7.3 Computing the Values of Trigonometric Functions of Acute Angles

OBJECTIVES
1. Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$ (p. 529)
2. Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$ (p. 530)
3. Use a Calculator to Approximate the Values of the Trigonometric Functions of Acute Angles (p. 532)
4. Model and Solve Applied Problems Involving Right Triangles (p. 532)

In the previous section, we developed ways to find the value of each trigonometric function of an acute angle when one of the functions is known. In this section, we discuss the problem of finding the value of each trigonometric function of an acute angle when the angle is given.

For three special acute angles, we can use some results from plane geometry to find the exact value of each of the six trigonometric functions.

1 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

EXAMPLE 1 Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

Find the exact values of the six trigonometric functions of $\frac{\pi}{4} = 45°$.

Solution Using the right triangle in Figure 27(a), in which one of the angles is $\frac{\pi}{4} = 45°$, it follows that the other acute angle is also $\frac{\pi}{4} = 45°$, and hence the triangle is isosceles. As a result, side a and side b are equal in length. Since the values of the trigonometric functions of an angle depend only on the angle and not on the size of the triangle, we may assign any values to a and b for which $a = b > 0$. We decide to use the triangle for which

$$a = b = 1$$

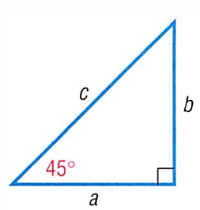
Then, by the Pythagorean Theorem,

$$c^2 = a^2 + b^2 = 1 + 1 = 2$$
$$c = \sqrt{2}$$

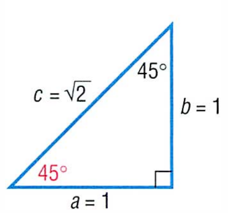
As a result, we have the triangle in Figure 27(b), from which we find

$$\sin\frac{\pi}{4} = \sin 45° = \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \cos 45° = \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Figure 27



(a)



(b)

Using Quotient and Reciprocal Identities, we find

$$\tan\frac{\pi}{4} = \tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \qquad \cot\frac{\pi}{4} = \cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

$$\sec\frac{\pi}{4} = \sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \qquad \csc\frac{\pi}{4} = \csc 45° = \frac{1}{\sin 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

EXAMPLE 2 Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) $(\sin 45°)(\tan 45°)$ (b) $\left(\sec\frac{\pi}{4}\right)\left(\cot\frac{\pi}{4}\right)$

Solution We use the results obtained in Example 1.

(a) $(\sin 45°)(\tan 45°) = \frac{\sqrt{2}}{2}\cdot 1 = \frac{\sqrt{2}}{2}$

(b) $\left(\sec\frac{\pi}{4}\right)\left(\cot\frac{\pi}{4}\right) = \sqrt{2}\cdot 1 = \sqrt{2}$

Now Work PROBLEMS 5 AND 17

2 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$

EXAMPLE 3 Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$

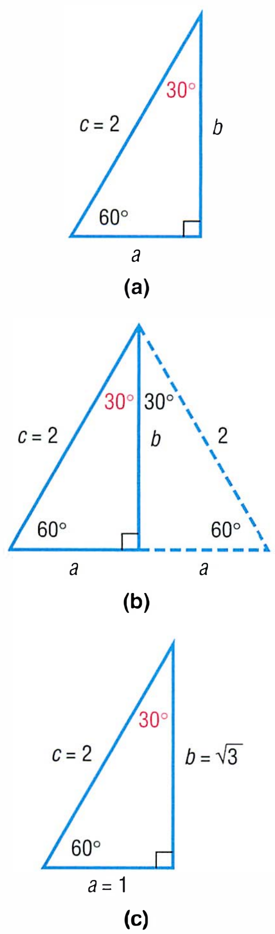
Find the exact values of the six trigonometric functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$.

Solution Form a right triangle in which one of the angles is $\frac{\pi}{6} = 30°$. It then follows that the third angle is $\frac{\pi}{3} = 60°$. Figure 28(a) illustrates such a triangle with hypotenuse of length 2. Our problem is to determine a and b.

We begin by placing next to the triangle in Figure 28(a) another triangle congruent to the first, as shown in Figure 28(b). Notice that we now have a triangle whose angles are each 60°. This triangle is therefore equilateral, so each side is of length 2. In particular, the base is $2a = 2$, so $a = 1$. By the Pythagorean Theorem, b satisfies the equation $a^2 + b^2 = c^2$, so we have

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \qquad a = 1, c = 2 \\ b^2 &= 4 - 1 = 3 \\ b &= \sqrt{3} \end{aligned}$$

Figure 28



Using the triangle in Figure 28(c) and the fact that $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$ are complementary angles, we find

$$\sin\frac{\pi}{6} = \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos\frac{\pi}{3} = \cos 60° = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{3} = \cot 60° = \frac{\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \csc 30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2 \qquad \sec\frac{\pi}{3} = \sec 60° = 2$$

$$\sec\frac{\pi}{6} = \sec 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \csc\frac{\pi}{3} = \csc 60° = \frac{2\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \cot 30° = \frac{1}{\tan 30°} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \qquad \tan\frac{\pi}{3} = \tan 60° = \sqrt{3}$$

Table 3 summarizes the information just derived for the angles $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$. Rather than memorize the entries in Table 3, you can draw the appropriate triangle to determine the values given in the table.

Table 3

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

EXAMPLE 4 Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) $\sin 45° \cos 30°$ (b) $\tan\frac{\pi}{4} - \sin\frac{\pi}{3}$ (c) $\tan^2\frac{\pi}{6} + \sin^2\frac{\pi}{4}$

Solution

(a) $\sin 45° \cos 30° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{4}$

(b) $\tan\frac{\pi}{4} - \sin\frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$

(c) $\tan^2\frac{\pi}{6} + \sin^2\frac{\pi}{4} = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

Now Work PROBLEMS 9 AND 19

The exact values of the trigonometric functions for the angles $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$ are relatively easy to calculate, because the triangles that contain such angles have "nice" geometric features. For most other angles, we can only approximate the value of each trigonometric function. To do this, we will need a calculator.

3 Use a Calculator to Approximate the Values of the Trigonometric Functions of Acute Angles

Before getting started, you must first decide whether to enter the angle in the calculator using radians or degrees and then set the calculator to the correct MODE. (Check your instruction manual to find out how your calculator handles degrees and radians.) Your calculator has the keys marked $\boxed{\text{sin}}$, $\boxed{\text{cos}}$, and $\boxed{\text{tan}}$. To find the values of the remaining three trigonometric functions (secant, cosecant, and cotangent), we use the reciprocal identities.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

EXAMPLE 5 Using a Calculator to Approximate the Value of Trigonometric Functions

Use a calculator to find the approximate value of:

(a) $\cos 48°$ (b) $\csc 21°$ (c) $\tan\frac{\pi}{12}$

Express your answer rounded to two decimal places.

Solution (a) First, we set the MODE to receive degrees. Rounded to two decimal places,

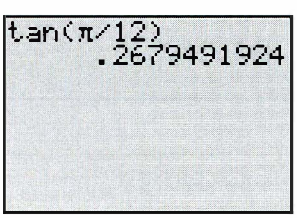
$$\cos 48° = 0.67$$

(b) Most calculators do not have a csc key. The manufacturers assume the user knows some trigonometry. To find the value of $\csc 21°$, we use the fact that $\csc 21° = \frac{1}{\sin 21°}$. Rounded to two decimal places, $\csc 21° = 2.79$.

(c) Set the MODE to receive radians. Figure 29 shows the solution using a TI-84 Plus graphing calculator. Rounded to two decimal places,

$$\tan\frac{\pi}{12} = 0.27$$

Figure 29



Now Work PROBLEM 29

4 Model and Solve Applied Problems Involving Right Triangles

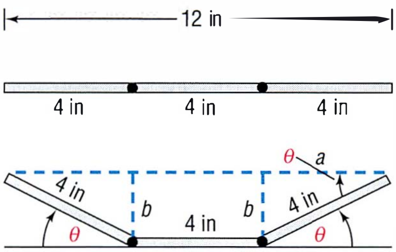
Right triangles can be used to model many types of situations, such as the optimal design of a rain gutter.*

* In applied problems, it is important that answers be reported with both justifiable accuracy and appropriate significant figures. We shall assume that the problem data are accurate to the number of significant digits, resulting in sides being rounded to two decimal places and angles being rounded to one decimal place.

EXAMPLE 6 Constructing a Rain Gutter

A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. See Figure 30.

Figure 30



(a) Express the area A of the opening as a function of θ.
[**Hint:** Let b denote the vertical height of the bend.]

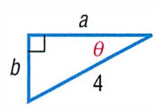
(b) Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 75°$.

(c) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

Solution

(a) Look again at Figure 30. The area A of the opening is the sum of the areas of two congruent right triangles and one rectangle. Look at Figure 31, showing one of the triangles in Figure 30 redrawn. We see that

Figure 31



$$\cos\theta = \frac{a}{4} \quad \text{so} \quad a = 4\cos\theta \qquad \sin\theta = \frac{b}{4} \quad \text{so} \quad b = 4\sin\theta$$

The area of the triangle is

$$\text{area of triangle} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ab = \frac{1}{2}(4\cos\theta)(4\sin\theta) = 8\sin\theta\cos\theta$$

So the area of the two congruent triangles is $16\sin\theta\cos\theta$.

The rectangle has length 4 and height b, so its area is

$$\text{area of rectangle} = 4b = 4(4\sin\theta) = 16\sin\theta$$

The area A of the opening is

$$A = \text{area of the two triangles} + \text{area of the rectangle}$$
$$A(\theta) = 16\sin\theta\cos\theta + 16\sin\theta = 16\sin\theta(\cos\theta + 1)$$

(b) For $\theta = 30°$: $\quad A(30°) = 16\sin 30°(\cos 30° + 1)$

$$= 16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + 1\right) = 4\sqrt{3} + 8 \approx 14.9$$

The area of the opening for $\theta = 30°$ is about 14.9 square inches.

For $\theta = 45°$: $\quad A(45°) = 16\sin 45°(\cos 45° + 1)$

$$= 16\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + 1\right) = 8 + 8\sqrt{2} \approx 19.3$$

The area of the opening for $\theta = 45°$ is about 19.3 square inches.

For $\theta = 60°$: $\quad A(60°) = 16\sin 60°(\cos 60° + 1)$

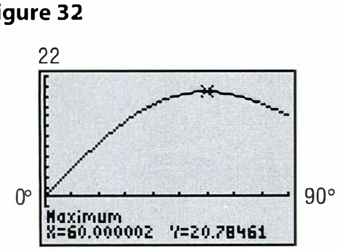
$$= 16\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} + 1\right) = 12\sqrt{3} \approx 20.8$$

The area of the opening for $\theta = 60°$ is about 20.8 square inches.

For $\theta = 75°$: $\quad A(75°) = 16\sin 75°(\cos 75° + 1) \approx 19.5$

The area of the opening for $\theta = 75°$ is about 19.5 square inches.

Figure 32



(c) Figure 32 shows the graph of $A = A(\theta)$. Using MAXIMUM, the angle θ that makes A largest is 60°.

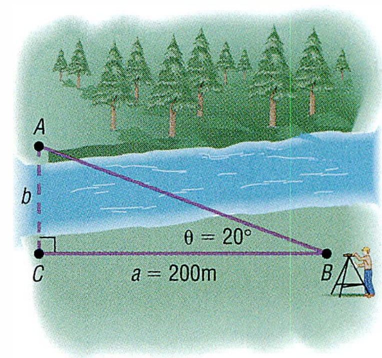
Now Work PROBLEM 51

In addition to developing models using right triangles, we can use right triangle trigonometry to measure heights and distances that are either awkward or impossible to measure by ordinary means. When using right triangles to solve these problems, pay attention to the known measures. This will indicate what trigonometric function to use. For example, if we know the measure of an angle and the length of the side adjacent to the angle, and wish to find the length of the opposite side, we would use the tangent function. Do you know why?

EXAMPLE 7 Finding the Width of a River

A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 33. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle θ is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

Figure 33



Solution We seek the length of side b. We know a and θ. So we use the fact that b is opposite θ and a is adjacent to θ and write

$$\tan \theta = \frac{b}{a}$$

which leads to

$$\tan 20° = \frac{b}{200}$$

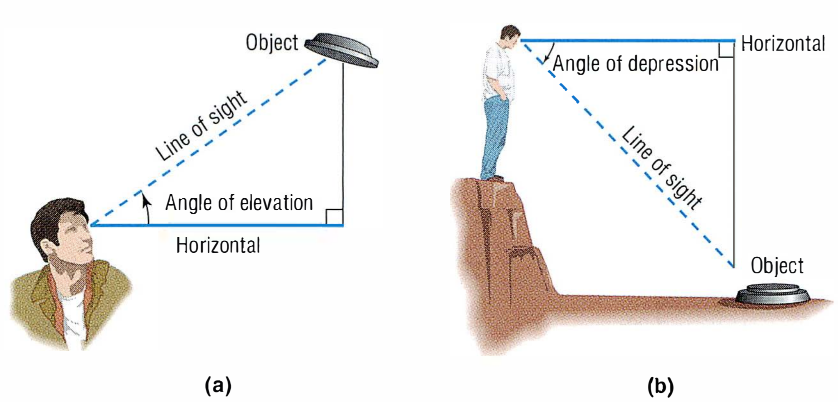
$$b = 200 \tan 20° \approx 72.79 \text{ meters}$$

The width of the river is 73 meters, rounded to the nearest meter.

Now Work PROBLEM 59

Vertical heights can sometimes be measured using either the *angle of elevation* or the *angle of depression*. If a person is looking up at an object, the acute angle measured from the horizontal to a line of sight to the object is called the **angle of elevation.** See Figure 34(a).

Figure 34



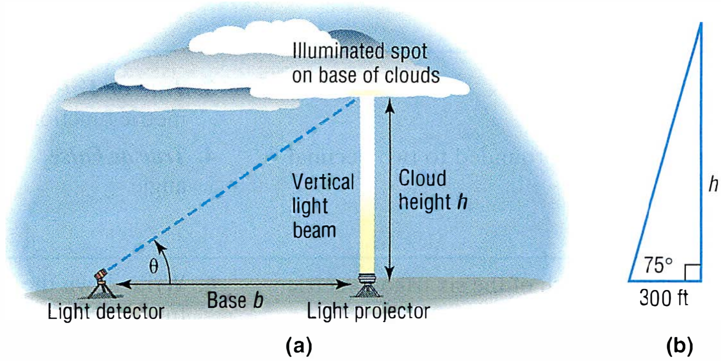
* An instrument used in surveying to measure angles.

If a person is standing on a cliff looking down at an object, the acute angle made by the line of sight to the object and the horizontal is called the **angle of depression.** See Figure 34(b).

EXAMPLE 8 Finding the Height of a Cloud

Meteorologists find the height of a cloud using an instrument called a **ceilometer.** A ceilometer consists of a **light projector** that directs a vertical light beam up to the cloud base and a **light detector** that scans the cloud to detect the light beam. See Figure 35(a). On December 1, 2006, at Midway Airport in Chicago, a ceilometer was employed to find the height of the cloud cover. It was set up with its light detector 300 feet from its light projector. If the angle of elevation from the light detector to the base of the cloud is 75°, what is the height of the cloud cover?

Figure 35



Solution Figure 35(b) illustrates the situation. To find the height h, we use the fact that $\tan 75° = \frac{h}{300}$, so

$$h = 300 \tan 75° \approx 1120 \text{ feet}$$

The ceiling (height to the base of the cloud cover) is approximately 1120 feet.

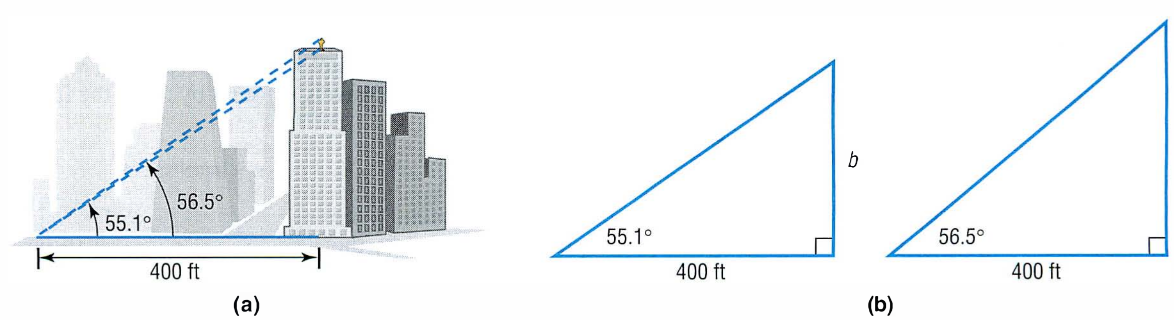
Now Work PROBLEM 61

The idea behind Example 8 can also be used to find the height of an object with a base that is not accessible to the horizontal.

EXAMPLE 9 Finding the Height of a Statue on a Building

Adorning the top of the Board of Trade building in Chicago is a statue of Ceres, the Roman goddess of wheat. From street level, two observations are taken 400 feet from the center of the building. The angle of elevation to the base of the statue is found to be 55.1° and the angle of elevation to the top of the statue is 56.5°. See Figure 36(a). What is the height of the statue?

Figure 36



Solution Figure 36(b) shows two triangles that replicate Figure 36(a). The height of the statue of Ceres will be $b' - b$. To find b and b', we refer to Figure 36(b).

$$\tan 55.1^\circ = \frac{b}{400} \qquad \tan 56.5^\circ = \frac{b'}{400}$$

$$b = 400 \tan 55.1^\circ \approx 573.39 \qquad b' = 400 \tan 56.5^\circ \approx 604.33$$

The height of the statue is approximately $604.33 - 573.39 = 30.94$ feet ≈ 31 feet.

Now Work PROBLEM 67

7.3 Assess Your Understanding

Concepts and Vocabulary

1. $\tan \dfrac{\pi}{4} + \sin 30^\circ =$ _____.

2. Using a calculator, $\sin 2 =$ _____, rounded to two decimal places.

3. ***True or False*** Exact values can be found for the trigonometric functions of 60°.

4. ***True or False*** Exact values can be found for the sine of any angle.

Skill Building

5. Write down the exact value of each of the six trigonometric functions of 45°.

6. Write down the exact value of each of the six trigonometric functions of 30° and of 60°.

In Problems 7–16, $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$. Find the exact value of each expression if $\theta = 60^\circ$. Do not use a calculator.

7. $f(\theta)$
8. $g(\theta)$
9. $f\left(\dfrac{\theta}{2}\right)$
10. $g\left(\dfrac{\theta}{2}\right)$
11. $[f(\theta)]^2$
12. $[g(\theta)]^2$
13. $2f(\theta)$
14. $2g(\theta)$
15. $\dfrac{f(\theta)}{2}$
16. $\dfrac{g(\theta)}{2}$

In Problems 17–28, find the exact value of each expression. Do not use a calculator.

17. $4\cos 45^\circ - 2\sin 45^\circ$
18. $2\sin 45^\circ + 4\cos 30^\circ$
19. $6\tan 45^\circ - 8\cos 60^\circ$
20. $\sin 30^\circ \cdot \tan 60^\circ$
21. $\sec\dfrac{\pi}{4} + 2\csc\dfrac{\pi}{3}$
22. $\tan\dfrac{\pi}{4} + \cot\dfrac{\pi}{4}$
23. $\sec^2\dfrac{\pi}{6} - 4$
24. $4 + \tan^2\dfrac{\pi}{3}$
25. $\sin^2 30^\circ + \cos^2 60^\circ$
26. $\sec^2 60^\circ - \tan^2 45^\circ$
27. $1 - \cos^2 30^\circ - \cos^2 60^\circ$
28. $1 + \tan^2 30^\circ - \csc^2 45^\circ$

In Problems 29–46, use a calculator to find the approximate value of each expression. Round the answer to two decimal places.

29. $\sin 28^\circ$
30. $\cos 14^\circ$
31. $\tan 21^\circ$
32. $\cot 70^\circ$
33. $\sec 41^\circ$
34. $\csc 55^\circ$
35. $\sin\dfrac{\pi}{10}$
36. $\cos\dfrac{\pi}{8}$
37. $\tan\dfrac{5\pi}{12}$
38. $\cot\dfrac{\pi}{18}$
39. $\sec\dfrac{\pi}{12}$
40. $\csc\dfrac{5\pi}{13}$
41. $\sin 1$
42. $\tan 1$
43. $\sin 1^\circ$
44. $\tan 1^\circ$
45. $\tan 0.3$
46. $\tan 0.1$

Applications and Extensions

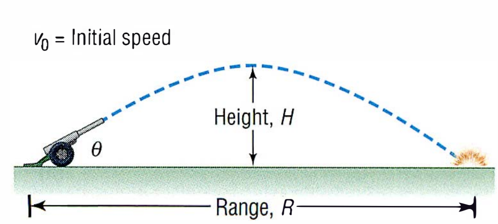
Problems 47–51 require the following discussion.

Projectile Motion The path of a projectile fired at an inclination θ to the horizontal with initial speed v_0 is a parabola. See the figure. The **range *R*** of the projectile, that is, the horizontal distance that the projectile travels, is found by using the function

$$R(\theta) = \frac{2v_0^2 \sin\theta\cos\theta}{g}$$

where $g \approx 32.2$ feet per second per second ≈ 9.8 meters per second per second is the acceleration due to gravity. The maximum height H of the projectile is given by the function

$$H(\theta) = \frac{v_0^2 \sin^2\theta}{2g}$$



In Problems 47–50, find the range R and maximum height H of the projectile. Round answers to two decimal places.

47. The projectile is fired at an angle of 45° to the horizontal with an initial speed of 100 feet per second.

48. The projectile is fired at an angle of 30° to the horizontal with an initial speed of 150 meters per second.

49. The projectile is fired at an angle of 25° to the horizontal with an initial speed of 500 meters per second.

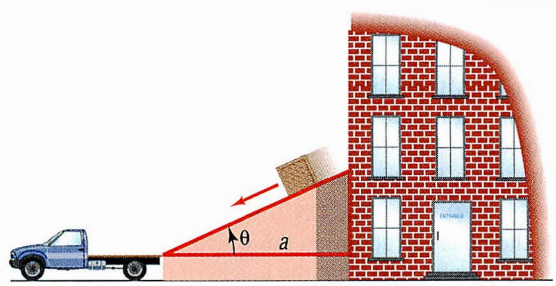
50. The projectile is fired at an angle of 50° to the horizontal with an initial speed of 200 feet per second.

51. Inclined Plane See the illustration. If friction is ignored, the time t (in seconds) required for a block to slide down an inclined plane is given by the function

$$t(\theta) = \sqrt{\frac{2a}{g \sin\theta \cos\theta}}$$

where a is the length (in feet) of the base and $g \approx 32$ feet per second per second is the acceleration due to gravity. How long does it take a block to slide down an inclined plane with base $a = 10$ feet when

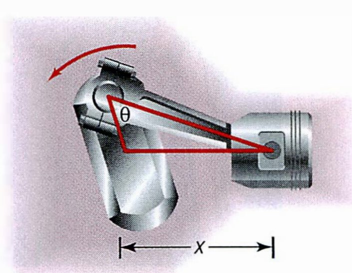
(a) $\theta = 30°$? (b) $\theta = 45°$? (c) $\theta = 60°$?



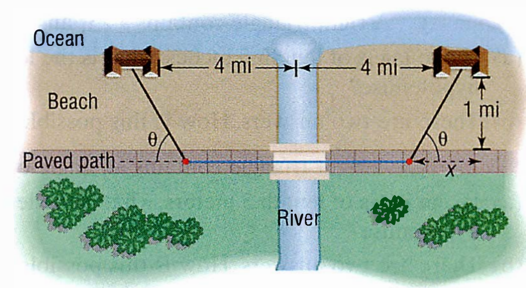
52. Piston Engines See the illustration. In a certain piston engine, the distance x (in inches) from the center of the drive shaft to the head of the piston is given by the function

$$x(\theta) = \cos\theta + \sqrt{16 + 0.5(2\cos^2\theta - 1)}$$

where θ is the angle between the crank and the path of the piston head. Find x when $\theta = 30°$ and when $\theta = 45°$.



53. Calculating the Time of a Trip Two oceanfront homes are located 8 miles apart on a straight stretch of beach, each a distance of 1 mile from a paved road that parallels the ocean. Sally can jog 8 miles per hour along the paved road, but only 3 miles per hour in the sand on the beach. Because a river flows between the two houses, it is necessary to jog on the sand to the road, continue on the road, and then jog on the sand to get from one house to the other. See the illustration.



(a) Express the time T to get from one house to the other as a function of the angle θ shown in the illustration.

(b) Calculate the time T for $\theta = 30°$. How long is Sally on the paved road?

(c) Calculate the time T for $\theta = 45°$. How long is Sally on the paved road?

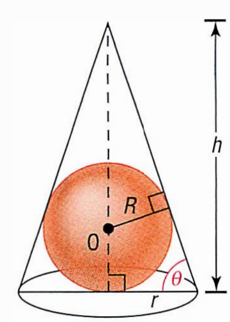
(d) Calculate the time T for $\theta = 60°$. How long is Sally on the paved road?

(e) Calculate the time T for $\theta = 90°$. Describe the path taken.

(f) Calculate the time T for $\tan\theta = \frac{1}{4}$. Describe the path taken. Explain why θ must be larger than 14°.

(g) Graph $T = T(\theta)$. What angle θ results in the least time? What is the least time? How long is Sally on the paved road?

54. Designing Fine Decorative Pieces A designer of decorative art plans to market solid gold spheres encased in clear crystal cones. Each sphere is of fixed radius R and will be enclosed in a cone of height h and radius r. See the illustration. Many cones can be used to enclose the sphere, each having a different slant angle θ.



(a) Express the volume V of the cone as a function of the slant angle θ of the cone.
[**Hint:** The volume V of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.]

(b) What volume V is required to enclose a sphere of radius 2 centimeters in a cone whose slant angle θ is 30°? 45°? 60°?

(c) What slant angle θ should be used for the volume V of the cone to be a minimum? (This choice minimizes the amount of crystal required and gives maximum emphasis to the gold sphere.)

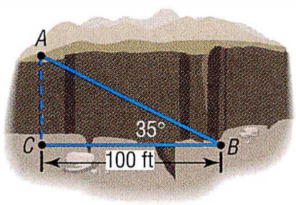
55. Geometry A right triangle has a hypotenuse of length 8 inches. If one angle is 35°, find the length of each leg.

56. Geometry A right triangle has a hypotenuse of length 10 centimeters. If one angle is 40°, find the length of each leg.

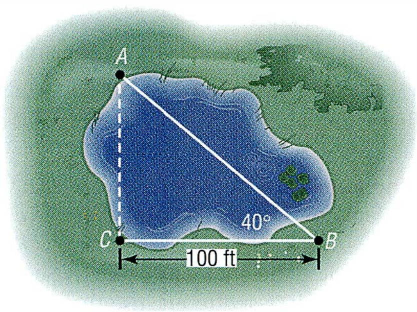
57. **Geometry** A right triangle contains a 25° angle.
(a) If one leg is of length 5 inches, what is the length of the hypotenuse?
(b) There are two answers. How is this possible?

58. **Geometry** A right triangle contains an angle of $\frac{\pi}{8}$ radian.
(a) If one leg is of length 3 meters, what is the length of the hypotenuse?
(b) There are two answers. How is this possible?

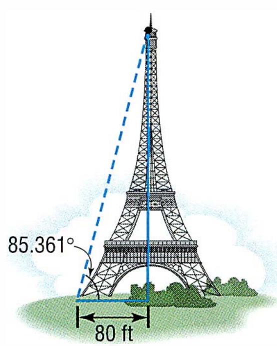
59. **Finding the Width of a Gorge** Find the distance from A to C across the gorge illustrated in the figure.



60. **Finding the Distance across a Pond** Find the distance from A to C across the pond illustrated in the figure.



61. **The Eiffel Tower** The tallest tower built before the era of television masts, the Eiffel Tower was completed on March 31, 1889. Find the height of the Eiffel Tower (before a television mast was added to the top) using the information given in the illustration.

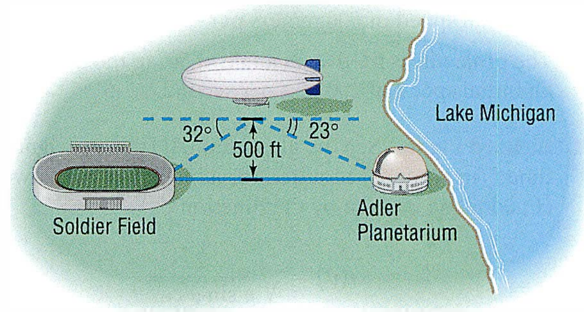


62. **Finding the Distance of a Ship from Shore** A person in a small boat, offshore from a vertical cliff known to be 100 feet in height, takes a sighting of the top of the cliff. If the angle of elevation is found to be 25°, how far offshore is the ship?

63. **Finding the Distance to a Plateau** Suppose that you are headed toward a plateau 50 meters high. If the angle of elevation to the top of the plateau is 20°, how far are you from the base of the plateau?

64. **Finding the Reach of a Ladder** A 22-foot extension ladder leaning against a building makes a 70° angle with the ground. How far up the building does the ladder touch?

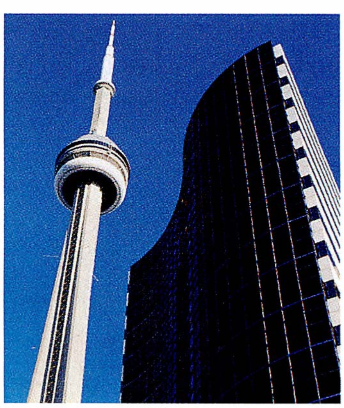
65. **Finding the Distance between Two Objects** A blimp, suspended in the air at a height of 500 feet, lies directly over a line from Soldier Field to the Adler Planetarium on Lake Michigan (see the figure). If the angle of depression from the blimp to the stadium is 32° and from the blimp to the planetarium is 23°, find the distance between Soldier Field and the Adler Planetarium.



66. **Hot-air Balloon** While taking a ride in a hot-air balloon in Napa Valley, Francisco wonders how high he is. To find out, he chooses a landmark that is to the east of the balloon and measures the angle of depression to be 54°. A few minutes later, after traveling 100 feet east, the angle of depression to the same landmark is determined to be 61°. Use this information to determine the height of the balloon.

67. **Mt. Rushmore** To measure the height of Lincoln's caricature on Mt. Rushmore, two sightings 800 feet from the base of the mountain are taken. If the angle of elevation to the bottom of Lincoln's face is 32° and the angle of elevation to the top is 35°, what is the height of Lincoln's face?

68. **The CN Tower** The CN Tower, located in Toronto, Canada, is the tallest structure in the world. While visiting Toronto, a tourist wondered what the height of the tower above the top of the Sky Pod is. While standing 4000 feet from the tower, she measured the angle to the top of the Sky Pod to be 20.1°. At this same distance, the angle of elevation to the top of the tower was found to be 24.4°. Use this information to determine the height of the tower above the Sky Pod.



69. **Finding the Length of a Guy Wire** A radio transmission tower is 200 feet high. How long should a guy wire be if it is to be attached to the tower 10 feet from the top and is to make an angle of 69° with the ground?

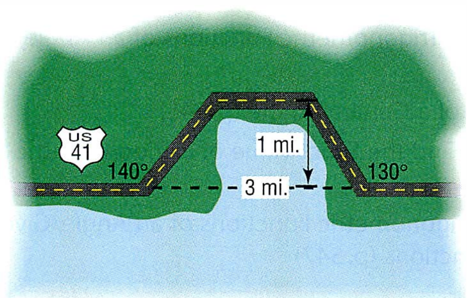
70. **Finding the Height of a Tower** A guy wire 80 feet long is attached to the top of a radio transmission tower, making an angle of 65° with the ground. How high is the tower?

71. **Washington Monument** The angle of elevation of the Sun is 35.1° at the instant the shadow cast by the Washington

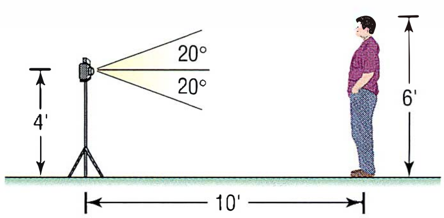
Monument is 789 feet long. Use this information to calculate the height of the monument.

72. Finding the Length of a Mountain Trail A straight trail with an inclination of 17° leads from a hotel at an elevation of 9000 feet to a mountain lake at an elevation of 11,200 feet. What is the length of the trail?

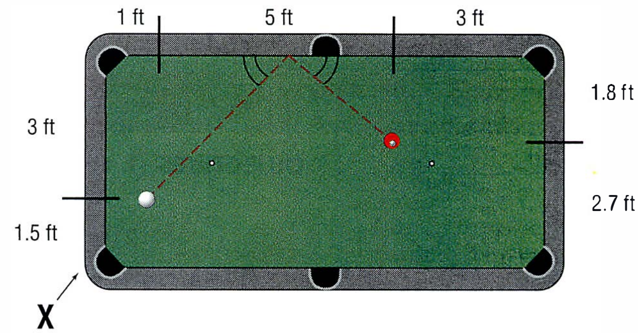
73. Constructing a Highway A highway whose primary directions are north–south is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?



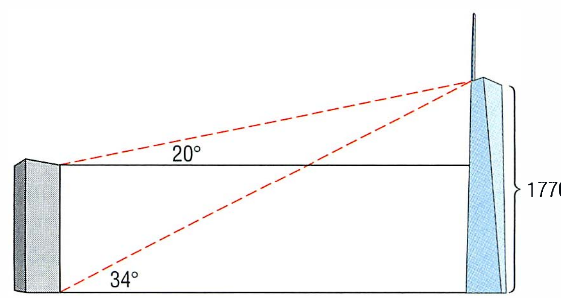
74. Photography A camera is mounted on a tripod 4 feet high at a distance of 10 feet from George, who is 6 feet tall. See the illustration. If the camera lens has angles of depression and elevation of 20°, will George's feet and head be seen by the lens? If not, how far back will the camera need to be moved to include George's feet and head?



75. Calculating Pool Shots A Pool player located at **X** wants to shoot the white ball off the top cushion and hit the red ball dead center. He knows from physics that the white ball will come off a cushion at the same angle as it hits a cushion. Where on the top cushion should he hit the white ball?



76. The Freedom Tower The Freedom Tower is to be the centerpiece of the rebuilding of the World Trade Center in New York City. The tower will be 1776 feet tall (not including a broadcast antenna). The angle of elevation from the base of an office building to the top of the tower is 34°. The angle of elevation from the helipad on the roof of the office building to the top of the tower is 20°.



(a) How far away is the office building from the Freedom Tower? Assume the side of the tower is vertical. Round to the nearest foot.

(b) How tall is the office building? Round to the nearest foot.

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77. Use a calculator set in radian mode to complete the following table. What can you conclude about the value of $f(\theta) = \dfrac{\sin\theta}{\theta}$ as θ approaches 0?

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$f(\theta) = \dfrac{\sin\theta}{\theta}$								

78. Use a calculator set in radian mode to complete the following table. What can you conclude about the value of $g(\theta) = \dfrac{\cos\theta - 1}{\theta}$ as θ approaches 0?

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$g(\theta) = \dfrac{\cos\theta - 1}{\theta}$								

79. Find the exact value of $\tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 89°$.

80. Find the exact value of $\cot 1° \cdot \cot 2° \cdot \cot 3° \cdot \ldots \cdot \cot 89°$.

81. Find the exact value of $\cos 1° \cdot \cos 2° \cdot \ldots \cdot \cos 45° \cdot \csc 46° \cdot \ldots \cdot \csc 89°$.

82. Find the exact value of $\sin 1° \cdot \sin 2° \cdot \ldots \cdot \sin 45° \cdot \sec 46° \cdot \ldots \cdot \sec 89°$.