

## 6. Definition of Derivative – Classwork

Let's review. For  $f(x) = x^2 - 3x + 1$ , find the slope of the tangent line at a)  $x = 2$ , b)  $x = -1$ .

$$\begin{aligned} \text{a) } m_{\tan} &= \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 3(2+h) + 1 - (-1)}{h} = \frac{4 + 4h + h^2 - 6 - 3h + 2}{h} = \frac{h(1+h)}{h} = 1 + h = 1 \\ \text{b) } m_{\tan} &= \frac{f(-1+h) - f(-1)}{h} = \frac{(h-1)^2 - 3(h-1) + 1 - 5}{h} = \frac{h^2 - 2h + 1 - 3h + 3 - 4}{h} = \frac{h(h-5)}{h} = h - 5 = -5 \end{aligned}$$

Notice how the process is the same – only the point at which we find the slope changes. With that in mind, we introduce a basic concept of calculus. The **derivative** of a function is a formula for the slope of the tangent line to that function at *any point*  $x$ . The process of taking derivatives is called **differentiation**.

The mathematical definition of a derivative of a function  $f(x)$  is  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  or  $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ .

This mimics the procedure above but calculates the slope of the tangent line at any value  $x$ . We now formally define a derivative in terms of a limit, which substitutes for the awkward “as  $h$  gets close to zero.”

Since derivatives are so important, we need some notation to denote them. There are two that we use frequently:

Function:  $f(x) = \dots$  Derivative:  $f'(x) =$  or  $f' =$       Function:  $y = \dots$  Derivative:  $y' =$  or  $\frac{dy}{dx} =$

The latter  $\frac{dy}{dx}$  appears as a fraction but for now, we think of it as one entity – the derivative. We say “ $dy \ dx$ ”

<sup>6a</sup>Example 1) For the function  $f(x) = x^2 - 3x + 1$ , find its derivative  $f'(x)$ . Then find  $f'(2)$ ,  $f'(0)$ , and  $f'(-1)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 1 - x^2 + 3x - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h} = \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3 \quad f'(2) = 4 - 3 = 1 \quad f'(0) = -3 \quad f'(-1) = -2 - 3 = -5 \end{aligned}$$

Example 2) If  $f(x) = 4x$ , find  $f'(x)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{4(x+h) - 4x}{h} = \lim_{h \rightarrow 0} \frac{4h}{h} = \lim_{h \rightarrow 0} 4 = 4$$

Example 3) If  $g(x) = x^2 + x$ , find  $g'(x)$

$$\begin{aligned} g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 + x + \Delta x - (x^2 + x)}{\Delta x} \\ g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x + 1)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 1) = 2x + 1 \end{aligned}$$

Example 4) If  $f(x) = 2x^2 - 5x + 6$ , find  $f'(x)$  and  $f'(3)$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 5(x+h) + 6 - (2x^2 - 5x + 6)}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{h(4x + 3h - 5)}{h} = 4x - 5 \quad f'(3) = 12 - 5 = 7 \end{aligned}$$

Example 5)  $y = \frac{4}{x}$ , find  $\frac{dy}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left( \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \right) \left[ \frac{x(x+h)}{x(x+h)} \right] = \lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{hx(x+h)} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{-4h}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{-4}{x(x+h)} = \frac{-4}{x^2} \end{aligned}$$

## 6. Definition of Derivative – Homework

For the following functions, find their derivatives and evaluate at  $x = 2, -4, 0,$  and  $\pi$ . Use proper notation.

1.  $y = 2x$

$$y' = \lim_{h \rightarrow 0} \frac{2(x+h) - 2x}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = 2$$

$$y'(2) = 2 \quad y'(-4) = 2 \quad y'(0) = 2 \quad y'(\pi) = 2$$

2.  $f(x) = x^2 - 5$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 5 - (x^2 - 5)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 5 - x^2 + 5}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{h(2x + \Delta x)}{h} = 2x$$

$$f'(2) = 4 \quad f'(-4) = -8 \quad f'(0) = 0 \quad f'(\pi) = 2\pi$$

3.  $f(x) = x^2 + 3x - 4$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) - 4 - (x^2 + 3x - 4)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - 4 - x^2 - 3x + 4}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(2x + h + 3)}{h} = 2x + 3$$

$$f'(2) = 7 \quad f'(-4) = -5 \quad f'(0) = 3 \quad f'(\pi) = 2\pi + 3$$

4.  $f(x) = 6x - 4x^2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{6(x+h) - 4(x+h)^2 - (6x - 4x^2)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{6x + 6h - 4x^2 - 8xh - 4h^2 - 6x + 4x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(6 - 8x - 4h)}{h} = 6 - 8x$$

$$f'(2) = -10 \quad f'(-4) = 38 \quad f'(0) = 6 \quad f'(\pi) = 6 - 8\pi$$

5.  $g(x) = x^3 + 2x$

$$g'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 + 2(x+h) - (x^3 + 2x)}{h}$$

$$g'(x) = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h}$$

$$g'(x) = \lim_{h \rightarrow 0} \frac{h(3x^2 + 2 + 3xh + h^2)}{h} = 3x^2 + 2$$

$$g'(2) = 14 \quad g'(-4) = 50 \quad g'(0) = 2 \quad g'(\pi) = 3\pi^2 + 2$$

6.  $h(x) = \frac{5}{x} + 1$

$$h'(x) = \lim_{h \rightarrow 0} \left[ \frac{\frac{5}{x+h} + 1 - (\frac{5}{x} + 1)}{h} \right] \left[ \frac{x(x+h)}{x(x+h)} \right]$$

$$h'(x) = \lim_{h \rightarrow 0} \frac{5x - 5(x+h)}{xh(x+h)} = \lim_{h \rightarrow 0} \frac{-5h}{xh(x+h)} = \frac{-5}{x^2}$$

$$h'(2) = -\frac{5}{4} \quad h'(-4) = -\frac{5}{16} \quad h'(0) = \text{DNE} \quad h'(\pi) = \frac{-5}{\pi^2}$$

7. An alternative method for finding  $f'(a)$  is to use the formula  $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ . Use the formula to find  $f'(5)$  for  $f(x) = 3x^2 - 8x - 10$ .

$$f'(5) = \lim_{x \rightarrow 5} \frac{3x^2 - 8x - 10 - 25}{x - 5} = f'(5) = \lim_{x \rightarrow 5} \frac{3x^2 - 8x - 35}{x - 5} = \lim_{x \rightarrow 5} \frac{(3x+7)(x-5)}{x-5} = \lim_{x \rightarrow 5} (3x+7) = 22$$

## 7. Derivatives Using Technology - Classwork

While the TI-84 calculator cannot find derivatives, it can approximate the value of a derivative of a function at a specific value of  $x$ . Only calculators like the TI-89 or N-Spire with CAS (Computer Algebra System) can find the actual derivative of a function because of its ability to work symbolically.

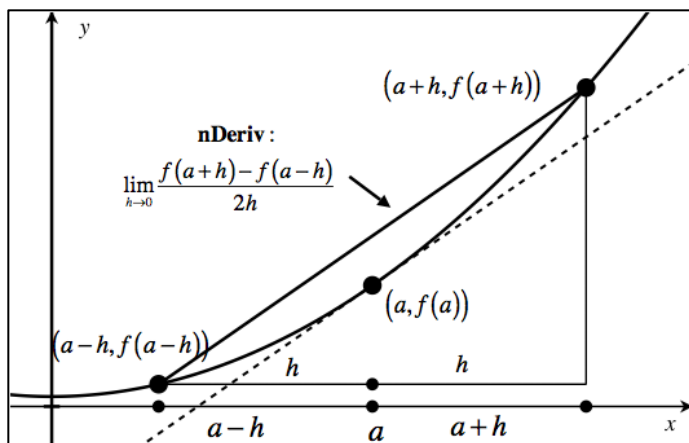
The derivative is defined as a formula for the slope of a tangent line to a function  $f(x)$ . Its formal definition is:

$$\boxed{f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}} \quad \text{and at a specific value } a, \quad \boxed{f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}}$$

The TI-84 (or equivalent) calculator uses another formula to approximate the slope of the tangent line to a function  $f(x)$  at a specific  $x$ -value. It is called the **numerical derivative**. The TI-84 abbreviates this command as **nDeriv**.<sup>7a</sup> To find the numerical derivative of a function  $f(x)$  at a specific value  $x = a$ , we calculate the slope of the secant line between 2 points the same distance from  $x = a$ , one to the left of  $a$ , and one to the right of  $a$ .

The definition of the numerical derivative (an approximation to the actual derivative) is

$$\boxed{f'(x) \approx \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}} \quad \text{and at a specific value } a, \quad \boxed{f'(a) \approx \lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{2h}}$$



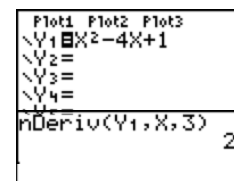
We want the derivative (the slope of the tangent line at  $(a, f(a))$ , shown by the dashed line). We start at  $x = a$ , move a distance  $h$  to the left and the same distance  $h$  to the right and draw the secant line. The slope of this line is rise over run. The rise is  $f(a+h) - f(a-h)$  and the run is  $2h$ . The slope is thus  $\frac{f(a+h) - f(a-h)}{2h}$ . When  $h$  gets very small this should be a very good approximation to  $f'(a)$ .

To use **nDeriv** with the TI-84: (to approximate  $f'(a)$ )

- 1) Place the function you wish to use in Y1.
- 2) **nDeriv is Math 8**. Use **nDeriv(Y1, X, a)**.  
Get Y1 through the VARS Y-VARS Function menu.  
You can also type the function in the **nDeriv** command.

TI-84 example: Find  $f'(3)$  for  $f(x) = x^2 - 4x + 1$ .

- 1)  $Y1 = x^2 - 4x + 1$
- 2) **nDeriv(Y1,X,3)**  
or **nDeriv(X<sup>2</sup>-4X+1,X,3)**



It is not necessary to use an  $h$  with **nDeriv**. The calculator assumes the value 0.001 as  $h$ . You can specify a smaller value of  $h$  using **nDeriv(Y1, X, a, h)**.



For the examples below, taking the derivative by the limit method is difficult algebraically, or for you at this point, impossible, because you haven't been taught differentiation methods. But if you are asked to find derivatives at specific values of  $x$ , the calculator works well. It is suggested that you set your calculator to 3-decimal place accuracy as **nDeriv** is only an approximation to the slope of the tangent line. On the calculator-active section on the AP, you are allowed to use the calculator to find derivatives without showing work.<sup>7b</sup>

Ex 1) If  $f(x) = x^4 + 3x^3 - 5x^2 - 6x$   
find  $f'(-2)$

```
nDeriv(Y1,X,-2)
18.000
```

Ex 2) If  $f(x) = e^{-x} - x$   
find  $f'(-1)$

```
nDeriv(Y1,X,-1)
-3.718
```

Ex 3) If  $f(x) = \frac{\sin x}{x}$  find  $f'(\pi)$

```
nDeriv(Y1,X,pi)
-.318
```

For the following functions  $f(x)$ , find the equation of the tangent line to  $f$  at  $x = a$ . Confirm by graphing.

Ex 4)  $f(x) = \sqrt{x+10}$ ,  $a = 6$

```
nDeriv(Y1,X,6)
.125
```

```
y - 4 = 0.125(x - 6)
```

Ex 5)  $f(x) = \sin 2x + \cos x$ ,  $a = \pi/2$

```
nDeriv(Y1,X,pi/2)
-3.000
```

```
y = -3(x - pi/2)
```

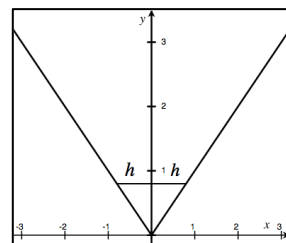
Ex 6)  $f(x) = \ln(x^2 + 2x + 1)$ ,  $a = 0$

```
nDeriv(Y1,X,0)
2.000
```

```
y = 2x
```

There are times when the **nDeriv** command will give a false answer. One of these is a point at which the derivative of a function is not defined. The best example of this is the absolute value function,  $f(x) = |x|$ , in the figure to the right. We have shown that there is no tangent line to  $|x|$  at  $x = 0$ . Yet, the **nDeriv** command gives the value zero as shown because **nDeriv** is defined as rise over run or  $0/2h = 0$ . This illustrates that **nDeriv** is an approximation to the derivative. But for most functions you will encounter, this approximation will be a good one.

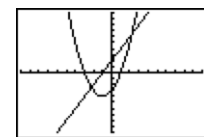
```
nDeriv(abs(X),X,0)
0.000
```



Setting up the calculator with a function such as  $f(x) = x^2 + 2x - 3$  and a numerical derivative lead to an interesting result. Placing  $f(x)$  in Y1 and defining Y2 as **nDeriv**(Y1,X,X) commands the calculator to take the numerical derivative of Y1 with respect to  $x$  at each value of  $x$  that is graphed. The resulting TI-84 table shows each point of the function ( $x$ , Y1) and Y2 is the slope of the tangent line at that point. For instance,  $f(x)$  passes through the point  $(-3, 0)$  and the slope of the tangent line to  $f$  at that point is  $-4$ .  $f(x)$  passes through the point  $(0, -3)$  and the slope to the tangent line to  $f$  at that point is  $2$ . If you graph all points  $(x, Y2)$ , you get the line  $y = 2x + 2$ , the result of taking the derivative of  $f(x) = x^2 + 2x - 3$ . This line is **not** tangent to  $f(x) = x^2 + 2x - 3$ . The  $y$ -values are the slopes of the tangent line to  $f(x)$  at each value  $x$ .

```
Plot1 Plot2 Plot3
Y1 X^2+2X-3
Y2 nDeriv(Y1,X,
X)
```

X	Y1	Y2
-3	0	-4
-2	-3	-2
-1	-4	0
0	-3	2
1	0	4
2	3	6
3	6	8



Remember that the calculator only finds derivatives numerically, not symbolically. Since the derivative is a formula for the slope of the tangent line, we must learn techniques to find this formula. Much of the semester will be devoted to that need and the calculator will not be of much use. But, if you need to find the slope of a tangent line at a specific point, the calculator will work well. Even if you have to show your work, the calculator will verify what you find by paper and pencil techniques.

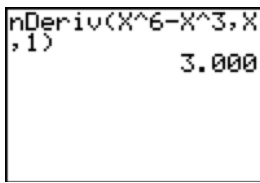


## 7. Derivatives Using Technology - Homework

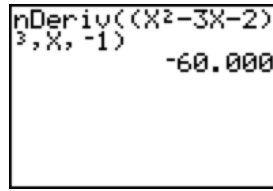


For each function  $f(x)$ , find  $f'(a)$  using the calculator.

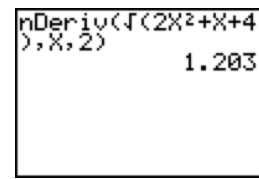
1.  $f(x) = x^6 - x^3, a = 1$



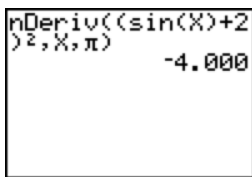
2.  $f(x) = (x^2 - 3x - 2)^3, a = -1$



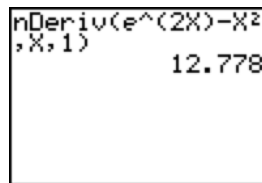
3.  $f(x) = \sqrt{2x^2 + x + 4}, a = 2$



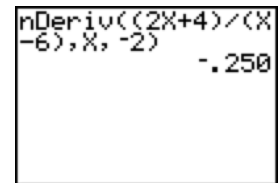
4.  $f(x) = (\sin x + 2)^2, a = \pi$



5.  $f(x) = e^{2x} - x^2, a = 1$

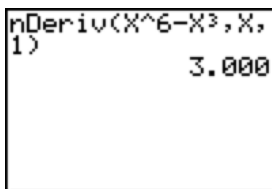


6.  $f(x) = \frac{2x+4}{x-6}, a = -2$



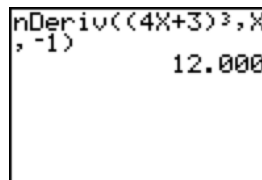
For each function  $f(x)$ , find the equation of the tangent line at  $x = a$  using the calculator.

7.  $f(x) = x^6 - x^3, a = 1$



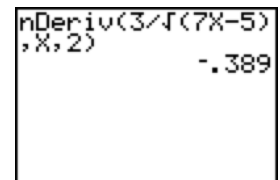
$y = 3(x-1) \Rightarrow y = 3x - 3$

8.  $f(x) = (4x+3)^3, a = -1$



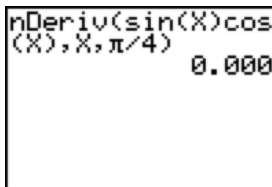
$y + 1 = 12(x + 1) \Rightarrow y = 12x + 11$

9.  $f(x) = \frac{3}{\sqrt{7x-5}}, a = 2$



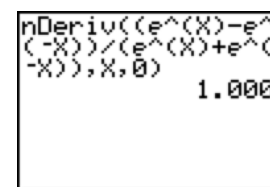
$y - 1 = -0.389(x - 2)$

10.  $f(x) = \sin x \cos x, a = \frac{\pi}{4}$



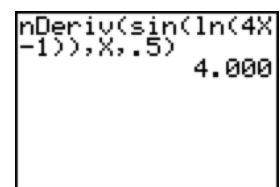
$y = \frac{1}{2}$

11.  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, a = 0$



$y = x$

12.  $f(x) = \sin(\ln(4x-1)), a = \frac{1}{2}$

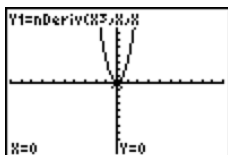


$y = 4\left(x - \frac{1}{2}\right) \Rightarrow y = 4x - 2$

Show the following using your calculator.

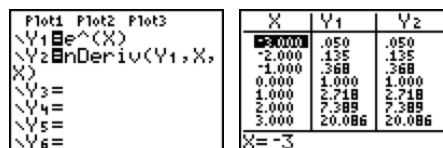
13. the derivative of  $y = x^3$

graphs a parabola.



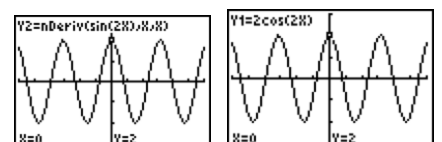
14. the derivative of  $y = e^x$

is also  $y = e^x$ .



15. the derivative of  $y = \sin 2x$

is  $y = 2 \cos 2x$ .



## 8. Techniques of Differentiation - Classwork

You might wonder why this concept of a derivative, a slope of a tangent line to a curve, is important enough to have an entire math course based on it. When you examine applications of the derivative later on in the course, you will understand its significance. But for now, trust that derivatives are important and we need to learn how to take them rather than go through the cumbersome limit process. We will spend some time learning how to take derivatives of all sorts of functions in order that when we learn applications, we will be able to use real-life situations. For the different rules below, for each problem, take the derivative using correct notation.

1) **The constant rule:** The derivative of a constant is 0. That is, if  $c$  is a real number, then  $\frac{d}{dx}[c] = 0$ . From a geometric point of view, the graph of  $y = c$  is a horizontal line and at any point along the line, its slope is 0.

a)  $y = 9$

$$y' = 0$$

b)  $f(x) = 0$

$$f'(x) = 0$$

c)  $s(t) = -8$

$$s'(t) = 0$$

d)  $y = \frac{1}{\pi^3}$

$$y' = 0$$

2) **The single variable rule:** The derivative of  $x = 1$ :  $\frac{d}{dx}[x] = 1$ . This is consistent with the fact that the slope of the line  $y = x$  is 1 at any point on the line.

a)  $y = x$

$$y' = 1$$

b)  $f(x) = x$

$$f'(x) = 1$$

c)  $s(t) = t$

$$s'(t) = 1$$

d)  $g(T) = T$

$$g'(T) = 1$$

Functions for which the derivative exists are called **differentiable**. A function may be differentiable at some  $x$ -values and not differentiable at others. If the derivative doesn't exist at a value  $c$ , it is non-differentiable at  $c$ .

3) **The Power Rule:** If  $n$  is a real number, then the function  $f(x) = x^n$  is differentiable and  $\frac{d}{dx}[x^n] = nx^{n-1}$ .

Usually, some work will need to be done to get the function in the proper form so the power rule can be used.

Determine the derivatives and determine any values of  $x$  for which the function is differentiable.

a)  $y = x^2$

$$y' = 2x$$

Diff. everywhere

b)  $f(x) = x^6$

$$f'(x) = 6x^5$$

Diff. everywhere

c)  $s(t) = t^{30}$

$$s'(t) = 30t^{29}$$

Diff. everywhere

d)  $y = \sqrt{x} = x^{1/2}$

$$y = x^{1/2}$$

$$y' = \frac{1}{2\sqrt{x}} \quad \text{Diff: } x > 0$$

In general, we prefer to leave answers with positive exponents.

e)  $y = \frac{1}{x}$

$$y = x^{-1}$$

$$y' = -1x^{-2} = \frac{-1}{x^2}$$

Not diff. at  $x = 0$

f)  $f(x) = \frac{1}{x^3}$

$$f(x) = x^{-3}$$

$$f'(x) = -3x^{-4} = \frac{-3}{x^4}$$

Not diff. at  $x = 0$

g)  $s(t) = \frac{1}{\sqrt[3]{t}}$

$$s(t) = t^{-1/3}$$

$$s'(t) = -\frac{1}{3}t^{-4/3} = \frac{-1}{3t^{4/3}}$$

Not diff. at  $t = 0$

h)  $y = \frac{1}{x^{3/4}}$

$$y = x^{-3/4}$$

$$y' = -\frac{3}{4}x^{-7/4} = \frac{-3}{4x^{7/4}}$$

Diff at  $x > 0$

4) **The constant multiple rule:** If  $f$  is a differentiable function and  $c$  is a real number,  $\frac{d}{dx}[c \cdot f(x)] = cf'(x)$ .

Take the derivative of each function. Use proper notation.

a)  $y = \frac{2}{x^2}$

$$y = 2x^{-2}$$

$$y' = -4x^{-3} = \frac{-4}{x^3}$$

b)  $f(x) = \frac{4x^3}{3}$

$$f'(x) = 4x^2$$

c)  $s(t) = -t^5$

$$s'(t) = -5t^4$$

d)  $y = 4\sqrt{x}$

$$y = 4x^{1/2}$$

$$y' = 2x^{-1/2} = \frac{2}{\sqrt{x}}$$

e)  $y = \frac{5}{3x^3}$

$$y = \frac{5}{3}x^{-3}$$

$$y' = -5x^{-4} = \frac{-5}{x^4}$$

f)  $f(x) = \frac{-5}{(3x)^3}$

$$f(x) = \frac{-5}{27}x^{-3}$$

$$f'(x) = \frac{15}{27}x^{-4} = \frac{5}{9x^4}$$

g)  $s(t) = \frac{40}{\sqrt{t}}$

$$s(t) = 40t^{-1/2}$$

$$s'(t) = -20t^{-3/2} = \frac{-20}{t^{3/2}}$$

h)  $y = \frac{12}{\sqrt[3]{x^5}}$

$$y = 12x^{-5/3}$$

$$y' = -20x^{-8/3} = \frac{-20}{x^{8/3}}$$

5) **The sum or difference rule.** The derivative of a sum or difference is the sum or difference of the derivatives.

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Take the derivative of each function. Use proper notation.<sup>8a</sup>

a)  $y = x^2 + 5x - 3$

$$y' = 2x + 5$$

b)  $f(x) = x^4 - \frac{3}{2}x^3 + 2x^2 + x - 6$

$$f'(x) = 4x^3 - \frac{9}{2}x^2 + 4x + 1$$

c)  $y = (2x - 3)^2$

$$y = 4x^2 - 12x + 9$$

$$y' = 8x - 12$$

d)<sup>8b</sup>  $y = \frac{4}{x} - \frac{4}{x^2} + \frac{4}{x^3}$

$$y = 4x^{-1} - 4x^{-2} + 4x^{-3}$$

$$y' = -4x^{-2} + 8x^{-3} - 12x^{-4}$$

$$y' = \frac{-4}{x^2} + \frac{8}{x^3} - \frac{12}{x^4}$$

e)  $f(x) = 6\sqrt{x}(2\sqrt{x} - 3)$

$$f(x) = 12x - 18x^{1/2}$$

$$f'(x) = 12 - 9x^{-1/2}$$

$$f'(x) = 12 - \frac{9}{\sqrt{x}}$$

f)  $y = \frac{(x^2 - x + 1)^2}{2}$

$$y = \frac{1}{2}(x^4 - 2x^3 + 3x^2 - 2x + 1)$$

$$y' = \frac{1}{2}(4x^3 - 6x^2 + 6x - 2)$$

$$y' = 2x^3 - 3x^2 + 3x - 1$$

g)  $y = \frac{8}{\sqrt{x}} - \frac{6}{\sqrt[3]{x}}$

$$y = 8x^{-1/2} - 6x^{-1/3}$$

$$y' = -4x^{-3/2} + 2x^{-4/3}$$

$$y' = \frac{-4}{x^{3/2}} + \frac{2}{x^{4/3}}$$

h)  $y = \frac{9x - 3\sqrt{x}}{x}$

$$f(x) = 9 - \frac{3}{\sqrt{x}} = 9 - 3x^{-1/2}$$

$$f'(x) = \frac{3}{2}x^{-3/2} = \frac{3}{2x^{3/2}}$$

i)  $y = \frac{x^2 - 6x - 16}{2x + 4}$

$$y = \frac{(x+2)(x-8)}{2(x+2)}$$

$$y = \frac{x-8}{2} = \frac{x}{2} - 4 \quad y' = \frac{1}{2}$$

6) **The Product Rule:** The derivative of the product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.<sup>8c</sup>

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

a) Find  $y'$  if  $y = (2x^2 - 4x)(3x - 5)$

b) Find  $f'(x)$  if  $f(x) = (x^2 - x - 1)(x^2 + 2)$

without product rule	with product rule
$y = 6x^3 - 22x^2 + 20x$	$y' = (2x^2 - 4x)3 + (3x - 5)(4x - 4)$
$y' = 18x^2 - 44x + 20$	$y' = 6x^2 - 12x + 12x^2 - 32x + 20$
Seems easier	$y' = 18x^2 - 44x + 20$

$f'(x) = (x^2 - x - 1)(2x) + (x^2 + 2)(2x - 1)$
$f'(x) = 2x^3 - 2x^2 - 2x + 2x^3 - x^2 + 4x - 2$
$f'(x) = 4x^3 - 3x^2 + 2x - 2$

c) Find  $f'(x)$  if  $f(x) = (3x^2 - 2x + 5)(-5x^4 + 2x^3 - 7x^2 + x + 2)$

$$f'(x) = (3x^2 - 2x + 5)(-20x^3 + 6x^2 - 14x + 1) + (-5x^4 + 2x^3 - 7x^2 + x + 2)(6x - 2)$$

Expressions this complicated are not expanded so the product rule is far superior than expanding  $f(x)$ .

7) **The Quotient Rule:** The derivative of the quotient of two functions is found using the following formula:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

a) Find  $y'$  if  $y = \frac{5}{3x - 2}$

$y' = \frac{0 - 5(3)}{(3x - 2)^2}$
$y' = \frac{-15}{(3x - 2)^2}$

b) Find  $f'(x)$  if  $f(x) = \frac{4x - 3}{2x + 1}$

$f'(x) = \frac{(2x + 1)4 - (4x - 3)2}{(2x + 1)^2}$
$f'(x) = \frac{8x + 4 - 8x + 6}{(2x + 1)^2} = \frac{10}{(2x + 1)^2}$

c) Find  $\frac{dy}{dx}$  if  $y = \frac{-x}{x^2 + 4x - 2}$

$\frac{dy}{dx} = \frac{-(x^2 + 4x - 2) + x(2x + 4)}{(x^2 + 4x - 2)^2}$
$\frac{dy}{dx} = \frac{x^2 + 2}{(x^2 + 4x - 2)^2}$

The quotient rule scares student for some reason, and many will try and avoid it. There are times you can avoid it and experience in taking derivatives will tell you when it is easier to attack a problem using another technique or biting the bullet and using the quotient rule.

d) Find  $y'$  if  $y = \frac{1}{x^2}$

e) Find  $y'$  if  $y = \frac{x^2 - 3x + 1}{x^2}$

power rule	quotient rule
$y = x^{-2}$	$y' = \frac{-2x}{x^4} = \frac{-2}{x^3}$
$y' = -2x^{-3} = \frac{-2}{x^3}$	A bit easier

quotient rule	separating
$y' = \frac{x^2(2x - 3) - (x^2 - 3x + 1)2x}{x^4}$	$y = 1 - \frac{3}{x} + \frac{1}{x^2}$
$y' = \frac{2x^3 - 3x^2 - 2x^3 + 6x^2 - 2x}{x^4}$	$y' = \frac{3}{x^2} - \frac{2}{x^3}$
$y' = \frac{x(3x - 2)}{x^4} = \frac{3x - 2}{x^3}$	Answers are equivalent. Separating is easier.

8) Find an equation of the tangent line to the graph of  $f$  at the point – use your calculator to confirm the result.

a)  $f(x) = 3x^3 - 2x^2 - x + 2$  at  $(1, 2)$

$$f'(x) = 9x^2 - 4x - 1 \quad f'(1) = 4$$

$$\text{Tangent line: } y - 2 = 4(x - 1)$$

$$y = 4x - 2$$

b)  $f(x) = (x^2 - 4x + 2)(4x - 1)$  at  $(1, -3)$

$$f'(x) = 4(x^2 - 4x + 2) + (4x - 1)(2x - 4)$$

$$f'(1) = 4(-1) + 3(-2) = -10$$

$$\text{Tangent line: } y + 3 = -10(x - 1)$$

$$y = -10x + 7$$

c)  $f(x) = \frac{8}{x} - \frac{8}{x^2}$  at  $(-2, -6)$

$$f'(x) = \frac{-8}{x^2} + \frac{16}{x^3}$$

$$f'(-2) = -2 - 2 = -4$$

$$\text{Tangent line: } y + 6 = -4(x + 2)$$

$$y = -4x - 14$$

d)  $f(x) = \frac{x-4}{x^2+4}$  at  $\left(2, \frac{-1}{4}\right)$

$$f'(x) = \frac{x^2 + 4 - (x-4)(2x)}{(x^2+4)^2} = \frac{4+8x-x^2}{(x^2+4)^2}$$

$$f'(2) = \frac{16}{64} = \frac{1}{4}$$

$$\text{Tangent line: } y + \frac{1}{4} = \frac{1}{4}(x - 2)$$

9) Find an equation of the line normal to the graph of  $f$  at the point – use your calculator to confirm the result.

a)  $f(x) = x^3 - 4x^2$  at  $(3, -9)$

$$f'(x) = 3x^2 - 8x$$

$$f'(3) = 3 \Rightarrow m_{\perp} = \frac{-1}{3}$$

$$\text{Normal line: } y + 9 = \frac{-1}{3}(x - 3) \Rightarrow y = \frac{-1}{3}x - 8$$

b)  $f(x) = \frac{-6}{\sqrt{x+1}}$  at  $(4, -2)$

$$f'(x) = \frac{6\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x+1})^2} \Rightarrow f'(4) = \frac{6}{9} = \frac{1}{6} \Rightarrow m_{\perp} = -6$$

$$\text{Normal line: } y + 2 = -6(x - 4) \Rightarrow y = -6x + 22$$

10) Determine the points at which the graphs of the following functions have horizontal tangents.

a)  $f(x) = x^2 + 2x - 24$

$$f'(x) = 2x + 2 = 0 \Rightarrow x = -1$$

$$\text{Point: } (-1, -25)$$

b)  $f(x) = x^4 - 4x^2$

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0$$

$$4x = 0 \Rightarrow x = 0 \quad x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$

$$\text{Points: } (0, 0), (\sqrt{2}, -4), (-\sqrt{2}, -4)$$

c)  $f(x) = \frac{x^2 - 3}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)2x - (x^2 - 3)2x}{(x^2 + 1)^2} = \frac{8x}{(x^2 + 1)^2}$$

$$f'(x) = 0 \Rightarrow x = 0 \quad \text{Point: } (0, -3)$$

d)  $f(x) = \frac{x-1}{x^2+3}$

$$f'(x) = \frac{(x^2+3) - (x-1)2x}{(x^2+3)^2} = \frac{-x^2+2x+3}{(x^2+3)^2}$$

$$-(x+1)(x-3) = 0 \Rightarrow x = -1, 3 \quad \text{Points: } \left(-1, -\frac{1}{2}\right), \left(3, \frac{1}{6}\right)$$

11) Use the chart to find  $f'(3)$ .

$g(3)$	$g'(3)$	$h(3)$	$h'(3)$
4	-2	3	$\pi$

a)  $f(x) = 4g(x) - \frac{1}{2}h(x) + 1$

$$\begin{aligned} f'(x) &= 4g'(x) - \frac{1}{2}h'(x) \\ f'(3) &= 4g'(3) - \frac{1}{2}h'(3) \\ f'(3) &= 4(-2) - \frac{1}{2}\pi = -8 - \frac{\pi}{2} \end{aligned}$$

b)  $f(x) = g(x)h(x) + x$

$$\begin{aligned} f'(x) &= g(x)h'(x) + h(x)g'(x) + 1 \\ f'(3) &= g(3)h'(3) + h(3)g'(3) + 1 \\ f'(3) &= 4\pi + 3(-2) + 1 \\ f'(3) &= 4\pi - 5 \end{aligned}$$

c)  $f(x) = \frac{g(x)}{2h(x)}$

$$\begin{aligned} f'(x) &= \frac{1}{2} \left[ \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \right] \\ f'(3) &= \frac{1}{2} \left[ \frac{h(3)g'(3) - g(3)h'(3)}{[h(3)]^2} \right] \\ f'(3) &= \frac{1}{2} \left[ \frac{3(-2) - 4\pi}{3^2} \right] \\ f'(3) &= \frac{1}{2} \left[ \frac{-6 - 4\pi}{9} \right] = \frac{-3 - 2\pi}{9} \end{aligned}$$

d)  $f(x) = \frac{g(x) - h(x)}{g(x)}$

$$\begin{aligned} f(x) &= 1 - \frac{h(x)}{g(x)} \\ f'(x) &= - \left[ \frac{g(x)h'(x) - h(x)g'(x)}{[g(x)]^2} \right] \\ f'(3) &= - \left[ \frac{4\pi - 3(-2)}{4^2} \right] \\ f'(3) &= - \left[ \frac{4\pi + 6}{16} \right] = \frac{-2\pi - 3}{8} \end{aligned}$$

The 2<sup>nd</sup> derivative of a function is the derivative of the derivative of the function. We will learn about the geometric meaning of the 2<sup>nd</sup> derivative later in the course.

Notation for the 2<sup>nd</sup> derivative and subsequent derivatives are shown in the chart.<sup>8d</sup>

2nd deriv.	$f''(x)$	$y''$	$\frac{d^2y}{dx^2}$
3rd deriv	$f'''(x)$	$y'''$	$\frac{d^3y}{dx^3}$
$n$ th deriv	$f^{(n)}$	$y^{(n)}$	$\frac{d^ny}{dx^n}$

12) For each of the following functions, find their second derivatives.

a)  $f(x) = \frac{8}{3}x^3 - 5x^2 - 7x - 1$

$$f'(x) = 8x^2 - 10x - 7 \quad f''(x) = 16x - 10$$

b)  $y = \frac{x^2 + 4x - 2}{x}$

$$y = x + 4 - \frac{2}{x} \Rightarrow y' = 1 + \frac{2}{x^2} \quad y'' = \frac{-4}{x^3}$$

c)  $y = 4\sqrt{x} - \frac{2}{\sqrt{x}}$

$$\begin{aligned} \frac{dy}{dx} &= 2x^{-1/2} + x^{-3/2} \\ \frac{d^2y}{dx^2} &= -x^{-3/2} - \frac{3}{2}x^{-5/2} = \frac{-1}{x^{3/2}} - \frac{3}{2x^{5/2}} \end{aligned}$$

d)  $f(x) = \frac{x}{x-1}$

$$\begin{aligned} f'(x) &= \frac{1}{(x-1)^2} = \frac{-1}{x^2 - 2x + 1} \\ f''(x) &= \frac{2x - 2}{(x^2 - 2x + 1)^2} = \frac{2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3} \end{aligned}$$

## 8. Techniques of Differentiation - Homework

For the following functions, find  $f'(x)$  and  $f'(c)$  at the indicated value of  $c$ . Use proper notation.

1.  $f(x) = x^2 - 6x + 1, c = 0$

$$\boxed{\begin{array}{l} f'(x) = 2x - 6 \\ f'(0) = -6 \end{array}}$$

2.  $f(x) = \frac{1}{x} - \frac{3}{x^2} + \frac{4}{x^3}, c = -1$

$$\boxed{\begin{array}{l} f'(x) = \frac{-1}{x^2} + \frac{6}{x^3} - \frac{12}{x^4} \\ f'(-1) = -1 - 6 - 12 = -19 \end{array}}$$

3.  $f(x) = 3\sqrt{x} - \frac{1}{\sqrt[3]{x}}, c = 1$

$$\boxed{\begin{array}{l} f'(x) = \frac{3}{2x^{1/2}} + \frac{1}{3x^{4/3}} \\ f'(1) = \frac{3}{2} + \frac{1}{3} = \frac{11}{6} \end{array}}$$

For the following functions, find the derivative using the power rule. Use proper notation.

4.  $y = \pi^5$

$$\boxed{y' = 0}$$

5.  $y = \frac{x^2 + 2}{6}$

$$\boxed{y' = \frac{x}{3}}$$

6.  $y = \frac{1}{a} \left( x^2 - \frac{1}{b^2} x + c \right), a, b, c$  constants

$$\boxed{y' = \frac{1}{a} \left( 2x - \frac{1}{b^2} \right)}$$

7.  $y = \frac{8}{3x^2}$

$$\boxed{\begin{array}{l} y = \frac{8}{3} x^{-2} \\ y' = \frac{-16}{3} x^{-3} = \frac{-16}{3x^3} \end{array}}$$

8.  $y = \frac{-9}{(3x^2)^3}$

$$\boxed{\begin{array}{l} y = \frac{-9}{27x^6} = \frac{-1}{3} x^{-6} \\ y' = 2x^{-7} = \frac{2}{x^7} \end{array}}$$

9.  $y = \frac{6x^{3/2}}{x}$

$$\boxed{\begin{array}{l} y = 6x^{1/2} \\ y' = 3x^{-1/2} = \frac{3}{x^{1/2}} \end{array}}$$

10.  $y = \frac{4x^2 - 5x + 6}{3}$

$$\boxed{y' = \frac{1}{3}(8x - 5)}$$

11.  $y = \frac{x^2 - 6x + 2}{2x}$

$$\boxed{\begin{array}{l} y = \frac{x}{2} - 3 + \frac{1}{x} \\ y' = \frac{1}{2} - \frac{1}{x^2} \end{array}}$$

12.  $y = \frac{x^3 + 8}{x + 2}$

$$\boxed{\begin{array}{l} y = \frac{(x+2)(x^2 - 2x + 4)}{x+2} \\ y' = 2x - 2 \end{array}}$$

13.  $y = x^4 - \frac{3}{2}x^3 + 5x^2 - 6x - 2$

$$\boxed{y' = 4x^3 - \frac{9}{2}x^2 + 10x - 6}$$

14.  $y = \frac{x^3 - 3x^2 + 10x - 5}{x^2}$

$$\boxed{\begin{array}{l} y = x - 3 + \frac{10}{x} - \frac{5}{x^2} \\ y' = 1 - \frac{10}{x^2} + \frac{10}{x^3} \end{array}}$$

15.  $y = (x^2 + 4x)(2x - 1)$

$$\boxed{\begin{array}{l} y = 2x^3 + 7x^2 - 4x \\ y' = 6x^2 + 14x - 4 \end{array}}$$



16.  $y = (x-2)^3$

$$y = x^3 - 6x^2 + 12x - 8$$

$$y' = 3x^2 - 12x + 12$$

17.  $y = \sqrt[3]{x} - \sqrt[3]{x^2}$

$$y = x^{1/3} - x^{2/3}$$

$$y' = \frac{1}{3x^{2/3}} - \frac{2}{3x^{1/3}}$$

18.  $y = \frac{x^4 - 2x^3 + 5x^2 - 4x + 4}{x}$

$$y = x^3 - 2x^2 + 5x - 4 + \frac{4}{x}$$

$$y' = 3x^2 - 4x + 5 - \frac{4}{x^2}$$

For the following functions, find the derivatives. Simplify all but # 19 fully. Use proper notation.

19.  $y = (x^2 - 4x - 6)(x^3 - 5x^2 - 3x)$

$$\frac{dy}{dx} = (x^2 - 4x - 6)(3x^2 - 10x - 3) + (x^3 - 5x^2 - 3x)(2x - 4)$$

20.  $y = \frac{3x-2}{2x+3}$

$$\frac{dy}{dx} = \frac{3(2x+3) - 2(3x-2)}{(2x+3)^2}$$

$$\frac{dy}{dx} = \frac{13}{(2x+3)^2}$$

21.  $y = \frac{x^2 - 4x - 2}{x^2 - 1}$

$$\frac{dy}{dx} = \frac{(x^2 - 1)(2x - 4) - (x^2 - 4x - 2)(2x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2x^3 - 4x^2 - 2x + 4 - 2x^3 + 8x^2 + 4x}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{4x^2 + 2x + 4}{(x^2 - 1)^2}$$

22.  $y = \frac{x-1}{\sqrt{x}}$

$$y = x^{1/2} - x^{-1/2}$$

$$\frac{dy}{dx} = \frac{1}{2x^{1/2}} + \frac{1}{2x^{3/2}}$$

23.  $y = \frac{x^2 - x + 1}{\sqrt[3]{x}}$

$$y = x^{5/3} - x^{2/3} + x^{-1/3}$$

$$\frac{dy}{dx} = \frac{5}{3}x^{2/3} - \frac{2}{3x^{1/3}} - \frac{1}{3x^{4/3}}$$

24.  $y = \left(\frac{x-3}{x-4}\right)(3x+2)$

$$y = \frac{3x^2 - 7x - 6}{x - 4}$$

$$\frac{dy}{dx} = \frac{(x-4)(6x-7) - (3x^2 - 7x - 6)}{(x-4)^2}$$

$$\frac{dy}{dx} = \frac{3x^2 - 24x + 34}{(x-4)^2}$$

25.  $y = \frac{1}{(x^2 - 5)^2}$

$$y' = \frac{-(4x^3 - 20x)}{(x^4 - 10x^2 + 25)^2}$$

$$y' = \frac{-4x(x^2 - 5)}{(x^2 - 5)^4} = \frac{-4x}{(x^2 - 5)^3}$$

26.  $y = \frac{x+k}{x-k}$ ,  $k$  is a constant

$$\frac{dy}{dx} = \frac{(x-k) - (x+k)}{(x-k)^2}$$

$$\frac{dy}{dx} = \frac{-2k}{(x-k)^2}$$

27.  $y = \frac{x^2 + k^2}{x^2 - k^2}$ ,  $k$  is a constant

$$\frac{dy}{dx} = \frac{(x^2 - k^2)2x - (x^2 + k^2)2x}{(x^2 - k^2)^2}$$

$$\frac{dy}{dx} = \frac{-4k^2x}{(x^2 - k^2)^2}$$

Find the equation of the tangent line to the graph of  $f$  at the indicated point and then use your calculator to confirm the results.

28.  $f(x) = \frac{x^2}{x-1}$  at  $(2, 4)$

29.  $f(x) = (x-2)(x^2 - 3x - 1)$  at  $(-1, -9)$

$$f'(x) = \frac{(x-1)2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$$f'(2) = 0 \quad \text{Tan line: } y - 4 = 0(x - 2) \Rightarrow y = 4$$

$$f'(x) = (x-2)(2x-3) + x^2 - 3x - 1$$

$$f'(-1) = -3(-5) + 1 + 3 - 1 = 18$$

$$\text{Tan line: } y + 9 = 18(x + 1) \Rightarrow y = 18x + 9$$

30.  $f(x) = \frac{x^2 - 4x + 2}{2x - 1}$  at  $(2, \frac{-2}{3})$

31.  $f(x) = \frac{-1}{(2x-3)^2}$  at  $(0, -\frac{1}{9})$

$$f'(x) = \frac{(2x-1)(2x-4) - 2(x^2 - 4x + 2)}{(2x-1)^2}$$

$$f'(2) = \frac{3(0) - 2(-2)}{9} = \frac{4}{9}$$

$$\text{Tan line: } y + \frac{2}{3} = \frac{4}{9}(x - 2) \Rightarrow y = \frac{4}{9}x - \frac{14}{9}$$

$$f'(x) = \frac{8x - 12}{(4x^2 - 12x + 9)^2}$$

$$f'(0) = \frac{-12}{81} = \frac{-4}{27}$$

$$\text{Tan line: } y + \frac{1}{9} = \frac{-4}{27}x \Rightarrow y = \frac{-4x - 3}{27}$$

Find the equation of the normal line to the graph of  $f$  at the indicated point and then use your calculator to confirm the results.

32.  $f(x) = 4x^2 - 8x - 3$  at  $(\frac{1}{2}, -6)$

33.  $f(x) = \frac{\sqrt{x} + 4}{\sqrt{x}}$  at  $(4, 3)$

$$f'(x) = 8x - 8$$

$$f'(\frac{1}{2}) = 4 - 8 = -4 \quad m_{\perp} = \frac{1}{4}$$

$$\text{Normal line: } y + 6 = \frac{1}{4}(x - \frac{1}{2})$$

$$f(x) = 1 + 4x^{-1/2} \Rightarrow f'(x) = -2x^{-3/2}$$

$$f'(4) = \frac{-2}{8} = \frac{-1}{4} \quad m_{\perp} = 4$$

$$\text{Normal line: } y - 3 = 4(x - 4) \Rightarrow y = 4x - 13$$

Determine the point(s) at which the graphs of the following functions have a horizontal tangent.

34.  $y = \frac{x^2}{x^2 - 4}$

35.  $f(x) = \frac{4x}{x^2 + 4}$

36.  $f(x) = x^3 - 9x$

$$f'(x) = \frac{(x^2 - 4)2x - x^2(2x)}{(x^2 - 4)^2}$$

$$\frac{-8x}{(x^2 - 4)^2} = 0$$

$$x = 0 \quad \text{Pt: } (0, 0)$$

$$f'(x) = \frac{(x^2 + 4)4 - 4x(2x)}{(x^2 + 4)^2}$$

$$\frac{-4(x^2 - 4)}{(x^2 + 4)^2} = \frac{-4(x+2)(x-2)}{(x^2 + 4)^2} = 0$$

$$x = \pm 2 \quad \text{Pts: } (2, 1), (-2, -1)$$

$$f'(x) = 3x^2 - 9 = 0$$

$$3(x^2 - 3) = 0 \Rightarrow x = \pm\sqrt{3}$$

$$\text{Pts: } (\sqrt{3}, -6\sqrt{3}), (-\sqrt{3}, 6\sqrt{3})$$

Use the chart to find  $h'(4)$ .

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
-8	3	$3\pi$	4

37.  $h(x) = 5f(x) - \frac{2}{3}g(x)$

$$h'(x) = 5f'(x) - \frac{2}{3}g'(x)$$

$$h'(4) = 5f'(4) - \frac{2}{3}g'(4)$$

$$h'(4) = 5(3) - \frac{2}{3}(4) = \frac{37}{3}$$

38.  $h(x) = x^2f(x)$

$$h'(x) = x^2f'(x) + 2x \cdot f(x)$$

$$h'(4) = 16f'(4) + 8f(4)$$

$$h'(4) = 16(3) + 8(-8) = -16$$

39.  $h(x) = f(x)g(x)$

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

$$h'(4) = f(4)g'(4) + g(4)f'(4)$$

$$h'(4) = -8(4) + 3\pi(3) = 9\pi - 32$$

40.  $h(x) = \frac{g(x)}{f(x)}$

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

$$h'(4) = \frac{f(4)g'(4) - g(4)f'(4)}{[f(4)]^2}$$

$$h'(4) = \frac{-8(4) - 3\pi(3)}{(-8)^2} = \frac{-9\pi - 32}{64}$$

41.  $h(x) = \frac{f(x) + g(x)}{x}$

$$h'(x) = \frac{x[f'(x) + g'(x)] - [f(x) + g(x)]}{x^2}$$

$$h'(4) = \frac{4[f'(4) + g'(4)] - [f(4) + g(4)]}{4^2}$$

$$h'(4) = \frac{4(3+4) - (-8+3\pi)}{16} = \frac{36-3\pi}{16}$$

For each of the following, find  $f''(x)$ .

42.  $f(x) = \frac{x^3 - 3x^2 - 4x - 1}{2x}$

$$f(x) = \frac{1}{2} \left( x^2 - 3x - 4 - \frac{1}{x} \right)$$

$$f'(x) = \frac{1}{2} \left( 2x - 3 + \frac{1}{x^2} \right)$$

$$f''(x) = \frac{1}{2} \left( 2 - \frac{2}{x^3} \right) = 1 - \frac{1}{x^3}$$

43.  $f(x) = \frac{x}{x-4}$

$$f'(x) = \frac{x-4-x}{(x-4)^2} = \frac{-4}{x^2-8x+16}$$

$$f''(x) = \frac{4(2x-8)}{(x^2-8x+16)^2}$$

$$f''(x) = \frac{8(x-4)}{(x-4)^4} = \frac{8}{(x-4)^3}$$

44.  $f(x) = \sqrt{x} - 4\sqrt[3]{x} + \frac{6}{\sqrt[4]{x}}$

$$f(x) = x^{1/2} - 4x^{1/3} + 6x^{-1/4}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-2/3} - \frac{3}{2}x^{-5/4}$$

$$f''(x) = \frac{-1}{4}x^{-3/2} + \frac{8}{9}x^{-5/3} + \frac{15}{8}x^{-9/4}$$

$$f''(x) = \frac{-1}{4x^{3/2}} + \frac{8}{9x^{5/3}} + \frac{15}{8x^{9/4}}$$

45. Find an equation of the line tangent to  $f(x) = x^2 - 6x + 7$  and

a. parallel to the line  $y = 2x + 4$

parallel means same slope so slope of  $f(x)$  = slope of line

$$f'(x) = 2x - 6 = 2 \Rightarrow 2x = 8 \Rightarrow x = 4$$

the point of tangency is  $(4, f(4)) = (4, -1)$

Parallel line:  $y + 1 = 2(x - 4) \Rightarrow y = 2x - 9$

b. perpendicular to the line  $y = 2x + 4$

perpendicular means slope of  $f(x) = \frac{-1}{\text{slope of line}}$

$$f'(x) = 2x - 6 = \frac{-1}{2} \Rightarrow 2x = \frac{11}{2} \Rightarrow x = \frac{11}{4}$$

point of tangency is  $\left( \frac{11}{4}, f\left(\frac{11}{4}\right) \right) = \left( \frac{11}{4}, \frac{-31}{16} \right)$

Perpendicular line:  $y + \frac{31}{16} = \frac{-1}{2} \left( x - \frac{11}{4} \right)$



$$5) f(x) = \frac{1}{4x-3}$$

a) quotient rule

$$f'(x) = \frac{(4x-3)(0) - 1(4)}{(4x-3)^2}$$

$$f'(x) = \frac{-4}{(4x-3)^2}$$

b) power rule

$$f(x) = (4x-3)^{-1}$$

$$f'(x) = -(4x-3)^2(4)$$

$$f'(x) = \frac{-4}{(4x-3)^2}$$

$$6) f(x) = \frac{3}{(3x^2 - 2x + 1)^3}$$

$$f(x) = 3(3x^2 - 2x + 1)^{-3}$$

$$f'(x) = -9(3x^2 - 2x + 1)^{-4}(6x - 2)$$

$$f'(x) = \frac{-18(3x - 1)}{(3x^2 - 2x + 1)^4}$$

$$7) f(x) = (7 - 4x^2)^{2/3}$$

$$f'(x) = \frac{2}{3}(7 - 4x^2)^{-1/3}(-8x)$$

$$f'(x) = \frac{-16x}{3(7 - 4x^2)^{1/3}}$$

$$8) f(x) = -5\sqrt{x^2 - 4x + 1}$$

$$f'(x) = \frac{-5}{2}(x^2 - 4x + 1)^{-1/2}(2x - 4)$$

$$f'(x) = \frac{-5(x - 2)}{(x^2 - 4x + 1)^{1/2}}$$

$$9) f(x) = \frac{-2}{\sqrt[4]{6x-1}}$$

$$f(x) = -2(6x-1)^{-1/4}$$

$$f'(x) = \frac{1}{2}(6x-1)^{-5/4}(6)$$

$$f'(x) = \frac{3}{(6x-1)^{5/4}}$$

We have 3 basic forms for differentiation: Other than constants, problems are written in these forms:

i) power rule: (expression)<sup>power</sup>    ii) product rule: (expression 1)(expression 2)    iii) quotient rule:  $\frac{\text{expression 1}}{\text{expression 2}}$

Note that the chain rule is not a basic form. The chain rule is always in effect. For instance, when you take the derivative of  $7x^2$ , you get  $14x$  times the derivative of  $x$  (which is 1). So the answer is  $14x$ .

Once you apply your derivative rules, you have to decide whether the expression can be simplified. In general, we prefer derivatives in factored form, if possible. This adds a degree of algebraic skill to the mix and unless your algebra skills are good, you can get problems incorrect, not because of the calculus, but for the algebra.<sup>9a</sup>

For the following, decide the form of  $f(x)$  and then find  $f'(x)$ . (\* adequate answer \*\* best answer)

$$10) f(x) = x^2(2x-3)^4 \quad \text{Product rule}$$

$$f'(x) = x^2(4)(2x-3)^3(2) + (2x-3)^4(2x)$$

$$* f'(x) = 8x^2(2x-3)^3 + 2x(2x-3)^4$$

$$f'(x) = 2x(2x-3)^3(4x+2x-3)$$

$$f'(x) = 2x(2x-3)^3(3)(2x-1)$$

$$** f'(x) = 6x(2x-3)^3(2x-1)$$

$$11) f(x) = x\sqrt{4-x^2} \quad \text{Product rule}$$

$$f'(x) = x\left(\frac{1}{2}\right)(4-x^2)^{-1/2}(-2x) + (4-x^2)^{1/2}$$

$$* f'(x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$$

$$f'(x) = \sqrt{4-x^2} \left( \frac{\sqrt{4-x^2}}{\sqrt{4-x^2}} \right) - \frac{x^2}{\sqrt{4-x^2}}$$

$$** f'(x) = \frac{4-x^2-x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

$$12) f(x) = \left(\frac{2x-1}{2x+1}\right)^5 \quad \text{Power rule}$$

$$f'(x) = 5 \left(\frac{2x-1}{2x+1}\right)^4 \left[ \frac{(2x+1)2 - (2x-1)2}{(2x+1)^2} \right]$$

$$* f'(x) = 5 \left(\frac{2x-1}{2x+1}\right)^4 \left[ \frac{4}{(2x+1)^2} \right]$$

$$** f'(x) = \frac{20(2x-1)^4}{(2x+1)^6}$$

$$13) f(x) = \frac{x}{\sqrt{x^2-1}} \quad \text{Quotient rule}$$

$$f'(x) = \frac{(x^2-1)^{1/2} - x \left[ \frac{2x}{2(x^2-1)^{1/2}} \right]}{x^2-1} \left[ \frac{(x^2-1)^{1/2}}{(x^2-1)^{1/2}} \right]$$

$$** f'(x) = \frac{x^2-1-x^2}{(x^2-1)^{3/2}} = \frac{-1}{(x^2-1)^{3/2}}$$

$$14) f(x) = \sqrt{\frac{x}{4x-1}} \quad \text{Power rule}$$

$$f'(x) = \frac{1}{2} \left(\frac{x}{4x-1}\right)^{-1/2} \left[ \frac{4x-1-4x}{(4x-1)^2} \right]$$

$$* f'(x) = \left[ \frac{-1}{2(4x-1)^2} \right] \left(\frac{x}{4x-1}\right)^{-1/2}$$

$$* f'(x) = \left[ \frac{-1}{2(4x-1)^2} \right] \left(\frac{4x-1}{x}\right)^{1/2}$$

$$** f'(x) = \frac{-1}{2x^{1/2}(4x-1)^{3/2}}$$

$$15) f(x) = (x^2-4)^{-3}(x^2+4)^3 \quad \text{Product rule}$$

$$f(x) = \left(\frac{x^2+4}{x^2-4}\right)^3 \quad \text{Power rule}$$

$$f'(x) = 3 \left(\frac{x^2+4}{x^2-4}\right)^2 \left[ \frac{(x^2-4)2x - (x^2+4)2x}{(x^2-4)^2} \right]$$

$$* f'(x) = 3 \left(\frac{x^2+4}{x^2-4}\right)^2 \left[ \frac{-16x}{(x^2-4)^2} \right]$$

$$** f'(x) = \frac{-48x(x^2+4)^2}{(x^2-4)^4}$$

Given that  $f(2) = -3$ ,  $f'(2) = 6$ ,  $g(2) = 3$ ,  $g'(2) = -2$ ,  $f'(3) = 4$ ,  $g'(-3) = -1$ , find  $h'(2)$ .

$$16) h(x) = [f(x)]^3$$

$$h'(x) = 3[f(x)]^2 f'(x)$$

$$h'(2) = 3[f(2)]^2 f'(2)$$

$$h'(2) = 3(-3)^2 (6) = 162$$

$$17) h(x) = [f(x) - g(x)]^2$$

$$h'(x) = 2[f(x) - g(x)][f'(x) - g'(x)]$$

$$h'(2) = 2[f(2) - g(2)][f'(2) - g'(2)]$$

$$h'(2) = 2(-3 - 3)(6 + 2) = -96$$

$$18) h(x) = \sqrt{4g(x) - x - 1}$$

$$h'(x) = \frac{4g'(x) - 1}{2\sqrt{4g(x) - x - 1}}$$

$$h'(2) = \frac{4g'(2) - 1}{2\sqrt{4g(2) - 2 - 1}}$$

$$h'(2) = \frac{4(-2) - 1}{2\sqrt{4(3) - 3}} = \frac{-9}{6} = \frac{-3}{2}$$

$$19) h(x) = f(g(x))$$

$$h'(x) = f'(g(x))(g'(x))$$

$$h'(2) = f'(g(2))(g'(2))$$

$$h'(2) = f'(3)(-2) = 4(-2) = -8$$

$$20) h(x) = g(f(x))$$

$$h'(x) = g'(f(x))(f'(x))$$

$$h'(2) = g'(f(2))(f'(2))$$

$$h'(2) = g'(-3)(6) = -1(6) = -6$$

$$21) h(x) = f(-f(x))$$

$$h'(x) = f'(-f(x))(-f'(x))$$

$$h'(2) = f'(-f(2))(-f'(2))$$

$$h'(2) = f'(3)(-6) = 4(-6) = -24$$

## 9. The Chain Rule - Homework

Find the derivative of the following functions. Use proper notation.

1.  $f(x) = (3x - 8)^4$

$$\begin{aligned} f'(x) &= 4(3x - 8)^3(3) \\ f'(x) &= 12(3x - 8)^3 \end{aligned}$$

2.  $h(x) = (3x^2 + 2)^5$

$$\begin{aligned} y' &= 5(3x^2 + 2)^4(6x) \\ y' &= 30x(3x^2 + 2)^4 \end{aligned}$$

3.  $y = -4(x^2 + x + 1)^{10}$

$$\begin{aligned} y' &= -4(10)(x^2 + x + 1)^9(2x + 1) \\ y' &= -40(x^2 + x + 1)^9(2x + 1) \end{aligned}$$

4.  $y = -6(4 - 9x)^{3/2}$

$$\begin{aligned} y' &= -6\left(\frac{3}{2}\right)(4 - 9x)^{1/2}(-9) \\ y' &= 81(4 - 9x)^{1/2} \end{aligned}$$

5.  $y = \frac{1}{ax - b}$

$$\begin{aligned} y &= (ax - b)^{-1} \\ y' &= -(ax - b)^{-2}(a) = \frac{-a}{(ax - b)^2} \end{aligned}$$

6.  $g(x) = \frac{-1}{(x^2 - 5x - 6)^2}$

$$\begin{aligned} g'(x) &= 2(x^2 - 5x - 6)^{-3}(2x - 5) \\ g'(x) &= \frac{4x - 10}{(x^2 - 5x - 6)^3} \end{aligned}$$

7.  $y = \left(\frac{2}{2-x}\right)^3$

$$\begin{aligned} y' &= 3\left(\frac{2}{2-x}\right)^2\left[\frac{2}{(2-x)^2}\right] \\ y' &= \frac{24}{(2-x)^4} \end{aligned}$$

8.  $s(t) = \sqrt{1-t}$

$$\begin{aligned} s'(t) &= \frac{1}{2}(1-t)^{-1/2}(-1) \\ s'(t) &= \frac{-1}{2\sqrt{1-t}} \end{aligned}$$

9.  $y = \sqrt[3]{3x^3 - 6x + 2}$

$$\begin{aligned} y' &= \frac{9x^2 - 6}{3(3x^3 - 6x + 2)^{2/3}} \\ y' &= \frac{3x^2 - 2}{(3x^3 - 6x + 2)^{2/3}} \end{aligned}$$

10.  $y = \frac{2}{\sqrt{2x+3}}$

$$\begin{aligned} y' &= 2\left(\frac{-1}{2}\right)(2x+3)^{-3/2}(2) \\ y' &= \frac{-2}{(2x+3)^{3/2}} \end{aligned}$$

11.  $y = \frac{-1}{\sqrt{2x+1}}$

$$y' = \frac{\frac{1}{2}(2x)^{-1/2}(2)}{(\sqrt{2x+1})^2} = \frac{1}{\sqrt{2x}(\sqrt{2x+1})^2}$$

12.  $y = \sqrt{\frac{1}{x^2+x}}$

$$\begin{aligned} y' &= \frac{-1}{2}(x^2+x)^{-3/2}(2x+1) \\ y' &= \frac{-(2x+1)}{2(x^2+x)^{3/2}} \end{aligned}$$

13.  $f(x) = x^3(5x-1)^4$

$$\begin{aligned} f'(x) &= x^3(4)(5x-1)^3(5) + (5x-1)^4(3x^2) \\ *f'(x) &= 20x^3(5x-1)^3 + 3x^2(5x-1)^4 \\ f'(x) &= x^2(5x-1)^3[20x+3(5x-1)] \\ **f'(x) &= x^2(5x-1)^3(35x-3) \end{aligned}$$

14.  $y = \frac{4x}{(2x+1)^2}$

$$\begin{aligned} y' &= \frac{4(2x+1)^2 - 4x(2)(2x+1)(2)}{(2x+1)^4} \\ *y' &= \frac{4(2x+1)^2 - 16x(2x+1)}{(2x+1)^4} \\ y' &= \frac{4(2x+1)(2x+1-4x)}{(2x+1)^4} \Rightarrow **y' = \frac{4(-2x+1)}{(2x+1)^3} \end{aligned}$$



$$15. y = \sqrt{\frac{3x}{2x-3}}$$

$$y' = \frac{1}{2} \left( \frac{3x}{2x-3} \right)^{-1/2} \left[ \frac{(2x-3)3 - 3x(2)}{(2x-3)^2} \right]$$

$$*y' = \frac{1}{2} \left( \frac{2x-3}{3x} \right)^{1/2} \left[ \frac{-9}{(2x-3)^2} \right]$$

$$**y' = \frac{-9}{2(3x)^{1/2} (2x-3)^{3/2}}$$

$$16. y = (x^2 + 1)^3 (x^3 + 1)^2$$

$$y' = (x^2 + 1)^3 (2)(x^3 + 1)(3x^2) + (x^3 + 1)^2 (3)(x^2 + 1)^2 (2x)$$

$$*y' = 6x^2 (x^2 + 1)^3 (x^3 + 1) + 6x (x^3 + 1)^2 (x^2 + 1)^2$$

$$y' = 6x (x^3 + 1)(x^2 + 1)^2 [x(x^2 + 1) + x^3 + 1]$$

$$**y' = 6x (x^3 + 1)(x^2 + 1)^2 (2x^3 + x + 1)$$

For each of the following, find the equation of the tangent line at the indicated point. Confirm by calculator.<sup>9b</sup>

$$17. f(x) = \sqrt{x^2 + 2x + 8} \text{ at } (2, 4)$$

$$18. f(x) = \sqrt[5]{3x^3 + 4x} \text{ at } (2, 2)$$

$$19. f(x) = \sqrt{\frac{3x-1}{2x+1}} \text{ at } (-1, 2)$$

$$f'(x) = \frac{2x+2}{2\sqrt{x^2+2x+8}}$$

$$f'(x) = \frac{x+1}{\sqrt{x^2+2x+8}} \quad f'(2) = \frac{3}{4}$$

$$y-4 = \frac{3}{4}(x-2) \Rightarrow y = \frac{3}{4}x + \frac{5}{2}$$

$$f'(x) = \frac{9x^2+4}{5(3x^3+4x)^{4/5}}$$

$$f'(2) = \frac{40}{5(16)} = \frac{1}{2}$$

$$y-2 = \frac{1}{2}(x-2) \Rightarrow y = \frac{1}{2}x + 1$$

$$f'(x) = \frac{1}{2} \sqrt{\frac{2x+1}{3x-1}} \left[ \frac{-5}{(2x+1)^2} \right]$$

$$f'(-1) = \frac{1}{2} \sqrt{\frac{-1}{-4}} \left[ \frac{-5}{(-1)^2} \right] = \frac{5}{4}$$

$$y-2 = \frac{5}{4}(x+1) \Rightarrow y = \frac{5x+13}{4}$$

For each of the following, find point(s) of horizontal tangency.

$$20. y = x^2(x-4)^2$$

$$21. y = \frac{x}{\sqrt{2x-1}}$$

$$22. f(x) = -3 \pm \sqrt{24+2x-x^2}$$

$$y' = x^2(2)(x-4) + 2x(x-4)^2$$

$$2x(x-4)(2x-4) = 0 \Rightarrow x = 0, 4, 2$$

Pts: (0,0), (4,0), (2,16)

$$y' = \frac{\sqrt{2x-1} - \frac{2x}{2\sqrt{2x-1}}}{2x-1} = 0$$

$$\sqrt{2x-1} = \frac{x}{\sqrt{2x-1}} \Rightarrow 2x-1 = x$$

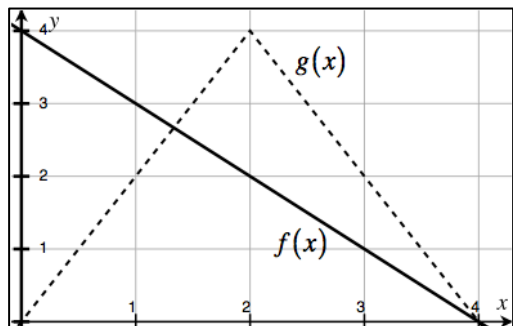
$x = 1$  Pt: (1,1)

$$f'(x) = \pm \frac{2-2x}{2\sqrt{24+2x-x^2}} = 0$$

$$\pm(2-2x) = 0 \Rightarrow x = 1$$

Pts: (1, -3±5) ⇒ (1, 2), (1, -8)

Below are the graphs of  $f(x)$  and  $g(x)$ . For each problem, find  $h'(1)$ .



$$23. h(x) = [f(x) + g(x)]^2$$

$$h'(x) = 2[f(x) + g(x)][f'(x) + g'(x)]$$

$$h'(1) = 2[f(1) + g(1)][f'(1) + g'(1)]$$

$$h'(1) = 2[3+2][-1+2] = 10$$

$$24. h(x) = f(g(3x))$$

$$h'(x) = f'(g(3x))g'(3x)(3)$$

$$h'(1) = 3f'(g(3))g'(3)$$

$$h'(1) = 3f'(2)(-2)$$

$$h'(1) = 3(-1)(-2) = 6$$

Given the following functions  $h(x)$ , find the values of  $h'(3)$  using the information in the chart to the right. These problems are important, so we have given you the answers. If you don't get these answers, redo the problem.<sup>9c</sup>

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	-6	5	0
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	$\pi$	4

25.  $h(x) = f(3x-1)$   $\boxed{3\pi}$

$$\begin{aligned} h'(x) &= f'(3x-1)(3) \\ h'(3) &= 3f'(8) \\ h'(3) &= 3\pi \end{aligned}$$

26.  $h(x) = [f(x)]^2$   $\boxed{-6}$

$$\begin{aligned} h'(x) &= 2f(x)f'(x) \\ h'(3) &= 2f(3)f'(3) \\ h'(3) &= 2(1)(-3) = -6 \end{aligned}$$

27.  $h(x) = \frac{1}{g(2x)} \frac{-5}{2}$

$$\begin{aligned} h'(3) &= \frac{-2g'(6)}{[g(6)]^2} \\ h'(3) &= \frac{-2(5)}{(-2)^2} = \frac{-5}{2} \end{aligned}$$

28.  $h(x) = \sqrt{xf(x)}$   $\boxed{\frac{-4}{\sqrt{3}}}$

$$\begin{aligned} h'(x) &= \frac{xf'(x) + f(x)}{2\sqrt{xf(x)}} \\ h'(3) &= \frac{3f'(3) + f(3)}{2\sqrt{3f(3)}} \\ h'(3) &= \frac{3(-3) + 1}{2\sqrt{3(1)}} = \frac{-4}{\sqrt{3}} \end{aligned}$$

29.  $h(x) = \left(\frac{f(x)}{x}\right)^3 \frac{-10}{27}$

$$\begin{aligned} h'(x) &= 3\left(\frac{f(x)}{x}\right)^2 \left[\frac{xf'(x) - f(x)}{x^2}\right] \\ h'(3) &= 3\left(\frac{f(3)}{3}\right)^2 \left[\frac{3f'(3) - f(3)}{3^2}\right] \\ h'(3) &= 3\left(\frac{1}{3}\right)^2 \left[\frac{3(-3) - 1}{9}\right] \\ h'(3) &= \frac{1}{3} \left(\frac{-10}{9}\right) = \frac{-10}{27} \end{aligned}$$

30.  $h(x) = \sqrt{f(2x-5)}$   $\boxed{\frac{5}{\sqrt{2}}}$

$$\begin{aligned} h'(x) &= \frac{2f'(2x-5)}{2[f(2x-5)]^{1/2}} \\ h'(3) &= \frac{f'(1)}{[f(1)]^{1/2}} \\ h'(3) &= \frac{5}{\sqrt{2}} \end{aligned}$$

31.  $h(x) = \sqrt{f(x)+g(x)}$   $\boxed{\frac{-4}{3}}$

$$\begin{aligned} h'(x) &= \frac{f'(x) + g'(x)}{2[f(x) + g(x)]^{1/2}} \\ h'(3) &= \frac{f'(3) + g'(3)}{2[f(3) + g(3)]^{1/2}} \\ h'(3) &= \frac{-3 - 5}{2(1+8)^{1/2}} = \frac{-4}{3} \end{aligned}$$

32.  $h(x) = [f(x)]^3 g(x)$   $\boxed{-77}$

$$\begin{aligned} h'(x) &= [f(x)]^3 g'(x) + g(x)(3)[f(x)]^2 f'(x) \\ h'(3) &= [f(3)]^3 g'(3) + g(3)(3)[f(3)]^2 f'(3) \\ h'(3) &= (1^3)(-5) + 3(8)(1^2)(-3) = -77 \end{aligned}$$

33.  $h(x) = \frac{-1}{\sqrt[3]{g(x)}} \frac{-5}{48}$

$$\begin{aligned} h'(x) &= \frac{-1}{3} [g(x)]^{-4/3} [g'(x)] \\ h'(3) &= \frac{-g'(3)}{3[g(3)]^{-4/3}} \\ h'(3) &= \frac{-5}{3(16)} = \frac{-5}{48} \end{aligned}$$

34.  $h(x) = f(g(x))$   $\boxed{-5\pi}$

$$\begin{aligned} h'(x) &= f'(g(x))(g'(x)) \\ h'(3) &= f'(g(3))(g'(3)) \\ h'(3) &= f'(8)(-5) \\ h'(3) &= -5\pi \end{aligned}$$

35.  $h(x) = g(f(x))$   $\boxed{0}$

$$\begin{aligned} h'(x) &= g'(f(x))(f'(x)) \\ h'(3) &= g'(f(3))(f'(3)) \\ h'(3) &= g'(1)(-3) \\ h'(3) &= 0(-3) = 0 \end{aligned}$$

36.  $h(x) = g(g(x))$   $\boxed{-20}$

$$\begin{aligned} h'(x) &= g'(g(x))(g'(x)) \\ h'(3) &= g'(g(3))(g'(3)) \\ h'(3) &= g'(8)(-5) \\ h'(3) &= 4(-5) = -20 \end{aligned}$$