# 6.5 Trigonometric Form of a Complex Number

### What you should learn

- Plot complex numbers in the complex plane and find absolute values of complex numbers.
- Write the trigonometric forms of complex numbers.
- Multiply and divide complex numbers written in trigonometric form.
- Use DeMoivre's Theorem to find powers of complex numbers.
- Find *n*th roots of complex numbers.

### Why you should learn it

You can use the trigonometric form of a complex number to perform operations with complex numbers. For instance, in Exercises 105–112 on page 480, you can use the trigonometric forms of complex numbers to help you solve polynomial equations.

## The Complex Plane

Just as real numbers can be represented by points on the real number line, you can represent a complex number

$$z = a + bi$$

as the point $(a, b)$ in a coordinate plane (the **complex plane**). The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis,** as shown in Figure 6.44.

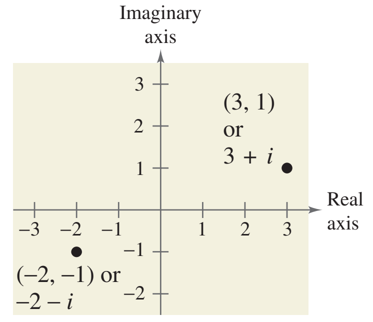



FIGURE 6.44

The **absolute value** of the complex number $a + bi$ is defined as the distance between the origin $(0, 0)$ and the point $(a, b)$.

### Definition of the Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

If the complex number $a + bi$ is a real number (that is, if $b = 0$), then this definition agrees with that given for the absolute value of a real number

$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|.$$

### Example 1 Finding the Absolute Value of a Complex Number

Plot $z = -2 + 5i$ and find its absolute value.

#### Solution

The number is plotted in Figure 6.45. It has an absolute value of

$$|z| = \sqrt{(-2)^2 + 5^2}$$
$$= \sqrt{29}.$$

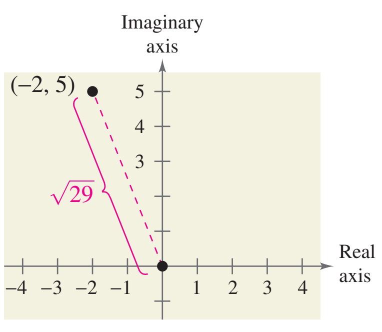



FIGURE 6.45

CHECKPOINT Now try Exercise 3.

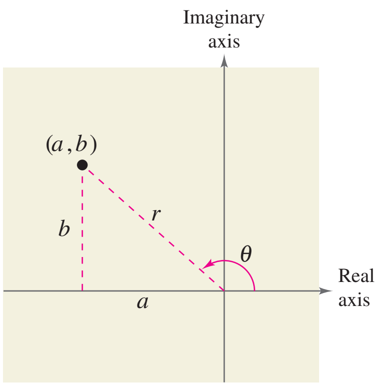



FIGURE 6.46

## Trigonometric Form of a Complex Number

In Section 2.4, you learned how to add, subtract, multiply, and divide complex numbers. To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in trigonometric form. In Figure 6.46, consider the nonzero complex number $a + bi$. By letting $\theta$ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point $(a, b)$, you can write

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r\cos\theta) + (r\sin\theta)i$$

from which you can obtain the **trigonometric form of a complex number.**

### Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number $r$ is the **modulus** of $z$, and $\theta$ is called an **argument** of $z$.

The trigonometric form of a complex number is also called the *polar form*. Because there are infinitely many choices for $\theta$, the trigonometric form of a complex number is not unique. Normally, $\theta$ is restricted to the interval $0 \le \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.

### Example 2 Writing a Complex Number in Trigonometric Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

#### Solution

The absolute value of $z$ is

$$r = \left|-2 - 2\sqrt{3}i\right| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

and the reference angle $\theta'$ is given by

$$\tan\theta' = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and because $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, you choose $\theta$ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$z = r(\cos\theta + i\sin\theta)$$
$$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

See Figure 6.47.

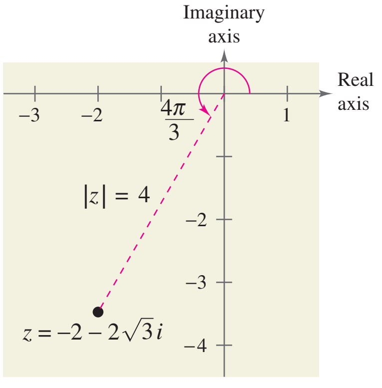



FIGURE 6.47

CHECKPOINT Now try Exercise 13.

**Example 3** **Writing a Complex Number in Standard Form**

Write the complex number in standard form $a + bi$.

$$z = \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

**Technology**

A graphing utility can be used to convert a complex number in trigonometric (or polar) form to standard form. For specific keystrokes, see the user's manual for your graphing utility.

**Solution**

Because $\cos(-\pi/3) = \frac{1}{2}$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can write

$$\begin{aligned} z &= \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \\ &= 2\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= \sqrt{2} - \sqrt{6}i. \end{aligned}$$

CHECKPOINT Now try Exercise 35.

## Multiplication and Division of Complex Numbers

The trigonometric form adapts nicely to multiplication and division of complex numbers. Suppose you are given two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

The product of $z_1$ and $z_2$ is given by

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]. \end{aligned}$$

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following rule. The second part is left for you to verify (see Exercise 117).

### Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

Note that this rule says that to *multiply* two complex numbers you multiply moduli and add arguments, whereas to *divide* two complex numbers you divide moduli and subtract arguments.

### Example 4 Multiplying Complex Numbers

Find the product $z_1 z_2$ of the complex numbers.

$$z_1 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \qquad z_2 = 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

**Solution**

$$\begin{aligned}
z_1 z_2 &= 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right) \\
&= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] && \text{Multiply moduli and add arguments.} \\
&= 16\left(\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right) \\
&= 16\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\
&= 16[0 + i(1)] \\
&= 16i
\end{aligned}$$

You can check this result by first converting the complex numbers to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically, as in Section 2.4.

$$\begin{aligned}
z_1 z_2 &= \left(-1 + \sqrt{3}i\right)\left(4\sqrt{3} - 4i\right) \\
&= -4\sqrt{3} + 4i + 12i + 4\sqrt{3} \\
&= 16i
\end{aligned}$$

CHECKPOINT Now try Exercise 47.

> **Technology**
>
> Some graphing utilities can multiply and divide complex numbers in trigonometric form. If you have access to such a graphing utility, use it to find $z_1 z_2$ and $z_1/z_2$ in Examples 4 and 5.

### Example 5 Dividing Complex Numbers

Find the quotient $z_1/z_2$ of the complex numbers.

$$z_1 = 24(\cos 300^\circ + i \sin 300^\circ) \qquad z_2 = 8(\cos 75^\circ + i \sin 75^\circ)$$

**Solution**

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{24(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 75^\circ + i \sin 75^\circ)} \\
&= \frac{24}{8}[\cos(300^\circ - 75^\circ) + i \sin(300^\circ - 75^\circ)] && \text{Divide moduli and subtract arguments.} \\
&= 3(\cos 225^\circ + i \sin 225^\circ) \\
&= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\
&= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i
\end{aligned}$$

CHECKPOINT Now try Exercise 53.

333202_0605.qxd 12/5/05 10:45 AM Page 474

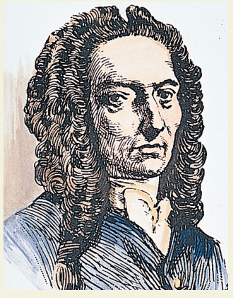


## Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta)\\
z^2 &= r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)\\
z^3 &= r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)\\
z^4 &= r^4(\cos 4\theta + i\sin 4\theta)\\
z^5 &= r^5(\cos 5\theta + i\sin 5\theta)\\
&\ \vdots
\end{aligned}
$$

This pattern leads to DeMoivre's Theorem, which is named after the French mathematician Abraham DeMoivre (1667–1754).



The Granger Collection

***Historical Note***
Abraham DeMoivre (1667–1754) is remembered for his work in probability theory and DeMoivre's Theorem. His book *The Doctrine of Chances* (published in 1718) includes the theory of recurring series and the theory of partial fractions.

### DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is a positive integer, then

$$
\begin{aligned}
z^n &= [r(\cos\theta + i\sin\theta)]^n\\
&= r^n(\cos n\theta + i\sin n\theta).
\end{aligned}
$$

### Example 6 Finding Powers of a Complex Number

Use DeMoivre's Theorem to find $\left(-1 + \sqrt{3}i\right)^{12}$.

#### Solution

First convert the complex number to trigonometric form using

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2 \text{ and } \theta = \arctan\frac{\sqrt{3}}{-1} = \frac{2\pi}{3}.$$

So, the trigonometric form is

$$z = -1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Then, by DeMoivre's Theorem, you have

$$
\begin{aligned}
(-1 + \sqrt{3}i)^{12} &= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12}\\
&= 2^{12}\left[\cos\frac{12(2\pi)}{3} + i\sin\frac{12(2\pi)}{3}\right]\\
&= 4096(\cos 8\pi + i\sin 8\pi)\\
&= 4096(1 + 0)\\
&= 4096.
\end{aligned}
$$

CHECKPOINT Now try Exercise 75.

## Roots of Complex Numbers

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree $n$ has $n$ solutions in the complex number system. So, the equation $x^6 = 1$ has six solutions, and in this particular case you can find the six solutions by factoring and using the Quadratic Formula.

$$\begin{aligned} x^6 - 1 &= (x^3 - 1)(x^3 + 1) \\ &= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = 0 \end{aligned}$$

Consequently, the solutions are

$$x = \pm 1, \qquad x = \frac{-1 \pm \sqrt{3}i}{2}, \qquad \text{and} \qquad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is a sixth root of 1. In general, the ***n*th root of a complex number** is defined as follows.

### Definition of the *n*th Root of a Complex Number

The complex number $u = a + bi$ is an ***n*th root** of the complex number $z$ if

$$z = u^n = (a + bi)^n.$$

*Exploration*

The $n$th roots of a complex number are useful for solving some polynomial equations. For instance, explain how you can use DeMoivre's Theorem to solve the polynomial equation

$$x^4 + 16 = 0.$$

[*Hint:* Write $-16$ as $16(\cos \pi + i \sin \pi)$.]

To find a formula for an $n$th root of a complex number, let $u$ be an $n$th root of $z$, where

$$u = s(\cos \beta + i \sin \beta)$$

and

$$z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by $r$, you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \qquad \text{and} \qquad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of $2\pi$, these last two equations have solutions if and only if the angles differ by a multiple of $2\pi$. Consequently, there must exist an integer $k$ such that

$$n\beta = \theta + 2\pi k$$

$$\beta = \frac{\theta + 2\pi k}{n}.$$

By substituting this value of $\beta$ into the trigonometric form of $u$, you get the result stated on the following page.

### Finding $n$th Roots of a Complex Number

For a positive integer $n$, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly $n$ distinct $n$th roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

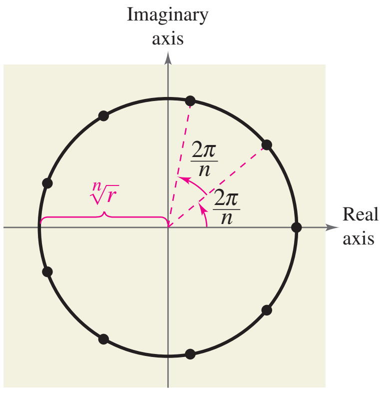



FIGURE 6.48

When $k$ exceeds $n - 1$, the roots begin to repeat. For instance, if $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

is coterminal with $\theta/n$, which is also obtained when $k = 0$.

The formula for the $n$th roots of a complex number $z$ has a nice geometrical interpretation, as shown in Figure 6.48. Note that because the $n$th roots of $z$ all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive $n$th roots have arguments that differ by $2\pi/n$, the $n$ roots are equally spaced around the circle.

You have already found the sixth roots of 1 by factoring and by using the Quadratic Formula. Example 7 shows how you can solve the same problem with the formula for $n$th roots.

### Example 7 Finding the $n$th Roots of a Real Number

Find all the sixth roots of 1.

#### Solution

First write 1 in the trigonometric form $1 = 1(\cos 0 + i\sin 0)$. Then, by the $n$th root formula, with $n = 6$ and $r = 1$, the roots have the form

$$\sqrt[6]{1}\left(\cos\frac{0 + 2\pi k}{6} + i\sin\frac{0 + 2\pi k}{6}\right) = \cos\frac{\pi k}{3} + i\sin\frac{\pi k}{3}.$$

So, for $k = 0, 1, 2, 3, 4,$ and $5$, the sixth roots are as follows. (See Figure 6.49.)

$$\cos 0 + i\sin 0 = 1$$

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \text{Increment by } \frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos\pi + i\sin\pi = -1$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

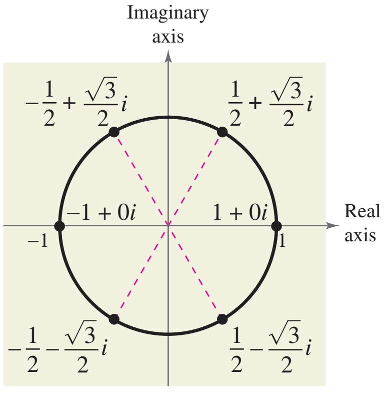



FIGURE 6.49

CHECKPOINT Now try Exercise 97.

**Activities**

1. Use DeMoivre's Theorem to find $\left(-2 - 2\sqrt{3}i\right)^3$.

   *Answer:* $64(\cos 4\pi + i \sin 4\pi) = 64$

2. Find the cube roots of

   $8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

   $8\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right).$

   *Answer:* $2\left(\cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9}\right)$

   $2\left(\cos \frac{8\pi}{9} + i \sin \frac{8\pi}{9}\right)$

   $2\left(\cos \frac{14\pi}{9} + i \sin \frac{14\pi}{9}\right)$

3. Find all of the solutions of the equation $x^4 + 1 = 0$.

   *Answer:*

   $\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

   $\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

   $\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

   $\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

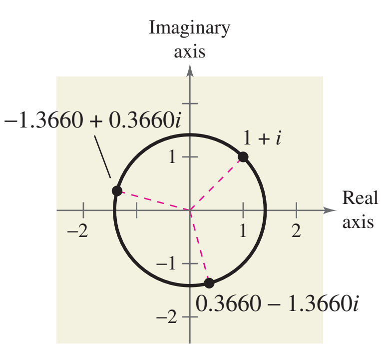



FIGURE 6.50

**STUDY TIP**

Note in Example 8 that the absolute value of $z$ is

$$r = |-2 + 2i|$$
$$= \sqrt{(-2)^2 + 2^2}$$
$$= \sqrt{8}$$

and the angle $\theta$ is given by

$$\tan \theta = \frac{b}{a} = \frac{2}{-2} = -1.$$

In Figure 8.49, notice that the roots obtained in Example 7 all have a magnitude of 1 and are equally spaced around the unit circle. Also notice that the complex roots occur in conjugate pairs, as discussed in Section 2.5. The $n$ distinct $n$th roots of 1 are called the ***n*th roots of unity.**

## Example 8 Finding the *n*th Roots of a Complex Number

Find the three cube roots of $z = -2 + 2i$.

### Solution

Because $z$ lies in Quadrant II, the trigonometric form of $z$ is

$$z = -2 + 2i$$
$$= \sqrt{8}\,(\cos 135° + i \sin 135°). \qquad \theta = \arctan(2/-2) = 135°$$

By the formula for $n$th roots, the cube roots have the form

$$\sqrt[6]{8}\left(\cos \frac{135° + 360°k}{3} + i \sin \frac{135° + 360°k}{3}\right).$$

Finally, for $k = 0, 1,$ and $2$, you obtain the roots

$$\sqrt[6]{8}\left(\cos \frac{135° + 360°(0)}{3} + i \sin \frac{135° + 360°(0)}{3}\right) = \sqrt{2}(\cos 45° + i \sin 45°)$$
$$= 1 + i$$

$$\sqrt[6]{8}\left(\cos \frac{135° + 360°(1)}{3} + i \sin \frac{135° + 360°(1)}{3}\right) = \sqrt{2}(\cos 165° + i \sin 165°)$$
$$\approx -1.3660 + 0.3660i$$

$$\sqrt[6]{8}\left(\cos \frac{135° + 360°(2)}{3} + i \sin \frac{135° + 360°(2)}{3}\right) = \sqrt{2}(\cos 285° + i \sin 285°)$$
$$\approx 0.3660 - 1.3660i.$$

See Figure 6.50.

CHECKPOINT Now try Exercise 103.

**WRITING ABOUT MATHEMATICS**

**A Famous Mathematical Formula** The famous formula

$$e^{a+bi} = e^a(\cos b + i \sin b)$$

is called Euler's Formula, after the Swiss mathematician Leonhard Euler (1707–1783). Although the interpretation of this formula is beyond the scope of this text, we decided to include it because it gives rise to one of the most wonderful equations in mathematics.

$$e^{\pi i} + 1 = 0$$

This elegant equation relates the five most famous numbers in mathematics—0, 1, $\pi$, $e$, and $i$—in a single equation. Show how Euler's Formula can be used to derive this equation.

## 6.5 Exercises

**VOCABULARY CHECK:** Fill in the blanks.

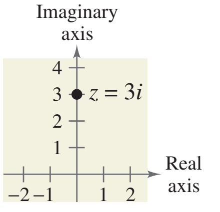
1. The ________ ________ of a complex number $a + bi$ is the distance between the origin $(0, 0)$ and the point $(a, b)$.
2. The ________ ________ of a complex number $z = a + bi$ is given by $z = r(\cos\theta + i\sin\theta)$, where $r$ is the ________ of $z$ and $\theta$ is the ________ of $z$.
3. ________ Theorem states that if $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is a positive integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
4. The complex number $u = a + bi$ is an ________ ________ of the complex number $z$ if $z = u^n = (a + bi)^n$.

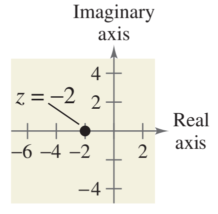
**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at **www.Eduspace.com.**

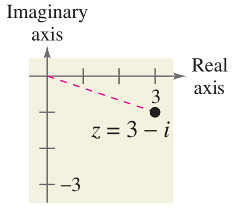
**In Exercises 1–6, plot the complex number and find its absolute value.**

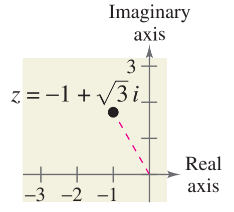
1. $-7i$
2. $-7$
3. $-4 + 4i$
4. $5 - 12i$
5. $6 - 7i$
6. $-8 + 3i$

**In Exercises 7–10, write the complex number in trigonometric form.**

7.



8.



9.



10.



**In Exercises 11–30, represent the complex number graphically, and find the trigonometric form of the number.**

11. $3 - 3i$
12. $2 + 2i$
13. $\sqrt{3} + i$
14. $4 - 4\sqrt{3}i$
15. $-2(1 + \sqrt{3}i)$
16. $\frac{5}{2}(\sqrt{3} - i)$
17. $-5i$
18. $4i$
19. $-7 + 4i$
20. $3 - i$
21. $7$
22. $4$
23. $3 + \sqrt{3}i$
24. $2\sqrt{2} - i$
25. $-3 - i$
26. $1 + 3i$
27. $5 + 2i$
28. $8 + 3i$
29. $-8 - 5\sqrt{3}i$
30. $-9 - 2\sqrt{10}i$

**In Exercises 31–40, represent the complex number graphically, and find the standard form of the number.**

31. $3(\cos 120° + i\sin 120°)$
32. $5(\cos 135° + i\sin 135°)$
33. $\frac{3}{2}(\cos 300° + i\sin 300°)$
34. $\frac{1}{4}(\cos 225° + i\sin 225°)$
35. $3.75\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
36. $6\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$
37. $8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
38. $7(\cos 0 + i\sin 0)$
39. $3[\cos(18°\,45') + i\sin(18°\,45')]$
40. $6[\cos(230°\,30') + i\sin(230°\,30')]$

**In Exercises 41–44, use a graphing utility to represent the complex number in standard form.**

41. $5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$
42. $10\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$
43. $3(\cos 165.5° + i\sin 165.5°)$
44. $9(\cos 58° + i\sin 58°)$

**In Exercises 45 and 46, represent the powers $z$, $z^2$, $z^3$, and $z^4$ graphically. Describe the pattern.**

45. $z = \frac{\sqrt{2}}{2}(1 + i)$
46. $z = \frac{1}{2}(1 + \sqrt{3}i)$

**In Exercises 47–58, perform the operation and leave the result in trigonometric form.**

47. $\left[2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\left[6\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]$
48. $\left[\frac{3}{4}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]$
49. $\left[\frac{5}{3}(\cos 140° + i\sin 140°)\right]\left[\frac{2}{3}(\cos 60° + i\sin 60°)\right]$
50. $[0.5(\cos 100° + i\sin 100°)] \times [0.8(\cos 300° + i\sin 300°)]$
51. $[0.45(\cos 310° + i\sin 310°)] \times [0.60(\cos 200° + i\sin 200°)]$
52. $(\cos 5° + i\sin 5°)(\cos 20° + i\sin 20°)$
53. $\dfrac{\cos 50° + i\sin 50°}{\cos 20° + i\sin 20°}$
54. $\dfrac{2(\cos 120° + i\sin 120°)}{4(\cos 40° + i\sin 40°)}$
55. $\dfrac{\cos(5\pi/3) + i\sin(5\pi/3)}{\cos\pi + i\sin\pi}$
56. $\dfrac{5(\cos 4.3 + i\sin 4.3)}{4(\cos 2.1 + i\sin 2.1)}$
57. $\dfrac{12(\cos 52° + i\sin 52°)}{3(\cos 110° + i\sin 110°)}$
58. $\dfrac{6(\cos 40° + i\sin 40°)}{7(\cos 100° + i\sin 100°)}$

**In Exercises 59–66, (a) write the trigonometric forms of the complex numbers, (b) perform the indicated operation using the trigonometric forms, and (c) perform the indicated operation using the standard forms, and check your result with that of part (b).**

59. $(2 + 2i)(1 - i)$
60. $(\sqrt{3} + i)(1 + i)$
61. $-2i(1 + i)$
62. $4(1 - \sqrt{3}i)$
63. $\dfrac{3 + 4i}{1 - \sqrt{3}i}$
64. $\dfrac{1 + \sqrt{3}i}{6 - 3i}$
65. $\dfrac{5}{2 + 3i}$
66. $\dfrac{4i}{-4 + 2i}$

**In Exercises 67–70, sketch the graph of all complex numbers *z* satisfying the given condition.**

67. $|z| = 2$
68. $|z| = 3$
69. $\theta = \dfrac{\pi}{6}$
70. $\theta = \dfrac{5\pi}{4}$

**In Exercises 71–88, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.**

71. $(1 + i)^5$
72. $(2 + 2i)^6$
73. $(-1 + i)^{10}$
74. $(3 - 2i)^8$
75. $2(\sqrt{3} + i)^7$
76. $4(1 - \sqrt{3}i)^3$
77. $[5(\cos 20° + i\sin 20°)]^3$
78. $[3(\cos 150° + i\sin 150°)]^4$
79. $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12}$
80. $\left[2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^8$
81. $[5(\cos 3.2 + i\sin 3.2)]^4$
82. $(\cos 0 + i\sin 0)^{20}$
83. $(3 - 2i)^5$
84. $(\sqrt{5} - 4i)^3$
85. $[3(\cos 15° + i\sin 15°)]^4$
86. $[2(\cos 10° + i\sin 10°)]^8$
87. $\left[2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right]^5$
88. $\left[2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\right]^6$

**In Exercises 89–104, (a) use the theorem on page 476 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.**

89. Square roots of $5(\cos 120° + i\sin 120°)$
90. Square roots of $16(\cos 60° + i\sin 60°)$
91. Cube roots of $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
92. Fifth roots of $32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
93. Square roots of $-25i$
94. Fourth roots of $625i$
95. Cube roots of $-\frac{125}{2}(1 + \sqrt{3}i)$
96. Cube roots of $-4\sqrt{2}(1 - i)$
97. Fourth roots of 16
98. Fourth roots of $i$
99. Fifth roots of 1
100. Cube roots of 1000
101. Cube roots of $-125$
102. Fourth roots of $-4$
103. Fifth roots of $128(-1 + i)$
104. Sixth roots of $64i$

**In Exercises 105–112, use the theorem on page 476 to find all the solutions of the equation and represent the solutions graphically.**

**105.** $x^4 + i = 0$

**106.** $x^3 + 1 = 0$

**107.** $x^5 + 243 = 0$

**108.** $x^3 - 27 = 0$

**109.** $x^4 + 16i = 0$

**110.** $x^6 + 64i = 0$

**111.** $x^3 - (1 - i) = 0$

**112.** $x^4 + (1 + i) = 0$

## Synthesis

***True or False?*** **In Exercises 113–116, determine whether the statement is true or false. Justify your answer.**

**113.** Although the square of the complex number $bi$ is given by $(bi)^2 = -b^2$, the absolute value of the complex number $z = a + bi$ is defined as

$$|a + bi| = \sqrt{a^2 + b^2}.$$

**114.** Geometrically, the $n$th roots of any complex number $z$ are all equally spaced around the unit circle centered at the origin.

**115.** The product of two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

is zero only when $r_1 = 0$ and/or $r_2 = 0$.

**116.** By DeMoivre's Theorem,

$$\left(4 + \sqrt{6}i\right)^8 = \cos(32) + i\sin\left(8\sqrt{6}\right).$$

**117.** Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $z_2 \neq 0$, show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

**118.** Show that $\bar{z} = r[\cos(-\theta) + i\sin(-\theta)]$ is the complex conjugate of $z = r(\cos\theta + i\sin\theta)$.

**119.** Use the trigonometric forms of $z$ and $\bar{z}$ in Exercise 118 to find (a) $z\bar{z}$ and (b) $z/\bar{z}$, $\bar{z} \neq 0$.

**120.** Show that the negative of $z = r(\cos\theta + i\sin\theta)$ is $-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)]$.

**121.** Show that $-\frac{1}{2}\left(1 + \sqrt{3}i\right)$ is a sixth root of 1.

**122.** Show that $2^{-1/4}(1 - i)$ is a fourth root of $-2$.

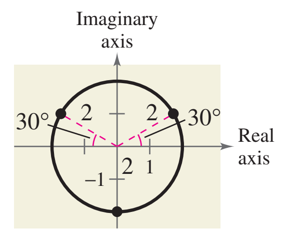
***Graphical Reasoning*** **In Exercises 123 and 124, use the graph of the roots of a complex number.**

**(a) Write each of the roots in trigonometric form.**

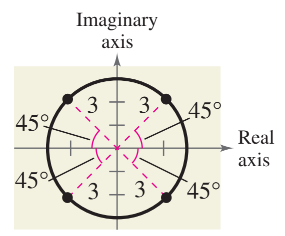
**(b) Identify the complex number whose roots are given.**

**(c) Use a graphing utility to verify the results of part (b).**

**123.**

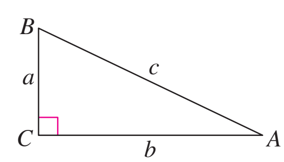



**124.**




## Skills Review

**In Exercises 125–130, solve the right triangle shown in the figure. Round your answers to two decimal places.**




**125.** $A = 22°$, $a = 8$

**126.** $B = 66°$, $a = 33.5$

**127.** $A = 30°$, $b = 112.6$

**128.** $B = 6°$, $b = 211.2$

**129.** $A = 42°\,15'$, $c = 11.2$

**130.** $B = 81°\,30'$, $c = 6.8$

***Harmonic Motion*** **In Exercises 131–134, for the simple harmonic motion described by the trigonometric function, find the maximum displacement and the least positive value of $t$ for which $d = 0$.**

**131.** $d = 16\cos\frac{\pi}{4}t$

**132.** $d = \frac{1}{8}\cos 12\pi t$

**133.** $d = \frac{1}{16}\sin\frac{5}{4}\pi t$

**134.** $d = \frac{1}{12}\sin 60\pi t$

**In Exercises 135 and 136, write the product as a sum or difference.**

**135.** $6\sin 8\theta\cos 3\theta$

**136.** $2\cos 5\theta\sin 2\theta$