8-3 Polynomials, Linear Factors, and Zeros

TEKS FOCUS

TEKS (7)(D) Determine the linear factors of a polynomial function of degree three and of degree four using algebraic methods.

TEKS (1)(G) Display, explain, and **justify** mathematical ideas and **arguments** using precise mathematical language in written or oral communication.

Additional TEKS (1)(A), (1)(D), (7)(B)

VOCABULARY

- **Multiple zero** – If a linear factor is repeated in the complete factored form of a polynomial, the zero related to that factor is a multiple zero.
- **Multiplicity** – The number of times the related linear factor is repeated in the factored form of the polynomial.
- **Relative maximum** – The value of the function at an up-to-down turning point.
- **Relative minimum** – The value of the function at a down-to-up turning point.
- **Root** – A root of an equation is a value that makes the equation true.

- **Argument** – a set of statements put forth to show the truth or falsehood of a mathematical claim
- **Justify** – explain with logical reasoning. You can justify a mathematical argument.

ESSENTIAL UNDERSTANDING

Finding the zeros of a polynomial function will help you factor the polynomial, graph the function, and solve the related polynomial equation.

take note — Key Concept: Roots, Zeros, and *x*-intercepts

The following are equivalent statements about a real number b and a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- b is a zero of the polynomial function $y = P(x)$.
- b is a root (or solution) of the polynomial equation $P(x) = 0$.
- b is an x-intercept of the graph of $y = P(x)$.

take note — Theorem: Factor Theorem

The expression $x - a$ is a factor of a polynomial if and only if the value a is a zero of the related polynomial function.

take note — Key Concept: How Multiple Zeros Affect a Graph

If a is a zero of multiplicity n in the polynomial function $y = P(x)$, then the behavior of the graph at the x-intercept a will be close to linear if $n = 1$, close to quadratic if $n = 2$, close to cubic if $n = 3$, and so on.

Problem 1

Plan

How do you write the factored form of a polynomial function? Write the polynomial function as a product of factors. Make sure each factor cannot be factored any further.

Writing a Polynomial Function in Factored Form

What is the factored form of $f(x) = x^3 - 2x^2 - 15x$?

	$x^3 - 2x^2 - 15x = x(x^2 - 2x - 15)$	Factor out the GCF, x.
	$= x(x - 5)(x + 3)$	Factor $x^2 - 2x - 15$.
Check	$x(x - 5)(x + 3) = x(x^2 - 2x - 15)$	Multiply $(x - 5)(x + 3)$.
	$= x^3 - 2x^2 - 15x$ ✔	Distributive Property

Problem 2

Finding Zeros of a Polynomial Function

What are the zeros of $f(x) = (x + 2)(x - 1)(x - 3)$? Graph the function.

Know	Need	Plan
Polynomial function	• Zeros • Additional points • End behavior	• Use the Zero-Product Property to find zeros. • Find points between the zeros. • Sketch the graph.

Think

Does knowing the zeros of a function give you enough information to sketch it? No; several different cubic functions could pass through (−2, 0), (1, 0), and (3, 0).

Step 1 Use the Zero-Product Property to find the zeros.

$$(x + 2)(x - 1)(x - 3) = 0$$

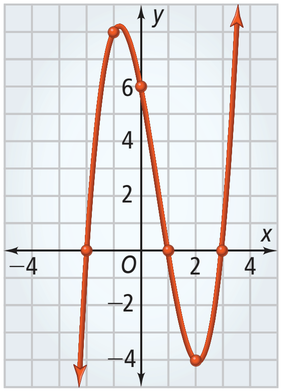
so $x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$.
The zeros of the function are −2, 1, and 3.

Step 2 Find points for x-values between the zeros.
Evaluate $f(x) = (x + 2)(x - 1)(x - 3)$ for $x = -1, 0$, and 2.

$(-1 + 2)(-1 - 1)(-1 - 3) = 8$ $(-1, 8)$

$(0 + 2)(0 - 1)(0 - 3) = 6$ $(0, 6)$

$(2 + 2)(2 - 1)(2 - 3) = -4$ $(2, -4)$



Step 3 Determine the end behavior.
The function $f(x) = (x + 2)(x - 1)(x - 3)$ is cubic. The coefficient of x^3 is +1, so the end behavior is *down and up*.

Step 4 Use the zeros: (−2, 0), (1, 0), (3, 0); the additional points: (−1, 8), (0, 6), (2, −4); and end behavior to sketch the graph.

Problem 3

TEKS Process Standard (1)(G)

Writing a Polynomial Function From Its Zeros

Plan

How can you use the zeros to find the function?
By the Factor Theorem, a is a zero means that $x - a$ is a factor of the related polynomial.

A **What is a cubic polynomial function in standard form with zeros -2, 2, and 3?**

$-2 \quad 2 \quad 3$ ↓ ↓ ↓	-2, 2, and 3 are zeros.
$f(x) = (x + 2)(x - 2)(x - 3)$	Write a linear factor for each zero.
$= (x + 2)(x^2 - 5x + 6)$	Multiply $(x-2)$ and $(x-3)$.
$= x(x^2 - 5x + 6) + 2(x^2 - 5x + 6)$	Distributive Property
$= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12$	Distributive Property
$= x^3 - 3x^2 - 4x + 12$	Simplify.

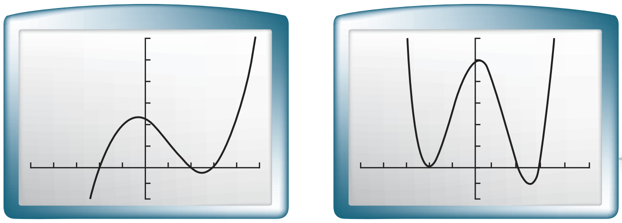
The cubic polynomial $f(x) = x^3 - 3x^2 - 4x + 12$ has zeros -2, 2, and 3.

B **What is a quartic polynomial function in standard form with zeros -2, -2, 2, and 3?**

$-2 \quad -2 \quad 2 \quad 3$ ↓ ↓ ↓ ↓	-2, -2, 2, and 3 are zeros.
$g(x) = (x + 2)(x + 2)(x - 2)(x - 3)$	Write a linear factor for each zero.
$= x^4 - x^3 - 10x^2 + 4x + 24$	Simplify.

The quartic polynomial $g(x) = x^4 - x^3 - 10x^2 + 4x + 24$ has zeros -2, -2, 2, and 3.

C **Graph both functions. How do the graphs differ? How are they similar?**



Both screens:
x-scale: 1
y-scale: 5

Both graphs have x-intercepts at -2, 2, and 3. The cubic has down-and-up end behavior with two turning points, and crosses the x-axis at -2. The quartic has up-and-up end behavior, three turning points, and touches the x-axis at -2 but does not cross it.

Problem 4

TEKS Process Standard (1)(D)

Finding the Multiplicity of a Zero

What is the factored form of $f(x) = x^4 - 2x^3 - 8x^2$? What are the zeros? What are the multiplicities of the zeros? How does the graph behave at these zeros?

Think

How can you find the multiplicities?
Factor the polynomial. Find the number of times each linear factor appears.

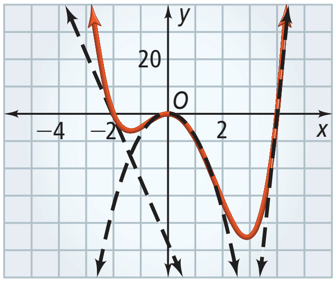
$f(x) = x^4 - 2x^3 - 8x^2$

$= x^2(x^2 - 2x - 8)$ Factor out the GCF, x^2.

$= x^2(x + 2)(x - 4)$ Factor $(x^2 - 2x - 8)$.

Since $x^2 = (x - 0)^2$, the number 0 is a zero of multiplicity 2. The numbers -2 and 4 are zeros of multiplicity 1.

The graph looks close to linear at the x-intercepts -2 and 4. It resembles a parabola at the x-intercept 0.



Problem 5

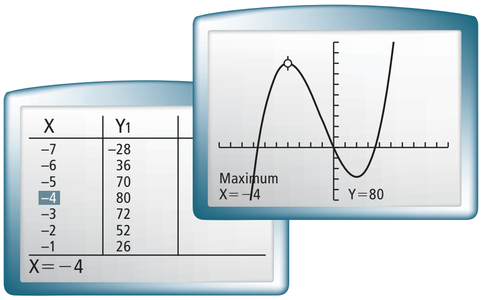
Identifying a Relative Maximum and Minimum

What are the relative maximum and minimum of $f(x) = x^3 + 3x^2 - 24x$?

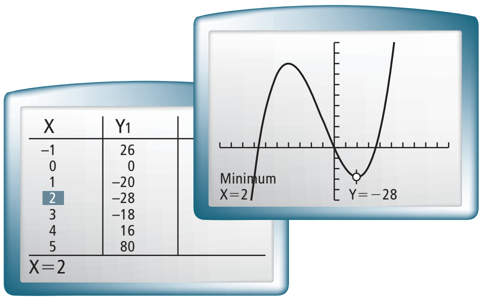
Think

How is a relative maximum different from a maximum at the vertex of a parabola?
A relative maximum is the greatest y-value in the "neighborhood" of its x-value. The maximum at the vertex of a parabola is the greatest y-value for *all* x-values.

Use a graphing calculator to find a relative maximum and a relative minimum.



Relative maximum



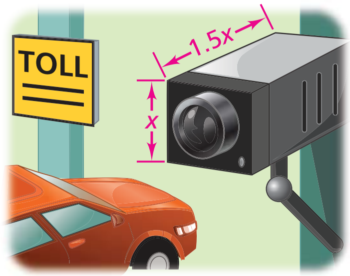
Relative minimum

The relative maximum is 80 at $x = -4$ and the relative minimum is -28 at $x = 2$.

Problem 6

Using a Polynomial Function to Maximize Volume

Technology **Digital box cameras connect to closed-circuit television and are used for security. Their design maximizes the volume while keeping the sum of the dimensions at 6 inches. If the length must be 1.5 times the height, what should each dimension be?**



Step 1 Define a variable x.

Let x = the height of the camera.

Step 2 Determine length and width.

length $= 1.5x$; width $= 6 - (x + 1.5x) = 6 - 2.5x$

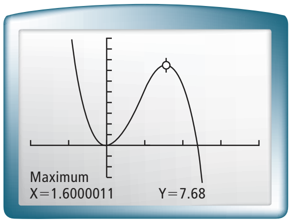
Think

What is the formula for the volume of a "box"?
$V = \ell wh$

Step 3 Model the volume.

$$V = (\text{length})(\text{width})(\text{height}) = (1.5x)(6 - 2.5x)(x)$$
$$= -3.75x^3 + 9x^2$$

Step 4 Graph the polynomial function. Use the **MAXIMUM** feature to find that the maximum volume is 7.68 in.^{3} for a height of 1.6 in.



height $= x = 1.6$

length $= 1.5x = 1.5(1.6) = 2.4$

width $= 6 - 2.5x = 6 - 2.5(1.6) = 2$

The dimensions of the camera should be 2.4 in. long by 2 in. wide by 1.6 in. high.

PRACTICE and APPLICATION EXERCISES

Scan page for a Virtual Nerd™ tutorial video.

For additional support when completing your homework, go to PearsonTEXAS.com.

Write each polynomial function in factored form. Check by multiplication.

1. $y = x^3 + 7x^2 + 10x$

2. $y = x^3 - 7x^2 - 18x$

3. $y = x^3 - 4x^2 - 21x$

4. $y = 3x^3 - 27x^2 + 24x$

5. $y = -2x^3 - 2x^2 + 40x$

6. $y = x^4 + 3x^3 - 4x^2$

Find the zeros of each function. Then graph the function.

7. $y = (x - 1)(x + 2)$

8. $y = (x - 2)(x + 9)$

9. $y = x(x + 5)(x - 8)$

10. $y = (x + 1)(x - 2)(x - 3)$

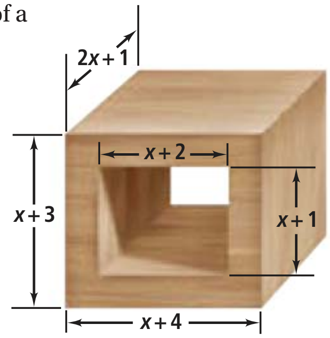
11. $y = (x + 1)(x - 1)(x - 2)$

12. $y = x(x + 2)(x + 3)$

13. **Apply Mathematics (1)(A)** A carpenter hollowed out the interior of a block of wood as shown at the right.

   a. Express the volume of the original block and the volume of the wood removed as polynomials in factored form.

   b. What polynomial represents the volume of the wood remaining?



Find the relative maximum and relative minimum of the graph of each function.

14. $f(x) = x^3 + 4x^2 - 5x$
15. $f(x) = -x^3 + 16x^2 - 76x + 96$
16. $f(x) = -4x^3 + 12x^2 + 4x - 12$
17. $f(x) = x^3 - 7x^2 + 7x + 15$

Find the zeros of each function. State the multiplicity of multiple zeros.

18. $y = (x + 3)^3$
19. $y = x(x - 1)^3$
20. $y = 2x^3 + x^2 - x$
21. $y = 3x^3 - 3x$
22. $y = (x - 4)^2$
23. $y = (x - 2)^2(x - 1)$
24. $y = (2x + 3)(x - 1)^2$
25. $y = (x + 1)^2(x - 1)(x - 2)$

26. **Use a Problem-Solving Model (1)(B)** A storage company needs to design a new storage box that has twice the volume of its largest box. Its largest box is 5 ft long, 4 ft wide, and 3 ft high. The new box must be formed by increasing each dimension by the same amount. Find the increase in each dimension.

Write a polynomial function in standard form with the given zeros.

27. $x = 5, 6, 7$
28. $x = -2, 0, 1$
29. $x = -5, -5, 1$
30. $x = 3, 3, 3$
31. $x = 1, -1, -2$
32. $x = 0, 4, -\frac{1}{2}$
33. $x = 0, 0, 2, 3$
34. $x = -1, -2, -3, -4$

35. Write a polynomial function with the following features: it has three distinct zeros; one of the zeros is 1; another zero has a multiplicity of 2.

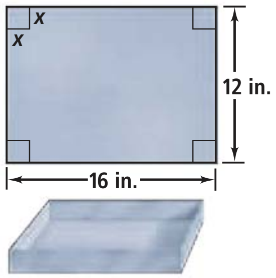
36. **Explain Mathematical Ideas (1)(G)** Explain how the graph of a polynomial function can help you factor the polynomial.

37. **Apply Mathematics (1)(A)** A metalworker wants to make an open box from a sheet of metal, by cutting equal squares from each corner as shown.

   a. Write expressions for the length, width, and height of the open box.

   b. Use your expressions from part (a) to write a function for the volume of the box. (*Hint:* Write the function in factored form.)

   c. Graph the function. Then find the maximum volume of the box and the side length of the cut-out squares that generates this volume.



38. **Apply Mathematics (1)(A)** A rectangular box is $2x + 3$ units long, $2x - 3$ units wide, and $3x$ units high. What is its volume, expressed as a polynomial?

39. **Apply Mathematics (1)(A)** The volume in cubic feet of a CD holder can be expressed as $V(x) = -x^3 - x^2 + 6x$, or, when factored, as the product of its three dimensions. The depth is expressed as $2 - x$. Assume that the height is greater than the width.

    a. Factor the polynomial to find linear expressions for the height and the width.

    b. Graph the function. Find the x-intercepts. What do they represent?

    c. What is a realistic domain for the function?

    d. What is the maximum volume of the CD holder?

40. Find a fourth-degree polynomial function with zeros 1, -1, i, and $-i$. Write the function in factored form.

41. **a.** Compare the graphs of $y = (x + 1)(x + 2)(x + 3)$ and $y = (x - 1)(x - 2)(x - 3)$. What transformation could you use to describe the change from one graph to the other?

    b. Compare the graphs of $y = (x + 1)(x + 3)(x + 7)$ and $y = (x - 1)(x - 3)(x - 7)$. Does the transformation that you chose in part (a) still hold true? Explain.

    c. What transformation could you use to describe the effect of changing the signs of the zeros of a polynomial function?

TEXAS Test Practice

42. The three most frequent letters in the English language are E, T, and A. They represent, on average, 30% of all letters. The most frequent letter, E, is 4% more frequent than the second most frequent letter, T. The combined frequency of T and A is 4% greater than the frequency of E. Approximately how many E's can you expect to encounter in a 500-letter paragraph?

    A. 49 **B.** 65 **C.** 72 **D.** 88

43. Which expression is the factored form of $x^3 + 2x^2 - 5x - 6$?

    F. $(x + 1)(x + 1)(x - 6)$

    G. $(x + 3)(x + 1)(x - 2)$

    H. $(x + 2)(2x - 5)(x - 6)$

    J. $(x - 3)(x - 1)(x + 2)$

44. A ball with a 3 in. radius has volume V_1. A second ball has a 9 in. radius and volume V_2. Which equation represents the volume of the second ball in terms of the first?

    A. $V_2 = 3V_1$ **B.** $V_2 = 27V_1$ **C.** $V_2 = {V_1}^2$ **D.** $V_2 = 9{V_1}^2$

45. What is the polynomial function, in factored form, whose zeros are -2, 5, and 6, and whose leading coefficient is -2? Graph this function and find any relative minimums or maximums.