50 years of Whitney elements

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Finite element exterior calculus

In the fundamental PDEs of math physics most quantities can be viewed as differential forms, and most operators built up from the exterior derivative $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$. To discretize we need finite element subspaces of the spaces

$$H\Lambda^k = \{ u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \},$$

compatible with exterior differentiation, i.e., with the de Rham complex

$$0 \longrightarrow H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^{n-1} \xrightarrow{d} H\Lambda^n \longrightarrow 0$$

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} \cdots \longrightarrow H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$

> “Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.”
>
> – James Clerk Maxwell, *Treatise on Electricity & Magnetism*, 1891

0-forms:	temperature; electric potential
1-forms:	temperature gradient; electric field; magnetic field
2-forms:	heat flux; magnetic flux
3-forms:	charge density; mass density

- $-\operatorname{div}\operatorname{grad} u = f$
- $(\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f$
- $\operatorname{div} u = f$, $\operatorname{curl} u = 0$
- $\operatorname{curl}\operatorname{curl} u = f$, $\operatorname{div} u = 0$
- Maxwell’s equations
- elasticity
- dynamic problems, eigenvalue problems, lower order-terms
- variable coefficients, nonlinearities...
- Hodge Laplacian on k-forms: $(dd^* + d^*d)u = f$

Compatible discretization

The key to compatibility turns out to be that the subspaces $\Lambda_h^k \subset H\Lambda^k$

1. form a *subcomplex*, which
2. admits a *bounded cochain projection*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H\Lambda^{k-1} & \xrightarrow{d} & H\Lambda^k & \xrightarrow{d} & H\Lambda^{k+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \pi_h^{k-1}} & & \downarrow{\scriptstyle \pi_h^k} & & \downarrow{\scriptstyle \pi_h^{k+1}} & & \\
\cdots & \longrightarrow & \Lambda_h^{k-1} & \xrightarrow{d} & \Lambda_h^k & \xrightarrow{d} & \Lambda_h^{k+1} & \longrightarrow & \cdots
\end{array}$$

From these two assumptions follows almost everything that is needed for the stability and convergence of mixed methods.

Finite element differential forms

There are two families of finite element subspaces that are by far the most natural for $H\Lambda^k$. They are built with respect to a simplicial triangulation $\mathcal{T}_h$ in any number of dimensions n, and are indexed by the polynomial degree $r \geq 1$ and form degree $0 \leq k \leq n$:

$$\mathcal{P}_r\Lambda^k(\mathcal{T}_h) \qquad \text{and} \qquad \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$

- For $k = 0$ the spaces coincide and give the familiar Lagrange elts.
- For $k = n$, $\mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h)$, the space of *all* pw polynomials of degree <r.
- For $0 < k < n$, $\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$
- The polynomial shape functions for $\mathcal{P}_r^-\Lambda^k$ on a triangle T are defined through the *Koszul* differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(T)$$

$$\begin{aligned}\kappa(\, dx_i \wedge dx_j \wedge dx_k \wedge \cdots \wedge d_l) = \quad & x_i\, dx_j \wedge dx^k \wedge \cdots \wedge dx_l \\ & - x_j\, dx_i \wedge dx^k \wedge \cdots \wedge dx_l \\ & + \cdots\end{aligned}$$

Degrees of freedom

DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex f of dimension d we associate

$$\omega \mapsto \int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$$

Theorem. These DOFs are unisolvent and the resulting finite element space satisfies

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\, \omega \in H\Lambda^k(\Omega) \;:\; \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}\,\}$$

DOF for $\mathcal{P}_r^-\Lambda^k(T)$:

$$\omega \mapsto \int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$$

+ similar theorem...

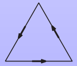
The early years of mixed FEM

In the mid-1960s Fraeijs de Veubeke and other engineers proposed the use of mixed formulations for elasticity, but didn't find any useful elements.

In the mid-1970s, Raviart and Thomas attacked the easier problem of elements for the mixed formulation of the Laplacian and invented the Raviart–Thomas elements.[1] $\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$ (2D)

Shape functions $\mathcal{P}_1^-\Lambda^1$: span $[dx, dy, \kappa(dx \wedge dy)]$ = span $\left[\binom{1}{0}, \binom{0}{1}, \binom{-y}{x}\right]$
$\kappa(dx \wedge dy) = -y\, dx + x\, dy$

DOFs: $u \mapsto= \int_e (\operatorname{tr}_e u) \wedge \eta, \quad \eta \in \mathcal{P}_0\Lambda^0(e), \qquad u \mapsto \int_e u \cdot s$

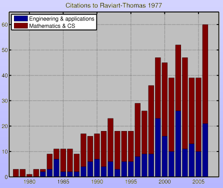


[1] Almost...

Citations to Raviart–Thomas 1977

A mixed FEM for 2nd order elliptic problems, Proc. conf. Math'l Aspects of the FEM, Rome 1975. Springer Lect. Notes in Math #606, 1977.

1300 Google scholar citations



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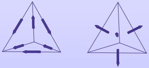
Progress in the 1980s

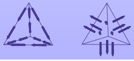
The next three major advances were published in Numer. Math.

- Nédélec 1980: *Mixed finite elements in* $\mathbb{R}^3$
  $\mathcal{P}_r^-\Lambda^1$ ($H(\text{curl})$ and $\mathcal{P}_r^-\Lambda^2$ ($H(\text{div})$) in 3D.



- Brezzi–Douglas–Marini 1985: *Two families of mixed finite elements for second order elliptic problems*, $\mathcal{P}_r\Lambda^1$ in 2D.

- Nédélec 1986: *A new family of mixed finite elements in* $\mathbb{R}^3$
  $\mathcal{P}_r\Lambda^1$ and $\mathcal{P}_r\Lambda^2$ in 3D.



Whitney forms

Bossavit 1988: *"Mixed elements are Whitney forms, rediscovered."*

Specifically, the lowest order elements of Raviart–Thomas '77 and Nédélec '80 were defined by Whitney in his 1957 book *Geometric Integration Theory*.

For $f \in \Delta_k(\mathcal{T}_h)$ let $\lambda_0, \dots, \lambda_k$ denote the hat functions associated to its vertices. Whitney defined an "elementary form" associated to f by

$$\omega_f = \sum_{j=0}^{k} (-1)^j \lambda_j \, d\lambda_0 \wedge \cdots \wedge \widehat{d\lambda_j} \wedge \cdots \wedge d\lambda_k$$

- $f \in H\Lambda^k(\Omega)$
- $f \in \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$
- $\int_g \omega_f = \delta_{fg}, \; f, g \in \Delta_k(\mathcal{T}_h)$

The space of Whitney k-forms $\operatorname{span}\{\omega_f \mid f \in \Delta_k(\mathcal{T}_h)\}$ is $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ in our notation.

What was a topologist doing with finite elements?

Topology 101: Betti numbers

The Betti numbers are the most basic topological invariants of a domain in $\mathbb{R}^n$ or, more generally, a smooth manifold. E.g., for $\Omega \subset \mathbb{R}^3$

$$b_k = \begin{cases} \#\text{ components of } \Omega, & k = 0 \\ \#\text{ tunnels thru } \Omega, & k = 1 \\ \#\text{ voids in } \Omega, & k = 2 \\ 0, & k = 3 \end{cases}$$

1, 1, 0, 0

1, 1, 0, 0

1, 2, 1, 0

1, 2 , 0, 0

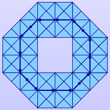
1, 0, 1, 0

Two approaches to compute Betti numbers

Combinatorial approach: simplicial homology (Poincaré 1890s)

$$0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0$$

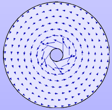
$$H^k_{\text{simp}} = \mathcal{N}(C_k \xrightarrow{\partial} C_{k-1}) / \mathcal{R}(C_{k+1} \xrightarrow{\partial} C_k)$$



PDE approach: de Rham cohomology (É. Cartan, de Rham 1930s)

$$0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$$

$$H^k_{\text{dR}} = \mathcal{N}(\Lambda^k \xrightarrow{d} \Lambda^{k+1}) / \mathcal{R}(\Lambda^{k-1} \xrightarrow{d} \Lambda^k)$$



De Rham's Theorem

- De Rham map:
$$\begin{array}{ccc} \Lambda^k(\Omega) & \longrightarrow & C^k(\mathcal{T}) := C_k(\mathcal{T})^* \\ \omega & \longmapsto & \left(\gamma \mapsto \int_\gamma \omega\right) \end{array}$$

- Stokes theorem says it's a cochain map, so induces a map from de Rham to simplicial cohomology.
$$\begin{tikzcd} \cdots \arrow[r, "d"] & \Lambda^k(\Omega) \arrow[r, "d"] \arrow[d] & \Lambda^{k+1}(\Omega) \arrow[r, "d"] \arrow[d] & \cdots \\ \cdots \arrow[r, "\partial^*"] & C_k^*(\mathcal{T}) \arrow[r, "\partial^*"] & C_{k+1}^*(\mathcal{T}) \arrow[r, "\partial^*"] & \cdots \end{tikzcd}$$

- De Rham's theorem: Induced map is an isomorphism on cohomology.
- An elegant proof can be giving by realizing cochains as differential forms via the Whitney forms.

Whitney as numerical analyst

Hassler Whitney
1907–1989

Whitney wished to compute a quantity described by a PDE (precisely, the multiplicity of 0 as an eigenvalue of the Hodge Laplacian). He replaced the solution space by a piecewise polynomial subspace and the differential operators by discrete analogues. In this sense he was using finite elements in the way we numerical people do.

Betti numbers are integers, so there is no convergence theorem. But the approximation properties of the Whitney forms does enter the proof. Along the way Whitney gave a refinement procedure that produces a sequence of refinements which remains shape regular as $h \to 0$.

Eigenvalue convergence and the Ray–Singer conjecture

De Rham's theorem equates two approaches to calculating Betti numbers, by *simplicial cohomology* defined discretely through triangulations, and by *de Rham cohomology* defined via PDE.

The Ray–Singer conjecture does this for another important topological invariant:

Rademeister-Franz torsion defined combinatorially using a triangulation	=	*analytic torsion* defined analytically using the Hodge Laplacian eigenvalues

Hoping to prove it, in 1976 Dodziuk and Patodi proved the convergence of the approximation of the eigenvalues of the Hodge Laplacian obtained using Whitney forms.

In 1978 Müller completed the program using Dodziuk and Patodi's convergence results to prove the Ray–Singer conjecture.

V.K. Patodi
1945–1976

Rates of convergence

Dodziuk and Patodi were not much concerned with rates of convergence. Their approach led to $O(h \log h)$ convergence of the eigenvalues. (optimal is $O(h^2)$)

Baker, who worked on Hodge theory but had a background in finite elements, set out to do better. Instead of the complex of Whitney forms

$$0 \to \mathcal{P}_1^-\Lambda^0 \xrightarrow{d} \mathcal{P}_1^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_1^-\Lambda^{n-1} \xrightarrow{d} \mathcal{P}_1^-\Lambda^n \to 0$$

he used the complex of "Sullivan–Whitney forms"

$$0 \to \mathcal{P}_{n+1}\Lambda^0 \xrightarrow{d} \mathcal{P}_n\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_2\Lambda^{n-1} \xrightarrow{d} \mathcal{P}_1\Lambda^n \to 0$$

and in '83 proved eigenvalue convergence of $O(h^2)$.

FEEC gives the sharp rates of convergence in all cases: $O(h^2)$ for the Whitney forms, $O(h^{2r})$ if the higher order Whitney forms $\mathcal{P}_r^-\Lambda^k$ or the Sullivan-Whitney forms $\mathcal{P}_r\Lambda^k$ are used.

Closing remarks

- Our FEEC approach differs from the topologists approach in several ways. In particular, we work with the *mixed formulation*

  $$\langle \sigma, \tau \rangle = \langle d\tau, u \rangle, \quad \langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle, \quad \forall \tau, v.$$

  For us, consistency is not a problem, and the concern is establishing stability. The topologists used the primal formulation $\langle du, dv \rangle + \langle d^* u, d^* v \rangle = \langle f, v \rangle$ and consistency was a big headache because none of the finite element spaces belong to the domain of d^*.
- We constructed *bounded cochain projections* and use them heavily. These are powerful tools, which simplify many arguments.
- The search has been on since the 1960s for *stable mixed finite elements for elasticity*. The first ones with polynomial shape functions were found in 2002 (joint with Winther) using tools of FEEC. Since then there has been a great deal further progress. But that is another story. . .

Primary references (joint with R. Falk and R. Winther)

Finite element exterior calculus, homological techniques, and applications, Acta Numerica 2006, p. 1–155

“Any young (or not so young) mathematician who spends the time to master this paper will have tools that will be useful for his or her entire career.” — Math Reviews

COMING SOON!

Finite element exterior calculus: From Hodge theory to numerical stability