UNIT 6 Modeling Exponential Growth

Now that students have had a chance to explore functions that grow at linear and quadratic rates, they have all the tools to analyze exponential growth and decay. You can introduce this concept to your students through the *Growth of a Smartphone App* problem, which offers an easy-to-understand model of how quickly things grow when they grow exponentially. The goal for this unit is for students to understand the difference between linear, quadratic, and exponential growth, and to know when each kind of growth applies to a given situation. At the end of the unit, students should also be able to understand and solve exponential growth problems as they appear on the HSE exam.

SKILLS DEVELOPED

- **Using tables, charts, and drawings to model exponential growth.**
- **Seeing exponential growth models in word problems.**
- **Distinguishing between situations that can be modeled with linear and exponential functions.**
- **Observing that a quantity increasing exponentially will eventually exceed a quantity increasing linearly or as a polynomial function.**
- **Noticing that in exponential functions, the variable is in the exponent position, and interpreting equations accordingly.**

KEY VOCABULARY

exponential growth: growth whose rate becomes increasingly rapid in proportion to a growing total number. In other words, a quantity may grow slowly at first, but it will eventually begin to grow at a very rapid rate. Some applications of exponential growth include calculating compound interest or population growth. In exponential growth problems on the HSE exam, the variable—or input—is in the exponent position.

exponential decay: a decrease whose rate is proportional to the size of the population. Exponential decay can be thought of as the opposite of exponential growth, i.e., the quantity will decrease very rapidly at first, but will eventually decrease at a smaller and smaller rate as the size of the quantity becomes smaller. Exponential decay is commonly applied in analyses of half-life of radioactive substances.

growth/decay factor: the constant factor by which a quantity multiplies itself over time. In mathematical functions, the growth factor is usually modeled by 1+ (percent of growth expressed as a decimal). For example, in the supplemental problem Observing a Mouse Population (below), the population is increasing annually by 8%. We see the population growth factor written in the formula as (1.08). The decay factor is similar. It is modeled by 1− (percent of decay expressed as a decimal).

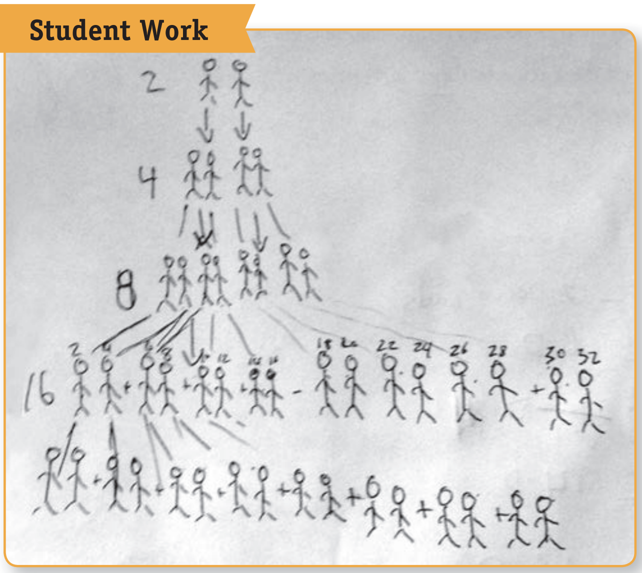
Core Problem Overview: Growth of a Smartphone App

This problem asks students to consider how many users will download a new Smartphone app given a specific rate of growth. Zach, the developer, creates the app and tells two friends about it. Those two friends download the app, and then each of them tells two of their friends about the app. These four friends also download the app, and then each of them tells two new friends. The pattern continues for one week. Through working on the problem, students should see that the number of downloads doubles each day, which models a simple form of exponential growth.

This activity starts with students making a prediction about the total number of downloads for the week. Students will usually predict a fairly small number of downloads given the fact that only two people downloaded it on the first day. If you have spent some time on linear growth, they may try to apply a linear growth model in their prediction, which will give them a very small number of downloads. The goal of this part of the activity is to get students thinking about the situation, and to get them in the habit of making predictions before they start working with the numbers. Before moving on to the second part of the problem, ask your students to talk about their predictions with the whole class. You might make note of the highest and lowest predictions, and record them on the board so that the class can analyze how close their answer was to the class's high and low predictions.

Now students should pick up their pencils and begin working on the problem. You should provide support by helping students settle on an appropriate strategy for solving the problem. What students should eventually see, through drawing a picture or making a table, is that the number of people who download the app doubles each day, or, in other words, that the rate of growth increases each day.

Student Work



As the sample of student work shows, drawing a picture is an excellent way of understanding the pattern, but it is ultimately an unsustainable way of solving the problem. That is, students would need to draw so many hash marks or stick figures that their picture would take a long time to draw and wouldn't fit on a single sheet of paper. We recommend allowing students to come to this conclusion on their own. It will be more meaningful if students try it as a strategy and then realize they will need to do something different in order to count all the downloads. Let them come to it, but make sure to make it explicit by asking them to talk about how they started, how/why their strategy shifted, and how their initial strategy helped inform their shift. Students will also need to understand that the problem requires them to add up the number of downloads each day in order to get the total for the week, which is 254 (excluding Zach).

Teaching the Core Problem

To begin the activity, you should ask students to put their pencils down and read through the scenario. Give students about five minutes to think about it, still with their pencils down, and ask volunteers (or each student) to provide a guess. You should record all of their predictions on chart paper or on the board so that the class can see all of the predictions together. Next, ask students which is the highest guess and which is the lowest, and make note of these. At this point, allow students to talk as a whole class about the predictions. You might ask some students to explain their guess, and then, based on these responses, ask if anyone would like to change their prediction.

Now, students should begin working on the second question. The teacher should let students work for a short time before intervening and asking questions. One common mistake that students make is to think that two new people download the app each day, which results in a total of 14 downloads at the end of the week. Another mistake is to think that two more people download the app each day, rather than twice as many. It's okay to let students make these mistake at first! This gives you the chance to talk to them about how they arrived at their answer and help them to see that the number of people who download the app doubles each day, rather than increases by two. To help them see this, you might ask if there is a way they could draw a picture that models the situation. If students are having trouble seeing this, you might also help them to see it by using people in the classroom. Some questions and prompts you could use are:

- *Let's say that you developed the app. Which students did you tell about it on the first day? On the next day, which other students did those two students tell? What do you notice about the pattern?*
- *I noticed you were drawing a picture but then you stopped. Why did you stop here? Did you notice a pattern before you stopped drawing? How could you keep the pattern going without drawing pictures?*
- *How could you organize all this information? Let's pick a day of the week when Zach told the first two people.*

It's important to note that guess-and-check won't work as a solution strategy for this problem. If students are trying to use this method, try to help them start small—with days 1 and 2—and build a table or chart from there. Once students begin to see the correct growth pattern in their drawing or chart, ask them to tell you more about it. Your goal is for them to see that the number of people who download the app doubles each day.

To get the correct answer, students will need to take all of the downloads from each day and add them up. Many students will stop at 128 downloads and forget to find the total for the week. You should ask them to talk about their answer of 128; specifically, does the 128 include all of the people who downloaded it on the other days? Here, some students will include Zach in their total and others won't. Rather than telling students that they should or should not include Zach, ask them to tell you more about it. This is something to discuss during the processing part of the activity.

If students finish early, you might ask them to analyze how the number of downloads would grow if each person who downloaded the app told *three* people rather than two. If your class has spent time working with exponents, you could ask students to look at the number of downloads from each day and think about how they could write those numbers as powers of two. Your students may or may not be able to do this on their own. In any case, you should plan to build this into the discussion after students have discussed their solution strategies.

Processing the Problem

Once all students have had time to work and found the answer on their own, you should lead a whole-class discussion of solution strategies. A good place to start would be to ask the class to look back at their predictions and talk about the differences between those predictions and their actual answer. Some questions for discussion include:

- *How close were our guesses to our actual answer? Why were we so far off?*
- *How did your thinking change once you started working with the actual numbers?*
- *What is the benefit of making predictions before solving a math problem like this one?*

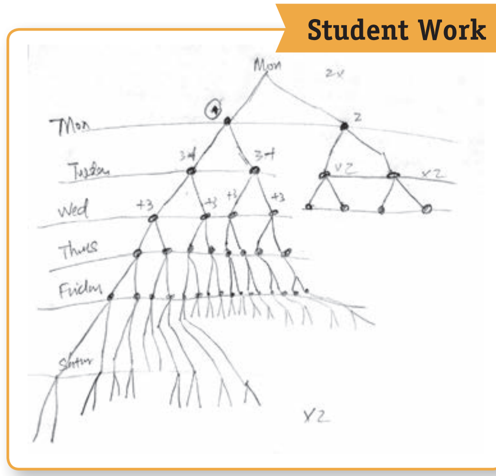
After you have had a chance to look at the predictions alongside the correct answer, explain to your students that you want to talk about the different ways of solving the problem. You should select three or four students to present their work on chart paper or on the board. One important aspect of whole-class discussions is to talk about common mistakes. You should find a student who had initially thought that two people downloaded the app each day, and ask them to explain what led them to make that mistake at first and how they corrected it. It would be a good idea to look carefully for different visual representations of the problem and ask students to present those. For example, some students will use hash marks or draw stick figures. You could ask these students to present first and show their work on the board exactly as they had written it on their paper. While they are presenting, ask some of the questions below:

- *Did anyone else try this strategy?*
- *What do you like about solving the problem this way?*
- *He or she stopped drawing after the fourth day. Why do you think he or she did that?*

Some students, like the sample of Ron Lee's work below shows, will draw a sort of tree diagram. If one of your students does this, you might ask them to present next, and ask them to talk about what the lines between each hash mark or stick figure represent.

Other students may have solved the problem correctly without drawing a picture at all, and instead focusing on a chart. You could ask these students to present last and explain how they were able to perform the calculations without a visual aid. At some point in the discussion, students will want to know if Zach should be included in the final tally. There is no correct or incorrect answer here, and one option would be to have your students present a case for why he should be included or not. You could also ask your students to vote on whether or not to include him.

Student Work



After all students have presented their work, you should take time to focus on the pattern that appears in the tables that students have generated, and help the class to see that the number of downloads on each day is actually a power of 2, as is shown in the table below. If some students have had time to think about this as an extension question, then they could lead the explanation of how each day's downloads can be written as a power of 2. Or, if no students have had time to try this, then you could get them started with days 1 and 2 and ask them to fill in the rest of the table.

Day	New Downloads On That Day	Written as Exponent
1	2	$2 = 2^1$
2	4	$2 \times 2 = 2^2$
3	8	$2 \times 2 \times 2 = 2^3$
4	16	$2 \times 2 \times 2 \times 2 = 2^4$
5	32	$2 \times 2 \times 2 \times 2 \times 2 = 2^5$
6	64	$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$
7	128	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$

Questions for discussion include:

- *In the third column, what is staying the same and what is changing?*
- *Do you see a relationship between the day and the number in the exponent position?*
- *Assuming this pattern continues, how could we find the number of new downloads that would occur on day 12?*
- *In mathematics (and in working with functions), how do we write a quantity that changes like this?*

When you feel like your students have a handle on how we could find the number of downloads on any given day by calculating 2^n, explain to your students that this kind of growth is called exponential. In exponential growth, the variable will always be in the exponent position, and it will often represent units of time.

Supplemental Problems

The supplemental problems in this unit will expose students to other scenarios that ask them to analyze exponential growth and decay. Two of them focus on applications to science, which resemble the types of problems that students might see on the math or science HSE subtests.

- **The Rice and Chess Problem**
  This problem is very similar to the core problem in this unit and can be used to reinforce the idea that when a quantity continues to double, it grows very rapidly and can produce large numbers. Because the numbers grow so quickly, students should be given calculators to work on this problem. Before they begin working, though, you might ask them to make a prediction about how many grains of rice the inventor would receive in total before they perform the calculations. In the end, they will find that the inventor would receive more grains of rice than actually exist on Earth!

- **Choosing Your Salary**
  This is another variation on the Rice and Chess problem that asks students whether they would choose a large salary that increases linearly or a small salary that increases exponentially. The problem underscores the idea that over time, exponential growth will always exceed linear growth. Students should be allowed to use calculators for this problem.

- **Observing a Mouse Population**
  This problem uses an exponential growth formula to model the increasing mouse population on an island. In order to use this problem in class, you need to have covered function notation in one of the previous units. You'll notice that this problem looks more like one that students might see on an HSE exam. The problem exposes students to three views of a function—a rule, a table, and a graph—and it opens the door to talk about science topics like population growth and carrying capacity.

Core Problem

Growth of a Smartphone App

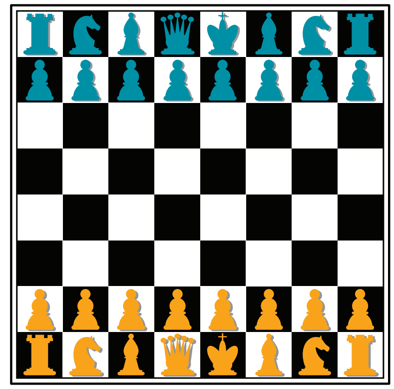
Zach developed a smartphone app. On Monday, he told two of his friends about his app and they downloaded it. The next day, those two friends each told two other friends about the app and they also downloaded it. Assume that this pattern continues, and each new person who downloads the app tells two of their friends about it. Make a prediction about how many people altogether will have downloaded the app by Sunday.

1. Write your prediction in the space below, and write a sentence or two to explain your thinking.

2. Let's see how close our guesses are. Exactly how many people will have downloaded the app after one week?

Supplemental Problem 1

The Rice and Chess Problem



There is a famous legend about the invention of chess that goes like this:

The inventor of the game showed it to a powerful ruler, who loved it so much that he told the inventor to name his reward—whatever he asked for the ruler promised to give him. The inventor said to the ruler: "I don't want much. I only ask that you give me one grain of rice for the first square on the chessboard, two grains for the next square, four for the next, eight for the next, and so on for all 64 squares."

The ruler agreed and laughed that the man had asked for such a small reward, but his treasurer—a mathematician—worried that the amount of rice would be more than the ruler could afford.

1. Create a table to find how many grains of rice the inventor would be given in total.

2. Were the mathematician's fears justified, or was the emperor correct in thinking that this reward is rather small?

Supplemental Problem 2

Choosing Your Salary

You are offered a job that last for only thirty days, and you get to choose your salary.

OPTION 1: You get $100 for the first day, $200 for the second day, $300 for the third day, and so on for each day of the seven weeks. Each day you are paid $100 more than you were the day before.

OPTION 2: You get paid only 1 cent for the first day. On the second day, you get paid 2 cents; on the third, you get paid 4 cents, and on the fourth you get paid 8 cents, and so on. Each day you get paid twice as much as you did the day before.

Which option do you choose? Why? (*Be prepared to explain your reasoning and help the class understand your way of thinking.)*

Supplemental Problem 3

Observing a Mouse Population

Over a period of ten years, scientists studied the population of mice living on a remote island in the Atlantic Ocean. They observed 120 mice on the island in the first year of their study, and they determined that they could use the following formula to calculate the population of mice on the island after a given time:

$$f(t) = 120(1.08)^t$$

1. Is the mouse population growing or decreasing? How do you know?

2. Complete the table below. If your answer is a decimal for any of the inputs, round it up to the nearest whole number.

Year *t*	Population *f(t)*
0	120
1	
2	
3	
4	
5	
6	
7	
8	
9	

3. Based on the data in your table, what can you conclude about the rate of increase in the mouse population? Is the growth constant, or not?

4. On graph paper, construct a horizontal axis for the time t and a vertical axis for the mouse population. Plot the points from your table and connect them. What do you notice about the shape of the graph?

> “Teachers prepared with support questions and extension questions can keep an entire class of “mixed-level” students working on the same problem and engaged in the productive struggle at the heart of each student’s proximal zone of development.”