5. Taylor and Laurent series

Complex sequences and series

An infinite sequence of complex numbers, denoted by $\{z_n\}$, can be considered as a function defined on a set of positive integers into the unextended complex plane. For example, we take $z_n = \dfrac{n+1}{2^n}$ so that the complex sequence is $\{z_n\} = \left\{\dfrac{1+i}{2}, \dfrac{2+i}{2^2}, \dfrac{3+i}{2^3}, \cdots\right\}$.

Convergence of complex sequences

Given a complex sequence $\{z_n\}$, if for each positive quantity ϵ, there exists a positive integer N such that

$$|z_n - z| < \epsilon \quad \text{whenever } n > N,$$

then the sequence is said to *converge* to the limt z. We write

$$\lim_{n\to\infty} z_n = z.$$

- In general, the choice of N depends on ϵ. The definition implies that every ϵ-neighborhood of z contains all but a finite number of members of the sequence. The limit of a convergent sequence is unique. If the sequence fails to converge, it is said to be *divergent*.

- Suppose we write $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\begin{aligned} |x_n - x| &\leq |z_n - z| \leq |x_n - x| + |y_n - y|, \\ |y_n - y| &\leq |z_n - z| \leq |x_n - x| + |y_n - y|. \end{aligned}$$

Suppose $\lim_{n\to\infty} = x$ and $\lim_{n\to\infty} y_n = y$ exist, then for any $\epsilon > 0$, there exist N_x and N_y such that

$$\begin{aligned} |x_n - x| < \frac{\epsilon}{2} &\quad \text{whenever } n > N_x \\ |y_n - y| < \frac{\epsilon}{2} &\quad \text{whenever } n > N_y. \end{aligned}$$

Choose $n > \max(N_x, N_y)$, then

$$|z_n - z| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- Using the above inequalities, it is easy to show that

$$\lim_{n\to\infty} z_n = z \Longleftrightarrow \lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

  Therefore, the study of the convergence of a complex sequence is equivalent to the consideration of two real sequences.

The above theorem enables us to write

$$\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or the one on the left exists.

For example, the sequence

$$z_n = \frac{1}{n^3} + i, \quad n = 1, 2, \cdots,$$

converges to i since $\lim\limits_{n\to\infty} \frac{1}{n^3}$ and $\lim\limits_{n\to\infty} 1$ exist, so

$$\lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i \lim_{n\to\infty} 1 = 0 + i \cdot 1 = i.$$

One can show that for each positive number ϵ

$$|z_n - i| < \epsilon \quad \text{whenever} \quad n > \frac{1}{\sqrt[3]{\epsilon}}.$$

Infinite series of complex numbers

- An infinite series of complex numbers $z_1, z_2, z_3, \cdots$ is the infinite sum of the sequence $\{z_n\}$ given by

$$z_1 + z_2 + z_3 + \cdots = \lim_{n\to\infty}\left(\sum_{k=1}^{n} z_k\right).$$

- To study the properties of an infinite series, we define the sequence of *partial sums* $\{S_n\}$ by

$$S_n = \sum_{k=1}^{n} z_k.$$

- If the limit of the sequence $\{S_n\}$ converges to S, then the series is said to be convergent and S is its sum; otherwise, the series is divergent.

- The consideration of an infinite series is relegated to that of an infinite sequence of partial sums.

Remainder after n terms

Suppose an infinite series converges. We define the remainder after n terms by

$$R_n = S - S_n$$

and obviously

$$\lim_{n\to\infty} R_n = 0.$$

Conversely, suppose $\lim_{n\to\infty} R_n = 0$, then for any $\epsilon > 0$, there exists $N(\epsilon)$ such that $|R_n| < \epsilon$ for $n > N(\epsilon)$. This is equivalent to

$$|S_n - S| < \epsilon \quad \text{for} \quad n > N(\epsilon),$$

hence

$$S = \lim_{n\to\infty} S_n.$$

A necessary condition for the convergence of a complex series is that

$$\lim_{n\to\infty} z_n = 0.$$

This is obvious since

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(R_{n-1} - R_n) = \lim_{n\to\infty} R_{n-1} - \lim_{n\to\infty} R_n = 0.$$

Comparison Test

If $\sum_{j=1}^{\infty} M_j$ is a convergent series with real nonnegative terms and for all j larger than some number $J, |z_j| \leq M_j$, then the series $\sum_{j=1}^{\infty} |z_j|$ converges also. This is easily seen since $S_n = \sum_{j=1}^{n} |z_j|$ is a bounded increasing sequence, so it must have a limit.

Absolute convergence

The complex series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges. Note that $|z_n| = \sqrt{x_n^2 + y_n^2}$ and since

$$|x_n| \leq \sqrt{x_n^2 + y_n^2} \quad \text{and} \quad |y_n| \leq \sqrt{x_n^2 + y_n^2},$$

then from the comparison test, the two series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

must converge. Thus, absolute convergence in a complex sequence implies convergence in that sequence.

The converse may not hold. If Σz_n converges but $\Sigma |z_n|$ does not, the series Σz_n is said to be *conditionally convergent*. For example,

$$-\text{Log}(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}, \quad \theta \neq 0,$$

is conditionally convergent.

Example

Show that the series $\sum_{j=1}^{\infty} (3+2i)/(j+1)^j$ converges.

Solution

We compare the series

$$\sum_{j=1}^{\infty} \frac{3+2i}{(j+1)^j} = \frac{(3+2i)}{9} + \frac{(3+2i)}{64} + \cdots \qquad (A)$$

with the *convergent* geometric series

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\ . \qquad (B)$$

Since $|3+2i| = \sqrt{13} < 4$, one can easily verify that for $j \geq 3$

$$\left|\frac{3+2i}{(j+1)^j}\right| < \frac{4}{(j+1)^j} \leq \frac{1}{2^j}.$$

The terms of (B) dominate those of (A), hence (A) converges.

Limit superior

Consider a sequence $\{x_n\}$ of real numbers, and let S denote the set of all of its limit points. The limit superior of $\{x_n\}$ is the supremum (least upper bound) of S. For example, $x_n = 3+(-1)^n, n = 1, 2, \cdots$, the limit points are 2 and 4 so that

$$\overline{\lim_{n\to\infty}} x_n = \max(2, 4) = 4.$$

Root test

Suppose the limit superior of $\{|z_n|^{1/n}\}$ equals L, the series Σz_n converges absolutely if $L < 1$ and diverges if $L > 1$. The root test fails when $L = 1$.

Ratio test

Suppose $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$ converges to L, then Σz_n is absolutely convergent if $L < 1$ and divergent if $L > 1$. When $L = 1$, the ratio test fails.

Gauss' test

Suppose $\left|\frac{z_{n+1}}{z_n}\right|$ admits the following asymptotic expansion

$$\left|\frac{z_{n+1}}{z_n}\right| = 1 - \frac{k}{n} + \frac{\alpha_n}{n^2} + \cdots ,$$

where $|\alpha_n|$ is bounded for all $n > N$ for some sufficiently large N, then Σz_n converges absolutely if $k > 1$, and diverges or converges conditionally if $k \leq 1$.

Proof of the Ratio Test

Suppose $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$ converges to $L, L < 1$, then we can choose an integer N such that for $n \geq N$, we have

$$\left|\frac{z_{n+1}}{z_n}\right| \leq r, \quad \text{where } L < r < 1.$$

Now, we have

$$|z_{N+1}| \leq r|z_N|, \quad |z_{N+2}| \leq r|z_{N+1}| \leq r^2|z_N|, \quad \text{and so forth,}$$
$$|z_{N+1}| + |z_{N+2}| + \cdots \leq |z_N|(r + r^2 + \cdots).$$

Thus, $\sum_{n=1}^{\infty} |z_n|$ converges absolutely by virtue of the comparison test.

Sequences of complex functions

Let $f_1(z), \cdots, f_n(z), \cdots$, denoted by $\{f_n(z)\}$, be a sequence of complex functions of z that are defined and single-valued in a region R in the complex plane.

For some point $z_0 \in R, \{f_n(z_0)\}$ becomes a sequence of complex numbers. Supposing $\{f_n(z_0)\}$ converges, the limit is unique. The value of the limit depends on z_0, and we write

$$f(z_0) = \lim_{n \to \infty} f_n(z_0).$$

If this holds for every $z \in R$, the sequence $\{f_n(z)\}$ defines a complex function $f(z)$ in R. We write

$$f(z) = \lim_{n \to \infty} f_n(z).$$

This is usually called *pointwise convergence*.

Definition

A sequence of complex functions $\{f_n(z)\}$ defined in a region R is said to *converge* to a complex function $f(z)$ defined in the same region if and only if, for any given small positive quantity ϵ, we can find a positive integer $N(\epsilon; z)$ [in general, $N(\epsilon; z)$ depends on ϵ and z] such that

$$|f(z) - f_n(z)| < \epsilon \quad \text{for all} \quad n > N(\epsilon; z).$$

The region R is called the *region of convergence* of the sequence of complex functions.

In general, we may not be able to find a single $N(\epsilon)$ that works for all point in R. However, when this is possible, $\{f_n(z)\}$ is said to *converge uniformly* to $f(z)$ in R.

Convergence of series of complex functions

An infinite series of complex functions

$$f_1(z) + f_2(z) + f_3(z) + \cdots = \sum_{k=1}^{\infty} f_k(z)$$

is related to the sequence of partial sum $\{S_n(z)\}$

$$S_n(z) = \sum_{k=1}^{n} f_k(z).$$

The infinite series is said to be *convergent* if

$$\lim_{n\to\infty} S_n(z) = S(z),$$

where $S(z)$ is called the *sum*; otherwise the series is *divergent*.

Many of the properties related to convergence of complex functions can be extended from their counterparts of complex numbers. For example, a necessary but not sufficient condition for the infinite series of complex functions to converge is that

$$\lim_{k \to \infty} f_k(z) = 0,$$

for all z in the region of convergence.

Example

Consider the complex series

$$\sum_{k=1}^{\infty} \frac{\sin kz}{k^2},$$

show that it is absolutely convergent when z is real but it becomes divergent when z is non-real.

Solution

(i) When z is real, we have

$$\left|\frac{\sin kz}{k^2}\right| \leq \frac{1}{k^2}, \quad \text{for all positive integer values of } k.$$

Since $\sum_{k=1}^{\infty} 1/k^2$ is known to be convergent, then $\sum_{k=1}^{\infty} \sin kz/k^2$ is absolutely convergent for all z by virtue of the comparison test.

(ii) When z is non-real, we let $z = x + iy, y \neq 0$. From the relation

$$\frac{\sin kz}{k^2} = \frac{e^{-ky}e^{ikx} - e^{ky}e^{-ikx}}{2k^2 i},$$

we deduce that

$$\left|\frac{\sin kz}{k^2}\right| \geq \frac{e^{k|y|} - e^{-k|y|}}{2k^2} \to \infty \text{ as } k \to \infty.$$

Since $\left|\sin kz/k^2\right|$ is unbounded as $k \to \infty$, the series is divergent.

Uniform convergence of an infinite series of complex functions

Let $R_n(z) = S(z) - \sum_{k=1}^{n} f_k(z) = S(z) - S_n(z)$.

The infinite series $\Sigma f_k(z)$ converges uniformly to $S(z)$ in some region R iff for any $\epsilon > 0$, there exists N which is independent of z such that for all $z \in R$

$$|R_n(z)| < \epsilon \quad \text{whenever } n > N.$$

Weierstrass M-test

If $|f_k(z)| \leq M_k$ where M_k is independent of z in $\mathcal{R}$ and the series ΣM_k converges, then $\Sigma f_k(z)$ is uniformly convergent in $\mathcal{R}$.

Proof of the Weierstrass M-test

The remainder of the series $\sum f_k(z)$ after n terms is

$$R_n(z) = f_{n+1}(z) + f_{n+2}(z) + \cdots .$$

Now,

$$|R_n(z)| \leq |f_{n+1}(z)| + |f_{n+2}(z)| + \cdots \leq M_{n+1} + M_{n+2} + \cdots .$$

Since $\sum M_k$ converges, $M_{n+1} + M_{n+2} + \cdots$ can be made less than $\epsilon > 0$ by choosing $n > N$ for some $N = N(\epsilon)$. As N is clearly independent of z, we have

$$|R_n(z)| < \epsilon \quad \text{for} \quad n > N,$$

so the series is uniformly convergent. We also have absolute convergence of the series.

Test for uniform convergence using the M-test

1. $\sum_{n=1}^{\infty} \frac{z_n}{n\sqrt{n+1}}, |z| \leq 1.$

   Note that $|f_n(z)| = \frac{|z|^n}{n\sqrt{n+1}} \leq \frac{1}{n^{3/2}}$ if $|z| \leq 1$. Take $M_n = \frac{1}{n^{3/2}}$ and note that $\sum M_n$ converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+z^2}, 1 < |z| < 2.$

   Omit the first two terms since it does not affect the uniform convergence property. For $n \geq 3$ and $1 < |z| < 2$, we have

   $$|n^2+z^2| \geq |n^2| - |z^2| \geq n^2 - 4 \geq \frac{n^2}{2} \text{ so that } \left|\frac{1}{n^2+z^2}\right| \leq \frac{2}{n^2}.$$

   Take $M_n = \frac{2}{n^2}$ and note that $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges.

Example

Prove that the series

$$z(1-z)+z^2(1-z)+z^3(1-z)+\cdots$$

converges for $|z|<1$ and find its sum.

Solution

$$\begin{aligned} S_n(z) &= z(1-z)+\cdots+z^n(1-z) \\ &= z-z^2+z^2-z^3+\cdots+z^n-z^{n+1}=z-z^{n+1}. \end{aligned}$$

For $|z|<1$, consider $|S_n(z)-z|=|-z^{n+1}|=|z|^{n+1}<\epsilon$. In order that this is true, we choose n such that

$$(n+1)\ln|z|<\ln\epsilon \quad \text{so that } n+1>\frac{\ln\epsilon}{\ln|z|} \text{ or } n>\frac{\ln\epsilon}{\ln|z|}-1, z\neq 0.$$

When $z=0, S_n(0)=0$ so $|S_n(0)-0|<\epsilon$ for all n.

Hence, $\lim_{n\to\infty} S_n(z)=z$ for all z such that $|z|<1$.

Questions

1. Does the series converge uniformly to z for $|z| \leq \dfrac{1}{2}$?

   Yes. If $|z| \leq \dfrac{1}{2}$, the largest value of $\dfrac{\ln \epsilon}{\ln |z|} - 1$ occurs when $|z| = \dfrac{1}{2}$ and is given by

$$\frac{\ln \epsilon}{\ln \frac{1}{2}} - 1.$$

   Take $N(\epsilon)$ to be the largest integer smaller than $\dfrac{\ln \epsilon}{\ln \frac{1}{2}} - 1$. It then follows that $|S_n(z) - z| < \epsilon$ for $n > N$ where N depends only on ϵ and not on the particular z in $|z| \leq \dfrac{1}{2}$. Hence, we have uniform convergence of the series for $|z| \leq \dfrac{1}{2}$.

2. Does the series converge uniformly for $|z| \le 1$?

The same argument given in Qn (1) serves to show uniform convergence for $|z| \le 0.9$ or $|z| \le 0.99$ by using

$$N = \mathrm{fl}\left(\frac{\ln \epsilon}{\ln 0.9} - 1\right) \quad \text{and} \quad N = \mathrm{fl}\left(\frac{\ln \epsilon}{\ln 0.99} - 1\right), \text{ respectively.}$$

However, if we apply the argument to $|z| \le 1$, this would require $N = \mathrm{fl}\left(\frac{\ln \epsilon}{\ln 1} - 1\right)$, which is infinite. The series does not converge uniformly for $|z| \le 1$.

Example

Show that the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly to $\dfrac{1}{1-z}$ on any closed subdisk $|z| \leq r < 1$ of the open unit disk $|z| < 1$.

Solution

To establish the uniform convergence of the series for $|z| \leq r < 1$, we apply the Weierstrass M-test. We have $|f_n(z)| = |z^n| \leq r^n = M_n$ for all $|z| \leq r$.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=0}^{\infty} r^n$ is convergent if $0 \leq r < 1$, we conclude that the series $\sum_{n=1}^{\infty} z_n$ converges uniformly for $|z| \leq r$.

For uniformly convergent infinite series of complex functions, properties such as continuity and analyticity of the continuous functions $f_k(z)$ are carried over to the sum $S(z)$. More precisely, suppose $f_k(z), k = 1, 2, \cdots$, are all continuous (analytic) in the region of convergence, then $\sum f_k(z)$ is also continuous (analytic) in the same region.

Further, an uniform convergent infinite series allows for termwise differentiation and integration, that is,

$$\int_C \sum_{k=1}^{\infty} f_k(z)\, dz = \sum_{k=1}^{\infty} \int_C f_k(z)\, dz$$

$$\frac{d}{dz} \sum_{k=1}^{\infty} f_k(z) = \sum_{k=1}^{\infty} f_k'(z).$$

Power series

The choice of $f_n(z) = a_n(z - z_0)^n$ leads to the power series expanded at $z = z_0$. A power series defines a function $f(z)$ for those points z at which it converges.

Given a power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$, there exists a non-negative real number R, R can be zero or infinity, such that the power series converges absolutely for $|z - z_0| < R$, and diverges for $|z - z_0| > R$.

R is called the radius of convergence

$|z - z_0| < R$ is called the circle of convergence.

Convergence of the power series must be determined for each point on the circle of convergence.

Ratio test as applied to the infinite power series

The radius of convergence R is given by $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, given that the limit exists.

Consider the ratio $\frac{|a_{n+1}||z-z_0|^{n+1}}{|a_n|\ |z-z_0|^n}$ and suppose $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, then the power series converges absolutely when $|z-z_0|$ satisfies

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\ |z-z_0| < 1 \quad \Leftrightarrow \quad |z-z_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

(i) $R = \infty$ when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. The series converges in the whole plane.

(ii) $R = 0$ when $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 0$. The series does not converges for any z other than z_0.

Root test as applied to the infinite power series

The radius of convergence can also be found by

$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}}$, which is a consequence of the root test.

Note that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}|z - z_0| < 1 \quad \Leftrightarrow \quad |z - z_0| < \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}.$$

Example

Find the circle of convergence for each of the following power series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}(z-i)^k$,

(b) $\displaystyle\sum_{k=1}^{\infty} k^{\ln k}(z-2)^k$,

(c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{z}{k}\right)^k$,

(d) $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2} z^k$.

Solution

(a) By the ratio test, we have

$$R = \lim_{k\to\infty} \frac{1/k}{1/(k+1)} = 1;$$

so the circle of convergence is $|z - i| = 1$.

(b) Using the root test, we have

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{k^{\ln k}}.$$

To evaluate the limit, we consider the logarithm,

$$\ln \lim_{k\to\infty} \sqrt[k]{k^{\ln k}} = \lim_{k\to\infty} \ln \sqrt[k]{k^{\ln k}} = \lim_{k\to\infty} \frac{(\ln k)^2}{k} = 0;$$

so

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[k]{k^{\ln k}} = e^0 = 1.$$

The circle of convergence is $|z - 2| = 1$.

(c) By the ratio test, we have

$$R = \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^k}{\left(\frac{1}{k+1}\right)^{k+1}} = \lim_{k\to\infty} (k+1)\left(1+\frac{1}{k}\right)^k = \infty;$$

so the circle of convergence is the whole complex plane.

(d) By the root test, we have

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e,$$

so the circle of convergence is $|z| = 1/e$.

Theorem

If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

then the power series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$.

Some useful results

Let $S(z)$ denote the sum of the infinite power series inside the circle of convergence. Then

(i) $S(z)$ represents a continuous function at each point interior to its circle of convergence.

(ii) $S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ at each point z inside the circle of convergence.

Power series of $f(z)$

Suppose a power series represents the function $f(z)$ inside the circle of convergence, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

It is known that a power series can be differentiated termwise so that

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \\ f''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}, \cdots . \end{aligned}$$

Putting $z = z_0$ successively, we obtain

$$\begin{aligned} a_n &= \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \\ f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \end{aligned}$$

A power series represents an analytic function inside its circle of convergence. Can we expand an analytic function in Taylor series and how is the domain of analyticity related to the circle of convergence?

Taylor series theorem

Let $f(z)$ be analytic in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ and $z_0 \in \mathcal{D}$. Determine R such that

$$R = \min\{|z - z_0|, \quad z \in \partial\mathcal{D}\}.$$

Then there exists a power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ which converges to $f(z)$ for $|z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)}{k!}, \quad k = 0, 1, 2, \cdots,$$

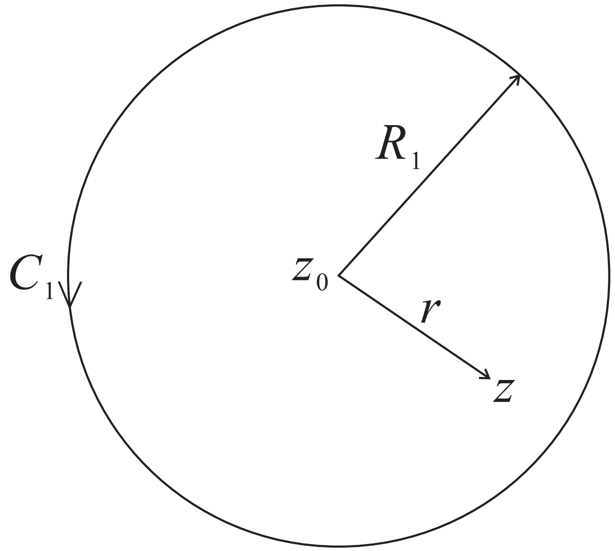
C is any closed contour around z_0 and lying completely inside $\mathcal{D}$.

Proof

Take a point z such that $|z - z_0| = r < R$.

A circle is drawn around z_0 with radius R_1, where $r < R_1 < R$. Since z lies inside C_1, by virtue of the Cauchy Integral Theorem, we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$



The circle C_1 lies completely inside $\mathcal{D}$.

The trick is to express the integrand $\dfrac{f(\zeta)}{\zeta - z}$ in powers of $\dfrac{z - z_0}{\zeta - z_0}$. Recall

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \frac{u^{n+1}}{1-u}.$$

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0}\left[1 + \frac{z - z_0}{\zeta - z_0} + \cdots + \frac{(z - z_0)^n}{(\zeta - z_0)^n} + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^{n+1}}{1 - \frac{z - z_0}{\zeta - z_0}}\right].
\end{aligned}$$

We then multiply by $\dfrac{f(\zeta)}{2\pi i}$ and integrate along C_1.

By observing the property

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta = \frac{f^{(k)}(z_0)}{k!},$$

we obtain

$$f(z) = \sum_{k=0}^{n}\frac{f^{(k)}(z)}{k!}(z - z_0)^k + R_n,$$

where the remainder is given by

$$R_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}\left(\frac{z - z_0}{\zeta - z_0}\right)^{n+1}\,d\zeta.$$

To complete the proof, it suffices to show that

$$\lim_{n\to\infty} R_n = 0.$$

(i) $|f(\zeta)| \le M$ for $\zeta \in C_1$ since $f(z)$ is continuous inside $\mathcal{D}$.
(ii) $|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \ge |\zeta - z_0| - |z - z_0| = R_1 - r$

so that

$$\left|\frac{f(\zeta)}{\zeta - z}\right|\left|\frac{z - z_0}{\zeta - z_0}\right|^{n+1} \leq \frac{M}{R_1 - r}\left(\frac{r}{R_1}\right)^{n+1}.$$

Finally

$$|R_n| \leq \left|\frac{1}{2\pi i}\right|\frac{M}{R_1 - r}\left(\frac{r}{R_1}\right)^{n+1} \underbrace{2\pi R_1}_{\text{arc length}} = \frac{MR_1}{R_1 - r}\left(\frac{r}{R_1}\right)^{n+1} \to 0 \text{ as } n \to \infty.$$

The Taylor coefficients are given by

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \qquad k = 0, 1, 2, \cdots \\ &= \frac{1}{2\pi i}\oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \end{aligned}$$

where C_1 is replaced by C and C is any simple closed contour enclosing z_0 and lying completely inside $\mathcal{D}$ [by virtue of Corollary 3 of the Cauchy-Goursat Theorem].

Example

Consider the function $\frac{1}{1-z}$, the Taylor series at $z = 0$ is given by

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

- The function has a singularity at $z = 1$. The maximum distance from $z = 0$ to the nearest singularity is one, so the radius of convergence is one.

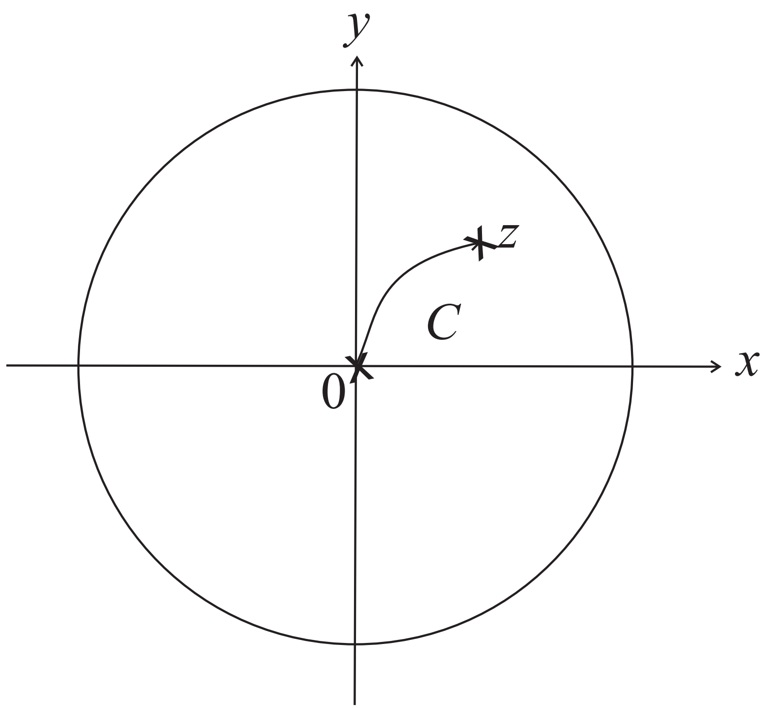
- Alternatively, the radius of convergence can be found by the ratio test, where

$$R = \lim_{k \to \infty} \left| \frac{a_k}{a_{k+1}} \right| = 1.$$

If we integrate along the contour C inside the circle of convergence $|z| < 1$ from the origin to an arbitrary point z, we obtain

$$\begin{aligned} \int_C \frac{1}{1-\zeta}\, d\zeta &= \sum_{n=0}^{\infty} \int_C \zeta^n\, d\zeta \\ -\mathrm{Log}(1-z) &= \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z^n}{n}. \end{aligned}$$

The radius of convergence is again one (checked by the ratio test).



Example

Consider the Taylor series of the real function: $\dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

What is the interval of convergence?

- The Taylor series converges only for $|x| < 1$. This is because the complex extension $\dfrac{1}{1+z^2}$ has singularities on the circle $|z| = 1$ so that the radius of convergence of the infinite series $\displaystyle\sum_{n=0}^{\infty} (-1)^n z^{2n}$ is one. The above infinite power series diverges for points outside the circle of convergence, including points on the real axis, where $|x| > 1$.
- When $|x| = 1$, the series becomes

  $$1 - 1 + 1 \cdots$$

  with alternating terms. It is known to be divergent.

Example

Find the Taylor expansion of $f(z) = \dfrac{1}{1+z^2}$ at $z = 1$. The function is analytic inside $|z-1| < \sqrt{2}$.

$$
\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) \\
&= \frac{1}{2i}\left(\frac{1}{1-i}\frac{1}{1+\frac{z-1}{1-i}} - \frac{1}{1+i}\frac{1}{1+\frac{z-1}{1+i}}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2i}\left[\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right](z-1)^n.
\end{aligned}
$$

By observing $1-i = \sqrt{2}e^{-i\pi/4}$ and $1+i = \sqrt{2}e^{i\pi/4}$, we obtain

$$
\begin{aligned}
\frac{1}{1+z^2} &= \sum_{n=0}^{\infty}(-1)^n \frac{e^{i(n+1)\pi/4} - e^{-i(n+1)\pi/4}}{(2i)2^{(n+1)/2}}(z-1)^n \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{\sin(n+1)\frac{\pi}{4}}{2^{(n+1)/2}}(z-1)^n.
\end{aligned}
$$

Example

Find the Maclaurin expansion of $f(z) = (z+1)^{1/2}$, where the principal branch of the function is used. Where is the expansion valid?

Solution

The required branch is identical to $e^{\frac{1}{2}\text{Log}(z+1)}$, whose derivative is given by

$$e^{\frac{1}{2}\text{Log}(z+1)}\frac{1}{2(z+1)} = \frac{(z+1)^{1/2}}{2(z+1)}.$$

Deductively, we have

$$f'(z) = \frac{1}{2}(z+1)^{\frac{1}{2}-1}, \quad f''(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)(z+1)^{\frac{1}{2}-2}, \cdots,$$
$$f^{(n)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left[\frac{1}{2}-(n-1)\right](z+1)^{\frac{1}{2}-n}, \quad n \geq 1,$$

where $(z+1)^{\frac{1}{2}-n} = \dfrac{(z+1)^{\frac{1}{2}}}{(z+1)^n} = \dfrac{e^{\frac{1}{2}\text{Log}(z+1)}}{(z+1)^n}$.

The principal branch of $\text{Log}(z+1)$ is taken to have branch cut along the negative real axis with branch points at $z = -1$ and $z = \infty$.

When $z = 0, e^{\frac{1}{2}\text{Log } 1} = 1$ so that the Maclaurin coefficients are

$$c_0 = 1, c_n = \frac{1}{n!}\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n-1\right)\right], \quad n \geq 1.$$

The singularity of $(z+1)^{1/2}$ nearest to the origin is the branch point $z = -1$. Hence, the circle of convergence is $|z| < 1$.

Remark

An analytic branch of a multi-valued function can be expanded in a Taylor series about any point within the domain of analyticity of the branch, provided that one takes care to use this branch consistently in obtaining the coefficients of the series.

Laurent series

Consider an infinite power series with negative power terms

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

how to find the region of convergence? Set $w = \dfrac{1}{z - z_0}$, the series becomes $\displaystyle\sum_{n=1}^{\infty} b_n w^n$, a Taylor series in w. Suppose $R' = \displaystyle\lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|$ exists, then $\displaystyle\sum_{n=1}^{\infty} b_n w^n$ converges for $|w| < R' \Leftrightarrow |z - z_0| > \dfrac{1}{R'}$.

Special cases: (i) $R' = 0$ and (ii) $\dfrac{1}{R'} = 0$.

1. When $R' = \lim_{n\to\infty} \left|\dfrac{b_n}{b_{n+1}}\right| = 0$, the infinite series $\sum_{n-1}^{n} b_n(z-z_0)^{-n}$ does not converge for any z, not even at $z = z_0$.

2. When $\dfrac{1}{R'} = \lim_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right| = 0$, we consider the ratio of successive terms in $\sum_{n=1}^{\infty} b_n w^n$ and observe that

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| |w| < 1$$

is satisfied for all w (except $w = \infty$ since the infinite series $\Sigma b_n(z-z_0)^{-n}$ is not defined at $z = z_0$). By virtue of the ratio test, the region of convergence is the whole complex plane except at $z = z_0$, that is, $|z - z_0| > 0$.

For the more general case (Laurent series at z_0)

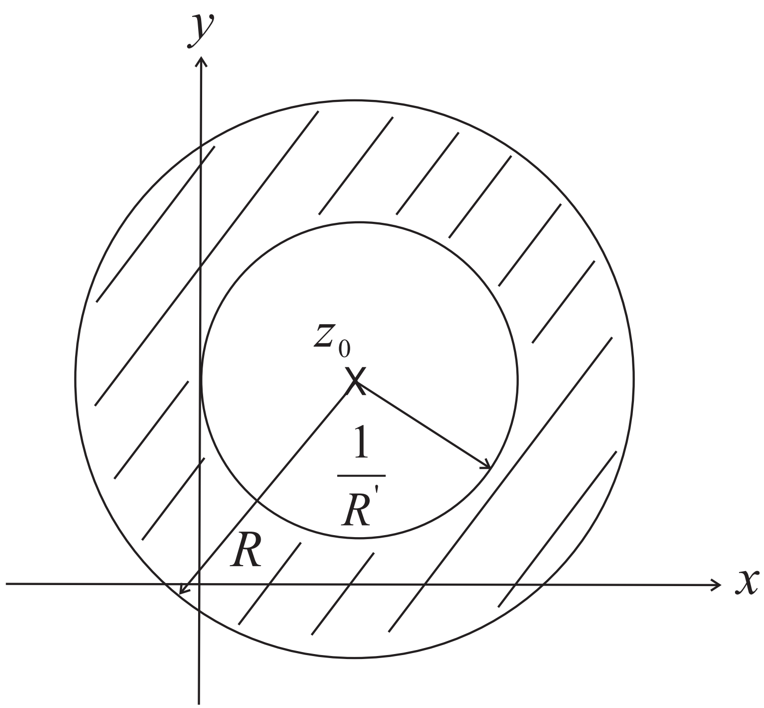
$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}}_{\text{principal part}},$$

suppose $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ and $R' = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ exists, and $RR' > 1$, then inside the annular domain

$$\left\{z : \quad \frac{1}{R'} < |z-z_0| < R\right\}$$

the Laurent series is convergent.

- When $RR' \leq 1$, the intersection of the two regions: $|z-z_0| < R$ and $|z-z_0| > \frac{1}{R'}$ is the empty set. This is because the combination of $|z-z_0| > \frac{1}{R'}$ and $RR' \leq 1$ implies $|z-z_0| \geq R$, which contradicts with $|z-z_0| < R$.



The annulus degenerates into

(i) hollow plane if $R = \infty$

(ii) punctured disc if $R' = \infty$.

Remarks

1. When $R = 0$, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ does not converge for any z other than the trivial point z_0. However, $z = z_0$ is a singularity for the principal part. Actually, when $R = 0, RR' > 1$ can never be satisfied.

2. A Laurent series defines a function $f(z)$ in its annular region of convergence. The Laurent series theorem states that a function analytic in an annulus can be expanded in a Laurent series expansion.

Laurent series theorem

Let $f(z)$ be analytic in the annulus $A : R_1 < |z - z_0| < R_2$, then $f(z)$ can be represented by the Laurent series,

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k,$$

which converges to $f(z)$ throughout the annulus. The Laurent coefficients are given by

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta, \quad k = 0, \pm 1, \pm 2, \cdots,$$

where C is any simple closed contour lying completely inside the annulus and going around the point z_0.

Remarks

1. Suppose $f(z)$ is analytic in the full disc: $|z - z_0| < R_2$ (without the punctured hole), then the integrand in calculating c_k for negative k becomes analytic in $|z - z_0| < R_2$. Hence, $c_k = 0$ for $k = -1, -2, \cdots$.

   The Laurent series is reduced to a Taylor series.

2. When $k = -1, c_{-1} = \dfrac{1}{2\pi i} \oint_C f(\zeta)\, d\zeta$. We may find c_{-1} by any means, so a contour integral can be evaluated without resort to direct integration.

Example

The Laurent expansion of $e^{1/z}$ at $z = 0$ is given by

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}.$$

The function $e^{1/z}$ is analytic everywhere except at $z = 0$ so that the annulus of convergence is $|z| > 0$. We observe

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = 0 \quad \text{so that } \frac{1}{R'} = 0.$$

Lastly, we consider $\oint_C e^{1/z}\, dz$, where the contour C is $|z| = 1$. Since C lies completely inside the punctured disc $|z| > 0$, we have

$$\oint_C e^{1/z}\, dz = 2\pi i(\text{coefficient of } \frac{1}{z} \text{ in Laurent expansion}) = 2\pi i$$

Example

Find the Laurent expansion of

$$f(z) = \sin\left(z - \frac{1}{z}\right).$$

The function has a singularity at $z = 0$ so that the annulus of convergence is $|z| > 0$. The Laurent coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{\sin\left(z - \frac{1}{z}\right)}{z^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2,$$

where C is chosen to be the unit circle $|z| = 1$.

Take $z = e^{i\theta}$ so that $dz = ie^{i\theta}\,d\theta$, then

$$
\begin{aligned}
c_n &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\sin(2i\sin\theta)ie^{i\theta}}{e^{i(n+1)\theta}}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} i\sinh(2\sin\theta)(\cos n\theta - i\sin n\theta)\,d\theta.
\end{aligned}
$$

Note that $\sinh(2\sin\theta)$ is an odd function in θ, hence

$$
\int_{-\pi}^{\pi} \sinh(2\sin\theta)\cos n\theta\,d\theta = 0.
$$

Finally,

$$
c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sinh(2\sin\theta)\sin n\theta\,d\theta, \qquad n = 0, \pm 1, \pm 2, \cdots .
$$

Example

Find the Laurent expansion of the function

$$f(z) = \frac{1}{z-k}$$

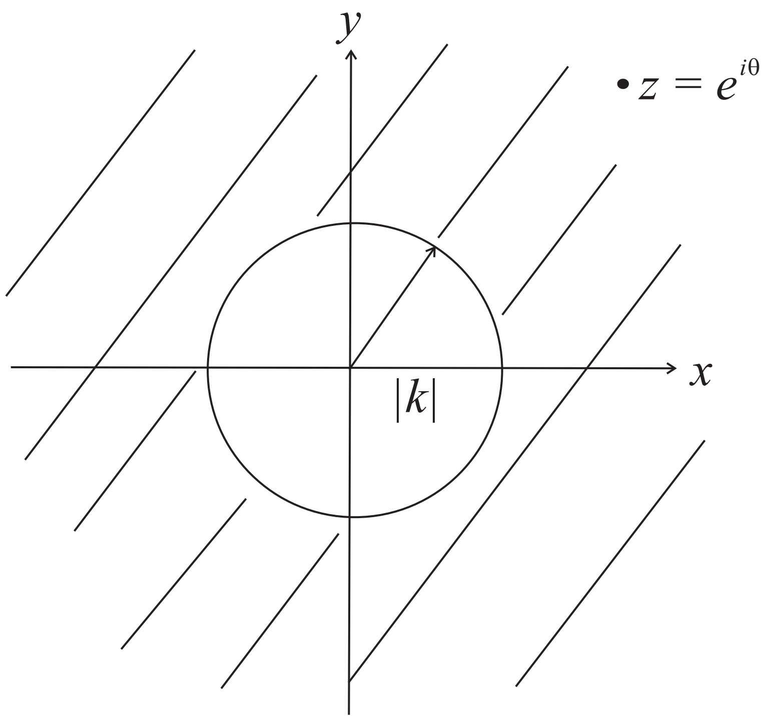
that is valid inside the domain $|z| > |k|$, where k is real and $|k| < 1$. Using the Laurent expansion, deduce that

$$\begin{aligned}
\sum_{n=1}^{\infty} k^n \cos n\theta &= \frac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}, \\
\sum_{n=1}^{\infty} k^n \sin n\theta &= \frac{k\sin\theta}{1 - 2k\cos\theta + k^2}.
\end{aligned}$$

Solution

For $|z| > |k|$, where $|k| < 1$, we have

$$\begin{aligned}\frac{1}{z-k} &= \frac{1}{z\left(1-\frac{k}{z}\right)} \\ &= \frac{1}{z}\left(1+\frac{k}{z}+\frac{k^2}{z^2}+\cdots\right) \quad \text{for} \quad \left|\frac{k}{z}\right| < 1.\end{aligned}$$



Take the point $z = e^{i\theta}$, which lies inside the region of convergence of the above Laurent series. Substituting into the infinite series

$$\frac{1}{e^{i\theta} - k} = \frac{1}{e^{i\theta}}(1 + ke^{-i\theta} + k^2 e^{-2i\theta} + \cdots + k^n e^{-in\theta} + \cdots).$$

Rearranging the terms

$$\begin{aligned}
\frac{e^{i\theta}(e^{-i\theta} - k)}{(e^{i\theta} - k)(e^{-i\theta} - k)} - 1 &= \frac{1 - k(\cos\theta + i\sin\theta) - (1 - 2k\cos\theta + k^2)}{1 - 2k\cos\theta + k^2} \\
&= \frac{k\cos\theta - k^2 - ik\sin\theta}{1 - 2k\cos\theta + k^2} \\
&= \sum_{n=1}^{\infty} k^n \cos n\theta - i\sum_{n=1}^{\infty} k^n \sin n\theta.
\end{aligned}$$

Equating the real and imaginary parts, we obtain the desired results.

Example

By evaluating the contour integral

$$\oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z},$$

show that

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = 2\pi \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}.$$

Solution

On $|z| = 1, z = e^{i\theta}, \left(z + \frac{1}{z}\right)^{2n} = 2^{2n}\cos^{2n}\theta, \frac{dz}{z} = i\, d\theta.$

$$I = \oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = 2^{2n} i \int_0^{2\pi} \cos^{2n}\theta\, d\theta.$$

On the other hand, the integrand can be expanded as

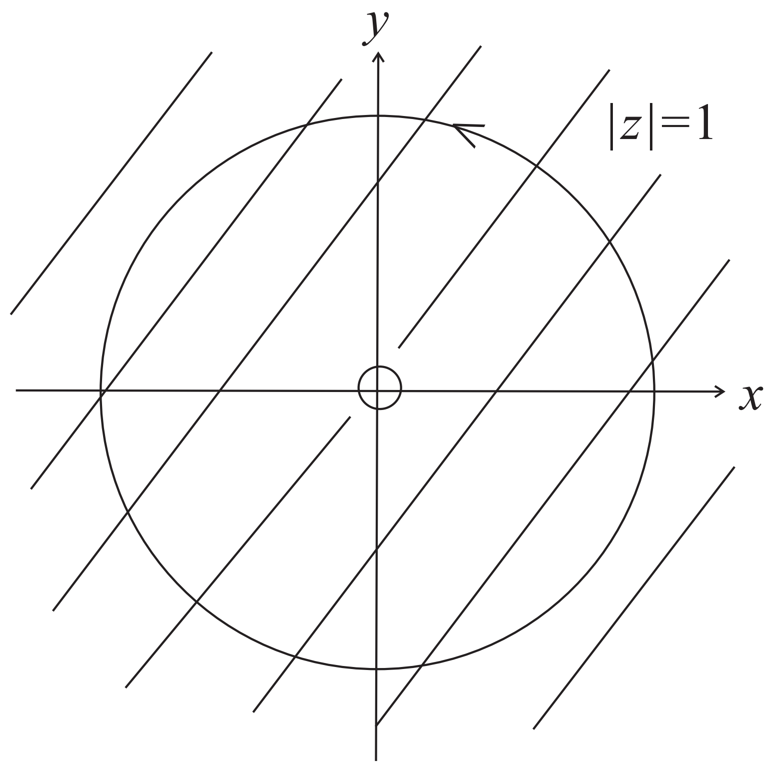
$$\frac{1}{z}\left(z^{2n} + {}_{2n}C_1 z^{2n-2} + \cdots + {}_{2n}C_n + \cdots + {}_{2n}C_{2n}\frac{1}{z^{2n}}\right), \quad 0 < |z| < \infty.$$

Since the integrand is analytic everywhere except at $z = 0$, the above expansion is the Laurent series of the integrand valid in the annulus: $|z| > 0$.

$$\begin{aligned} \oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} &= 2\pi i \left(\text{coefficient of } \frac{1}{z} \text{ in the Laurent series}\right) \\ &= 2\pi i \, {}_{2n}C_n. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^{2\pi} \cos^{2n}\theta \, d\theta &= 2\pi \frac{{}_{2n}C_n}{2^{2n}} \\ &= 2\pi \frac{(2n)!}{(n!)^2 2^{2n}} \\ &= 2\pi \frac{1 \cdot 3 \cdots (2n-1)(2^n n!)}{(2^n n!)(2^n n!)} \\ &= 2\pi \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}. \end{aligned}$$



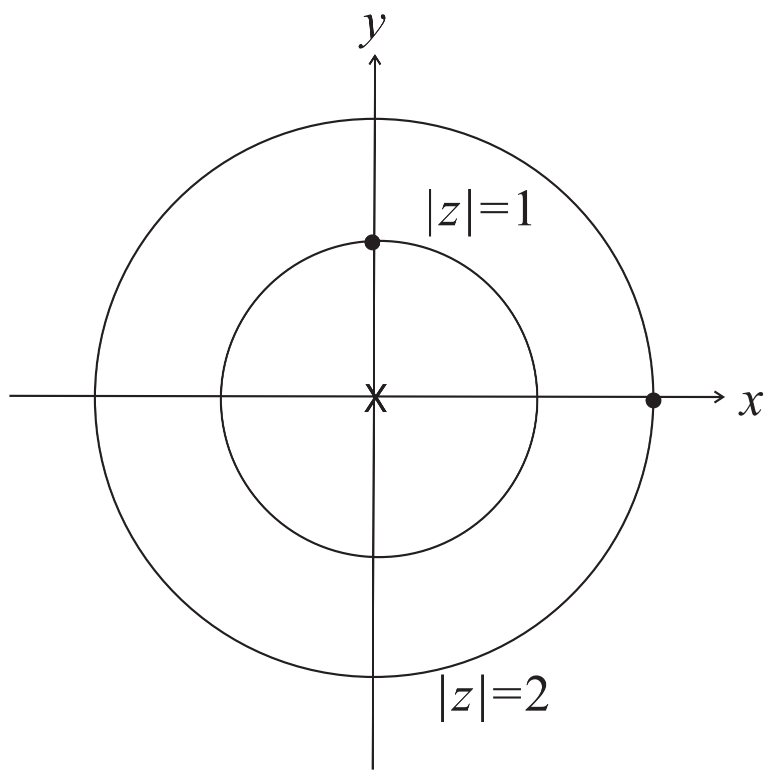
The shaded region is $|z| > 0$. The circle $C : |z| = 1$ lies completely inside the annulus of convergence and goes around $z = 0$. This is a punctured complex plane with a single deleted point.

Example

Find all the possible Taylor and Laurent series expansions of

$$f(z) = \frac{1}{(z-i)(z-2)} \quad \text{at } z_0 = 0.$$

Specify the region of convergence (solid disc or annulus) of each of the above series.



Solution

There are two isolated singularities, namely, at $z = i$ and $z = 2$. The possible circular or annular regions of analyticity are

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

(i) For $|z| < 1$

$$\begin{aligned}
\frac{1}{z-i} &= \frac{i}{1-\frac{z}{i}} = i\left[1+\frac{z}{i}+\cdots+\left(\frac{z^n}{i^n}\right)+\right] \quad \text{for} \quad |z|<1 \\
\frac{1}{z-2} &= \left(-\frac{1}{2}\right)\frac{1}{1-\frac{z}{2}} = \left(-\frac{1}{2}\right)\left[1+\frac{z}{2}+\cdots+\left(\frac{z}{2}\right)^n+\cdots\right] \quad \text{for} \quad |z|<2 \\
f(z) &= \frac{1}{i-2}\left(\frac{1}{z-i}-\frac{1}{z-2}\right) = \frac{1}{i-2}\left[i\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right] \\
&= \frac{1}{i-2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{i}\right)^{n-1}+\frac{1}{2^{n+1}}\right]z^n.
\end{aligned}$$

This is a Taylor series which converges inside the solid disc $|z| < 1$.

(ii) For $1 < |z| < 2$

$$\frac{1}{z-i} = \frac{1}{z}\frac{1}{1-\frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n \qquad \text{valid for } |z| > 1$$

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \qquad \text{valid for } |z| < 2$$

$$f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}\left(\frac{i^n}{z^{n+1}} + \frac{z^n}{2^{n+1}}\right)\right] \qquad \text{valid for } 1 < |z| < 2.$$

(iii) For $|z| > 2$

$$\frac{1}{z-i} = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} \qquad \text{valid for } |z| > 1$$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \qquad \text{valid for } |z| > 2$$

$$f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}(i^n - 2^n)\frac{1}{z^{n+1}}\right] \qquad \text{valid for } |z| > 2.$$

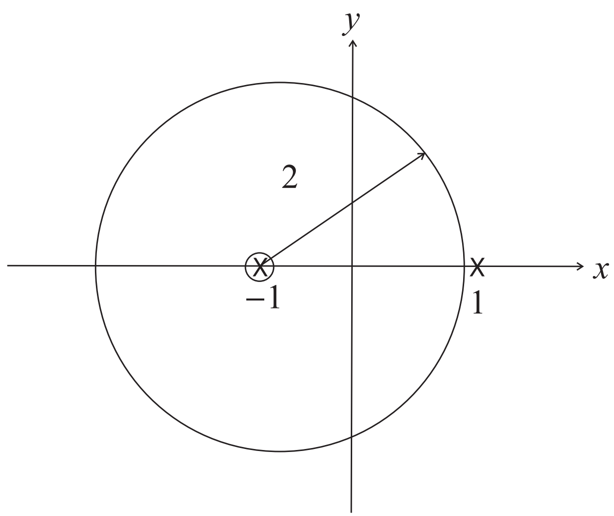
Example

Find all possible Laurent series of

$$f(z) = \frac{1}{1 - z^2} \quad \text{at the point } \alpha = -1.$$

Solution

The function has two isolated singular points at $z = 1$ and $z = -1$. There exist two annular regions (i) $0 < |z + 1| < 2$ and (ii) $|z + 1| > 2$ where the function is analytic everywhere inside the respective region.



(i) $0 < |z+1| < 2$

$$\begin{aligned}
\frac{1}{1-z^2} &= \frac{1}{2(z+1)\left(1-\frac{z+1}{2}\right)} \\
&= \frac{1}{2(z+1)} \sum_{k=0}^{\infty} \frac{(z+1)^k}{2^k} \\
&= \frac{1}{2(z+1)} + \frac{1}{4} + \frac{1}{8}(z+1) + \frac{1}{16}(z+1)^2 + \cdots .
\end{aligned}$$

The above expansion is valid provided $\left|\dfrac{z+1}{2}\right| < 1$ and $z+1 \neq 0$.

Given that $0 < |z+1| < 2$, the above requirement is satisfied.

For any simple closed curve C_1 lying completely inside $|z+1| < 2$ and encircling the point $z = -1$, we have

$$c_{-1} = \frac{1}{2\pi i} \oint_{C_1} \frac{1}{1-z^2}\, dz = \frac{1}{2}.$$

(ii) $|z+1| > 2$

$$\begin{aligned}
\frac{1}{1-z^2} &= -\frac{1}{(z+1)^2\left(1-\frac{2}{z+1}\right)} \\
&= -\frac{1}{(z+1)^2}\sum_{k=0}^{\infty}\frac{2^k}{(z+1)^k} \text{ since } \left|\frac{2}{z+1}\right| < 1 \\
&= -\left[\frac{1}{(z+1)^2}+\frac{2}{(z+1)^3}+\frac{4}{(z+1)^4}+\cdots\right].
\end{aligned}$$

For any simple closed curve C_2 encircling the circle: $|z+1| = 2$, we have

$$c_1 = \frac{1}{2\pi i}\oint_{C_2}\frac{1}{1-z^2}\,dz = 0.$$

Example

Suppose $f(z)$ is analytic inside the annulus: $r < |z| < \dfrac{1}{r}, r < 1$ and satisfies $f\left(\dfrac{1}{\overline{z}}\right) = \overline{f(z)}$. Write the Laurent expansion of $f(z)$ at $z = 0$ as $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$. Show that

(a) $\overline{a}_k = a_{-k}$,

(b) $f(z)$ is real on $|z| = 1$.

Solution

(a) Consider

$$\begin{aligned} f\left(\frac{1}{\overline{z}}\right) &= \sum_{k=-\infty}^{\infty} a_k \overline{z}^{-k} = \sum_{k=-\infty}^{\infty} a_{-k} \overline{z}^{k} \\ \overline{f(z)} &= \sum_{k=-\infty}^{\infty} \overline{a}_k \overline{z}^{k} \end{aligned}$$

$$f\left(\frac{1}{\overline{z}}\right) = \overline{f(z)} \Rightarrow \overline{a}_k = a_{-k}.$$

(b) It suffices to show $\overline{f(z)} = f(z)$ when z satisfies $z\overline{z} = 1$. This is obviously satisfied from the given property $f\left(\frac{1}{\overline{z}}\right) = \overline{f(z)}$, for all z lying inside the annulus: $r < |z| < \frac{1}{r}$.

Example

From the Maclaurin series expansion of e^z, it follows that the Laurent series expansion of $e^{1/z}$ about 0 is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}, \quad |z| > 0.$$

The Laurent coefficients are

$$b_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{1/z}}{z^{1-n}}\, dz$$

where $\mathcal{C}$ is chosen to be the circle $|z| = 1$. Such choice of $\mathcal{C}$ is feasible since the circle $|z| = 1$ lies completely inside the annulus region of convergence of $e^{1/z}$ and it goes around the point $z = 0$. By comparing the coefficients, we obtain

$$\oint_{\mathcal{C}} \frac{e^{1/z}}{z^{1-n}}\, dz = \frac{2\pi i}{n!}, \quad n \in \mathbb{N}.$$

On $|z| = 1, z = e^{i\theta}, -\pi \le \theta \le \pi$, so that the complex integral can be expressed as

$$\int_{-\pi}^{\pi} \frac{\exp(e^{-i\theta})}{e^{(1-n)i\theta}} ie^{i\theta}\, d\theta = \int_{-\pi}^{\pi} ie^{(\cos\theta - i\sin\theta + in\theta)}\, d\theta = \frac{2\pi i}{n!}$$

giving

$$\int_{-\pi}^{\pi} e^{\cos\theta}[\cos(n\theta - \sin\theta) + i\sin(n\theta - \sin\theta)]\, d\theta = \frac{2\pi}{n!}.$$

Comparing the real parts on both sides, we obtain

$$\int_{-\pi}^{\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}, \quad \text{for all } n \in \mathbb{N}.$$

Equating the imaginary parts gives

$$\int_{-\pi}^{\pi} e^{\cos\theta}\sin(n\theta - \sin\theta)\, d\theta = 0.$$

The above result is obvious since the integrand $e^{\cos\theta}\sin(n\theta - \sin\theta)$ is an odd function.

Classification of singularities and residue calculus

Singular point of a function $f(z)$ is a point at which $f(z)$ is not analytic.

Isolated singularity Existence of a neighborhood of z_0 in which z_0 is the only singular point of $f(z)$.

e.g. $f(z) = \dfrac{1}{z^2+1}$ has $\pm i$ as isolated singularities.

Singularities of $\csc z$ are all isolated and they are simply the zeros of $\sin z$, namely, $z = k\pi, k$ is any integer.

Non-isolated singularities

$f(z) = \overline{z}$ is nowhere analytic so that every point in $\mathbb{C}$ is a non-isolated singularity.

Consider $f(z) = \csc\frac{\pi}{z}$, all points $z = 1/n, n = \pm 1, \pm 2, \cdots$, are isolated singularities, while the origin is a non-isolated singularity.

Suppose z_0 is an isolated singularity, then there exists a positive number R such that $f(z)$ is analytic inside the deleted neighborhood $0 < |z - z_0| < R$.

This forms an annular domain within which the Laurent series theorem is applicable, where

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

The isolated singularity z_0 is classified according to the property of the principal part of the Laurent series expanded in the deleted neighborhood of z_0 where f is analytic. A singular point must be either a removable singularity, an essential singularity or a pole.

Removable singularity

The principal part vanishes and the Laurent series is essentially a Taylor series. The series represents an analytic function in $|z - z_0| < R$.

For example, $\dfrac{1-\cos z}{z^2}$ is undefined at $z = 0$. The Laurent expansion of $\dfrac{1-\cos z}{z^2}$ in a deleted neighborhood of $z = 0$ is

$$
\begin{aligned}
\frac{1-\cos z}{z^2} &= \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!}\cdots\right)\right] \\
&= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!}\cdots +, \quad 0 < |z| < \infty,
\end{aligned}
$$

Note that $\lim_{z\to 0} \dfrac{1-\cos z}{z^2} = \dfrac{1}{2!}$. The singularity at $z = 0$ can be removed by defining $f(0) = \dfrac{1}{2}$. The function $f(z) = \begin{cases} (1-\cos z)/z^2, & z \neq 0 \\ 1/2, & z = 0 \end{cases}$ admits the above Taylor series expansion valid for $|z| < \infty$.

Essential singularity

The principal part has infinitely many non-zero terms. For example, consider $z^2 e^{1/z}$. Inside the annular region $0 < |z| < \infty$, the Laurent series of $z^2 e^{1/z}$ is

$$z^2 e^{1/z} = z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots, \quad 0 < |z| < \infty.$$

Pole of order k

The Laurent expansion in the deleted neighborhood of z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_k}{(z - z_0)^k},$$

with $b_k \neq 0$. It is called a simple pole when $k = 1$. For example, $z = \pm i$ are simple poles of $\frac{1}{1 + z^2}$.

Example

Observe that

$$\frac{1-\cos z}{z^5} = \frac{1}{z^3}\left[\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots\right], \quad 0 < |z| < \infty,$$

so that $(1-\cos z)/z^5$ has a pole of order 3 at $z = 0$.

Example

The function $f(z) = \dfrac{1}{z(z-1)^2}$ has a pole of order 2 at $z = 1$. It admits the following Laurent expansion

$$\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^3} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^5} - \cdots + \cdots, \quad |z-1| > 1.$$

It is incorrect to conclude that $z = 1$ is an essential singularity by observing that the above Laurent expansion has infinitely many negative power terms. This is because the annulus of convergence of the series is **not** a deleted neighborhood of $z = 1$.

Example

The Laurent expansion

$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

contains infinitely many negative power terms of z. Is $z = 0$ an essential singularity of the function represented by the series?

The first sum converges to $\dfrac{1/z}{1 - 1/z} = \dfrac{1}{z-1}$ for $|z| > 1$ and the second sum converges to $\dfrac{1/2}{1 - z/2} = \dfrac{1}{2-z}$ for $|z| < 2$.

Hence, the Laurent series converges to

$$f(z) = \frac{1}{z-1} + \frac{1}{2-z} = -\frac{1}{z^2 - 3z + 2} \quad \text{in} \quad 1 < |z| < 2.$$

The annulus of convergence is **not** a deleted neighborhood of $z = 0$. Hence, we cannot make any claim about whether $z = 0$ is an essential singularity of $f(z)$ simply through the examination of the above series.

Simple method of finding the order of a pole

If z_0 is a pole of order k, then

$$\lim_{z \to z_0} (z - z_0)^k f(z) = b_k, \quad b_k \neq 0.$$

In general, if we multiply $f(z)$ by $(z - z_0)^m$ and take the limit $z \to z_0$, then

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \begin{cases} b_k & m = k \\ 0 & m > k \\ \infty & m < k \end{cases}.$$

For example, take $f(z) = \dfrac{1 - \cos z}{z^5}$.

$$\lim_{z \to 0} z^m \frac{1 - \cos z}{z^5} = \begin{cases} \frac{1}{2} & m = 3 \\ 0 & m > 3 \\ \infty & m < 3 \end{cases}.$$

Hence, $z = 0$ is a pole of order 3 of $(1 - \cos z)/z^5$.

Example

Suppose $z = 0$ is a pole of order n and m, respectively, of the functions $f_1(z)$ and $f_2(z)$, find the possible order of $z = 0$ as a pole for $f_1(z) + f_2(z)$. Without loss of generality, we assume $n \geq m$.

(i) When $n > m$

$$\begin{aligned} f_1(z) &= \sum_{k=1}^{n} b_k^{(1)} z^{-k} + \sum_{k=0}^{\infty} a_k^{(1)} z^k, \quad b_n^{(1)} \neq 0, \\ f_2(z) &= \sum_{k=1}^{m} b_k^{(2)} z^{-k} + \sum_{k=0}^{\infty} a_k^{(2)} z^k, \quad b_m^{(2)} \neq 0, \end{aligned}$$

so $f_1(z) + f_2(z)$ must contain negative power terms up to z^{-n}.

(ii) When $n = m$, in general, $z = 0$ is a pole of order n for $f_1(z) + f_2(z)$. However, it is possible that $f_1(z) = -f_2(z) + \dfrac{1}{z^\ell}, 0 \leq \ell \leq n$. In this case, $z = 0$ is a pole of order ℓ. When $\ell = 0, z = 0$ is **not** an isolated singularity.

Behavior of f near isolated singularities

1. If z_0 is a pole of f, then

$$\lim_{z \to z_0} f(z) = \infty.$$

To show the claim, suppose f has a pole of order m at z_0, then

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0. Since

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{\lim_{z \to z_0} (z - z_0)^m}{\lim_{z \to z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0,$$

hence $\lim_{z \to z_0} f(z) = \infty$.

2. If z_0 is a removable singularity of f, then f admits a Taylor series in some deleted neighborhood $0 < |z - z_0| < \delta$ of z_0, so it is analytic and bounded in that neighborhood.

3. In each deleted neighborhood of an essential singularity, the function assumes values that come arbitrarily close to any given number.

Casorati-Weierstrass Theorem

Suppose that z_0 is an essential singularity of a function f, and let w_0 be any complex number. Then, for any positive number ϵ,

$$|f(z) - w_0| < \epsilon \qquad (A)$$

is satisfied at some point z in each deleted neighborhood of z_0.

Proof

Take any deleted neighborhood of z_0. Since z_0 is an isolated singularity, the neighborhood itself or a subset of which will have the property that f is analytic inside $0 < |z - z_0| < \delta$ for some value of δ. Assume that inequality (A) is not satisfied for any z in the neighborhood $0 < |z - z_0| < \delta$ (inside which f is analytic). Thus,

$$|f(z) - w_0| \geq \epsilon \text{ when } 0 < |z - z_0| < \delta.$$

Define

$$g(z) = \frac{1}{f(z) - w_0}, \quad 0 < |z - z_0| < \delta,$$

which is bounded and analytic. Hence, z_0 is a removable singularity of $g^\star$. We let g be defined at z_0 such that it is analytic there.

$\star$ We use the following result: Suppose f is analytic and bounded in $0 < |z - z_0| < \epsilon$. If f is not analytic at z_0, then it has a removable singularity at z_0.

We consider the following two cases:

(i) If $g(z_0) \neq 0$, then

$$f(z) = \frac{1}{g(z)} + w_0, \quad 0 < |z - z_0| < \delta, \qquad (B)$$

becomes analytic at z_0 if $f(z_0) = \dfrac{1}{g(z_0)} + w_0$. This means that z_0 is a removable singularity of f, not an essential one. We have a contradiction.

(ii) If $g(z_0) = 0$, then g must have a zero of some finite order m at z_0 since $g(z)$ is not identically equal to zero in the neighborhood $|z - z_0| < \delta$. Write $g(z) = (z - z_0)^m \psi(z_0)$, where $\psi(z_0) \neq 0$. In this case, f has a pole of order m since

$$\lim_{z \longrightarrow z_0} f(z)(z - z_0)^m = \lim_{z \longrightarrow z_0} \frac{(z - z_0)^m}{g(z)} = \frac{1}{\psi(z_0)}$$

which is finite. Again, there is a contradiction.

Example

Classify the zeros and singularities of the function $\sin(1 - z^{-1})$.

Solution

Since the zeros of $\sin w$ occur only when w is an integer multiple of π, the function $\sin(1 - z^{-1})$ has zeros when

$$1 - z^{-1} = n\pi,$$

i.e., at

$$z = \frac{1}{1 - n\pi}, \quad n = 0, \pm 1, \pm 2, \cdots .$$

Furthermore, the zeros are simple because the derivative at these points is

$$\begin{aligned}\frac{d}{dz}\sin(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}} &= \frac{1}{z^2}\cos(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}} \\ &= (1-n\pi)^2\cos n\pi \neq 0.\end{aligned}$$

The only singularity of $\sin(1-z^{-1})$ appears at $z=0$. Suppose we let z approach 0 through positive values, $\sin(1-z^{-1})$ oscillates between ± 1. Such behavior can only characterize an essential singularity.