# 07 Algebraic Fractions

**By studying this lesson you will acquire knowledge about the following :**

- Addition of algebraic expressions
- Subtraction of algebraic expressions
- Multiplication of algebraic expressions
- Division of algebraic expressions

In the addition and subtraction of vulgar fraction we used the least common multiple (L.C.M) of the denominators of the fraction as the common denominator. In the addition and subtraction of algebraic fractions too, we use the L.C.M. of the denominators of the given fractions as the common denominator.

Let us consider a method of finding the L.C.M. of algebraic expressions.

## 7.1 Finding the L.C.M. of two algebraic expressions

The smallest algebraic expression exactly divisible by two given expressions is the L.C.M. of the two expressions.

To find the L.C.M., first factorise the expressions if they can be factorised. Then find the product of the common factors and the rest of the factors. If there are no common factors, then the LEM is the product of all the factors of the two exressions. This product is the L.C.M. of the given expressions. If the expressions are distinct and cannot be factorised, then the

**Example 1**

Find the L.C.M. of $x^2 + x - 2$ and $x^2 - 4x + 3$

$x^2 + x - 2 = (x + 2)(x - 1)$ [Resolving into factors]

$x^2 - 4x + 3 = (x - 3)(x - 1)$

The common factor is $(x - 1)$

The remaining factors are $(x + 2)$ and $(x - 3)$

$$\therefore \text{ L.C.M. } = (x-1)(x+2)(x-3)$$

**Example 2**

Find the L.C.M. of $x^2+x-6$ and $(x-2)^2$

$$x^2+x-6=(x+3)(x-2)$$

$$(x-2)^2=(x-2)(x-2)$$

The common factor is $(x-2)$

The remaining factors are $(x+3)$ and $(x-2)$

$$\therefore \text{ L.C.M.} = \underline{\underline{(x-2)(x+3)(x-2)}}$$

## 7.2 Finding the L.C.M. of three algebraic expressions

First of all factorise the given expressions. Then obtain the factors which are common to the three expressions, and after that obtain the factors common to two expressions. Thereafter obtain the rest of the factors. The product of all the above factors is the L.C.M. of the given expressions.

**Example 3** Find the L.C.M. of the expressions,

$$x^2+2x,\quad x^2+x-2 \quad \text{and} \quad (x+2)(x^2-2x+4)$$

$$x^2+2x = x(x+2)$$

$$x^2+x-2 = (x+2)(x-1)$$

$(x+2)(x^2-2x+4)$ cannot be factorized further.

The factor common to the three expressions $=(x+2)$

There are no factors common to only two expressions.

The rest of the factors are $x, (x-1)$ and $(x^2-2x+4)$

$$\underline{\underline{\therefore \text{ L.C.M.} = x(x+2)(x-1)(x^2-2x+4)}}$$

**Example 4** Find the L.C.M. of $(x-1)^2$ , $x^2-1$ , $x^2+4x+3$

$(x-1)^2 \quad = (x-1)(x-1)$

$x^2-1 \quad = x^2-1^2 = (x-1)(x+1)$

$x^2+4x+3 = (x+3)(x+1)$

There are no common factors for the three expressions.

The factors common to only two expressions are $(x-1)$ and $(x+1)$ .

The rest of the factors are, $(x-1)$and $(x+3)$.

$\therefore$ L.C.M. $= (x-1)(x-1)(x+1)(x+3)$

$= \underline{\underline{(x-1)^2(x+1)(x+3)}}$

**Exercise 7.1**

Find the L.C.M.

(1). $x^2, x^2+3x$

(2). $a^2-a,\ \ a^3-a$

(3). $b^2-9,\ \ b^2+3b$

(4). $x^2+3x-10,\ \ x^2-3x+2$

(5). $(a-3)^2;\ \ a^2-2a-3$

(6). $x^2-2x-3,\ \ x^2-5x+6$

(7). $m^2+3m+2,\ \ m^2-m-6$

(8). $a^2+a-12,\ \ a^2+2a-8$

(9). $a^2$ - 1, $a$ - 1, $a^2+2a+1$

(10). $ab,\ \ a(a-b),\ \ b(a-b)$

(11). $3m^2-2m-5,\ \ 3m^2+2m-5,\ \ m^2-1$

(12). $2a^2+3a-2,\ \ 6a^2-a-1,\ \ 3a^2+7a+2$

(13). $(a-1)^2;\ (a-1),\ \ (a+1)$

(14). $(1+x),\ (1$ -$x),\ (1$ -$x^2)$

(15). $x^2+x-2,\ \ 2x^2+3x-2,\ \ 2x^2-3x+1$

## 7.3 Addition and subtraction of algebraic fractions

Let us consider the addition of the following algebraic fractions.

$$\frac{3}{x^2-4} + \frac{1}{(x-2)^2}$$

First of all let us factorise the denominators.

$$\frac{3}{x^2-4} + \frac{1}{(x-2)^2} = \frac{3}{(x-2)(x+2)} + \frac{1}{(x-2)(x-2)}$$

Let us find the L.C.M. of the denominators.

$(x\text{-}2)$ is a common factor and,

the other factors are $(x+2)$ and $(x-2)$

$$\therefore \text{ L.C.M. } = (x-2)(x+2)(x-2)$$

Taking the L.C.M. as the denominator, let us form the fractions equivalent to the above fractions.

$$\frac{3}{(x-2)(x+2)} + \frac{1}{(x-2)(x-2)}$$

$$= \frac{3(x-2)}{(x-2)(x+2)(x-2)} + \frac{(x+2)}{(x-2)(x+2)(x-2)}$$

Because the denominators of the two fractions are equal, let us take it as the common denominator and add these fractions in the same way as adding vulgar fractions.

$$= \frac{3x-6+x+2}{(x-2)(x+2)(x-2)}$$

$$= \frac{4x-4}{(x-2)(x+2)(x-2)}$$

$$= \underline{\underline{\frac{4(x-1)}{(x-2)^2(x+2)}}}$$

**Example 5** Add the two fractions given below.

$$\frac{1}{x^2-5x+4} + \frac{x}{x^2+x-2}$$

| |
|---|
| Common factor $(x-1)$ |
| Other factors : $(x-4), (x+2)$ |
| $\therefore$ L.C.M. $= (x-1)(x-4)(x+2)$ |

$$= \frac{1}{(x-4)(x-1)} + \frac{x}{(x+2)(x-1)}$$

$$= \frac{(x+2)}{(x-1)(x+2)(x-4)} + \frac{x(x-4)}{(x-1)(x+2)(x-4)}$$

$$= \frac{x+2+x^2-4x}{(x-1)(x+2)(x-4)}$$

$$= \frac{x^2-3x+2}{(x-1)(x+2)(x-4)}$$

$$= \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+2)(x-4)}$$

$$= \frac{(x-2)}{(x+2)(x-4)}$$

**Example 6**

Simplify

$$\frac{a-3}{a^2-3a-4} - \frac{a-1}{a^2-a-2}$$

| |
|---|
| Common factor : $(a+1)$ |
| Other factors : $(a-4)(a-2)$ |
| L.C.M. $= (a+1)(a-2)(a-4)$ |

$$= \frac{a-3}{(a-4)(a+1)} - \frac{a-1}{(a-2)(a+1)}$$

$$= \frac{(a-3)(a-2)}{(a+1)(a-2)(a-4)} - \frac{(a-1)(a-4)}{(a+1)(a-2)(a-4)}$$

$$= \frac{a^2-5a+6-(a^2-5a+4)}{(a+1)(a-2)(a-4)}$$

$$= \frac{\cancel{a^2} - \cancel{5a} + 6 - \cancel{a^2} + \cancel{5a} - 4}{(a+1)(a-2)(a-4)}$$

$$= \frac{2}{(a+1)(a-2)(a-4)}$$

**Example 7**

Simplify

$$\frac{2x-1}{x^2+x-2} - \frac{x}{1-x^2} - \frac{1}{x-1}$$

$$= \frac{2x-1}{(x-1)(x+2)} - \frac{x}{(1-x)(1+x)} - \frac{1}{x-1}$$

By writing the factor (1-$x$) in the denominator of the second fraction as (-1) ($x$-1), the factor($x$-1) can be converted to a common factor of the denominators. This negative sign with the negative sign outside the bracket will change the sign to positive.

$$= \frac{(2x-1)}{(x-1)(x+2)} + \frac{x}{(x-1)(x+1)} - \frac{1}{(x-1)}$$

L.C.M. of the denominators = ($x$ - 1) ($x$ + 1) ($x$ + 2)

L.C.M. $= (x-1)(x+1)(x+2)$

$$= \frac{(2x-1)(x+1) + x(x+2) - 1(x+2)(x+1)}{(x-1)(x+2)(x+1)}$$

$$= \frac{2x^2 + x - 1 + x^2 + 2x - x^2 - 3x - 2}{(x-1)(x+2)(x+1)}$$

$$= \frac{2x^2 - 3}{(x-1)(x+2)(x+1)}$$

**Exercise 7.2**

Simplify the following expressions including algebraic fraction

(1). $\frac{1}{x+2}+\frac{1}{x+3}$

(2). $\frac{x}{x-y}-\frac{y}{x^2-y^2}$

(3). $\frac{2a}{a^2-9}+\frac{1}{a+3}$

(4). $\frac{x-y}{x^2+xy}+\frac{2(x+3y)}{x^2-y^2}$

(5). $\frac{6x}{x^2-4x+3}+\frac{2}{4x-12}$

(6). $\frac{a-3}{a^2-3a-4}-\frac{a-1}{a^2-a-2}$

(7). $\frac{1}{p^2+3p-10}-\frac{1}{4-p^2}$

(8). $\frac{3y^2+8}{y^3-1}-\frac{5y+7}{y^2+y+1}+\frac{2}{y-1}$

(9). $\frac{3}{x^2+13x+30}+\frac{1}{x^2+5x+6}$

(10). $\frac{1}{3x^2-2x-1}-\frac{1}{3x^2+7x+2}-\frac{1}{6x^2-x-1}$

(11). $\frac{a-1}{a-2}-\frac{a+1}{a+2}-\frac{2a}{4-a^2}$

(12). $\frac{3}{y^2+y-2}-\frac{5}{2y^2+3y-2}-\frac{1}{2y^2-3y+1}$

## 7.4 Multiplication of algebraic fractions

Multiplication of algebraic fractions can be done in the same way that we mutiply vulgar fractions. First, the expressions in the denominators as well as the numerators are resolved into factors. If there are any factors common to the numerator and the denominator, then the numerator and the denominator are divided by these common factors. Then the final answer is obtained by multiplying the terms in the numerator and the denominator separately.

Let us do the following examples.

**Example 8**

Simplify

$$\frac{x^2-4a^2}{ax+a^2}\times\frac{2a}{x^2-2ax}$$

$$=\frac{(x-2a)(x+2a)}{a(x+a)}\times\frac{2a}{x(x-2a)}$$

• Factorisation of numerators and denominators

$$=\frac{\overset{1}{\cancel{(x-2a)}}(x+2a)}{{}_1\cancel{a}(x+a)}\times\frac{2\cancel{a}^{\,1}}{x\underset{1}{\cancel{(x-2a)}}}$$

• Because $a$ and ($x$-2$a$) are common factors of the denominator and the numerator, divide by them.

$$=\frac{2(x+2a)}{x(x+a)}$$

• Multiplying the rest of the expressions in the denominators and the umerators separately.

**Example 9**

Simplify

$$\frac{2x^2+5x+2}{x^2-9}\times\frac{x^2+3x}{2x^2+9x+4}$$

$$=\frac{(2x+1)(x+2)}{(x-3)(x+3)}\times\frac{x(x+3)}{(2x+1)(x+4)}$$

• Factorisation

$$=\frac{\overset{1}{\cancel{(2x+1)}}(x+2)}{(x-3)\underset{1}{\cancel{(x+3)}}}\times\frac{x\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(2x+1)}}(x+4)}$$

• Division by common factors

$$=\frac{x(x+2)}{(x-3)(x+4)}$$

• Multiplication of the rest of the expressions in the denominators and the numerators separately.

## 7.5 Division of an algebraic fraction by another algebraic fraction

You will remember that in the division of vulgar fractions, we converted the division into a multiplication by using the latter reciprocal of the fraction. The same method could be applied in the division of algebraic fractions. When an algebraic fraction is to be divided by another algebraic fraction, we can convert the division into a multiplication by multiplying by the reciprocal of the latter fraction. Then the simplification can be done in the same way as we do multiplications.

**Example 10**

Simplify

$$\frac{x^2-3x+2}{x^2-4x-12} \div \frac{x^2-4}{x^2-7x+6}$$

$$= \frac{x^2-3x+2}{x^2-4x-12} \times \frac{x^2-7x+6}{x^2-4}$$

- Multiplication by the reciprocal of the fraction which appears after the division sign.

$$= \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-6)}(x+2)} \times \frac{\cancel{(x-6)}(x-1)}{\cancel{(x-2)}(x+2)}$$

- Resolving into factors and dividing by common factors.

$$= \frac{(x-1)^2}{(x+2)^2} = \left[\frac{x-1}{x+2}\right]^2$$

- Multiplication of the expressions in the denominators and numerators separately

**Example 11**

Simplify

$$\frac{x^2+x-12}{x^3-64} \div \frac{x^2-x-6}{x^2+4x+16}$$

$$= \frac{x^2+x-12}{x^3-4^3} \times \frac{x^2+4x+16}{x^2-x-6}$$

Multiplication by the reciprocal

$$= \frac{(x+4)\cancel{(x-3)}}{(x-4)\cancel{(x^2+4x+16)}} \times \frac{\cancel{(x^2+4x+16)}}{\cancel{(x-3)}(x+2)}$$

Resolving into factors and dividing by common factors

$$= \frac{(x+4)}{(x-4)(x+2)}$$

**Example 12**

Simplify

$$\frac{x^2 - 5x + 6}{x^2 + 5x + 4} \div \frac{x^2 - 4x + 3}{2x^2 + 3x + 1} \times \frac{x^2 + 3x - 4}{2x^2 - 3x - 2}$$

$$= \frac{x^2 - 5x + 6}{x^2 + 5x + 4} \times \frac{2x^2 + 3x + 1}{x^2 - 4x + 3} \times \frac{x^2 + 3x - 4}{2x^2 - 3x - 2}$$

$$= \frac{\overset{1}{\cancel{(x-3)}}\,\overset{1}{\cancel{(x-2)}}}{\underset{1}{\cancel{(x+4)}}\,\underset{1}{\cancel{(x+1)}}} \times \frac{\overset{1}{\cancel{(2x+1)}}\,\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x-3)}}\,\underset{1}{\cancel{(x-1)}}} \times \frac{\overset{1}{\cancel{(x+4)}}\,\overset{1}{\cancel{(x-1)}}}{\underset{1}{\cancel{(2x+1)}}\,\underset{1}{\cancel{(x-2)}}}$$

(Factorisation and division by common factors)

$$= \underline{\underline{1}}$$

In the above example, after dividing by the common factors, the denominator as well as the numerator will be left with only 1. When 1 is divided by 1 the final answer is 1.

**Exercise 7.3**

Simplify

(1). $\dfrac{x}{2y+5} \times \dfrac{4y^2+10y}{3x^2}$

(2). $\dfrac{x^2-4}{x+1} \times \dfrac{x^2+2x+1}{x+2}$

(3). $\dfrac{a^2-121}{a^2-4} \div \dfrac{a+11}{a+2}$

(4). $\dfrac{x^2+5x+6}{x^2-1} \times \dfrac{x^2-2x-3}{x^2-9}$

(5). $\dfrac{(p+2)(p^2-2p-3)}{p^2-9} \div \dfrac{p+2}{p^2+2p-3}$

(6). $\dfrac{a^2-4}{3ab} \times \dfrac{12b^2}{a^2+3a-10}$

(7). $\dfrac{x^2-y^2}{x^2-2xy+y^2} \times \dfrac{x^2-y^2}{x^2+xy}$

(8). $\dfrac{25a^2-b^2}{9a^2x^2-4x^2} \times \dfrac{x(3a+2)}{5a+b}$

(9). $\dfrac{x^2-5x}{x^2-4x-5} \div \dfrac{x^3-x^2-2x}{x^2+2x+1}$

(10). $\dfrac{a^2b^2+3ab}{4a^2-1} \div \dfrac{ab+3}{2a+1}$

## 7.4 Miscellaneous Exercises

Simplify

(1). $\dfrac{1}{(x-a)^2} - \dfrac{1}{x^2-a^2}$

(2). $\dfrac{x+4}{x^2+3x-10} - \dfrac{x-4}{x^2-5x+6}$

(3). $\dfrac{a}{3+a} - \dfrac{a}{a-3} + \dfrac{2a^2}{9-a^2}$

(4). $\dfrac{x^2-14x-15}{x^2-4x-45} \div \dfrac{x^2-12x-45}{x^2-6x-27}$

(5). $\dfrac{b^2-8b}{b^2-4b-5} \times \dfrac{b^2+2b+1}{b^3-8b^2} \div \dfrac{b^2+2b-3}{b-5}$

(6). $\dfrac{1}{(x-2)^2} - \dfrac{3}{x^2-4} + \dfrac{3}{(x+2)^2}$

(7). $\dfrac{1}{(a-b)(b-c)} + \dfrac{1}{(c-a)(a-b)} + \dfrac{1}{(b-c)(c-a)}$