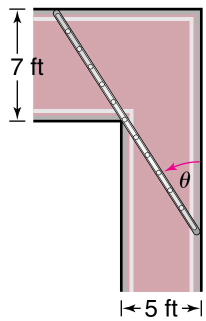
7-2 Verifying Trigonometric Identities

OBJECTIVES
- Use the basic trigonometric identities to verify other identities.
- Find numerical values of trigonometric functions.

Real World Application

PROBLEM SOLVING While working on a mathematics assignment, a group of students derived an expression for the length of a ladder that, when held horizontally, would turn from a 5-foot wide corridor into a 7-foot wide corridor. They determined that the maximum length ℓ of a ladder that would fit was given by $\ell(\theta) = \dfrac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta}$, where θ is the angle that the ladder makes with the outer wall of the 5-foot wide corridor. When their teacher worked the problem, she concluded that $\ell(\theta) = 7 \sec \theta + 5 \csc \theta$. Are the two expressions for $\ell(\theta)$ equivalent? *This problem will be solved in Example 2.*



Verifying trigonometric identities algebraically involves transforming one side of the equation into the same form as the other side by using the basic trigonometric identities and the properties of algebra. Either side may be transformed into the other side, or both sides may be transformed separately into forms that are the same.

Suggestions for Verifying Trigonometric Identities

- Transform the more complicated side of the equation into the simpler side.
- Substitute one or more basic trigonometric identities to simplify expressions.
- Factor or multiply to simplify expressions.
- Multiply expressions by an expression equal to 1.
- Express all trigonometric functions in terms of sine and cosine.

You cannot add or subtract quantities from each side of an unverified identity, nor can you perform any other operation on each side, as you often do with equations. An unverified identity is not an equation, so the properties of equality do not apply.

Example 1 **Verify that $\sec^2 x - \tan x \cot x = \tan^2 x$ is an identity.**

Since the left side is more complicated, transform it into the expression on the right.

$$\sec^2 x - \tan x \cot x \stackrel{?}{=} \tan^2 x$$

$$\sec^2 x - \tan x \cdot \frac{1}{\tan x} \stackrel{?}{=} \tan^2 x \qquad \textit{cot } x = \frac{1}{\tan x}$$

$$\sec^2 x - 1 \stackrel{?}{=} \tan^2 x \qquad \textit{Multiply.}$$

$$\tan^2 x + 1 - 1 \stackrel{?}{=} \tan^2 x \qquad \sec^2 x = \tan^2 x + 1$$

$$\tan^2 x = \tan^2 x \qquad \textit{Simplify.}$$

We have transformed the left side into the right side. The identity is verified.

Examples **2** **PROBLEM SOLVING** **Verify that the two expressions for $\ell(\theta)$ in the application at the beginning of the lesson are equivalent. That is, verify that $\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} = 7 \sec \theta + 5 \csc \theta$ is an identity.**

Real World Application

Begin by writing the right side in terms of sine and cosine.

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} 7 \sec \theta + 5 \csc \theta$$

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} \frac{7}{\cos \theta} + \frac{5}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta}, \csc \theta = \frac{1}{\sin \theta}$$

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} \frac{7 \sin \theta}{\sin \theta \cos \theta} + \frac{5 \cos \theta}{\sin \theta \cos \theta} \qquad \textit{Find a common denominator.}$$

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} = \frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \qquad \textit{Simplify.}$$

The students and the teacher derived equivalent expressions for $\ell(\theta)$, the length of the ladder.

3 **Verify that $\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} = \csc^2 A - \cot^2 A$ is an identity.**

Since the two sides are equally complicated, we will transform each side independently into the same form.

$$\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} \stackrel{?}{=} \csc^2 A - \cot^2 A$$

$$\frac{\sin A}{\frac{1}{\sin A}} + \frac{\cos A}{\frac{1}{\cos A}} \stackrel{?}{=} (1 + \cot^2 A) - \cot^2 A \qquad \textit{Quotient identities; Pythagorean identity}$$

$$\sin^2 A + \cos^2 A \stackrel{?}{=} 1 \qquad \textit{Simplify.}$$

$$1 = 1 \qquad \sin^2 A + \cos^2 A = 1$$

The techniques that you use to verify trigonometric identities can also be used to simplify trigonometric equations. Sometimes you can change an equation into an equivalent equation involving a single trigonometric function.

Example **4** **Find a numerical value of one trigonometric function of x if $\frac{\cot x}{\cos x} = 2$.**

You can simplify the trigonometric expression on the left side by writing it in terms of sine and cosine.

$$\frac{\cot x}{\cos x} = 2$$

$$\frac{\frac{\cos x}{\sin x}}{\cos x} = 2 \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} = 2 \qquad \textit{Definition of division}$$

$$\frac{1}{\sin x} = 2 \quad \textit{Simplify.}$$

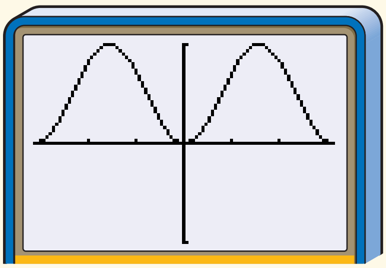
$$\csc x = 2 \quad \frac{1}{\sin x} = \csc x$$

Therefore, if $\frac{\cot x}{\cos x} = 2$, then $\csc x = 2$.

You can use a graphing calculator to investigate whether an equation may be an identity.

GRAPHING CALCULATOR EXPLORATION

- Graph both sides of the equation as two separate functions. For example, to test $\sin^2 x = (1 - \cos x)(1 + \cos x)$, graph $y_1 = \sin^2 x$ and $y_2 = (1 - \cos x)(1 + \cos x)$ on the same screen.
- If the two graphs do not match, then the equation is not an identity.
- If the two sides appear to match in every window you try, then the equation may be an identity.



$[-\pi, \pi]$ scl:1 by $[-1, 1]$ scl:1

TRY THESE **Determine whether each equation could be an identity. Write *yes* or *no*.**

1. $\sin x \csc x - \sin^2 x = \cos^2 x$

2. $\sec x + \csc x = 1$

3. $\sin x - \cos x = \frac{1}{\csc x - \sec x}$

WHAT DO YOU THINK?

4. If the two sides appear to match in every window you try, does that prove that the equation is an identity? Justify your answer.

5. Graph the function $f(x) = \frac{\sec x - \cos x}{\tan x}$. What simpler function could you set equal to $f(x)$ in order to obtain an identity?

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a trigonometric equation that is not an identity. Explain how you know it is not an identity.
2. **Explain** why you cannot square each side of the equation when verifying a trigonometric identity.
3. **Discuss** why both sides of a trigonometric identity are often rewritten in terms of sine and cosine.

4. *Math Journal* **Create** your own trigonometric identity that contains at least three different trigonometric functions. Explain how you created it. Give it to one of your classmates to verify. Compare and contrast your classmate's approach with your approach.

Guided Practice

Verify that each equation is an identity.

5. $\cos x = \dfrac{\cot x}{\csc x}$

6. $\dfrac{1}{\tan x + \sec x} = \dfrac{\cos x}{\sin x + 1}$

7. $\csc \theta - \cot \theta = \dfrac{1}{\csc \theta + \cot \theta}$

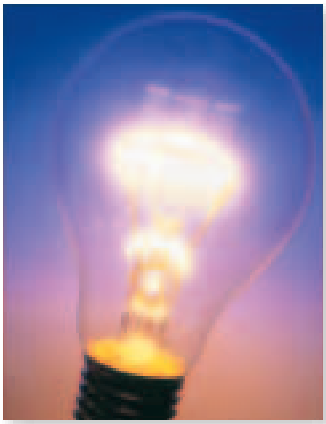
8. $\sin \theta \tan \theta = \sec \theta - \cos \theta$

9. $(\sin A - \cos A)^2 = 1 - 2 \sin^2 A \cot A$

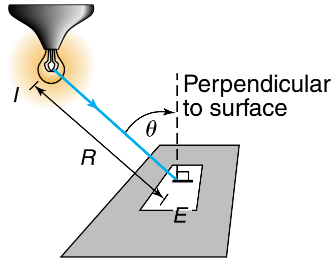
Find a numerical value of one trigonometric function of *x*.

10. $\tan x = \dfrac{1}{4} \sec x$

11. $\cot x + \sin x = -\cos x \cot x$



12. **Optics** The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface that is R feet from a source of light with intensity I candelas is $E = \dfrac{I \cos \theta}{R^2}$, where θ is the measure of the angle between the direction of the light and a line perpendicular to the surface being illuminated. Verify that $E = \dfrac{I \cot \theta}{R^2 \csc \theta}$ is an equivalent formula.



EXERCISES

Practice

Verify that each equation is an identity.

13. $\tan A = \dfrac{\sec A}{\csc A}$

14. $\cos \theta = \sin \theta \cot \theta$

15. $\sec x - \tan x = \dfrac{1 - \sin x}{\cos x}$

16. $\dfrac{1 + \tan x}{\sin x + \cos x} = \sec x$

17. $\sec x \csc x = \tan x + \cot x$

18. $\sin \theta + \cos \theta = \dfrac{2 \sin^2 \theta - 1}{\sin \theta - \cos \theta}$

19. $(\sin A + \cos A)^2 = \dfrac{2 + \sec A \csc A}{\sec A \csc A}$

20. $(\sin \theta - 1)(\tan \theta + \sec \theta) = -\cos \theta$

21. $\dfrac{\cos y}{1 - \sin y} = \dfrac{1 + \sin y}{\cos y}$

22. $\cos \theta \cos (-\theta) - \sin \theta \sin (-\theta) = 1$

23. $\csc x - 1 = \dfrac{\cot^2 x}{\csc x + 1}$

24. $\cos B \cot B = \csc B - \sin B$

25. $\sin \theta \cos \theta \tan \theta + \cos^2 \theta = 1$

26. $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}$

27. $\sin x + \cos x = \dfrac{\cos x}{1 - \tan x} + \dfrac{\sin x}{1 - \cot x}$

28. Show that $\sin \theta + \cos \theta + \tan \theta \sin \theta = \sec \theta + \cos \theta \tan \theta$.

Find a numerical value of one trigonometric function of *x*.

29. $\dfrac{\csc x}{\cot x} = \sqrt{2}$

30. $\dfrac{1 + \tan x}{1 + \cot x} = 2$

31. $\dfrac{1}{\cot x} - \dfrac{\sec x}{\csc x} = \cos x$

32. $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = 4$

33. $\cos^2 x + 2 \sin x - 2 = 0$

34. $\csc x = \sin x \tan x + \cos x$

35. If $\dfrac{\tan^3 \theta - 1}{\tan \theta - 1} - \sec^2 \theta - 1 = 0$, find $\cot \theta$.

Graphing Calculator

Use a graphing calculator to determine whether each equation could be an identity.

36. $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = 1$

37. $\cos \theta(\cos \theta - \sec \theta) = -\sin^2 \theta$

38. $2 \sin A + (1 - \sin A)^2 = 2 - \cos^2 A$

39. $\dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \sin^2 x + \cos^2 x$

Applications and Problem Solving

Real World Application

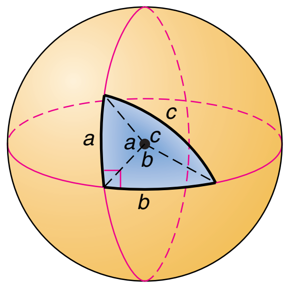
40. **Electronics** When an alternating current of frequency *f* and peak current I_0 passes through a resistance *R*, then the power delivered to the resistance at time *t* seconds is $P = I_0^2 R \sin^2 2\pi ft$.

a. Write an expression for the power in terms of $\cos^2 2\pi ft$.

b. Write an expression for the power in terms of $\csc^2 2\pi ft$.

41. **Critical Thinking** Let $x = \dfrac{1}{2} \tan \theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Write $f(x) = \dfrac{x}{\sqrt{1 + 4x^2}}$ in terms of a single trigonometric function of θ.

42. **Spherical Geometry** Spherical geometry is the geometry that takes place on the surface of a sphere. A line segment on the surface of the sphere is measured by the angle it subtends at the center of the sphere. Let *a*, *b*, and *c* be the sides of a right triangle on the surface of the sphere. Let the angles opposite those sides be α, β, and $\gamma = 90°$, respectively. The following equations are true:



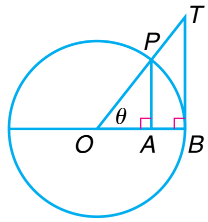
β is the Greek letter beta and γ is the Greek letter gamma.

$$\sin a = \sin \alpha \sin c$$
$$\cos b = \frac{\cos \beta}{\sin \alpha}$$
$$\cos c = \cos a \cos b.$$

Show that $\cos \beta = \tan a \cot c$.

43. **Physics** When a projectile is fired from the ground, its height *y* and horizontal displacement *x* are related by the equation $y = \dfrac{-gx^2}{2{v_0}^2 \cos^2 \theta} + \dfrac{x \sin \theta}{\cos \theta}$, where v_0 is the initial velocity of the projectile, θ is the angle at which it was fired, and *g* is the acceleration due to gravity. Rewrite this equation so that $\tan \theta$ is the only trigonometric function that appears in the equation.

44. **Critical Thinking** Consider a circle O with radius 1. $\overline{PA}$ and $\overline{TB}$ are each perpendicular to $\overline{OB}$. Determine the area of $ABTP$ as a product of trigonometric functions of θ.



45. **Geometry** Let a, b, and c be the sides of a triangle. Let α, β, and γ be the respective opposite angles. Show that the area A of the triangle is given by $A = \dfrac{a^2 \sin \beta \sin \gamma}{2 \sin (\beta + \gamma)}$.

Mixed Review

46. Simplify $\dfrac{\tan x + \cos x + \sin x \tan x}{\sec x + \tan x}$. *(Lesson 7-1)*

47. Write an equation of a sine function with amplitude 2, period 180°, and phase shift 45°. *(Lesson 6-5)*

48. Change $\dfrac{15\pi}{16}$ radians to degree measure to the nearest minute. *(Lesson 6-1)*

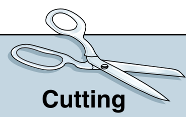
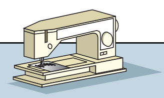
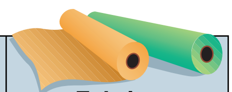
49. Solve $\sqrt[3]{3y - 1} - 2 = 0$. *(Lesson 4-7)*

50. Determine the equations of the vertical and horizontal asymptotes, if any, of $f(x) = \dfrac{3x}{x + 1}$. *(Lesson 3-7)*

51. **Manufacturing** The Simply Sweats Corporation makes high quality sweatpants and sweatshirts. Each garment passes through the cutting and sewing departments of the factory. The cutting and sewing departments have 100 and 180 worker-hours available each week, respectively. The fabric supplier can provide 195 yards of fabric each week. The hours of work and yards of fabric required for each garment are shown in the table below. If the profit from a sweatshirt is $5.00 and the profit from a pair of sweatpants is $4.50, how many of each should the company make for maximum profit? *(Lesson 2-7)*

Simply Sweats Corporation

"Quality Sweatpants and Sweatshirts"

Clothing	Cutting	Sewing	Fabric
Shirt	1 h	2.5 h	1.5 yd
Pants	1.5 h	2 h	3 yd

52. State the domain and range of the relation {(16,−4),(16, 4)}. Is this relation a function? Explain. *(Lesson 1-1)*

53. **SAT/ACT Practice** Divide $\dfrac{a - b}{a + b}$ by $\dfrac{b - a}{b + a}$.

A 1

B $\dfrac{(a - b)^2}{(a + b)^2}$

C $\dfrac{1}{a^2 - b^2}$

D −1

E 0

Extra Practice See p. A38.

CONTENTS