Lecture Five
Stochastic Calculus

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5 Stochastic Calculus

5.1 Itô Integral for a Simple Integrand

The Itô integral problem

Definition

Let W be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $\Delta(s, \omega)$, a function of $s \geq 0$ and $\omega \in \Omega$, is adapted if the dependence of $\Delta(s, \omega)$ on ω is as a function of the initial path fragment $W(u, \omega), 0 \leq u \leq s$. In particular, $\Delta(s)$ is independent of $W(t) - W(s)$ whenever $0 \leq s \leq t$.

We want to make sense of

$$\int_0^t \Delta(s)\, dW(s), \quad 0 \leq t \leq T.$$

Remark

If $g(s)$ is a differentiable function, then we can define

$$\int_0^t \Delta(s)\, dg(s) = \int_0^t \Delta(u) g'(s)\, ds.$$

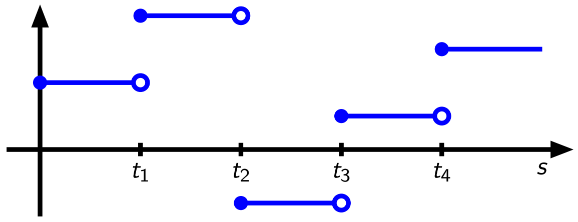
This won't work for Brownian motion, however, because the paths of Brownian motion are not differentiable.

Simple Integrand

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$, i.e.,

$$0 = t_0 \le t_1 \le \cdots \le t_n = T.$$

Assume that $\Delta(s)$ is constant in s on each subinterval $[t_k, t_{k+1})$.
We call such a Δ a simple process.



One path of Δ

Example

$$\Delta(s) = W(t_k), \quad t_k \le s < t_{k+1}$$

Interpretation of Simple Integrand

- Think of $W(s)$ as the price per share of an asset at time s.
- Think of $t_0, t_1, \dots, t_{n-1}$ as the trading dates in the asset.
- Think of $\Delta(t_0), \Delta(t_1), \dots, \Delta(t_{n-1})$ as the number of shares of the asset acquired at each trading date and held to the next trading date.

Gain from trading.

$$\begin{aligned} I(t) &= \Delta(t_0)[W(t) - W(t_0)] = \Delta(t_0)W(t), \quad 0 \le t \le t_1, \\ I(t) &= \Delta(t_0)[W(t_1) - W(t_0)] + \Delta(t_1)[W(t) - W(t_1)], \\ & \qquad t_1 \le t \le t_2, \\ I(t) &= \Delta(t_0)[W(t_1) - W(t_0)] + \Delta(t_1)[W(t_2) - W(t_1)] \\ & \quad + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \le t \le t_3. \end{aligned}$$

The process I is the Itô integral of the simple process Δ, i.e.,

$$I(t) = \int_0^t \Delta(s)\, dW(s), \quad 0 \le t \le T.$$

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5.2 Properties for Simple Integrands

Expectation of Itô integral

Theorem

The Itô integral of a simple process has expectation zero.

PROOF: By definition

$$I(T) = \sum_{j=0}^{n-1} \Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big).$$

Compute expectation term by term. Because $\Delta(t_j)$ is independent of $W(t_{j+1}) - W(t_j)$, we have

$$\begin{aligned} \mathbb{E}\big[\Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big)\big] &= \mathbb{E}\Delta(t_j) \cdot \mathbb{E}\big[W(t_{j+1}) - W(t_j)\big] \\ &= \mathbb{E}\Delta(t_j) \cdot 0 \\ &= 0. \end{aligned}$$

Exercise (5.1)

Suppose $Y(t), 0 \le t \le T$, is a stochastic process (a function of t and ω) such that if $0 \le s \le t$, then the increment $Y(t) - Y(s)$ is independent of the path of Y up to time s and has expectation zero. Let $\{\Delta(s)\}_{0 \le s \le T}$ be a simple process adapted to Y, i.e., there is a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$ such that $\Delta(s)$ is constant in s in each subinterval $[t_j, t_{j+1})$, and for each $s \in [0, T]$, the random variable $\Delta(s)$ depends on ω only through the path of Y up to time s, and hence $\Delta(s)$ is independent of $Y(t) - Y(s)$ for all $t \in [s, T]$. Define the Itô integral

$$I(T) = \sum_{j=0}^{n-1} \Delta(t_j)\big(Y(t_{j+1}) - Y(t_j)\big).$$

(i) *Show that $\mathbb{E}I(T) = 0$.*

(ii) *A simple arbitrage is a simple process Δ such that $I(T) \ge 0$ almost surely and $\mathbb{P}\{I(T) > 0\} > 0$. Show that there is no simple arbitrage under the assumptions of this exercise.*

Quadratic Variation of Itô Integral

Theorem

The simple Itô integral

$$I(t) = \int_0^t \Delta(u)\, dW(u)$$

has quadratic variation

$$[I, I](T) = \int_0^T \Delta^2(u)\, du \qquad (QV)$$

and variance

$$\mathbb{E}\big[I^2(T)\big] = \mathbb{E}\int_0^T \Delta^2(u)\, du. \qquad (VAR)$$

Remark

Both sides of (QV) are random, but the expressions in (VAR) are not. (VAR) is called Itô's Isometry.

Proof of (QV)

For $s \in [t_j, t_{j+1}]$, we have $\Delta(s) = \Delta(t_j)$ and

$$\begin{aligned} I(s) &= I(t_j) + \Delta(t_j)\Big[W(s) - W(t_j)\Big] \\ &= \Big[I(t_j) - \Delta(t_j)W(t_j)\Big] + \Delta(t_j)W(s). \end{aligned}$$

On this subinterval, quadratic variation of I comes from the quadratic variation of W, which is scaled by $\Delta(t_j)$. Therefore

$$\begin{aligned} [I, I](t_{j+1}) - [I, I](t_j) &= \Delta^2(t_j)\Big([W, W](t_{j+1}) - [W, W](t_j)\Big) \\ &= \Delta^2(t_j)(t_{j+1} - t_j) \\ &= \int_{t_j}^{t_{j+1}} \Delta^2(s)\, ds. \end{aligned}$$

Summing over subintervals, we obtain

$$[I, I](T) = \sum_{j=0}^{n-1} \Big([I, I](t_{j+1}) - [I, I](t_j)\Big) = \int_0^T \Delta^2(s)\, ds.$$

Proof of (VAR)

$$I(T) = \sum_{j=0}^{n-1} \Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big).$$

Squaring and taking expectations, we obtain

$$\begin{aligned} \mathbb{E}\big[I^2(T)\big) &= \sum_{k=j}^{n-1} \mathbb{E}\left[\Delta^2(t_j)\big(W(t_{j+1}) - W(t_j)\big)^2\right] \\ &+ 2\sum_{j<k} \mathbb{E}\left[\Delta(t_j)\Delta(t_k)\big(W(t_{j+1}) - W(t_j)\big)\big(W(t_{k+1}) - W(t_k)\big)\right]. \end{aligned}$$

We use independence to simplify the pure square terms:

$$\begin{aligned} \mathbb{E}\big[\Delta^2(t_j)\big(W(t_{j+1}) - W(t_j)\big)^2\big] &= \mathbb{E}\big[\Delta^2(t_j)\big] \cdot \mathbb{E}\big[\big(W(t_{j+1}) - W(t_j)\big)^2\big] \\ &= \mathbb{E}\big[\Delta^2(t_j)\big] \cdot (t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \mathbb{E}\Delta^2(s)\, ds. \end{aligned}$$

The sum of the pure square terms is $\mathbb{E}\int_0^T \Delta^2(s) ds$.

Proof of (VAR) (continued)

It remains to show that the cross-terms have zero expectation. For $j < k$, the increment $W(t_{k+1}) - W(t_k)$ is independent of $\Delta(t_j)\Delta(t_k)\Big(W(t_{j+1}) - W(t_j)\Big)$, and hence

$$\begin{aligned}
&\mathbb{E}\Big[\Delta(t_j)\Delta(t_k)\Big(W(t_{j+1}) - W(t_j)\Big)\Big(W(t_{k+1}) - W(t_k)\Big)\Big] \\
&= \mathbb{E}\Big[\Delta(t_j)\Delta(t_k)\Big(W(t_{j+1}) - W(t_j)\Big)\Big] \cdot \mathbb{E}\Big(W(t_{k+1}) - W(t_k)\Big) \\
&= \mathbb{E}\Big[\Delta(t_j)\Delta(t_k)\Big(W(t_{j+1}) - W(t_j)\Big)\Big] \cdot 0 \\
&= 0.
\end{aligned}$$

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5.3 Construction for General Integrands

Outline of construction for general integrands

- Given $\Delta(s), 0 \leq s \leq T$, satisfying
$$\mathbb{E}\int_0^T \Delta^2(s)\,ds < \infty,$$
construct an approximating sequence of simple processes $\Delta_n(s)$, $0 \leq s \leq T$, such that
$$\lim_{n\to\infty} \mathbb{E}\int_0^T \big(\Delta(s) - \Delta_n(s)\big)^2\,ds = 0.$$
- Set $I_n(T) = \int_0^T \Delta_n(s)\,dW(s)$. Itô's isometry implies that
$$\mathbb{E}\big[(I_n(T) - I_m(T))^2\big] = \mathbb{E}\int_0^T \big(\Delta_n(s) - \Delta_m(s)\big)^2\,ds.$$
- Because the sequence $\{\Delta_n\}_{n=1}^\infty$ converges in $L_2(\Omega \times [0,T], \mathcal{F} \otimes \text{ Borel}([0,T]), \mathbb{P} \times \text{ Lebesgue})$, it is Cauchy in this space. Therefore, $\{I_n(T)\}_{n=1}^\infty$ is Cauchy in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.

Outline of construction (continued)

- $L_2(\Omega, \mathcal{F}, \mathbb{P})$ is complete, and so the sequence $\{I_n(T)\}_{n=1}^\infty$ has a limit $I(T)$ in this space.
- We define
$$\int_0^T \Delta(s)\,dW(s) = I(T) = \lim_{n\to\infty} I_n(T).$$
This limit does not depend on the approximating sequence $\{\Delta_n\}_{n=1}^\infty$.
- By choosing approximating sequences that converge rapidly, we can in fact make the convergence of $I_n(T)$ to $I(T)$ be almost sure (almost everywhere with respect to $\mathbb{P}$) rather than in L_2.
- With additional work, one can choose the approximating sequence so that the paths of $I_n(t)$, $0 \leq t \leq T$, converge uniformly in $t \in [0,T]$ almost surely. This guarantees that there is a limit $I(t)$, $0 \leq t \leq T$, that is a continuous function of $t \in [0,T]$ for $\mathbb{P}$-almost every ω.

Theorem

Under the assumption $\mathbb{E}[\int_0^T \Delta^2(s)\, ds] < \infty$, *the Itô integral*

$$I(t) = \int_0^t \Delta(s)\, dW(s), \quad 0 \le t \le T,$$

is defined and continuous in $t \in [0, T]$. *We have*

$$\mathbb{E}I(t) = 0, \quad 0 \le t \le T.$$

The quadratic variation of the Itô integral is

$$[I, I](t) = \int_0^t \Delta^2(s)\, ds, \quad 0 \le t \le T,$$

and the Itô integral satisfies Itô's Isometry

$$Var[I(t)] = \mathbb{E}\big[I^2(t)\big] = \mathbb{E}\left[\int_0^t \Delta^2(s)\, ds\right], \quad 0 \le t \le T.$$

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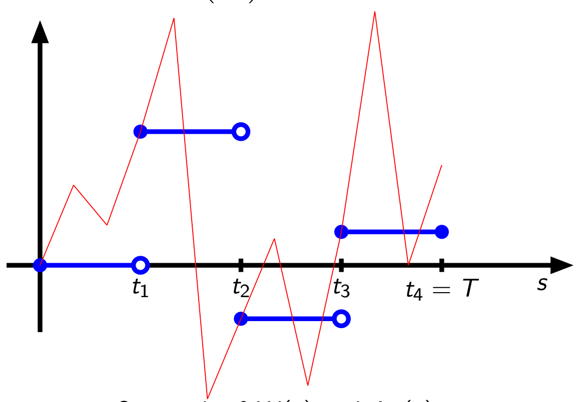
5.4 Example of an Itô Integral

$\int_0^T W(s)\, dW(s)$

Divide $[0, T]$ into n equal subintervals. Define

$$\Delta_n(s) = W\left(\frac{jT}{n}\right) \text{ for } \frac{jT}{n} \le s < \frac{(j+1)T}{n}.$$



One path of $W(s)$ and $\Delta_4(s)$

By definition,

$$\int_0^T W(s)\, dW(s)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right].$$

To simplify notation, we denote $W_j = W\left(\frac{jT}{n}\right)$. Then $W_0 = W(0) = 0$, $W_n = W(T)$, and

$$\int_0^T W(s)\, dW(s) = \lim_{n\to\infty} \sum_{j=0}^{n-1} W_j(W_{j+1} - W_j).$$

$$\begin{aligned}\frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2 &= \frac{1}{2}\sum_{j=0}^{n-1}W_{j+1}^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2\\ &= \frac{1}{2}\sum_{k=1}^{n}W_k^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2\\ &= \frac{1}{2}W_n^2 + \frac{1}{2}\sum_{k=0}^{n-1}W_k^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2\\ &= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1}W_j^2 - \sum_{j=0}^{n-1}W_jW_{j+1}\\ &= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1}W_j(W_j - W_{j+1}).\end{aligned}$$

$$\sum_{j=0}^{n-1}W_j(W_{j+1}-W_j) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2.$$

From the previous page, we have

$$\sum_{j=0}^{n-1}W_j(W_{j+1}-W_j) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2.$$

Letting $n \to \infty$, we get

$$\int_0^T W(s)\,dW(s) = \frac{1}{2}W^2(T) - \frac{1}{2}[W,W](T) = \frac{1}{2}W^2(T) - \frac{1}{2}T.$$

Remark

If g is a differentiable function with $g(0) = 0$, then

$$\int_0^T g(s)\,dg(s) = \int_0^T g(s)g'(s)\,ds = \frac{1}{2}g^2(s)\Big|_0^T = \frac{1}{2}g^2(T).$$

The extra term $\frac{1}{2}T$ in $\int_0^T W(s)dW(s)$ comes from the nonzero quadratic variation of Brownian motion.

Exercise (5.2)

Show that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}W\left(\frac{(j+1)T}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right] = \frac{1}{2}W^2(T) + \frac{1}{2}T.$$

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5.5 Itô's Formula for One Process

Along the path of a Brownian motion, we want to "differentiate" $f(W(t))$, where $f(x)$ is a differentiable function. If the path of the Brownian motion $W(t)$ could be differentiated with respect to t, then the ordinary chain rule would give

$$\frac{d}{dt} f(W(t)) = f'(W(t)) W'(t),$$

which could be written in differential notation as

$$df(W(t)) = f'(W(t))\, W'(t)\, dt = f'(W(t))\, dW(t).$$

Because W has nonzero quadratic variation, the correct formula has an extra term, namely,

$$df(W(t)) = f'(W(t))\, dW(t) + \frac{1}{2} f''(W(t)) \underbrace{dt}_{dW(t)dW(t)}.$$

This is Itô's formula in differential form.

Remark

The mathematically meaningful form of Itô's formula is Itô's formula in integral form:

$$f(W(T)) - f(W(0)) = \int_0^t f'(W(t))\, dW(t) + \frac{1}{2}\int_0^T f''(W(t))\, dt.$$

This is because we have definitions for both integrals appearing on the right-hand side. The first,

$$\int_0^T f'(W(t))\, dW(t)$$

is an Itô integral. The second

$$\int_0^T f''(W(t))\, dt$$

is a Riemann integral with respect to time, computed path by path.

Application of Itô's Formula

Consider $f(x) = \frac{1}{2}x^2$, so that

$$f'(x) = x, \quad f''(x) = 1.$$

Itô's formula in integral form

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(s))\, dW(s) + \frac{1}{2}\int_0^T f''(W(s))\, ds$$

becomes

$$\frac{1}{2}W^2(T) = \int_0^T W(s)\, dW(s) + \frac{1}{2}\int_0^T 1\, ds = \int_0^T W(s)\, dW(s) + \frac{1}{2}T,$$

or equivalently,

$$\int_0^T W(u)\, dW(u) = \frac{1}{2}W^2(T) - \frac{1}{2}T.$$

Derivation of Itô's Formula

Consider $f(x) = \frac{1}{2}x^2$, so that

$$f'(x) = x, \quad f''(x) = 1.$$

Let x_{j+1} and x_j be numbers. Taylor's formula implies

$$f(x_{j+1}) - f(x_j) = (x_{j+1} - x_j) f'(x_j) + \frac{1}{2}(x_{j+1} - x_j)^2 f''(x_j).$$

In this case, Taylor's formula to second order is *exact* because f is a *quadratic function.*

In the general case, the above equation is only approximate, and the error is of the order of $(x_{k+1} - x_k)^3$. The total error will have limit zero in the last step of the following argument (see Exercise 4.6(iii) of Lecture 4).

Fix $T > 0$ and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$.

$$
\begin{aligned}
&f(W(T)) - f(W(0)) \\
&= \sum_{j=0}^{n-1} \Big[f(W(t_{j+1})) - f(W(t_j)) \Big] \\
&= \sum_{j=0}^{n-1} \Big[W(t_{j+1}) - W(t_j) \Big] f'(W(t_j)) \\
&\qquad + \frac{1}{2} \sum_{j=0}^{n-1} \Big[W(t_{j+1}) - W(t_j) \Big]^2 f''(W(t_j)) \\
&= \sum_{j=0}^{n-1} W(t_j) \Big[W(t_{j+1}) - W(t_j) \Big] + \frac{1}{2} \sum_{j=0}^{n-1} \Big[W(t_{j+1}) - W(t_j) \Big]^2 .
\end{aligned}
$$

From the previous page, we have

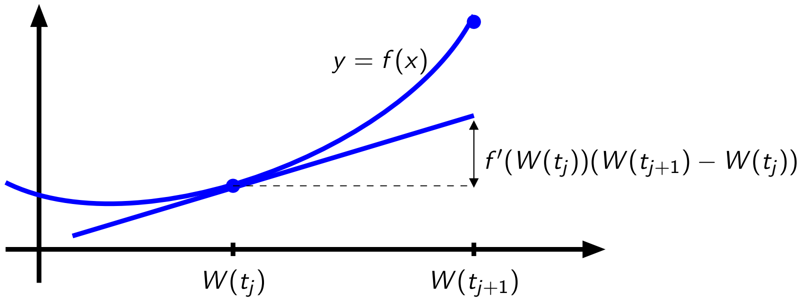
$$
\begin{aligned}
&f(W(T)) - f(W(0)) \\
&= \sum_{j=0}^{n-1} W(t_j) \Big[W(t_{j+1}) - W(t_j) \Big] + \frac{1}{2} \sum_{k=0}^{n-1} \Big[W(t_{j+1}) - W(t_j) \Big]^2 .
\end{aligned}
$$

We let $\|\Pi\| \to 0$, to obtain

$$
\begin{aligned}
&f(W(T)) - f(W(0)) \\
&= \int_0^T W(s)\, dW(s) + \frac{1}{2} \underbrace{[W, W](T)}_{T} \\
&= \int_0^T f'(W(s))\, dW(s) + \frac{1}{2} \int_0^T \underbrace{f''(W(s))}_{1}\, ds.
\end{aligned}
$$

This is Itô's formula in integral form for the special case

$$f(x) = \frac{1}{2} x^2.$$



$$
\begin{aligned}
f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) &= f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \text{Small Error} \\
f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) &= f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \frac{1}{2} f''\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)^2 \\
&\quad + \text{Smaller Error}
\end{aligned}
$$

We need the higher accuracy before summing. Otherwise, the accumulated small errors do not vanish as the step-size goes to zero.

Exercise (5.3)

Let $u \in \mathbb{R}$ be constant and define $f(x) = e^{ux}$. Use Itô's formula applied to $f(W(t))$ to obtain the moment-generating function formula

$$\mathbb{E} e^{uW(T)} = e^{\frac{1}{2} u^2 T}.$$

(Compare with Exercise 4.4 of Lecture 4.)

Exercise (5.4)

Let $f(x) = x^4$. Use Itô's formula applied to $f(W(t))$ to obtain the fourth-moment formula

$$\mathbb{E} W^4(T) = 3T^2.$$

(Compare with Exercise 4.5 of Lecture 4.)

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5.7 Solution to Exercise

Solution to Exercise 5.1

(i) As in the proof of the theorem preceding the exercise, we use independence to compute $\mathbb{E}I(T)$ term by term:

$$\begin{aligned}\mathbb{E}\big[\Delta(t_j)\big(Y(t_{j+1}) - Y(t_j)\big)\big] &= \mathbb{E}\Delta(t_j) \cdot \mathbb{E}\big[Y(t_{j+1}) - Y(t_j)\big] \\ &= \mathbb{E}\Delta(t_j) \cdot 0 \;=\; 0.\end{aligned}$$

(ii) If $I(T) \geq 0$ almost surely and $\mathbb{P}\{I(T) > 0\} > 0$, then $\mathbb{E}I(T) > 0$. This contradicts part (i).

Solution to Exercise 5.2

Let $t_j = \frac{jT}{n}$. The quadratic variation result for Brownian motion is

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)^2 = T.$$

The Example shows that

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_j)(W(t_{j+1}) - W(t_j)) = \frac{1}{2}W^2(T) - \frac{1}{2}T.$$

Adding these two equations, we obtain

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_{j+1})\big(W(t_{j+1}) - W(t_j)\big) = \frac{1}{2}W^2(T) + \frac{1}{2}T.$$

This is the desired result.

Solution to Exercise 5.3

We have $f'(x) = uf(x)$ and $f''(x) = u^2 f(x)$. Therefore, Itô's formula becomes

$$e^{uW(T)} = e^{uW(0)} + u\int_0^T e^{uW(t)}\,dW(t) + \frac{1}{2}u^2\int_0^T e^{uW(t)}\,dt.$$

Taking expectations and using the fact that the expectation of the Itô integral is zero, we obtain

$$\mathbb{E}e^{uW(T)} = 1 + \frac{1}{2}u^2\int_0^T \mathbb{E}e^{uW(t)}\,dt.$$

We differentiate both sides with respect to T to obtain

$$\frac{d}{dT}\mathbb{E}e^{uW(T)} = \frac{1}{2}u^2\mathbb{E}e^{uW(T)}.$$

The unique solution to this ordinary differential equation satisfying $\mathbb{E}e^{uW(0)} = 1$ is

$$\mathbb{E}e^{uW(T)} = e^{\frac{1}{2}u^2T}.$$

Exercise 5.4

With $f(x) = x^4$, we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula becomes

$$W^4(T) = 4\int_0^T W^3(t)\,dW(t) + 6\int_0^T W^2(t)\,dt.$$

Taking expectations of both sides and using the fact that the Itô integral has expectation zero, we obtain

$$\begin{aligned}\mathbb{E}W^4(T) &= 6\int_0^T \mathbb{E}W^2(t)\,dt \\ &= 6\int_0^T t\,dt \\ &= 3T^2.\end{aligned}$$