the culture as $t \to \infty$ is 1.25 grams.
Figure 5.22

5.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Derivatives for Bases Other than e** What are the values of a and b?

$$\frac{d}{dx}[6^{4x}] = a(\ln b)6^{4x}$$

2. **Integration for Bases Other than e** What are two options for finding the indefinite integral below?

$$\int 5^t\, dt$$

3. **Logarithmic Differentiation** Explain when it is necessary to use logarithmic differentiation to find the derivative of an exponential function.

4. **Compound Interest Formulas** Explain how to choose which compound interest formula to use to find the balance of a deposit.

Evaluating a Logarithmic Expression **In Exercises 5–10, evaluate the expression without using a calculator.**

5. $\log_2 \frac{1}{8}$
6. $\log_3 81$
7. $\log_7 1$
8. $\log_a \dfrac{1}{a}$
9. $\log_{64} 32$
10. $\log_{27} \frac{1}{9}$

Exponential and Logarithmic Forms of Equations **In Exercises 11–14, write the exponential equation as a logarithmic equation or vice versa.**

11. (a) $2^3 = 8$
    (b) $3^{-1} = \frac{1}{3}$
12. (a) $27^{2/3} = 9$
    (b) $16^{3/4} = 8$
13. (a) $\log_{10} 0.01 = -2$
    (b) $\log_{0.5} 8 = -3$
14. (a) $\log_3 \frac{1}{9} = -2$
    (b) $49^{1/2} = 7$

Sketching a Graph **In Exercises 15–20, sketch the graph of the function.**

15. $y = 2^x$
16. $y = 4^{x-1}$
17. $y = \left(\frac{1}{3}\right)^x$
18. $y = 2^{x^2}$
19. $h(x) = 5^{x-2}$
20. $y = 3^{-|x|}$

Solving an Equation **In Exercises 21–26, solve for x.**

21. (a) $\log_{10} 1000 = x$
    (b) $\log_{10} 0.1 = x$
22. (a) $\log_3 \frac{1}{81} = x$
    (b) $\log_6 36 = x$
23. (a) $\log_3 x = -1$
    (b) $\log_2 x = -4$
24. (a) $\log_4 x = -2$
    (b) $\log_5 x = 3$
25. (a) $x^2 - x = \log_5 25$
    (b) $3x + 5 = \log_2 64$
26. (a) $\log_3 x + \log_3(x - 2) = 1$
    (b) $\log_{10}(x + 3) - \log_{10} x = 1$

Solving an Equation **In Exercises 27–36, solve the equation accurate to three decimal places.**

27. $3^{2x} = 75$
28. $6^{-2x} = 74$
29. $2^{3-z} = 625$
30. $3(5^{x-1}) = 86$
31. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$
32. $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$
33. $\log_2(x - 1) = 5$
34. $\log_{10}(t - 3) = 2.6$
35. $\log_7 x^3 = 1.9$
36. $\log_5 \sqrt{x - 4} = 3.2$

Inverse Functions **In Exercises 37 and 38, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.**

37. $f(x) = 4^x$
    $g(x) = \log_4 x$
38. $f(x) = 3^x$
    $g(x) = \log_3 x$

Finding a Derivative **In Exercises 39–60, find the derivative of the function.**

39. $f(x) = 4^x$
40. $f(x) = 3^{4x}$
41. $y = 5^{-4x}$
42. $y = 6^{3x-4}$
43. $f(x) = x\,9^x$
44. $y = -7x(8^{-2x})$
45. $f(t) = \dfrac{-2t^2}{8^t}$
46. $f(t) = \dfrac{3^{2t}}{t}$
47. $h(\theta) = 2^{-\theta}\cos \pi\theta$
48. $g(\alpha) = 5^{-\alpha/2}\sin 2\alpha$
49. $y = \log_4(6x + 1)$
50. $y = \log_3(x^2 - 3x)$
51. $h(t) = \log_5(4 - t)^2$
52. $g(t) = \log_2(t^2 + 7)^3$
53. $y = \log_5 \sqrt{x^2 - 1}$
54. $f(x) = \log_2 \sqrt[3]{2x + 1}$
55. $f(x) = \log_2 \dfrac{x^2}{x - 1}$
56. $y = \log_{10} \dfrac{x^2 - 1}{x}$
57. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$
58. $g(x) = \log_5 \dfrac{4}{x^2\sqrt{1 - x}}$
59. $g(t) = \dfrac{10 \log_4 t}{t}$
60. $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

Finding an Equation of a Tangent Line **In Exercises 61–64, find an equation of the tangent line to the graph of the function at the given point.**

61. $y = 2^{-x}$, $(-1, 2)$
62. $y = 5^{x-2}$, $(2, 1)$
63. $y = \log_3 x$, $(27, 3)$
64. $y = \log_{10} 2x$, $(5, 1)$

Logarithmic Differentiation **In Exercises 65–68, use logarithmic differentiation to find dy/dx.**

65. $y = x^{2/x}$
66. $y = x^{x-1}$
67. $y = (x - 2)^{x+1}$
68. $y = (1 + x)^{1/x}$

Finding an Indefinite Integral **In Exercises 69–76, find the indefinite integral.**

69. $\int 3^x \, dx$

70. $\int 2^{-x} \, dx$

71. $\int (x^2 + 2^{-x}) \, dx$

72. $\int (x^4 + 5^x) \, dx$

73. $\int x(5^{-x^2}) \, dx$

74. $\int (4 - x)6^{(4-x)^2} \, dx$

75. $\int \frac{3^{2x}}{1 + 3^{2x}} \, dx$

76. $\int 2^{\sin x} \cos x \, dx$

Evaluating a Definite Integral **In Exercises 77–80, evaluate the definite integral. Use a graphing utility to verify your result.**

77. $\int_{-1}^{2} 2^x \, dx$

78. $\int_{-4}^{4} 3^{x/4} \, dx$

79. $\int_{0}^{1} (5^x - 3^x) \, dx$

80. $\int_{1}^{3} (4^{x+1} + 2^x) \, dx$

Area **In Exercises 81 and 82, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

81. $y = \frac{\log_4 x}{x}, \; y = 0, \; x = 1, \; x = 5$

82. $y = 3^{\cos x} \sin x, \; y = 0, \; x = 0, \; x = \pi$

EXPLORING CONCEPTS

83. **Exponential Function** What happens to the rate of change of the exponential function $y = a^x$ as a becomes larger?

84. **Logarithmic Function** What happens to the rate of change of the logarithmic function $y = \log_a x$ as a becomes larger?

85. **Analyzing a Logarithmic Equation** Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) Let x be a real number between 1000 and 10,000. Determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) When $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

86. **Comparing Rates of Growth** Order the functions

$$f(x) = \log_2 x, \; g(x) = x^x, \; h(x) = x^2, \text{ and } k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the least rate of growth for large values of x.

87. **Inflation** When the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) The price of an oil change for your car is presently \$24.95. Estimate the price 10 years from now.

(b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

88. **Depreciation** After t years, the value of a car purchased for \$25,000 is

$$V(t) = 25{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 89–92, complete the table by determining the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous Compounding
A						

89. $P = \$1000$
$r = 3\tfrac{1}{2}\%$
$t = 10$ years

90. $P = \$2500$
$r = 6\%$
$t = 20$ years

91. $P = \$7500$
$r = 4.8\%$
$t = 30$ years

92. $P = \$4000$
$r = 4\%$
$t = 15$ years

Compound Interest **In Exercises 93–96, complete the table by determining the amount of money P (present value) that should be invested at rate r to produce a balance of \$100,000 in t years.**

t	1	10	20	30	40	50
P						

93. $r = 4\%$
Compounded continuously

94. $r = 0.6\%$
Compounded continuously

95. $r = 5\%$
Compounded monthly

96. $r = 2\%$
Compounded daily

97. Compound Interest Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now (b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

98. Compound Interest Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a graphing utility to graph the exponential functions describing the growth of the investment over the 20 years for the following interest rates. Compare the ending balances.

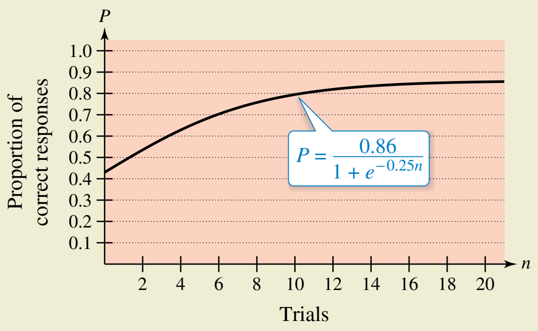
(a) $r = 3\%$ (b) $r = 5\%$ (c) $r = 6\%$

99. Timber Yield The yield V (in millions of cubic feet per acre) for a stand of timber at age t is $V = 6.7e^{-48.1/t}$, where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ and $t = 60$.

100. HOW DO YOU SEE IT? The graph shows the proportion P of correct responses after n trials in a group project in learning theory.



(a) What is the limiting proportion of correct responses as n approaches infinity?

(b) What happens to the rate of change of the proportion in the long run?

101. Population Growth A lake is stocked with 500 fish, and the population p is growing according to the logistic curve

$$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) Find the fish populations after 6 months, 12 months, 24 months, 36 months, and 48 months. What is the limiting size of the fish population?

(c) Find the rates at which the fish population is changing after 1 month and after 10 months.

(d) After how many months is the population increasing most rapidly?

102. Modeling Data

The breaking strengths B (in tons) of steel cables of various diameters d (in inches) are shown in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rates of growth of the model when $d = 0.8$ and $d = 1.5$.

103. Comparing Models The total numbers y of AIDS cases by year of diagnosis in Canada for the years 2005 through 2014 are shown in the table, with $x = 5$ corresponding to 2005. *(Source: Public Health Agency of Canada)*

x	5	6	7	8	9
y	434	398	371	367	296

x	10	11	12	13	14
y	276	234	223	226	188

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$

$y_2 = a + b \ln x$

$y_3 = ab^x$

$y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Find the rate of change of each of the models in part (a) for the year 2012. Which model is decreasing at the greatest rate in 2012?

104. An Approximation of *e* Complete the table to demonstrate that e can also be defined as

$$\lim_{x \to 0^+} (1 + x)^{1/x}.$$

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

pidjoe/iStock/Getty Images

Modeling Data **In Exercises 105 and 106, find an exponential function that fits the experimental data collected over time t.**

105.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

106.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

Using Properties of Exponents **In Exercises 107–110, find the exact value of the expression.**

107. $5^{1/\ln 5}$

108. $6^{(\ln 10)/\ln 6}$

109. $9^{1/\ln 3}$

110. $32^{1/\ln 2}$

111. Comparing Functions

(a) Show that $(2^3)^2 \neq 2^{(3^2)}$.

(b) Are

$$f(x) = (x^x)^x \quad \text{and} \quad g(x) = x^{(x^x)}$$

the same function? Why or why not?

(c) Find $f'(x)$ and $g'(x)$.

112. Finding an Inverse Function Let

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for $a > 0$, $a \neq 1$. Show that f has an inverse function. Then find f^{-1}.

113. Logistic Differential Equation Show that solving the logistic differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

results in the logistic growth function in Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

114. Using Properties of Exponents Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

115. Tangent Lines

(a) Determine y' given $y^x = x^y$.

(b) Find the slope of the tangent line to the graph of $y^x = x^y$ at each of the following points.

(i) (c, c) (ii) $(2, 4)$ (iii) $(4, 2)$

(c) At what points on the graph of $y^x = x^y$ does the tangent line not exist?

PUTNAM EXAM CHALLENGE

116. Which is greater

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} \quad \text{or} \quad \left(\sqrt{n+1}\right)^{\sqrt{n}}$$

where $n > 8$?

117. Show that if x is positive, then

$$\log_e\left(1 + \frac{1}{x}\right) > \frac{1}{1 + x}.$$

SECTION PROJECT

Using Graphing Utilities to Estimate Slope

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \leq x \leq 3$, $-2 \leq y \leq 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer using a graphing utility.

FOR FURTHER INFORMATION For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

5.6 Indeterminate Forms and L'Hôpital's Rule

- **Recognize limits that produce indeterminate forms.**
- **Apply L'Hôpital's Rule to evaluate a limit.**

Indeterminate Forms

Recall from Chapters 1 and 3 that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist. When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	Limit	Algebraic Technique
$\frac{0}{0}$	$\lim_{x\to -1} \frac{2x^2 - 2}{x + 1} = \lim_{x\to -1} 2(x - 1) = -4$	Divide numerator and denominator by $(x + 1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty} \frac{3x^2 - 1}{2x^2 + 1} = \lim_{x\to\infty} \frac{3 - (1/x^2)}{2 + (1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^x - 1}$$

produces the indeterminate form $0/0$. Factoring and then dividing produces

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x\to 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1} = \lim_{x\to 0} (e^x + 1) = 2.$$

Not all indeterminate forms, however, can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved. For instance, the limit

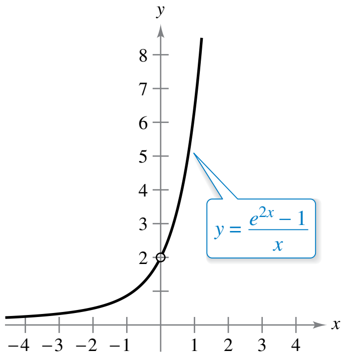
$$\lim_{x\to 0} \frac{e^{2x} - 1}{x}$$

produces the indeterminate form $0/0$. Rewriting the expression to obtain

$$\lim_{x\to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x}\right)$$

merely produces another indeterminate form, $\infty - \infty$. Of course, you could use technology to estimate the limit, as shown in the table and in Figure 5.23. From the table and the graph, the limit appears to be 2. (This limit will be verified in Example 1.)

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x} - 1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389



The limit as x approaches 0 appears to be 2.
Figure 5.23

GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "... I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. ... I have made free use of their discoveries ...," said L'Hôpital.

See LarsonCalculus.com to read more of this biography.

L'Hôpital's Rule

To find the limit illustrated in Figure 5.23, you can use a theorem called **L'Hôpital's Rule.** This theorem states that under certain conditions, the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem.**

THEOREM 5.16 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

A proof of this theorem is given in Appendix A.

To see why Theorem 5.16 is called the Extended Mean Value Theorem, consider the special case in which $g(x) = x$. For this case, you obtain the "standard" Mean Value Theorem as presented in Section 3.2.

THEOREM 5.17 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

A proof of this theorem is given in Appendix A.

FOR FURTHER INFORMATION To enhance your understanding of the necessity of the restriction that $g'(x)$ be nonzero for all x in (a, b), except possibly at c, see the article "Counterexamples to L'Hôpital's Rule" by R. P. Boas in *The American Mathematical Monthly.* To view this article, go to *MathArticles.com.*

People occasionally use L'Hôpital's Rule incorrectly by applying the Quotient Rule to $f(x)/g(x)$. Be sure you see that the rule involves

$$\frac{f'(x)}{g'(x)}$$

not the derivative of $f(x)/g(x)$.

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

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Exploration

Numerical and Graphical Approaches Use a numerical or a graphical approach to approximate each limit.

a. $\displaystyle\lim_{x\to 0}\frac{2^{2x}-1}{x}$

b. $\displaystyle\lim_{x\to 0}\frac{3^{2x}-1}{x}$

c. $\displaystyle\lim_{x\to 0}\frac{4^{2x}-1}{x}$

d. $\displaystyle\lim_{x\to 0}\frac{5^{2x}-1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for determining these limits? If so, explain your reasoning.

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{x}$.

Solution Because direct substitution results in the indeterminate form 0/0

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} \quad \begin{matrix} \nearrow \lim_{x\to 0}(e^{2x}-1)=0 \\ \searrow \lim_{x\to 0} x = 0 \end{matrix}$$

you can apply L'Hôpital's Rule, as shown below.

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}[e^{2x}-1]}{\frac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to 0}\frac{2e^{2x}}{1} \qquad \text{Differentiate numerator and denominator.}$$

$$= 2 \qquad \text{Evaluate the limit.}$$

In the solution to Example 1, note that you actually do not know that the first limit is equal to the second limit until you have shown that the second limit exists. In other words, if the second limit had not existed, then it would not have been permissible to apply L'Hôpital's Rule.

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form 0/0 or ∞/∞, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to\infty}\frac{1}{x} \qquad \text{Differentiate numerator and denominator.}$$

$$= 0. \qquad \text{Evaluate the limit.}$$

TECHNOLOGY Use a graphing utility to graph $y_1 = \ln x$ and $y_2 = x$ in the same viewing window. Which function grows faster as x approaches ∞? How is this observation related to Example 2?

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form, as shown in Example 3.

FOR FURTHER INFORMATION To read about the connection between Leonhard Euler and Guillaume L'Hôpital, see the article "When Euler Met l'Hôpital" by William Dunham in *Mathematics Magazine*. To view this article, go to *MathArticles.com.*

EXAMPLE 3 Applying L'Hôpital's Rule More than Once

Evaluate $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This limit yields the indeterminate form $(-\infty)/(-\infty)$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0.$$

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0 \cdot \infty$, 1^∞, ∞^0, 0^0, and $\infty - \infty$. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x\to0}\left(\frac{1}{x}\right)(x)}_{\text{Limit is 1.}}, \quad \underbrace{\lim_{x\to0}\left(\frac{2}{x}\right)(x)}_{\text{Limit is 2.}}, \quad \underbrace{\lim_{x\to\infty}\left(\frac{1}{e^x}\right)(x)}_{\text{Limit is 0.}}, \quad \underbrace{\lim_{x\to\infty}\left(\frac{1}{x}\right)(e^x)}_{\text{Limit is }\infty.}$$

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The remaining examples in this section show methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

EXAMPLE 4 Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

Solution Because direct substitution produces the indeterminate form $0 \cdot \infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞. In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} \qquad \text{Differentiate numerator and denominator.}$$

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x} \qquad \text{Simplify.}$$

$$= 0. \qquad \text{Evaluate the limit.}$$

When rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work, try the other form. For instance, in Example 4, you can write the limit as

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}}$$

which yields the indeterminate form $0/0$. As it happens, applying L'Hôpital's Rule to this limit produces

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$.

The indeterminate forms 1^∞, ∞^0, and 0^0 arise from limits of functions that have variable bases and variable exponents. When you previously encountered this type of function, you used logarithmic differentiation to find the derivative. You can use a similar procedure when taking limits, as shown in the next example.

EXAMPLE 5 Indeterminate Form 1^∞

Evaluate $\displaystyle\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

REMARK Note that the solution to Example 5 is an alternate proof of Theorem 5.15.

Solution Because direct substitution yields the indeterminate form 1^∞, you can proceed as follows. To begin, assume that the limit exists and is equal to y.

$$y = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$$

Taking the natural logarithm of each side produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right].$$

Because the natural logarithmic function is continuous, you can write

$$\ln y = \lim_{x\to\infty}\left[x \ln\left(1 + \frac{1}{x}\right)\right] \qquad \text{Indeterminate form } \infty \cdot 0$$

$$= \lim_{x\to\infty}\left(\frac{\ln[1 + (1/x)]}{1/x}\right) \qquad \text{Indeterminate form } 0/0$$

$$= \lim_{x\to\infty}\left(\frac{(-1/x^2)\{1/[1 + (1/x)]\}}{-1/x^2}\right) \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x\to\infty}\frac{1}{1 + (1/x)}$$

$$= 1.$$

Now, because you have shown that

$$\ln y = 1$$

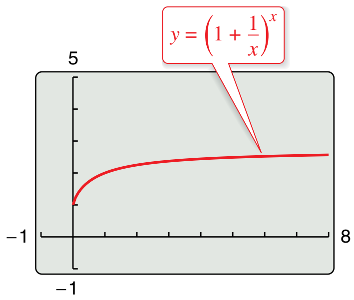
you can conclude that

$$y = e$$

and obtain

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 5.24.



The limit of $[1 + (1/x)]^x$ as x approaches infinity is e.
Figure 5.24

L'Hôpital's Rule can also be applied to one-sided limits, as demonstrated in Examples 6 and 7.

EXAMPLE 6 Indeterminate Form 0^0

See LarsonCalculus.com for an interactive version of this type of example.

Evaluate $\lim_{x\to 0^+} (\sin x)^x$.

Solution Because direct substitution produces the indeterminate form 0^0, you can proceed as shown below. To begin, assume that the limit exists and is equal to y.

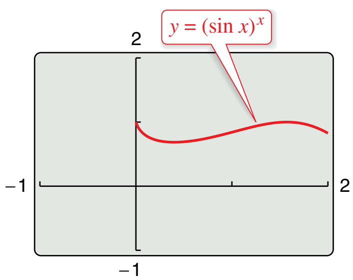
$$\begin{aligned}
y &= \lim_{x\to 0^+} (\sin x)^x && \text{Indeterminate form } 0^0 \\
\ln y &= \ln\Big[\lim_{x\to 0^+} (\sin x)^x\Big] && \text{Take natural log of each side.} \\
&= \lim_{x\to 0^+} [\ln(\sin x)^x] && \text{Continuity} \\
&= \lim_{x\to 0^+} [x \ln(\sin x)] && \text{Indeterminate form } 0 \cdot (-\infty) \\
&= \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} && \text{Indeterminate form } -\infty/\infty \\
&= \lim_{x\to 0^+} \frac{\cot x}{-1/x^2} && \text{L'Hôpital's Rule} \\
&= \lim_{x\to 0^+} \frac{-x^2}{\tan x} && \text{Indeterminate form } 0/0 \\
&= \lim_{x\to 0^+} \frac{-2x}{\sec^2 x} && \text{L'Hôpital's Rule} \\
&= 0
\end{aligned}$$

Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x\to 0^+} (\sin x)^x = 1.$$

TECHNOLOGY When evaluating complicated limits such as the one in Example 6, it is helpful to check the reasonableness of the solution with a graphing utility. For instance, the calculations in the table and the graph in the figure (see below) are consistent with the conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

x	1	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999



The limit of $(\sin x)^x$ is 1 as x approaches 0 from the right.

Use a graphing utility to estimate the limits $\lim_{x\to 0} (1 - \cos x)^x$ and $\lim_{x\to 0^+} (\tan x)^x$. Then try to verify your estimates analytically.

EXAMPLE 7 Indeterminate Form $\infty - \infty$

Evaluate $\displaystyle\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution Because direct substitution yields the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x\to1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to1^+}\frac{x-1-\ln x}{(x-1)\ln x}.$$

Now, because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to1^+}\frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to1^+}\frac{\frac{d}{dx}[x-1-\ln x]}{\frac{d}{dx}[(x-1)\ln x]}$$

$$= \lim_{x\to1^+}\frac{1-(1/x)}{(x-1)(1/x)+\ln x}$$

$$= \lim_{x\to1^+}\frac{x-1}{x-1+x\ln x}.$$

This limit also yields the indeterminate form $0/0$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to1^+}\frac{x-1}{x-1+x\ln x} = \lim_{x\to1^+}\frac{1}{1+x(1/x)+\ln x} = \frac{1}{2}.$$

You can check the reasonableness of this solution using a table, as shown at the left.

x	$\frac{1}{\ln x} - \frac{1}{x-1}$
2	0.44270
1.5	0.46630
1.1	0.49206
1.01	0.49917
1.001	0.49992
1.0001	0.49999
1.00001	0.50000

The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as "determinate."

$\infty + \infty \to \infty$ Limit is positive infinity.

$-\infty - \infty \to -\infty$ Limit is negative infinity.

$0^\infty \to 0$ Limit is zero.

$0^{-\infty} \to \infty$ Limit is positive infinity.

REMARK You are asked to verify the last two forms in Exercises 110 and 111.

As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞. For instance, the application of L'Hôpital's Rule shown below is *incorrect*.

$$\lim_{x\to0}\frac{e^x}{x} = \lim_{x\to0}\frac{e^x}{1} = 1$$ Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

Exploration

In each of the examples presented in this section, L'Hôpital's Rule is used to find a limit that exists. It can also be used to conclude that a limit is infinite. For instance, try using L'Hôpital's Rule to show that $\displaystyle\lim_{x\to\infty} e^x/x = \infty$.

5.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **L'Hopital's Rule** Explain the benefit of L'Hôpital's Rule.

2. **Indeterminate Forms** For each limit, use direct substitution. Then identify the form of the limit as either indeterminate or not.

(a) $\lim_{x\to 0} \dfrac{x^2}{\sin 2x}$ (b) $\lim_{x\to\infty} (e^x + x^2)$

(c) $\lim_{x\to\infty} (\ln x - e^x)$ (d) $\lim_{x\to 0^+} \left(\ln x^2 - \dfrac{1}{x}\right)$

Numerical and Graphical Analysis **In Exercises 3–6, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

3. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 3x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x\to 0} \dfrac{1 - e^x}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

5. $\lim_{x\to\infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

6. $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2 - 2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

Using Two Methods **In Exercises 7–14, evaluate the limit (a) using techniques from Chapters 1 and 3 and (b) using L'Hôpital's Rule.**

7. $\lim_{x\to 4} \dfrac{3(x-4)}{x^2 - 16}$

8. $\lim_{x\to -4} \dfrac{2x^2 + 13x + 20}{x + 4}$

9. $\lim_{x\to 6} \dfrac{\sqrt{x+10} - 4}{x - 6}$

10. $\lim_{x\to -1} \left(\dfrac{1 - \sqrt{x+2}}{x+1}\right)$

11. $\lim_{x\to 0} \left(\dfrac{2 - 2\cos x}{6x}\right)$

12. $\lim_{x\to 0} \dfrac{\sin 6x}{4x}$

13. $\lim_{x\to\infty} \dfrac{5x^2 - 3x + 1}{3x^2 - 5}$

14. $\lim_{x\to\infty} \dfrac{x^3 + 2x}{4 - x}$

Evaluating a Limit **In Exercises 15–42, evaluate the limit, using L'Hôpital's Rule if necessary.**

15. $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x - 3}$

16. $\lim_{x\to -2} \dfrac{x^2 - 3x - 10}{x + 2}$

17. $\lim_{x\to 0} \dfrac{\sqrt{25 - x^2} - 5}{x}$

18. $\lim_{x\to 5^-} \dfrac{\sqrt{25 - x^2}}{x - 5}$

19. $\lim_{x\to 0^+} \dfrac{e^x - (1 + x)}{x^3}$

20. $\lim_{x\to 1} \dfrac{\ln x^3}{x^2 - 1}$

21. $\lim_{x\to 1} \dfrac{x^{11} - 1}{x^4 - 1}$

22. $\lim_{x\to 1} \dfrac{x^a - 1}{x^b - 1}$, where $a, b \neq 0$

23. $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$

24. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$, where $a, b \neq 0$

25. $\lim_{x\to\infty} \dfrac{7x^3 - 2x + 1}{6x^3 + 1}$

26. $\lim_{x\to\infty} \dfrac{8 - x}{x^3}$

27. $\lim_{x\to\infty} \dfrac{x^2 + 4x + 7}{x - 6}$

28. $\lim_{x\to\infty} \dfrac{x^3}{x + 2}$

29. $\lim_{x\to\infty} \dfrac{x^3}{e^{x/2}}$

30. $\lim_{x\to\infty} \dfrac{e^{x^2}}{1 - x^3}$

31. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}}$

32. $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x^2 + 1}}$

33. $\lim_{x\to\infty} \dfrac{\cos x}{x}$

34. $\lim_{x\to\infty} \dfrac{\sin x}{x - \pi}$

35. $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$

36. $\lim_{x\to\infty} \dfrac{\ln x^4}{x^3}$

37. $\lim_{x\to\infty} \dfrac{e^x}{x^4}$

38. $\lim_{x\to\infty} \dfrac{e^{2x-9}}{3x}$

39. $\lim_{x\to 0} \dfrac{\sin 5x}{\tan 9x}$

40. $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x}$

41. $\lim_{x\to\infty} \dfrac{\int_1^x \ln(e^{4t-1})\,dt}{x}$

42. $\lim_{x\to 1^+} \dfrac{\int_1^x \cos\theta\,d\theta}{x - 1}$

Evaluating a Limit **In Exercises 43–62, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).**

43. $\lim_{x\to\infty} x \ln x$

44. $\lim_{x\to 0^+} x^3 \cot x$

45. $\lim_{x\to\infty} x \sin \dfrac{1}{x}$

46. $\lim_{x\to\infty} x \tan \dfrac{1}{x}$

47. $\lim_{x\to 0^+} (e^x + x)^{2/x}$

48. $\lim_{x\to 0^+} \left(1 + \dfrac{1}{x}\right)^x$

49. $\lim_{x\to\infty} x^{1/x}$

50. $\lim_{x\to 0^+} x^{1/x}$

51. $\lim_{x\to 0^+} (1 + x)^{1/x}$

52. $\lim_{x\to\infty} (1 + x)^{1/x}$

53. $\lim_{x\to 0^+} 3x^{x/2}$

54. $\lim_{x\to 4^+} [3(x-4)]^{x-4}$

55. $\lim_{x\to 1^+} (\ln x)^{x-1}$

56. $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$

57. $\lim_{x\to 2^+} \left(\frac{8}{x^2-4}-\frac{x}{x-2}\right)$

58. $\lim_{x\to 2^+} \left(\frac{1}{x^2-4}-\frac{\sqrt{x-1}}{x^2-4}\right)$

59. $\lim_{x\to 1^+} \left(\frac{3}{\ln x}-\frac{2}{x-1}\right)$

60. $\lim_{x\to 0^+} \left(\frac{10}{x}-\frac{3}{x^2}\right)$

61. $\lim_{x\to\infty} (e^x - x)$

62. $\lim_{x\to\infty} \left(x-\sqrt{x^2+1}\right)$

EXPLORING CONCEPTS

63. Finding Functions Find differentiable functions f and g that satisfy the specified condition such that

$$\lim_{x\to 5} f(x) = 0 \quad\text{and}\quad \lim_{x\to 5} g(x) = 0.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

64. Finding Functions Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \text{ and } \lim_{x\to\infty} [f(x) - g(x)] = 25.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

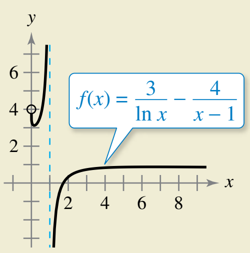
65. L'Hôpital's Rule Determine which of the following limits can be evaluated using L'Hôpital's Rule. Explain your reasoning. Do not evaluate the limit.

(a) $\lim_{x\to 2} \frac{x-2}{x^3-x-6}$ (b) $\lim_{x\to 0} \frac{x^2-4x}{2x-1}$

(c) $\lim_{x\to\infty} \frac{x^3}{e^x}$ (d) $\lim_{x\to 3} \frac{e^{x^2}-e^9}{x-3}$

(e) $\lim_{x\to 1} \frac{\cos \pi x}{\ln x}$ (f) $\lim_{x\to 1} \frac{1+x(\ln x - 1)}{(x-1)\ln x}$

66. HOW DO YOU SEE IT? Use the graph of f to find each limit.



(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

67. Numerical Analysis Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

68. Numerical Analysis Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

Comparing Functions **In Exercises 69–74, use L'Hôpital's Rule to determine the comparative rates of increase of the functions $f(x) = x^m$, $g(x) = e^{nx}$, and $h(x) = (\ln x)^n$, where $n > 0$, $m > 0$, and $x \to \infty$.**

69. $\lim_{x\to\infty} \frac{x^2}{e^{5x}}$

70. $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

71. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

72. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$

73. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

74. $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

Asymptotes and Relative Extrema **In Exercises 75–78, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function.**

75. $y = x^{1/x}$, $x > 0$

76. $y = x^x$, $x > 0$

77. $y = 2xe^{-x}$

78. $y = \frac{\ln x}{x}$

Think About It **In Exercises 79–82, L'Hôpital's Rule is used incorrectly. Describe the error.**

79. $\lim_{x\to 2} \frac{3x^2+4x+1}{x^2-x-2} = \lim_{x\to 2} \frac{6x+4}{2x-1} = \lim_{x\to 2} \frac{6}{2} = 3$

80. $\lim_{x\to 0} \frac{e^{2x}-1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x}$

$= \lim_{x\to 0} 2e^x$

$= 2$

81. $\lim_{x\to\infty} \frac{e^{-x}}{1+e^{-x}} = \lim_{x\to\infty} \frac{-e^{-x}}{-e^{-x}}$

$= \lim_{x\to\infty} 1$

$= 1$

82. $\lim_{x\to\infty} x\cos\frac{1}{x} = \lim_{x\to\infty} \frac{\cos(1/x)}{1/x}$

$= \lim_{x\to\infty} \frac{[-\sin(1/x)](1/x^2)}{-1/x^2}$

$= \lim_{x\to\infty} \sin\frac{1}{x}$

$= 0$

Analytic and Graphical Analysis **In Exercises 83 and 84, (a) explain why L'Hôpital's Rule cannot be used to find the limit, (b) find the limit analytically, and (c) use a graphing utility to graph the function and approximate the limit from the graph. Compare the result with that in part (b).**

83. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ **84.** $\lim_{x\to\pi/2^-} \dfrac{\tan x}{\sec x}$

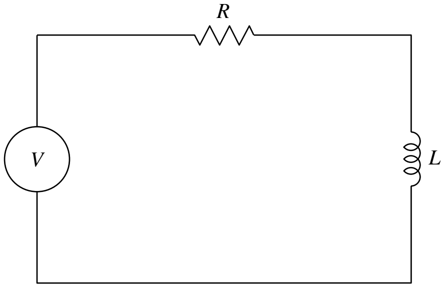
Graphical Analysis **In Exercises 85 and 86, graph $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$. What do you notice about these ratios as $x \to 0$? How does this illustrate L'Hôpital's Rule?**

85. $f(x) = \sin 3x,\ g(x) = \sin 4x$ **86.** $f(x) = e^{3x} - 1,\ g(x) = x$

87. Electric Circuit The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor. If voltage V is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}\left(1 - e^{-Rt/L}\right)$$

where L is the inductance and R is the resistance. Use L'Hôpital's Rule to find the formula for the current by fixing V and L and letting R approach 0 from the right.



88. Velocity in a Resisting Medium The velocity v of an object falling through a resisting medium such as air or water is given by

$$v = \frac{32}{k}\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)$$

where v_0 is the initial velocity, t is the time in seconds, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling body in a vacuum by fixing v_0 and t and letting k approach zero. (Assume that the downward direction is positive.)

89. The Gamma Function The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function given by $f(x) = x^{n-1}e^{-x}$, $n > 0$. Show that for any fixed value of n, the limit of $f(x)$ as x approaches infinity is zero.

90. Compound Interest The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year approaches infinity is given by $A = Pe^{rt}$.

Extended Mean Value Theorem **In Exercises 91–94, verify that the Extended Mean Value Theorem can be applied to the functions f and g on the closed interval $[a, b]$. Then find all values c in the open interval (a, b) such that**

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

	Functions	**Interval**
91.	$f(x) = x^3,\ g(x) = x^2 + 1$	$[0, 1]$
92.	$f(x) = \dfrac{1}{x},\ g(x) = x^2 - 4$	$[1, 2]$
93.	$f(x) = \sin x,\ g(x) = \cos x$	$\left[0, \dfrac{\pi}{2}\right]$
94.	$f(x) = \ln x,\ g(x) = x^3$	$[1, 4]$

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. A limit of the form $\infty/0$ is indeterminate.

96. A limit of the form $\infty \cdot \infty$ is indeterminate.

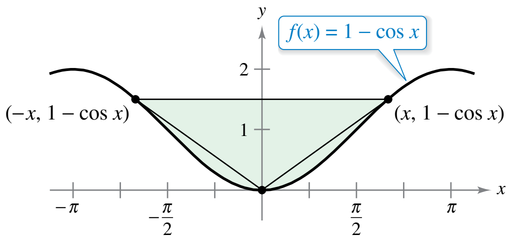
97. An indeterminate form does not guarantee the existence of a limit.

98. $\lim_{x\to 0} \dfrac{x^2 + x + 1}{x} = \lim_{x\to 0} \dfrac{2x + 1}{1} = 1$

99. If $p(x)$ is a polynomial, then $\lim_{x\to\infty} \dfrac{p(x)}{e^x} = 0$.

100. If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

101. Area Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



102. Finding a Limit In Section 1.3, a geometric argument (see figure) was used to prove that

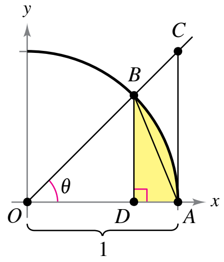
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

(a) Write the area of $\triangle ABD$ in terms of θ.

(b) Write the area of the shaded region in terms of θ.

(c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.



Continuous Function **In Exercises 103 and 104, find the value of c that makes the function continuous at $x = 0$.**

103. $f(x) = \begin{cases} \dfrac{4x - 2\sin 2x}{2x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$

104. $f(x) = \begin{cases} (e^x + x)^{1/x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

105. Finding Values Find the values of a and b such that

$$\lim_{x\to 0} \frac{a - \cos bx}{x^2} = 2.$$

106. Evaluating a Limit Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01. Then evaluate the limit

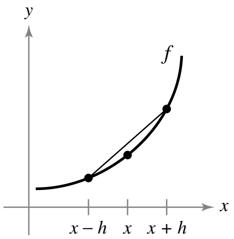
$$\lim_{k\to 0^+} \frac{x^k - 1}{k}.$$

107. Finding a Derivative

(a) Let $f'(x)$ be continuous. Show that

$$\lim_{h\to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).$$

(b) Explain the result of part (a) graphically.



108. Finding a Second Derivative Let $f''(x)$ be continuous. Show that

$$\lim_{h\to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).$$

109. Evaluating a Limit Consider the limit $\lim_{x\to 0^+} (-x \ln x)$.

(a) Describe the type of indeterminate form that is obtained by direct substitution.

(b) Evaluate the limit. Use a graphing utility to verify the result.

FOR FURTHER INFORMATION For a geometric approach to this exercise, see the article "A Geometric Proof of $\lim_{d\to 0^+} (-d \ln d) = 0$" by John H. Mathews in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

110. Proof Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

111. Proof Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$.

112. Think About It Use two different methods to find the limit

$$\lim_{x\to\infty} \frac{\ln x^m}{\ln x^n}$$

where $m > 0$, $n > 0$, and $x > 0$.

113. Indeterminate Forms Show that the indeterminate forms 0^0, ∞^0, and 1^∞ do not always have a value of 1 by evaluating each limit.

(a) $\lim_{x\to 0^+} x^{(\ln 2)/(1 + \ln x)}$

(b) $\lim_{x\to\infty} x^{(\ln 2)/(1 + \ln x)}$

(c) $\lim_{x\to 0} (x + 1)^{(\ln 2)/x}$

114. Calculus History In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

115. Finding a Limit Consider the function

$$h(x) = \frac{x + \sin x}{x}.$$

(a) Use a graphing utility to graph the function. Then use the *zoom* and *trace* features to investigate $\lim_{x\to\infty} h(x)$.

(b) Find $\lim_{x\to\infty} h(x)$ analytically by writing

$$h(x) = \frac{x}{x} + \frac{\sin x}{x}.$$

(c) Can you use L'Hôpital's Rule to find $\lim_{x\to\infty} h(x)$? Explain your reasoning.

116. Evaluating a Limit Let $f(x) = x + x \sin x$ and $g(x) = x^2 - 4$.

(a) Show that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.

(b) Show that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$.

(c) Evaluate the limit

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

What do you notice?

(d) Do your answers to parts (a) through (c) contradict L'Hôpital's Rule? Explain your reasoning.

PUTNAM EXAM CHALLENGE

117. Evaluate $\lim_{x\to\infty} \left[\dfrac{1}{x} \cdot \dfrac{a^x - 1}{a - 1}\right]^{1/x}$ where $a > 0$, $a \neq 1$.

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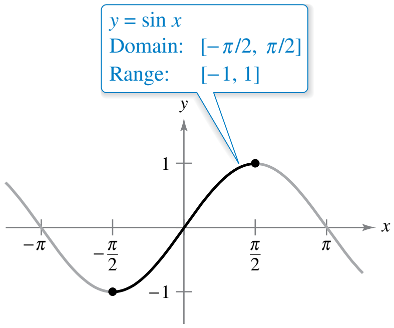
5.7 Inverse Trigonometric Functions: Differentiation

- **Develop properties of the six inverse trigonometric functions.**
- **Differentiate an inverse trigonometric function.**
- **Review the basic differentiation rules for elementary functions.**



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.25

Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section, you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains.*

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

as shown in Figure 5.25. On this interval, you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the next definition. (Note that the term "iff" is used to represent the phrase "if and only if.")

REMARK The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$."

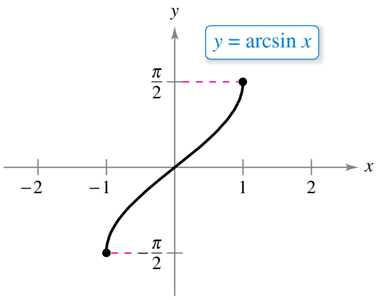
Definitions of Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$

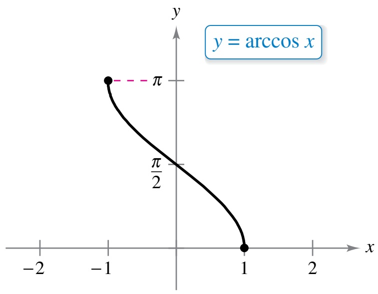
Exploration

The Inverse Secant Function In the definitions of the inverse trigonometric functions, the inverse secant function is defined by restricting the domain of the secant function to the intervals $[0, \pi/2) \cup (\pi/2, \pi]$. Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

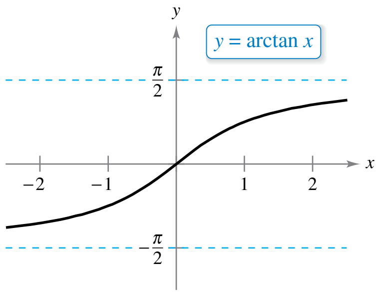
The graphs of the six inverse trigonometric functions are shown in Figure 5.26.



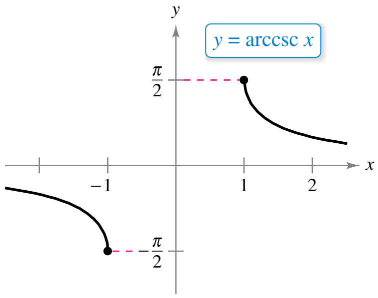
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



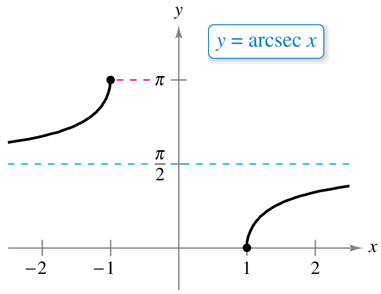
Domain: $[-1, 1]$
Range: $[0, \pi]$



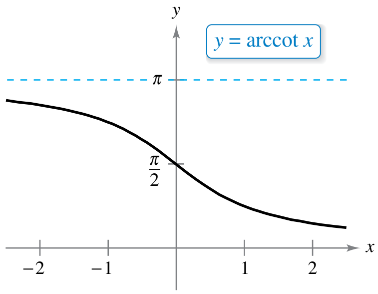
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Figure 5.26

When evaluating inverse trigonometric functions, remember that they denote angles in *radian measure*.

EXAMPLE 1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.305.$$

Inverse functions have the properties $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 2 Solving an Equation

$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Original equation}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4} \qquad \text{Take tangent of each side.}$$

$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 3.

EXAMPLE 3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

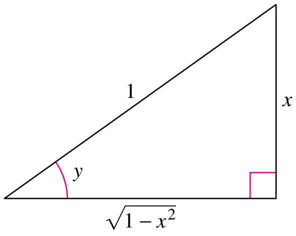
b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in the figure at the right.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

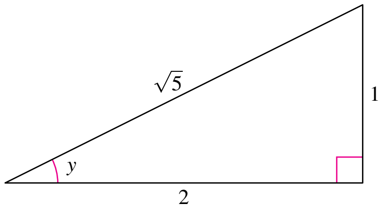
(This result is also valid for $-\pi/2 < y < 0$.)



$y = \arcsin x$

b. Use the right triangle shown in the figure at the left.

$$\tan y = \tan\left(\operatorname{arcsec}\frac{\sqrt{5}}{2}\right) = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$



$y = \operatorname{arcsec}\frac{\sqrt{5}}{2}$

Derivatives of Inverse Trigonometric Functions

In Section 5.1, you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The next theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

REMARK There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\operatorname{arcsec} x = \arccos \frac{1}{x}.$$

One consequence of this definition is that its graph has a positive slope at every x-value in its domain. (See Figure 5.26.) This accounts for the absolute value sign in the formula for the derivative of arcsec x.

THEOREM 5.18 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Proofs for arcsin u and arccos u are given in Appendix A. [The proofs for the other rules are left as an exercise (see Exercise 94).]

TECHNOLOGY If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \operatorname{arcsec} x = \arccos \frac{1}{x}.$$

EXAMPLE 4 Differentiating Inverse Trigonometric Functions

a. $\frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

c. $\frac{d}{dx}[\arcsin \sqrt{x}] = \frac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

d. $\frac{d}{dx}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$.

EXAMPLE 5 A Derivative That Can Be Simplified

$$y = \arcsin x + x\sqrt{1-x^2}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$= \sqrt{1-x^2} + \sqrt{1-x^2}$$

$$= 2\sqrt{1-x^2}$$

From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right).$$

EXAMPLE 6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$y' = 2(\arctan x)\left(\frac{1}{1 + x^2}\right)$$

$$= \frac{2 \arctan x}{1 + x^2}$$

you can see that the only critical number is $x = 0$. By the First Derivative Test, this value corresponds to a relative minimum. From the second derivative

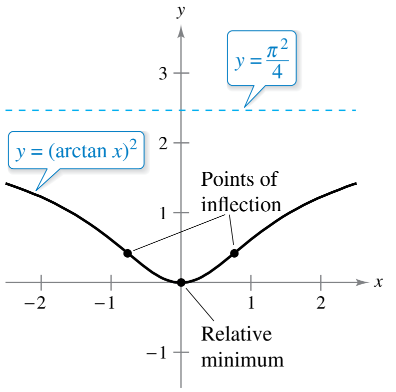
$$y'' = \frac{(1 + x^2)\left(\frac{2}{1 + x^2}\right) - (2 \arctan x)(2x)}{(1 + x^2)^2}$$

$$= \frac{2(1 - 2x \arctan x)}{(1 + x^2)^2}$$

it follows that points of inflection occur when $2x \arctan x = 1$. Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x \to \pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 5.27.

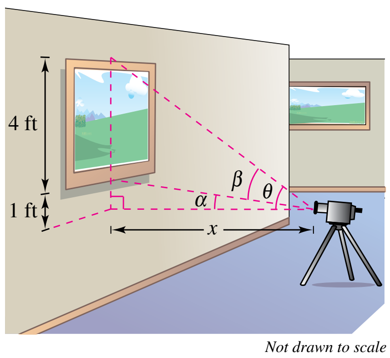


The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.27

EXAMPLE 7 Maximizing an Angle

See LarsonCalculus.com for an interactive version of this type of example.

A photographer is taking a picture of a painting hung in an art gallery. The height of the painting is 4 feet. The camera lens is 1 foot below the lower edge of the painting, as shown in the figure at the right. How far should the camera be from the painting to maximize the angle subtended by the camera lens?



The camera should be 2.236 feet from the painting to maximize the angle β.

Solution In the figure, let β be the angle to be maximized.

$$\beta = \theta - \alpha$$

$$= \operatorname{arccot} \frac{x}{5} - \operatorname{arccot} x$$

Differentiating produces

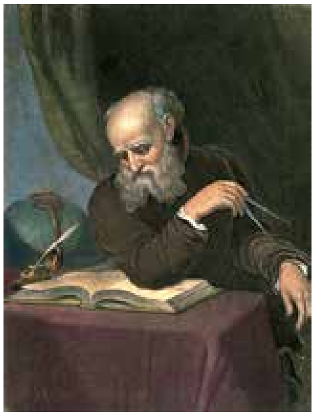
$$\frac{d\beta}{dx} = \frac{-1/5}{1 + (x^2/25)} - \frac{-1}{1 + x^2}$$

$$= \frac{-5}{25 + x^2} + \frac{1}{1 + x^2}$$

$$= \frac{4(5 - x^2)}{(25 + x^2)(1 + x^2)}.$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative Test that this distance yields a maximum value of β. So, the distance is $x \approx 2.236$ feet and the angle is $\beta \approx 0.7297$ radian $\approx 41.81°$.

REMARK In Example 7, you could also let $\theta = \arctan(5/x)$ and $\alpha = \arctan(1/x)$. Although these expressions are more difficult to use than those in Example 7, you should obtain the same answer. Try verifying this.



GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities*, such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities*, such as time, distance, force, and mass. *See LarsonCalculus.com to read more of this biography.*

Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	**Transcendental Functions**
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'),\ u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$
22. $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1 + u^2}$
23. $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

FOR FURTHER INFORMATION For more on the derivative of the arctangent function, see the article "Differentiating the Arctangent Directly" by Eric Key in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

* Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.

The Granger Collection

5.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

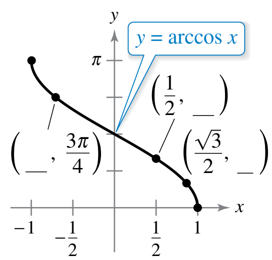
CONCEPT CHECK

1. **Inverse Trigonometric Function** Describe the meaning of arccos x in your own words.
2. **Restricted Domain** What is a restricted domain? Why are restricted domains necessary to define inverse trigonometric functions?
3. **Inverse Trigonometric Functions** Which inverse trigonometric function has a range of $0 < y < \pi$?
4. **Finding a Derivative** What is the missing value?

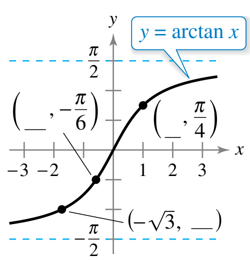
$$\frac{d}{dx}[\text{arccsc } x^3] = \frac{\square}{|x^3|\sqrt{x^6 - 1}}$$

Finding Coordinates **In Exercises 5 and 6, determine the missing coordinates of the points on the graph of the function.**

5.



6.



Evaluating Inverse Trigonometric Functions **In Exercises 7–14, evaluate the expression without using a calculator.**

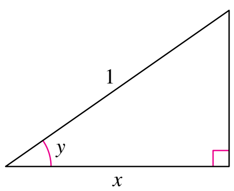
7. arcsin $\frac{1}{2}$
8. arcsin 0
9. arccos $\frac{1}{2}$
10. arccos(-1)
11. arctan $\frac{\sqrt{3}}{3}$
12. arccot$(-\sqrt{3})$
13. arccsc$(-\sqrt{2})$
14. arcsec 2

Approximating Inverse Trigonometric Functions **In Exercises 15–18, use a calculator to approximate the value. Round your answer to two decimal places.**

15. arccos(0.051)
16. arcsin(-0.39)
17. arcsec 1.269
18. arccsc(-4.487)

Using a Right Triangle **In Exercises 19–24, use the figure to write the expression in algebraic form given y = arccos x, where $0 < y < \pi/2$.**

19. cos y
20. sin y
21. tan y
22. cot y
23. sec y
24. csc y



Evaluating an Expression **In Exercises 25–28, evaluate each expression without using a calculator. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)**

25. (a) $\sin\left(\arctan \frac{3}{4}\right)$ (b) $\sec\left(\arcsin \frac{4}{5}\right)$
26. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$ (b) $\cos\left(\arcsin \frac{5}{13}\right)$
27. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$ (b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$
28. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$ (b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

Simplifying an Expression Using a Right Triangle **In Exercises 29–36, write the expression in algebraic form. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)**

29. $\cos(\arcsin 2x)$
30. $\sec(\arctan 6x)$
31. $\sin(\text{arcsec } x)$
32. $\cos(\text{arccot } x)$
33. $\tan\left(\text{arcsec } \frac{x}{3}\right)$
34. $\sec[\arcsin(x - 1)]$
35. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$
36. $\cos\left(\arcsin \frac{x - h}{r}\right)$

Solving an Equation **In Exercises 37–40, solve the equation for x.**

37. $\arcsin(3x - \pi) = \frac{1}{2}$
38. $\arctan(2x - 5) = -1$
39. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$
40. $\arccos x = \text{arcsec } x$

Finding a Derivative **In Exercises 41–56, find the derivative of the function.**

41. $f(x) = \arcsin(x - 1)$
42. $f(t) = \text{arccsc}(-t^2)$
43. $g(x) = 3 \arccos \frac{x}{2}$
44. $f(x) = \text{arcsec } 2x$
45. $f(x) = \arctan e^x$
46. $f(x) = \text{arccot } \sqrt{x}$
47. $g(x) = \frac{\arcsin 3x}{x}$
48. $h(x) = x^2 \arctan 5x$
49. $h(t) = \sin(\arccos t)$
50. $f(x) = \arcsin x + \arccos x$
51. $y = 2x \arccos x - 2\sqrt{1 - x^2}$
52. $y = x \arctan 2x - \frac{1}{4} \ln(1 + 4x^2)$
53. $y = \frac{1}{2}\left(\frac{1}{2} \ln \frac{x + 1}{x - 1} + \arctan x\right)$
54. $y = \frac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin \frac{x}{2}\right]$
55. $y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2}$
56. $y = \arctan x + \frac{x}{1 + x^2}$

Finding an Equation of a Tangent Line **In Exercises 57–62, find an equation of the tangent line to the graph of the function at the given point.**

57. $y = 2 \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{3}\right)$

58. $y = -\frac{1}{4} \arccos x, \quad \left(-\frac{1}{2}, -\frac{\pi}{6}\right)$

59. $y = \arctan \frac{x}{2}, \quad \left(2, \frac{\pi}{4}\right)$

60. $y = \operatorname{arcsec} 4x, \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

61. $y = 4x \arccos(x - 1), \quad (1, 2\pi)$

62. $y = 3x \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$

Finding Relative Extrema **In Exercises 63–66, find any relative extrema of the function.**

63. $f(x) = \operatorname{arcsec} x - x$

64. $f(x) = \arcsin x - 2x$

65. $f(x) = \arctan x - \arctan(x - 4)$

66. $h(x) = \arcsin x - 2 \arctan x$

Analyzing an Inverse Trigonometric Graph **In Exercises 67–70, analyze and sketch a graph of the function. Identify any relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.**

67. $f(x) = \arcsin(x - 1)$

68. $f(x) = \arctan x + \frac{\pi}{2}$

69. $f(x) = \operatorname{arcsec} 2x$

70. $f(x) = \arccos \frac{x}{4}$

Implicit Differentiation **In Exercises 71–74, use implicit differentiation to find an equation of the tangent line to the graph of the equation at the given point.**

71. $x^2 + x \arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$

72. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

73. $\arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

74. $\arctan(x + y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$

75. **Finding Values**

(a) Use a graphing utility to evaluate $\arcsin(\arcsin 0.5)$ and $\arcsin(\arcsin 1)$.

(b) Let

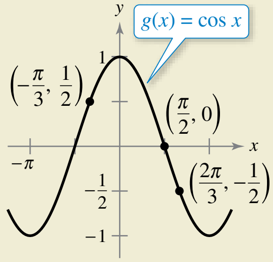
$$f(x) = \arcsin(\arcsin x).$$

Find the values of x in the interval $-1 \le x \le 1$ such that $f(x)$ is a real number.

76. **HOW DO YOU SEE IT?** The graph of $g(x) = \cos x$ is shown below. Explain whether the points

$$\left(-\frac{1}{2}, \frac{2\pi}{3}\right), \quad \left(0, \frac{\pi}{2}\right), \quad \text{and} \quad \left(\frac{1}{2}, -\frac{\pi}{3}\right)$$

lie on the graph of $y = \arccos x$.



EXPLORING CONCEPTS

77. **Inverse Trigonometric Functions** Determine whether

$$\frac{\arcsin x}{\arccos x} = \arctan x.$$

78. **Inverse Trigonometric Functions** Determine whether each inverse trigonometric function can be defined as shown. Explain.

(a) $y = \operatorname{arcsec} x$, Domain: $x > 1$, Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

(b) $y = \operatorname{arccsc} x$, Domain: $x > 1$, Range: $0 < y < \pi$

79. **Inverse Trigonometric Functions** Explain why $\sin 2\pi = 0$ does not imply that $\arcsin 0 = 2\pi$.

80. **Inverse Trigonometric Functions** Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

Verifying Identities **In Exercises 81 and 82, verify each identity.**

81. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, \quad |x| \ge 1$

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

82. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \le 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \le 1$

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

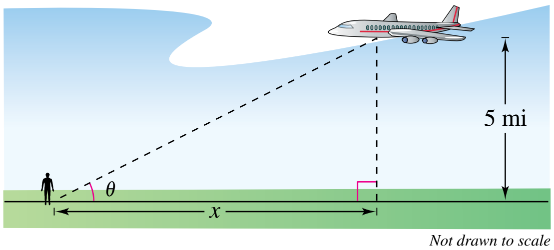
83. The slope of the graph of the inverse tangent function is positive for all x.

84. The range of $y = \arcsin x$ is $[0, \pi]$.

85. $\frac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

86. $\arcsin^2 x + \arccos^2 x = 1$

87. Angular Rate of Change An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.

88. Writing Repeat Exercise 87 for an altitude of 3 miles and describe how the altitude affects the rate of change of θ.

89. Angular Rate of Change In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).

(a) Find the position function that yields the height of the object at time t, assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

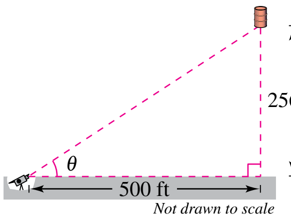


Figure for 89

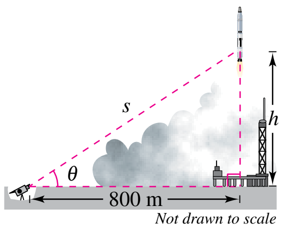


Figure for 90

90. Angular Rate of Change A television camera at ground level is filming the lift-off of a rocket at a point 800 meters from the launch pad. Let θ be the angle of elevation of the rocket and let s be the distance between the camera and the rocket (see figure). Write θ as a function of s for the period of time when the rocket is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

91. Maximizing an Angle A billboard 85 feet wide is perpendicular to a straight road and is 40 feet from the road (see figure). Find the point on the road at which the angle θ subtended by the billboard is a maximum.

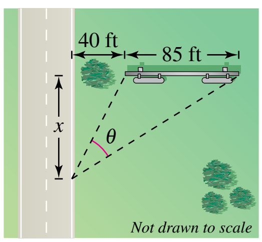


Figure for 91

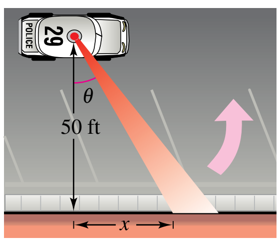


Figure for 92

92. Angular Speed A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. Write θ as a function of x. How fast is the light beam moving along the wall when the beam makes an angle of $\theta = 45°$ with the line perpendicular from the light to the wall?

93. Proof

(a) Prove that $\arctan x + \arctan y = \arctan \dfrac{x + y}{1 - xy}$, $xy \neq 1$.

(b) Use the formula in part (a) to show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}.$$

94. Proof Prove each differentiation formula.

(a) $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$

(b) $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1 + u^2}$

(c) $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$

(d) $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$

95. Describing a Graph Use a graphing utility to graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$.

(a) Describe the graph of f.

(b) Verify the result of part (a) analytically.

96. Think About It Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Explain why the graph of g is not the line $y = x$.

(b) Determine the extrema of g.

97. Maximizing an Angle In the figure, find the value of c in the interval $[0, 4]$ on the x-axis that maximizes angle θ.

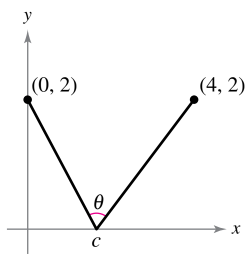


Figure for 97

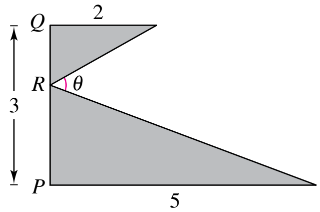


Figure for 98

98. Finding a Distance In the figure, find PR such that $0 \leq PR \leq 3$ and $m\angle\theta$ is a maximum.

99. Proof Prove that $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1 - x^2}}\right)$, $|x| < 1$.

100. Inverse Secant Function Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup [\pi, 3\pi/2)$.

(a) Sketch the graph $y = \operatorname{arcsec} x$ using this range.

(b) Show that $y' = \dfrac{1}{x\sqrt{x^2 - 1}}$.

5.8 Inverse Trigonometric Functions: Integration

- **Integrate functions whose antiderivatives involve inverse trigonometric functions.**
- **Use the method of completing the square to integrate a function.**
- **Review the basic integration rules involving elementary functions.**

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1 - x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you (see Exercises 73–75).

FOR FURTHER INFORMATION For a detailed proof of rule 2 of Theorem 5.19, see the article "A Direct Proof of the Integral Formula for Arctangent" by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

THEOREM 5.19 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

EXAMPLE 1 Integration with Inverse Trigonometric Functions

a. $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$ $u = x, a = 2$

b. $\displaystyle\int \frac{dx}{2 + 9x^2} = \frac{1}{3}\int \frac{3\,dx}{(\sqrt{2})^2 + (3x)^2}$ $u = 3x, a = \sqrt{2}$

$$= \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$$

c. $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2 - 3^2}}$ $u = 2x, a = 3$

$$= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$$

The integrals in Example 1 are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

EXAMPLE 2 **Integration by Substitution**

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution As it stands, this integral does not fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces

$$u = e^x \implies du = e^x\,dx \implies dx = \frac{du}{e^x} = \frac{du}{u}.$$

With this substitution, you can integrate as shown.

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} && \text{Write } e^{2x} \text{ as } (e^x)^2.\\
&= \int \frac{du/u}{\sqrt{u^2-1}} && \text{Substitute.}\\
&= \int \frac{du}{u\sqrt{u^2-1}} && \text{Rewrite to fit Arcsecant Rule.}\\
&= \operatorname{arcsec}\frac{|u|}{1} + C && \text{Apply Arcsecant Rule.}\\
&= \operatorname{arcsec} e^x + C && \text{Back-substitute.}
\end{aligned}$$

TECHNOLOGY PITFALL A symbolic integration utility can be useful for integrating functions such as the one in Example 2. In some cases, however, the utility may fail to find an antiderivative for two reasons. First, some elementary functions do not have antiderivatives that are elementary functions. Second, every utility has limitations—you might have entered a function that the utility was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan\sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to the one found in Example 2.

EXAMPLE 3 **Rewriting as the Sum of Two Quotients**

Find $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx\\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C\\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C
\end{aligned}$$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

EXAMPLE 4 Completing the Square

See LarsonCalculus.com for an interactive version of this type of example.

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares, as shown.

$$x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7 = (x - 2)^2 + 3 = u^2 + a^2$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x - 2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x - 2}{\sqrt{3}} + C$$

When the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square of $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$3x - x^2 = -(x^2 - 3x) = -\left[x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right] = \left(\tfrac{3}{2}\right)^2 - \left(x - \tfrac{3}{2}\right)^2$$

EXAMPLE 5 Completing the Square

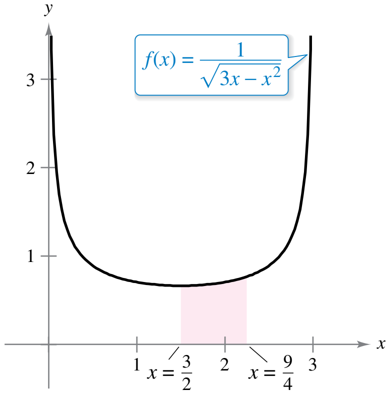
Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution In Figure 5.28, you can see that the area is

$$\begin{aligned} \text{Area} &= \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx \\ &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \qquad \text{Use completed square form derived above.} \\ &= \arcsin \frac{x - (3/2)}{3/2} \Bigg]_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524. \end{aligned}$$



The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.
Figure 5.28

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \arctan \dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on during the development of differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More important, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* solve with the techniques and rules you now know.

EXAMPLE 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

b. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

c. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx\\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C\\ &= \sqrt{x^2-1} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. (You should scan the list of basic integration rules to verify this conclusion.)

EXAMPLE 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x \ln x}$

b. $\displaystyle\int \frac{\ln x\,dx}{x}$

c. $\displaystyle\int \ln x\,dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1/x}{\ln x}\,dx\\ &= \ln|\ln x| + C\end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{\ln x\,dx}{x} &= \int \left(\frac{1}{x}\right)(\ln x)^1\,dx\\ &= \frac{(\ln x)^2}{2} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far.

REMARK Note in Examples 6 and 7 that the *simplest* functions are the ones that you cannot yet integrate.

5.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Integration Rules** Decide whether you can find each integral using the formulas and techniques you have studied so far. Explain.

(a) $\int \frac{2\,dx}{\sqrt{x^2+4}}$ (b) $\int \frac{dx}{x\sqrt{x^2-9}}$

2. **Completing the Square** In your own words, describe the process of completing the square of a quadratic function. Explain when completing the square is useful for finding an integral.

Finding an Indefinite Integral **In Exercises 3–22, find the indefinite integral.**

3. $\int \frac{dx}{\sqrt{9-x^2}}$

4. $\int \frac{dx}{\sqrt{1-4x^2}}$

5. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

6. $\int \frac{12}{1+9x^2}\,dx$

7. $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

8. $\int \frac{7}{4+(3-x)^2}\,dx$

9. $\int \frac{t}{\sqrt{1-t^4}}\,dt$

10. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

11. $\int \frac{t}{t^4+25}\,dt$

12. $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

13. $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

14. $\int \frac{5}{x\sqrt{9x^2-11}}\,dx$

15. $\int \frac{-\csc x\cot x}{\sqrt{25-\csc^2 x}}\,dx$

16. $\int \frac{\sin x}{7+\cos^2 x}\,dx$

17. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

18. $\int \frac{3}{2\sqrt{x}(1+x)}\,dx$

19. $\int \frac{x-3}{x^2+1}\,dx$

20. $\int \frac{x^2+8}{x\sqrt{x^2-4}}\,dx$

21. $\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$

22. $\int \frac{x-2}{(x+1)^2+4}\,dx$

Evaluating a Definite Integral **In Exercises 23–34, evaluate the definite integral.**

23. $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$

24. $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$

25. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$

26. $\int_{\sqrt{3}}^{3} \frac{1}{x\sqrt{4x^2-9}}\,dx$

27. $\int_1^7 \frac{1}{9+(x+2)^2}\,dx$

28. $\int_1^4 \frac{1}{x\sqrt{16x^2-5}}\,dx$

29. $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

30. $\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

31. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

32. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

33. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

34. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

Completing the Square **In Exercises 35–42, find or evaluate the integral by completing the square.**

35. $\int_0^2 \frac{dx}{x^2-2x+2}$

36. $\int_{-2}^{3} \frac{dx}{x^2+4x+8}$

37. $\int \frac{dx}{\sqrt{-2x^2+8x+4}}$

38. $\int \frac{dx}{3x^2-6x+12}$

39. $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$

40. $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

41. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

42. $\int_3^4 \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

Integration by Substitution **In Exercises 43–46, use the specified substitution to find or evaluate the integral.**

43. $\int \sqrt{e^t-3}\,dt$

$u = \sqrt{e^t-3}$

44. $\int \frac{\sqrt{x-2}}{x+1}\,dx$

$u = \sqrt{x-2}$

45. $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$

$u = \sqrt{x}$

46. $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

$u = \sqrt{x+1}$

Comparing Integration Problems **In Exercises 47–50, find the indefinite integrals, if possible, using the formulas and techniques you have studied so far in the text.**

47. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

(b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

(c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

48. (a) $\int e^{x^2}\,dx$

(b) $\int xe^{x^2}\,dx$

(c) $\int \frac{1}{x^2}e^{1/x}\,dx$

49. (a) $\int \sqrt{x-1}\,dx$

(b) $\int x\sqrt{x-1}\,dx$

(c) $\int \frac{x}{\sqrt{x-1}}\,dx$

50. (a) $\int \frac{1}{1+x^4}\,dx$

(b) $\int \frac{x}{1+x^4}\,dx$

(c) $\int \frac{x^3}{1+x^4}\,dx$

EXPLORING CONCEPTS

Comparing Antiderivatives **In Exercises 51 and 52, show that the antiderivatives are equivalent.**

51. $\displaystyle\int \frac{3x^2}{\sqrt{1-x^6}}\,dx = \arcsin x^3 + C \text{ or } \arccos\sqrt{1-x^6} + C$

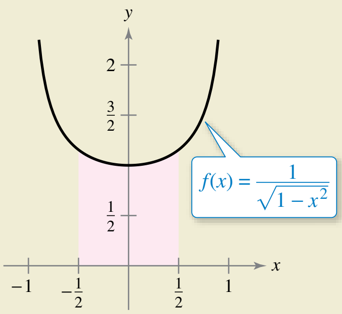
52. $\displaystyle\int \frac{6}{4+9x^2}\,dx = \arctan\frac{3x}{2} + C \text{ or } \operatorname{arccsc}\frac{\sqrt{4+9x^2}}{3x} + C$

53. Inverse Trigonometric Functions The antiderivative of

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

can be either $\arcsin x + C$ or $-\arccos x + C$. Does this mean that $\arcsin x = -\arccos x$? Explain.

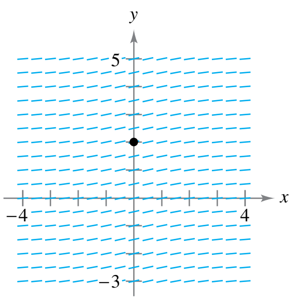
54. HOW DO YOU SEE IT? Using the graph, which value best approximates the area of the region between the x-axis and the function over the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$? Explain.



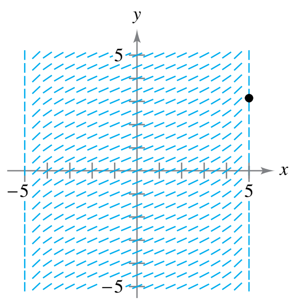
(a) -3 (b) $\frac{1}{2}$ (c) 1 (d) 2 (e) 4

Slope Field **In Exercises 55 and 56, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

55. $\dfrac{dy}{dx} = \dfrac{2}{9+x^2}$, $(0, 2)$



56. $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{25-x^2}}$, $(5, \pi)$



Slope Field **In Exercises 57–60, use a graphing utility to graph the slope field for the differential equation and graph the particular solution satisfying the specified initial condition.**

57. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2-1}}$

$y(3) = 0$

58. $\dfrac{dy}{dx} = \dfrac{1}{12+x^2}$

$y(4) = 2$

59. $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16-x^2}}$

$y(0) = 2$

60. $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1+x^2}$

$y(0) = 4$

Differential Equation **In Exercises 61 and 62, find the particular solution of the differential equation that satisfies the initial condition.**

61. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-x^2}}$

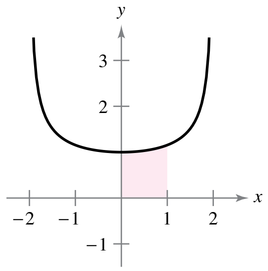
$y(0) = \pi$

62. $\dfrac{dy}{dx} = \dfrac{1}{4+x^2}$

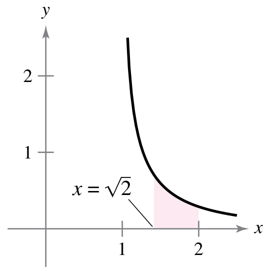
$y(2) = \pi$

Area **In Exercises 63–66, find the area of the given region. Use a graphing utility to verify your result.**

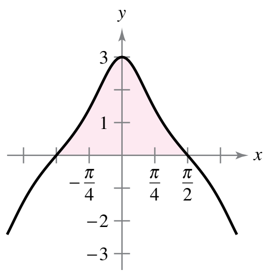
63. $y = \dfrac{2}{\sqrt{4-x^2}}$



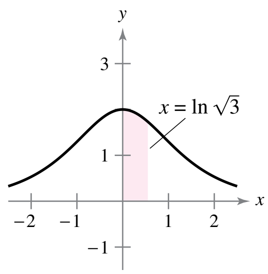
64. $y = \dfrac{1}{x\sqrt{x^2-1}}$



65. $y = \dfrac{3\cos x}{1+\sin^2 x}$



66. $y = \dfrac{4e^x}{1+e^{2x}}$



67. Area

(a) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x\,dx.$$

(b) Use the integration capabilities of a graphing utility to approximate the area.

(c) Find the exact area analytically.

68. Approximating Pi

(a) Show that

$$\int_0^1 \frac{4}{1+x^2}\,dx = \pi.$$

(b) Approximate the number π by using the integration capabilities of a graphing utility.

69. Investigation Consider the function

$$F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2+1}\,dt.$$

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function

$$f(x) = \frac{2}{x^2+1}.$$

Use what you have written to guess the value of x that will make F maximum.

(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

70. Comparing Integrals Consider the integral

$$\int \frac{1}{\sqrt{6x-x^2}}\,dx.$$

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) The antiderivatives in parts (a) and (b) appear to be significantly different. Use a graphing utility to graph each antiderivative in the same viewing window and determine the relationship between them. Find the domain of each.

True or False? **In Exercises 71 and 72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{4}\,\text{arcsec}\,\frac{3x}{4} + C$

72. $\displaystyle\int \frac{dx}{25+x^2} = \frac{1}{25}\arctan\frac{x}{25} + C$

Verifying an Integration Rule **In Exercises 73–75, verify the rule by differentiating. Let $a > 0$.**

73. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

74. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

75. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\text{arcsec}\,\frac{|u|}{a} + C$

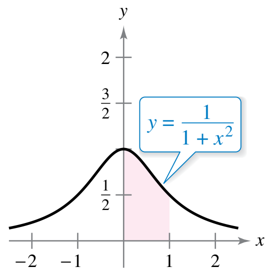
76. Proof Graph $y_1 = \dfrac{x}{1+x^2}$, $y_2 = \arctan x$, and $y_3 = x$ on $[0, 10]$. Prove that $\dfrac{x}{1+x^2} < \arctan x < x$ for $x > 0$.

77. Numerical Integration

(a) Write an integral that represents the area of the region in the figure.

(b) Use the Midpoint Rule with $n = 8$ to estimate the area of the region.

(c) Explain how you can use the results of parts (a) and (b) to estimate π.



78. Vertical Motion An object is projected upward from ground level with an initial velocity of 500 feet per second. In this exercise, the goal is to analyze the motion of the object during its upward flight.

(a) If air resistance is neglected, find the velocity of the object as a function of time. Use a graphing utility to graph this function.

(b) Use the result of part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where 32 feet per second per second is the acceleration due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32+kv^2} = -\int dt.$$

(d) Use a graphing utility to graph the velocity function $v(t)$ in part (c) for $k = 0.001$. Use the graph to approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t)\,dt$$

where $v(t)$ and t_0 are those found in part (d). This is the approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) and (e).

FOR FURTHER INFORMATION For more information on this topic, see the article "What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?" by John Lekner in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

5.9 Hyperbolic Functions

- **Develop properties of hyperbolic functions.**
- **Differentiate and integrate hyperbolic functions.**
- **Develop properties of inverse hyperbolic functions.**
- **Differentiate and integrate functions involving inverse hyperbolic functions.**

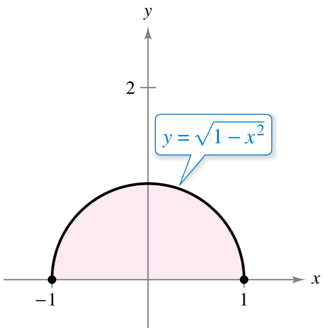
JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

See LarsonCalculus.com to read more of this biography.

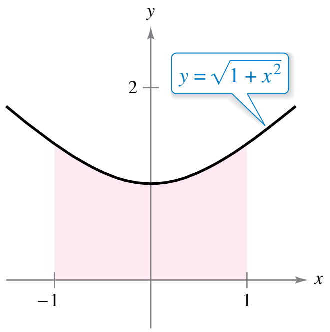
Hyperbolic Functions

In this section, you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.29, with the area of a region under a hyperbola, as shown in Figure 5.30.



Circle: $x^2 + y^2 = 1$

Figure 5.29



Hyperbola: $-x^2 + y^2 = 1$

Figure 5.30

The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{1 - x^2} + \arcsin x\right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1 + x^2}\, dx = \frac{1}{2}\left[x\sqrt{1 + x^2} + \sinh^{-1} x\right]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.

REMARK The notation $\sinh x$ is read as "the hyperbolic sine of x," $\cosh x$ as "the hyperbolic cosine of x," and so on.

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article "An Introduction to Hyperbolic Functions in Elementary Calculus" by Jerome Rosenthal in *Mathematics Teacher*. To view this article, go to *MathArticles.com*.

AIP Emilio Segre Visual Archives, Physics Today

The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.31. Note that the graph of $\sinh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph of $\cosh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.

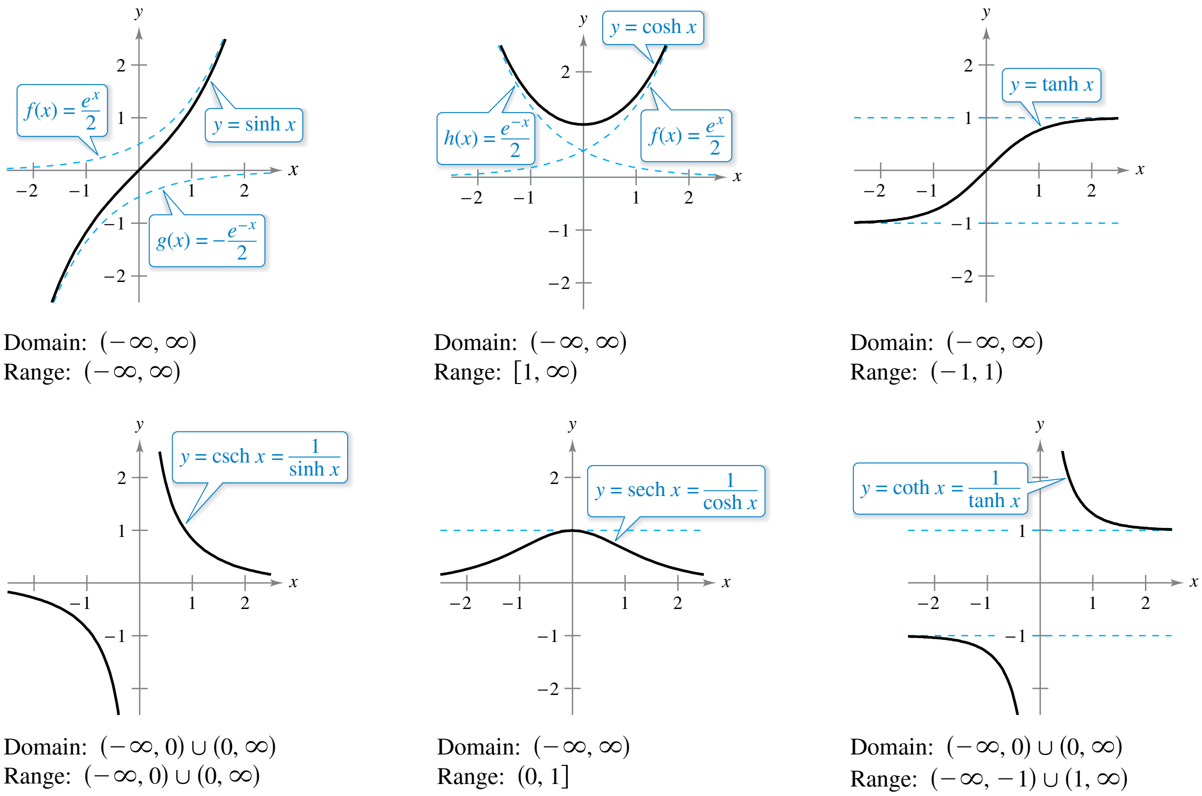


Figure 5.31

Many of the trigonometric identities have corresponding *hyperbolic identities*. For instance,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\\ &= \frac{4}{4}\\ &= 1.\end{aligned}$$

FOR FURTHER INFORMATION To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review*.

HYPERBOLIC IDENTITIES

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

$\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The next theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.20 Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\,du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\,du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\text{sech}^2\, u)u' \qquad \int \text{sech}^2\, u\,du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\text{csch}^2\, u)u' \qquad \int \text{csch}^2\, u\,du = -\coth u + C$$

$$\frac{d}{dx}[\text{sech}\, u] = -(\text{sech}\, u \tanh u)u' \qquad \int \text{sech}\, u \tanh u\,du = -\text{sech}\, u + C$$

$$\frac{d}{dx}[\text{csch}\, u] = -(\text{csch}\, u \coth u)u' \qquad \int \text{csch}\, u \coth u\,du = -\text{csch}\, u + C$$

Proof Here is a proof of two of the differentiation rules. (You are asked to prove some of the other differentiation rules in Exercises 99–101.)

$$\begin{aligned}\frac{d}{dx}[\sinh x] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right]\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[\tanh x] &= \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right]\\ &= \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \text{sech}^2\, x\end{aligned}$$

EXAMPLE 1 Differentiation of Hyperbolic Functions

a. $\frac{d}{dx}[\sinh(x^2 - 3)] = 2x\cosh(x^2 - 3)$

b. $\frac{d}{dx}[\ln(\cosh x)] = \frac{\sinh x}{\cosh x} = \tanh x$

c. $\frac{d}{dx}[x \sinh x - \cosh x] = x\cosh x + \sinh x - \sinh x = x\cosh x$

d. $\frac{d}{dx}[(x - 1)\cosh x - \sinh x] = (x - 1)\sinh x + \cosh x - \cosh x = (x - 1)\sinh x$

EXAMPLE 2 Finding Relative Extrema

Find the relative extrema of

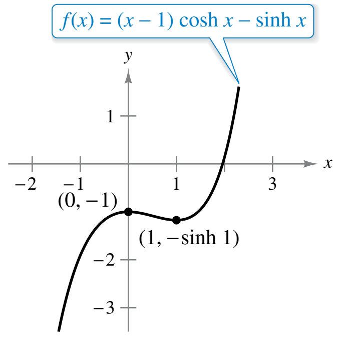
$$f(x) = (x - 1)\cosh x - \sinh x.$$

Solution Using the result of Example 1(d), set the first derivative of f equal to 0.

$$(x - 1)\sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, you can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh 1)$ yields a relative minimum, as shown in Figure 5.32. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions, as shown.

$$\begin{aligned} f(x) &= (x - 1)\left(\frac{1}{2}\right)(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) \\ &= \frac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \frac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.32

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a *catenary*, as discussed in Example 3.

EXAMPLE 3 Hanging Power Cables

See LarsonCalculus.com for an interactive version of this type of example.

Power cables are suspended between two towers, forming the catenary shown in Figure 5.33. The equation for this catenary is

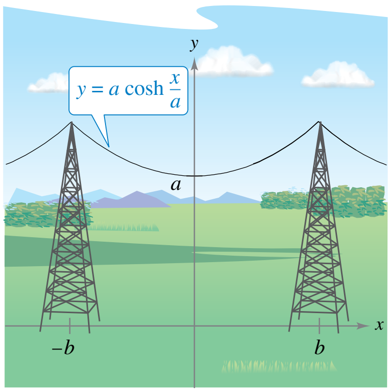
$$y = a\cosh\frac{x}{a}.$$

The distance between the two towers is $2b$. Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right)\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

At the point $(b, a\cosh(b/a))$, the slope (from the left) is $m = \sinh\frac{b}{a}$.



Catenary
Figure 5.33

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article "Reexamining the Catenary" by Paul Cella in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 4 Integrating a Hyperbolic Function

Find $\int \cosh 2x \sinh^2 2x\, dx$.

Solution

$$\begin{aligned} \int \cosh 2x \sinh^2 2x\, dx &= \frac{1}{2}\int (\sinh 2x)^2(2\cosh 2x)\, dx \qquad u = \sinh 2x \\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C \\ &= \frac{\sinh^3 2x}{6} + C \end{aligned}$$

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.31, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one when their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in the next theorem.

THEOREM 5.21 Inverse Hyperbolic Functions

Function	Domain
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$	$(-1, 1)$
$\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln \dfrac{1 + \sqrt{1 - x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

Proof The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, for

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

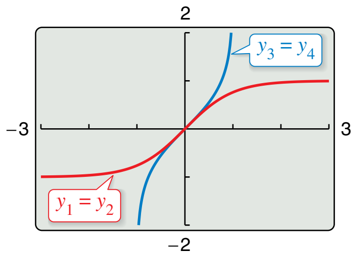
and

$$g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

you can show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

which implies that g is the inverse function of f. ■



Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function

Figure 5.34

TECHNOLOGY You can use a graphing utility to confirm graphically the results of Theorem 5.21. For instance, graph the following functions.

$y_1 = \tanh x$ — Hyperbolic tangent

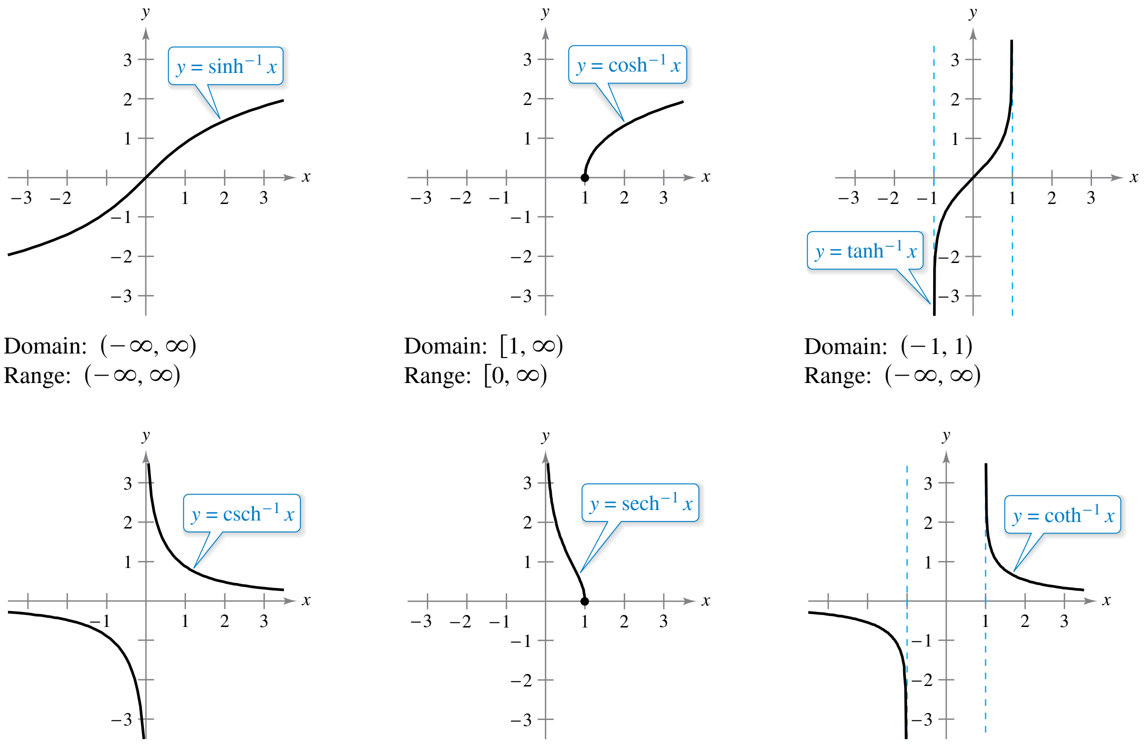
$y_2 = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ — Definition of hyperbolic tangent

$y_3 = \tanh^{-1} x$ — Inverse hyperbolic tangent

$y_4 = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$ — Definition of inverse hyperbolic tangent

The resulting display is shown in Figure 5.34. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

The graphs of the inverse hyperbolic functions are shown in Figure 5.35.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Domain: $[1, \infty)$
Range: $[0, \infty)$

Domain: $(-1, 1)$
Range: $(-\infty, \infty)$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(0, 1]$
Range: $[0, \infty)$

Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 5.35

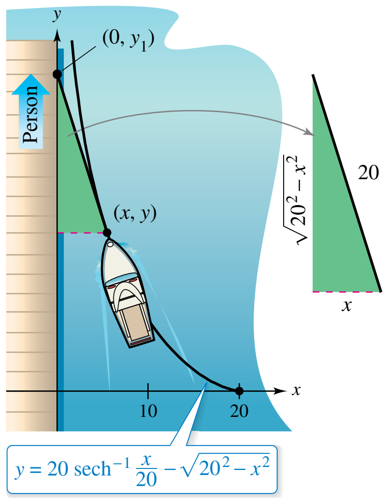
The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.

EXAMPLE 5 A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.36. As the person walks along the dock, the boat travels along a **tractrix,** given by the equation

$$y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$$

where a is the length of the rope. For $a = 20$ feet, find the distance the person must walk to bring the boat to a position 5 feet from the dock.



A person must walk about 41.27 feet to bring the boat to a position 5 feet from the dock.
Figure 5.36

Solution In Figure 5.36, notice that the distance the person has walked is

$$\begin{aligned} y_1 &= y + \sqrt{20^2 - x^2} \\ &= \left(20 \operatorname{sech}^{-1} \frac{x}{20} - \sqrt{20^2 - x^2}\right) + \sqrt{20^2 - x^2} \\ &= 20 \operatorname{sech}^{-1} \frac{x}{20}. \end{aligned}$$

When $x = 5$, this distance is

$$y_1 = 20 \operatorname{sech}^{-1} \frac{5}{20} = 20 \ln \frac{1 + \sqrt{1 - (1/4)^2}}{1/4} = 20 \ln\left(4 + \sqrt{15}\right) \approx 41.27 \text{ feet.}$$

So, the person must walk about 41.27 feet to bring the boat to a position 5 feet from the dock.

Inverse Hyperbolic Functions: Differentiation and Integration

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.22 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 102–104.)

THEOREM 5.22 Differentiation and Integration Involving Inverse Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1-u^2} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1-u^2}} \qquad \frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a+\sqrt{a^2 \pm u^2}}{|u|} + C$$

EXAMPLE 6 Differentiation of Inverse Hyperbolic Functions

a. $$\frac{d}{dx}[\sinh^{-1}(2x)] = \frac{2}{\sqrt{(2x)^2+1}} = \frac{2}{\sqrt{4x^2+1}}$$

b. $$\frac{d}{dx}[\tanh^{-1}(x^3)] = \frac{3x^2}{1-(x^3)^2} = \frac{3x^2}{1-x^6}$$

EXAMPLE 7 Integration Using Inverse Hyperbolic Functions

REMARK Let $a = 2$ and $u = 3x$.

a. $$\int \frac{dx}{x\sqrt{4-9x^2}} = \int \frac{3\,dx}{(3x)\sqrt{2^2-(3x)^2}} \qquad \int \frac{du}{u\sqrt{a^2-u^2}}$$

$$= -\frac{1}{2}\ln\frac{2+\sqrt{4-9x^2}}{|3x|} + C \qquad -\frac{1}{a}\ln\frac{a+\sqrt{a^2-u^2}}{|u|} + C$$

REMARK Let $a = \sqrt{5}$ and $u = 2x$.

b. $$\int \frac{dx}{5-4x^2} = \frac{1}{2}\int \frac{2\,dx}{(\sqrt{5})^2-(2x)^2} \qquad \int \frac{du}{a^2-u^2}$$

$$= \frac{1}{2}\left(\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x}{\sqrt{5}-2x}\right|\right) + C \qquad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$= \frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x}{\sqrt{5}-2x}\right| + C$$

5.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Hyperbolic Functions** Describe how the name *hyperbolic function* arose.
2. **Domains of Hyperbolic Functions** Which hyperbolic functions have domains that are not all real numbers?
3. **Hyperbolic Identities** Which hyperbolic identity corresponds to the trigonometric identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}?$$

4. **Derivatives of Inverse Hyperbolic Functions** What is the missing value?

$$\frac{d}{dx}[\text{sech}^{-1}(3x)] = \frac{\square}{3x\sqrt{1 - 9x^2}}$$

Evaluating a Function **In Exercises 5–10, evaluate the function. If the value is not a rational number, round your answer to three decimal places.**

5. (a) $\sinh 3$ (b) $\tanh(-2)$
6. (a) $\cosh 0$ (b) $\text{sech } 1$
7. (a) $\text{csch}(\ln 2)$ (b) $\coth(\ln 5)$
8. (a) $\sinh^{-1} 0$ (b) $\tanh^{-1} 0$
9. (a) $\cosh^{-1} 2$ (b) $\text{sech}^{-1} \frac{2}{3}$
10. (a) $\text{csch}^{-1} 2$ (b) $\coth^{-1} 3$

Verifying an Identity **In Exercises 11–18, verify the identity.**

11. $\sinh x + \cosh x = e^x$
12. $\cosh x - \sinh x = e^{-x}$
13. $\tanh^2 x + \text{sech}^2 x = 1$
14. $\coth^2 x - \text{csch}^2 x = 1$
15. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$
16. $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$
17. $\sinh 2x = 2 \sinh x \cosh x$
18. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

Finding Values of Hyperbolic Functions **In Exercises 19 and 20, use the value of the given hyperbolic function to find the values of the other hyperbolic functions.**

19. $\sinh x = \dfrac{3}{2}$
20. $\tanh x = \dfrac{1}{2}$

Finding a Limit **In Exercises 21–24, find the limit.**

21. $\lim_{x\to\infty} \sinh x$
22. $\lim_{x\to-\infty} \tanh x$
23. $\lim_{x\to 0} \dfrac{\sinh x}{x}$
24. $\lim_{x\to 0^-} \coth x$

Finding a Derivative **In Exercises 25–34, find the derivative of the function.**

25. $f(x) = \sinh 9x$
26. $f(x) = \cosh(8x + 1)$
27. $y = \text{sech } 5x^2$
28. $f(x) = \tanh(4x^2 + 3x)$
29. $f(x) = \ln(\sinh x)$
30. $y = \ln\left(\tanh \dfrac{x}{2}\right)$
31. $h(t) = \dfrac{t}{6} \sinh(-3t)$
32. $y = (x^2 + 1) \coth \dfrac{x}{3}$
33. $f(t) = \arctan(\sinh t)$
34. $g(x) = \text{sech}^2 3x$

Finding an Equation of a Tangent Line **In Exercises 35–38, find an equation of the tangent line to the graph of the function at the given point.**

35. $y = \sinh(1 - x^2), \quad (1, 0)$
36. $y = x^{\cosh x}, \quad (1, 1)$
37. $y = (\cosh x - \sinh x)^2, \quad (0, 1)$
38. $y = e^{\sinh x}, \quad (0, 1)$

Finding Relative Extrema **In Exercises 39–42, find the relative extrema of the function. Use a graphing utility to confirm your result.**

39. $g(x) = x \,\text{sech } x$
40. $h(x) = 2 \tanh x - x$
41. $f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$
42. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

Catenary **In Exercises 43 and 44, a model for a power cable suspended between two towers is given. (a) Graph the model. (b) Find the heights of the cable at the towers and at the midpoint between the towers. (c) Find the slope of the cable at the point where the cable meets the right-hand tower.**

43. $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$
44. $y = 18 + 25 \cosh \dfrac{x}{25}, \quad -25 \le x \le 25$

Finding an Indefinite Integral **In Exercises 45–54, find the indefinite integral.**

45. $\int \cosh 4x \, dx$
46. $\int \text{sech}^2 3x \, dx$
47. $\int \sinh(1 - 2x) \, dx$
48. $\int \dfrac{\cosh \sqrt{x}}{\sqrt{x}} \, dx$
49. $\int \cosh^2(x - 1) \sinh(x - 1) \, dx$
50. $\int \dfrac{\sinh x}{1 + \sinh^2 x} \, dx$

51. $\displaystyle\int \frac{\cosh x}{\sinh x}\,dx$ **52.** $\displaystyle\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\,dx$

53. $\displaystyle\int x\operatorname{csch}^2\frac{x^2}{2}\,dx$ **54.** $\displaystyle\int \operatorname{sech}^3 x\tanh x\,dx$

Evaluating a Definite Integral **In Exercises 55–60, evaluate the definite integral.**

55. $\displaystyle\int_0^{\ln 2} \tanh x\,dx$ **56.** $\displaystyle\int_0^1 \cosh^2 x\,dx$

57. $\displaystyle\int_3^4 \operatorname{csch}^2(x-2)\,dx$ **58.** $\displaystyle\int_{1/2}^1 \operatorname{sech}^2(2x-1)\,dx$

59. $\displaystyle\int_{5/3}^2 \operatorname{csch}(3x-4)\coth(3x-4)\,dx$

60. $\displaystyle\int_0^{\ln 2} 2e^{-x}\cosh x\,dx$

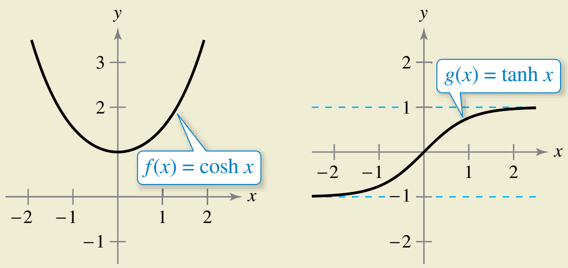
EXPLORING CONCEPTS

61. Using a Graph Explain graphically why there is no solution to $\cosh x = \sinh x$.

62. Hyperbolic Functions Use the graphs on page 391 to determine whether each hyperbolic function is even, odd, or neither.

63. Think About It Verify the results of Exercise 62 algebraically.

64. HOW DO YOU SEE IT? Use the graphs of f and g shown in the figures to answer the following.



(a) Identify the open interval(s) on which the graphs of f and g are increasing or decreasing.

(b) Identify the open interval(s) on which the graphs of f and g are concave upward or concave downward.

Finding a Derivative **In Exercises 65–74, find the derivative of the function.**

65. $y = \cosh^{-1}(3x)$ **66.** $y = \operatorname{csch}^{-1}(1-x)$

67. $y = \tanh^{-1}\sqrt{x}$ **68.** $f(x) = \coth^{-1}(x^2)$

69. $y = \sinh^{-1}(\tan x)$ **70.** $y = \tanh^{-1}(\sin 2x)$

71. $y = \operatorname{sech}^{-1}(\sin x),\ 0 < x < \pi/2$

72. $y = \coth^{-1}(e^{2x})$

73. $y = 2x\sinh^{-1}(2x) - \sqrt{1+4x^2}$

74. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2}$

Finding an Indefinite Integral **In Exercises 75–82, find the indefinite integral using the formulas from Theorem 5.22.**

75. $\displaystyle\int \frac{1}{3-9x^2}\,dx$ **76.** $\displaystyle\int \frac{1}{2x\sqrt{1-4x^2}}\,dx$

77. $\displaystyle\int \frac{1}{\sqrt{1+e^{2x}}}\,dx$ **78.** $\displaystyle\int \frac{x}{9-x^4}\,dx$

79. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1+x}}\,dx$ **80.** $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx$

81. $\displaystyle\int \frac{-1}{4x-x^2}\,dx$ **82.** $\displaystyle\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

Evaluating a Definite Integral **In Exercises 83–86, evaluate the definite integral using the formulas from Theorem 5.22.**

83. $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2-4}}\,dx$ **84.** $\displaystyle\int_1^3 \frac{1}{x\sqrt{4+x^2}}\,dx$

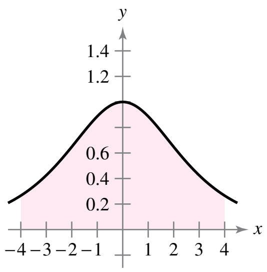
85. $\displaystyle\int_{-1}^1 \frac{1}{16-9x^2}\,dx$ **86.** $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2+1}}\,dx$

Differential Equation **In Exercises 87 and 88, find the general solution of the differential equation.**

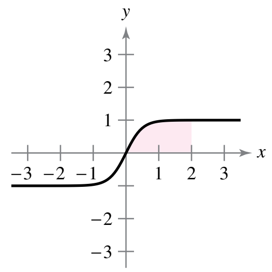
87. $\displaystyle\frac{dy}{dx} = \frac{x^3-21x}{5+4x-x^2}$ **88.** $\displaystyle\frac{dy}{dx} = \frac{1-2x}{4x-x^2}$

Area **In Exercises 89–92, find the area of the given region.**

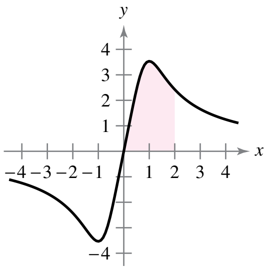
89. $y = \operatorname{sech}\dfrac{x}{2}$



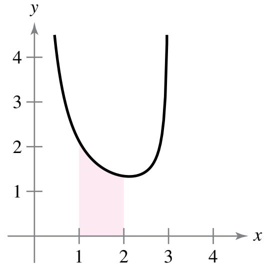
90. $y = \tanh 2x$



91. $y = \dfrac{5x}{\sqrt{x^4+1}}$



92. $y = \dfrac{6}{x\sqrt{9-x^2}}$



93. Tractrix Consider the equation of a tractrix

$$y = a\,\text{sech}^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}, \quad a > 0.$$

(a) Find dy/dx.

(b) Let L be the tangent line to the tractrix at the point P. When L intersects the y-axis at the point Q, show that the distance between P and Q is a.

94. Tractrix Show that the boat in Example 5 is always pointing toward the person.

95. Proof Prove that

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1.$$

96. Proof Prove that

$$\sinh^{-1} t = \ln\left(t + \sqrt{t^2 + 1}\right).$$

97. Using a Right Triangle Show that

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

98. Integration Let $x > 0$ and $b > 0$. Show that

$$\int_{-b}^{b} e^{xt}\,dt = \frac{2\sinh bx}{x}.$$

Proof In Exercises 99–101, prove the differentiation formula.

99. $\dfrac{d}{dx}[\cosh x] = \sinh x$

100. $\dfrac{d}{dx}[\coth x] = -\text{csch}^2\, x$

101. $\dfrac{d}{dx}[\text{sech}\, x] = -\text{sech}\, x \tanh x$

Verifying a Differentiation Formula In Exercises 102–104, verify the differentiation formula.

102. $\dfrac{d}{dx}[\cosh^{-1} x] = \dfrac{1}{\sqrt{x^2 - 1}}$

103. $\dfrac{d}{dx}[\sinh^{-1} x] = \dfrac{1}{\sqrt{x^2 + 1}}$

104. $\dfrac{d}{dx}[\text{sech}^{-1} x] = \dfrac{-1}{x\sqrt{1 - x^2}}$

PUTNAM EXAM CHALLENGE

105. From the vertex $(0, c)$ of the catenary $y = c\cosh(x/c)$ a line L is drawn perpendicular to the tangent to the catenary at point P. Prove that the length of L intercepted by the axes is equal to the ordinate y of the point P.

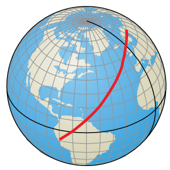
106. Prove or disprove: there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

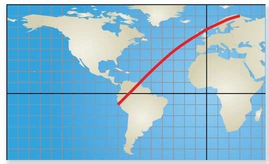
SECTION PROJECT

Mercator Map

When flying or sailing, pilots expect to be given a steady compass course to follow. On a standard flat map, this is difficult because a steady compass course results in a curved line, as shown below.

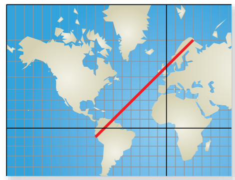


Globe: flight with constant 45° bearing



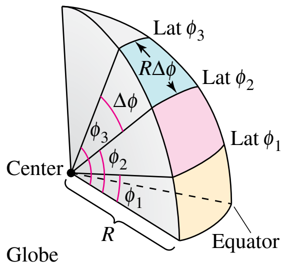
Standard flat map: flight with constant 45° bearing

For curved lines to appear as straight lines on a flat map, Flemish geographer Gerardus Mercator (1512-1594) realized that latitude lines must be stretched horizontally by a scaling factor of $\sec\phi$, where ϕ is the angle (in radians) of the latitude line. The Mercator map has latitude lines that are not equidistant, as shown at the right.

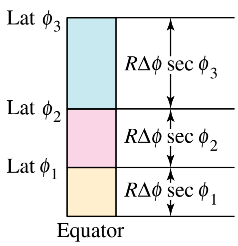


Mercator map: flight with constant 45° bearing

To calculate these vertical lengths, imagine a globe with radius R and latitude lines marked at angles of every $\Delta\phi$ radians, with $\Delta\phi = \phi_i - \phi_{i-1}$, as shown in the figure on the left below. The arc length of consecutive latitude lines is $R\Delta\phi$. On the corresponding Mercator map, the vertical distance between the ith and $(i-1)$st latitude lines is $R\Delta\phi\sec\phi_i$, and the total vertical distance from the equator to the nth latitude line is approximately $\sum_{i=1}^{n} R\Delta\phi\sec\phi_i$, as shown in the figure on the right below.



Globe



Mercator map

Mercator maps are still used by websites to display the world.

(a) Explain how to calculate the total vertical distance on a Mercator map from the equator to the nth latitude line using calculus.

(b) Using a globe radius of $R = 6$ inches, find the total vertical distances on a Mercator map from the equator to the latitude lines whose angles are 30°, 45°, and 60°.

(c) Explain what happens when you attempt to find the total vertical distance on a Mercator map from the equator to the North Pole.

(d) The *Gudermannian function* $\text{gd}(y) = \displaystyle\int_0^y \frac{dt}{\cosh t}$ expresses the latitude $\phi(y) = \text{gd}(y)$ in terms of the vertical position y on a Mercator map. Show that $\text{gd}(y) = \arctan(\sinh y)$.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Sketching a Graph In Exercises 1 and 2, sketch the graph of the function and state its domain.

1. $f(x) = \ln x - 3$
2. $f(x) = \ln(x + 3)$

Using Properties of Logarithms In Exercises 3 and 4, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 4 \approx 1.3863$ and $\ln 5 \approx 1.6094$.

3. (a) $\ln 20$ (b) $\ln \frac{4}{5}$ (c) $\ln 625$ (d) $\ln \sqrt{5}$
4. (a) $\ln 0.0625$ (b) $\ln \frac{5}{4}$ (c) $\ln 16$ (d) $\ln \sqrt[3]{80}$

Expanding a Logarithmic Expression In Exercises 5 and 6, use the properties of logarithms to expand the logarithmic expression.

5. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$
6. $\ln[(x^2 + 1)(x - 1)]$

Condensing a Logarithmic Expression In Exercises 7 and 8, write the expression as the logarithm of a single quantity.

7. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$
8. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

Finding a Derivative In Exercises 9–16, find the derivative of the function.

9. $g(x) = \ln \sqrt{2x}$
10. $f(x) = \ln(3x^2 + 2x)$
11. $f(x) = x\sqrt{\ln x}$
12. $f(x) = [\ln(2x)]^3$
13. $y = \ln \sqrt{\dfrac{x^2 + 4}{x^2 - 4}}$
14. $y = \ln \dfrac{4x}{x - 6}$
15. $y = \dfrac{1}{\ln(1 - 7x)}$
16. $y = \dfrac{\ln 5x}{1 - x}$

Finding an Equation of a Tangent Line In Exercises 17 and 18, find an equation of the tangent line to the graph of the function at the given point.

17. $y = \ln(2 + x) + \dfrac{2}{2 + x}$, $(-1, 2)$
18. $y = 2x^2 + \ln x^2$, $(1, 2)$

Logarithmic Differentiation In Exercises 19 and 20, use logarithmic differentiation to find dy/dx.

19. $y = x^2\sqrt{x - 1}$, $x > 1$
20. $y = \dfrac{x + 2}{\sqrt{3x - 2}}$, $x > \dfrac{2}{3}$

Finding an Indefinite Integral In Exercises 21–26, find the indefinite integral.

21. $\displaystyle\int \frac{1}{7x - 2}\,dx$
22. $\displaystyle\int \frac{x^2}{x^3 + 1}\,dx$
23. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx$
24. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\,dx$
25. $\displaystyle\int \frac{x^2 - 6x + 1}{x^2 + 1}\,dx$
26. $\displaystyle\int \frac{dx}{\sqrt{x}(2\sqrt{x} + 5)}$

Evaluating a Definite Integral In Exercises 27–30, evaluate the definite integral.

27. $\displaystyle\int_1^4 \frac{2x + 1}{2x}\,dx$
28. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$
29. $\displaystyle\int_0^{\pi/3} \sec \theta \, d\theta$
30. $\displaystyle\int_0^{\pi} \tan \frac{\theta}{3}\, d\theta$

Area In Exercises 31 and 32, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.

31. $y = \dfrac{6x^2}{x^3 - 2}$, $x = 3$, $x = 5$, $y = 0$
32. $y = x + \csc \dfrac{\pi x}{12}$, $x = 2$, $x = 6$, $y = 0$

Finding an Inverse Function In Exercises 33–38, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$, and (d) state the domains and ranges of f and f^{-1}.

33. $f(x) = \frac{1}{2}x - 3$
34. $f(x) = 5x - 7$
35. $f(x) = \sqrt{x + 1}$
36. $f(x) = x^3 + 2$
37. $f(x) = \sqrt[3]{x + 1}$
38. $f(x) = x^2 - 5$, $x \geq 0$

Evaluating the Derivative of an Inverse Function In Exercises 39–42, verify that f has an inverse function. Then use the function f and the given real number a to find $(f^{-1})'(a)$. (*Hint:* Use Theorem 5.9.)

39. $f(x) = x^3 + 2$, $a = -1$
40. $f(x) = x\sqrt{x - 3}$, $a = 4$
41. $f(x) = \tan x$, $-\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}$, $a = \dfrac{\sqrt{3}}{3}$
42. $f(x) = \cos x$, $0 \leq x \leq \pi$, $a = 0$

Solving an Exponential or Logarithmic Equation In Exercises 43–46, solve for x accurate to three decimal places.

43. $e^{3x} = 30$

44. $-4 + 3e^{-2x} = 6$

45. $\ln \sqrt{x+1} = 2$

46. $\ln x + \ln(x - 3) = 0$

Finding a Derivative In Exercises 47–52, find the derivative of the function.

47. $g(t) = t^2 e^t$

48. $g(x) = \ln \dfrac{e^x}{1 + e^x}$

49. $y = \sqrt{e^{2x} + e^{-2x}}$

50. $h(z) = e^{-z^2/2}$

51. $g(x) = \dfrac{x^3}{e^{2x}}$

52. $y = 3e^{-3/t}$

Finding an Equation of a Tangent Line In Exercises 53 and 54, find an equation of the tangent line to the graph of the function at the given point.

53. $f(x) = e^{6x}$, $(0, 1)$

54. $h(x) = -xe^{2-x}$, $(2, -2)$

Finding Extrema and Points of Inflection In Exercises 55 and 56, find the extrema and points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

55. $f(x) = (x + 1)e^{-x}$

56. $g(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x-5)^2/2}$

Finding an Indefinite Integral In Exercises 57–60, find the indefinite integral.

57. $\displaystyle\int xe^{1-x^2}\,dx$

58. $\displaystyle\int x^2 e^{x^3+1}\,dx$

59. $\displaystyle\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$

60. $\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

Evaluating a Definite Integral In Exercises 61–64, evaluate the definite integral.

61. $\displaystyle\int_0^1 xe^{-3x^2}\,dx$

62. $\displaystyle\int_{1/2}^2 \frac{e^{1/x}}{x^2}\,dx$

63. $\displaystyle\int_1^3 \frac{e^x}{e^x - 1}\,dx$

64. $\displaystyle\int_{1/4}^5 \frac{e^{4x} + 1}{4x + e^{4x}}\,dx$

65. **Area** Find the area of the region bounded by the graphs of

$y = 2e^{-x}$, $y = 0$, $x = 0$, and $x = 2$.

66. **Depreciation** The value V of an item t years after it is purchased is $V = 9000e^{-0.6t}$, $0 \le t \le 5$.
   (a) Use a graphing utility to graph the function.
   (b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.
   (c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 4$.

Sketching a Graph In Exercises 67 and 68, sketch the graph of the function.

67. $y = 3^{x/2}$

68. $y = \left(\dfrac{1}{4}\right)^x$

Solving an Equation In Exercises 69–74, solve the equation accurate to three decimal places.

69. $4^{1-x} = 52$

70. $2(3^{x+2}) = 17$

71. $\left(1 + \dfrac{0.03}{12}\right)^{12t} = 3$

72. $\left(1 + \dfrac{0.06}{365}\right)^{365t} = 2$

73. $\log_6(x + 1) = 2$

74. $\log_5 x^2 = 4.1$

Finding a Derivative In Exercises 75–82, find the derivative of the function.

75. $f(x) = 3^{x-1}$

76. $f(x) = 5^{3x}$

77. $g(t) = \dfrac{2^{3t}}{t^2}$

78. $f(x) = x(4^{-3x})$

79. $g(x) = \log_3 \sqrt{1 - x}$

80. $h(x) = \log_5 \dfrac{x}{x - 1}$

81. $y = x^{2x+1}$

82. $y = (3x + 5)^x$

Finding an Indefinite Integral In Exercises 83 and 84, find the indefinite integral.

83. $\displaystyle\int (x + 1)5^{(x+1)^2}\,dx$

84. $\displaystyle\int \frac{2^{-1/t}}{t^2}\,dt$

Evaluating a Definite Integral In Exercises 85 and 86, evaluate the definite integral.

85. $\displaystyle\int_1^2 6^x\,dx$

86. $\displaystyle\int_{-4}^0 9^{x/2}\,dx$

87. **Compound Interest**
   (a) A deposit of \$550 is made in a savings account that pays an annual interest rate of 1% compounded monthly. What is the balance after 11 years?
   (b) How large a deposit, at 5% interest compounded continuously, must be made to obtain a balance of \$10,000 in 15 years?
   (c) A deposit earns interest at a rate of r percent compounded continuously and doubles in value in 10 years. Find r.

88. **Climb Rate** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50 \log_{10} \frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is as high as the plane can fly.
   (a) Determine the domain of the function appropriate for the context of the problem.
   (b) Use a graphing utility to graph the function and identify any asymptotes.
   (c) Find the time when the altitude is increasing at the greatest rate.

Evaluating a Limit In Exercises 89–96, use L'Hôpital's Rule to evaluate the limit.

89. $\lim_{x\to 1} \frac{(\ln x)^2}{x-1}$

90. $\lim_{x\to 0} \frac{\sin \pi x}{\sin 5\pi x}$

91. $\lim_{x\to\infty} \frac{e^{2x}}{x^2}$

92. $\lim_{x\to\infty} xe^{-x^2}$

93. $\lim_{x\to\infty} (\ln x)^{2/x}$

94. $\lim_{x\to 1^+} (x-1)^{\ln x}$

95. $\lim_{n\to\infty} 1000\left(1+\frac{0.09}{n}\right)^n$

96. $\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^x$

Evaluating an Expression In Exercises 97 and 98, evaluate each expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

97. (a) $\sin\left(\arcsin \frac{1}{2}\right)$

(b) $\cos\left(\arcsin \frac{1}{2}\right)$

98. (a) $\tan(\operatorname{arccot} 2)$

(b) $\cos\left(\operatorname{arcsec} \sqrt{5}\right)$

Finding a Derivative In Exercises 99–104, find the derivative of the function.

99. $y = \operatorname{arccsc} 2x^2$

100. $y = \frac{1}{2}\arctan e^{2x}$

101. $y = x \operatorname{arcsec} x$

102. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\frac{x}{2},\quad 2 < x < 4$

103. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

104. $y = \tan(\arcsin x)$

Finding an Indefinite Integral In Exercises 105–110, find the indefinite integral.

105. $\int \frac{1}{e^{2x}+e^{-2x}}\,dx$

106. $\int \frac{1}{3+25x^2}\,dx$

107. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

108. $\int \frac{1}{x\sqrt{9x^2-49}}\,dx$

109. $\int \frac{\arctan(x/2)}{4+x^2}\,dx$

110. $\int \frac{\arcsin 2x}{\sqrt{1-4x^2}}\,dx$

Evaluating a Definite Integral In Exercises 111–114, evaluate the definite integral.

111. $\int_0^{1/7} \frac{dx}{\sqrt{1-49x^2}}$

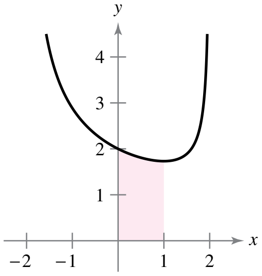
112. $\int_0^1 \frac{2x^2}{\sqrt{4-x^6}}\,dx$

113. $\int_{-1}^2 \frac{10e^{2x}}{25+e^{4x}}\,dx$

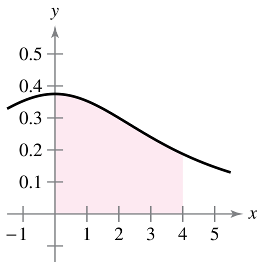
114. $\int_{\pi/3}^{\pi/2} \frac{\cos x}{(\sin x)\sqrt{\sin^2 x - (1/4)}}\,dx$

Area In Exercises 115 and 116, find the area of the given region.

115. $y = \frac{4-x}{\sqrt{4-x^2}}$



116. $y = \frac{6}{16+x^2}$



Verifying an Identity In Exercises 117 and 118, verify the identity.

117. $\cosh 2x = \cosh^2 x + \sinh^2 x$

118. $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

Finding a Derivative In Exercises 119–124, find the derivative of the function.

119. $y = \operatorname{sech}(4x-1)$

120. $y = 2x - \cosh\sqrt{x}$

121. $y = \coth 8x^2$

122. $y = \ln(\coth x)$

123. $y = \sinh^{-1}(4x)$

124. $y = x\tanh^{-1}(2x)$

Finding an Indefinite Integral In Exercises 125–130, find the indefinite integral.

125. $\int x^2 \operatorname{sech}^2 x^3\,dx$

126. $\int \sinh 6x\,dx$

127. $\int \frac{\operatorname{sech}^2 x}{\tanh x}\,dx$

128. $\int \operatorname{csch}^4 3x \coth 3x\,dx$

129. $\int \frac{1}{9-4x^2}\,dx$

130. $\int \frac{x}{\sqrt{x^4-1}}\,dx$

Evaluating a Definite Integral In Exercises 131–134, evaluate the definite integral.

131. $\int_1^2 \operatorname{sech} 2x \tanh 2x\,dx$

132. $\int_0^1 \sinh^2 x\,dx$

133. $\int_0^1 \frac{3}{\sqrt{9x^2+16}}\,dx$

134. $\int_{-1}^0 \frac{2}{49-4x^2}\,dx$

P.S. Problem Solving

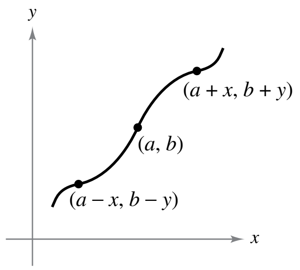
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Approximation To approximate e^x, you can use a function of the form

$$f(x) = \frac{a + bx}{1 + cx}.$$

(This function is known as a **Padé approximation.**) The values of $f(0)$, $f'(0)$, and $f''(0)$ are equal to the corresponding values of e^x. Show that these values are equal to 1 and find the values of a, b, and c such that $f(0) = f'(0) = f''(0) = 1$. Then use a graphing utility to compare the graphs of f and e^x.

2. Symmetry Recall that the graph of a function $y = f(x)$ is symmetric with respect to the origin if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. The graph of the function $y = f(x)$ is **symmetric with respect to the point (a, b)** if, whenever $(a - x, b - y)$ is a point on the graph, $(a + x, b + y)$ is also a point on the graph, as shown in the figure.



(a) Sketch the graph of $y = \sin x$ on the interval $[0, 2\pi]$. Write a short paragraph explaining how the symmetry of the graph with respect to the point $(\pi, 0)$ allows you to conclude that

$$\int_0^{2\pi} \sin x \, dx = 0.$$

(b) Sketch the graph of $y = \sin x + 2$ on the interval $[0, 2\pi]$. Use the symmetry of the graph with respect to the point $(\pi, 2)$ to evaluate the integral

$$\int_0^{2\pi} (\sin x + 2) \, dx.$$

(c) Sketch the graph of $y = \arccos x$ on the interval $[-1, 1]$. Use the symmetry of the graph to evaluate the integral

$$\int_{-1}^{1} \arccos x \, dx.$$

(d) Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}} \, dx.$

3. Finding a Value Find the value of the positive constant c such that

$$\lim_{x \to \infty} \left(\frac{x + c}{x - c}\right)^x = 9.$$

4. Finding a Value Find the value of the positive constant c such that

$$\lim_{x \to \infty} \left(\frac{x - c}{x + c}\right)^x = \frac{1}{4}.$$

5. Finding Limits Use a graphing utility to estimate each limit. Then calculate each limit using L'Hôpital's Rule. What can you conclude about the form $0 \cdot \infty$?

(a) $\displaystyle\lim_{x \to 0^+} \left(\cot x + \frac{1}{x}\right)$ (b) $\displaystyle\lim_{x \to 0^+} \left(\cot x - \frac{1}{x}\right)$

(c) $\displaystyle\lim_{x \to 0^+} \left[\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right)\right]$

6. Areas and Angles

(a) Let $P(\cos t, \sin t)$ be a point on the unit circle $x^2 + y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded circular sector AOP.

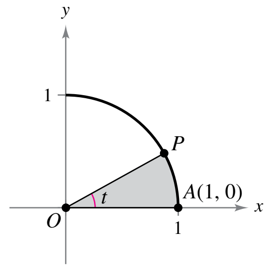


Figure for part (a)

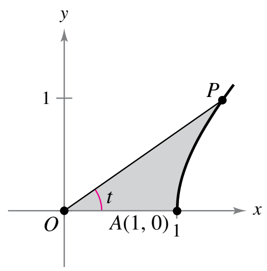


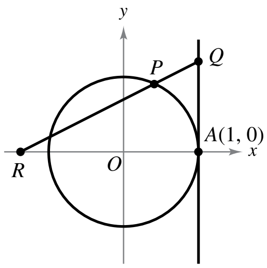
Figure for part (b)

(b) Let $P(\cosh t, \sinh t)$ be a point on the unit hyperbola $x^2 - y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded region AOP. Begin by showing that the area of the shaded region AOP is given by the formula

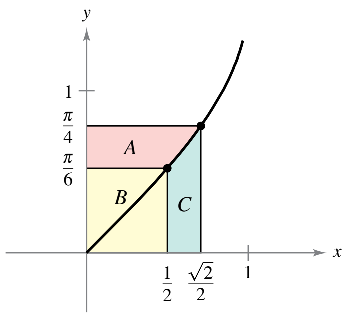
$$A(t) = \frac{1}{2} \cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} \, dx.$$

7. Intersection Graph the exponential function $y = a^x$ for $a = 0.5$, 1.2, and 2.0. Which of these curves intersects the line $y = x$? Determine all positive numbers a for which the curve $y = a^x$ intersects the line $y = x$.

8. Length The line $x = 1$ is tangent to the unit circle at A. The length of segment QA equals the length of the circular arc $\widehat{PA}$ (see figure). Show that the length of segment OR approaches 2 as P approaches A.



9. Area Consider the three regions A, B, and C determined by the graph of $f(x) = \arcsin x$, as shown in the figure.



(a) Calculate the areas of regions A and B.

(b) Use your answers in part (a) to evaluate the integral

$$\int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx.$$

(c) Use the methods in part (a) to evaluate the integral

$$\int_{1}^{3} \ln x \, dx.$$

(d) Use the methods in part (a) to evaluate the integral

$$\int_{1}^{\sqrt{3}} \arctan x \, dx.$$

10. Distance Let L be the tangent line to the graph of the function $y = \ln x$ at the point (a, b), where c is the y-intercept of the tangent line, as shown in the figure. Show that the distance between b and c is always equal to 1.

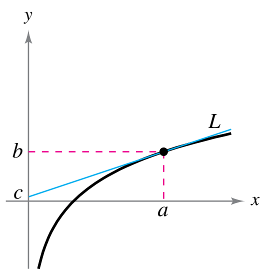


Figure for 10

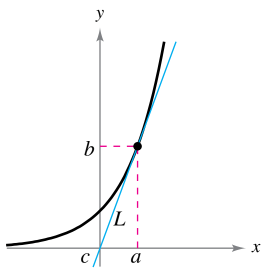


Figure for 11

11. Distance Let L be the tangent line to the graph of the function $y = e^x$ at the point (a, b), where c is the y-intercept of the tangent line, as shown in the figure. Show that the distance between a and c is always equal to 1.

12. Gudermannian Function The **Gudermannian function** of x is $\text{gd}(x) = \arctan(\sinh x)$.

(a) Graph gd using a graphing utility.

(b) Show that gd is an odd function.

(c) Show that gd is monotonic and therefore has an inverse.

(d) Find the point of inflection of gd.

(e) Verify that $\text{gd}(x) = \arcsin(\tanh x)$.

13. Decreasing Function Show that $f(x) = \dfrac{\ln x^n}{x}$ is a decreasing function for $x > e$ and $n > 0$.

14. Area Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sin^2 x + 4\cos^2 x}$$

between $x = 0$ and $x = \dfrac{\pi}{4}$.

15. Area Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sqrt{x} + x}$$

between $x = 1$ and $x = 4$.

16. Mortgage A \$120,000 home mortgage for 35 years at $9\frac{1}{2}\%$ has a monthly payment of \$985.93. Part of the monthly payment goes for the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes for interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the amount of the mortgage, r is the interest rate (in decimal form), M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Use the graphs in part (a) to make a conjecture about the relationship between the slopes of the tangent lines to the two curves for a specified value of t. Give an analytical argument to verify your conjecture. Find $u'(15)$ and $v'(15)$.

(d) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$1118.56$). What can you conclude?

17. Approximating a Function

(a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each given function.

(i) $y_1 = 1 + \dfrac{x}{1!}$

(ii) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(iii) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

(b) Identify the pattern of successive polynomials in part (a), extend the pattern one more term, and compare the graph of the resulting polynomial function with the graph of $y = e^x$.

(c) What do you think this pattern implies?