8: The Black-Scholes Model

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Outline

We will examine the following issues:

1. The Wiener Process and its Properties
2. The Black-Scholes Market Model
3. The Black-Scholes Call Option Pricing Formula
4. The Black-Scholes Partial Differential Equation
5. Random Walk Approximations

PART 1

THE WIENER PROCESS AND ITS PROPERTIES

The Origin of the Wiener Process

- The **Brownian motion** is a mathematical model used to describe the random mouvements of particles. It was named after Scottish botanist Robert Brown (1773-1858) who has published in 1827 a paper in which the chaotic mouvements of pollen suspended in water were examined.
- The Brownian motion was used by Louis Bachelier in his PhD thesis completed in 1900 and devoted to pricing of options.
- The Brownian motion was also used by physicists to describe the diffusion mouvements of particles, in particular, by Albert Einstein (1879-1955) in his famous paper published in 1905.
- The Brownian motion is also known as the **Wiener process** in honour of the famous American mathematician Norbert Wiener (1894-1964).
- The Brownian motion is nowadays widely used to model uncertainty in engineering, economics and finance.

Wiener Process: Definition

Definition (Wiener Process)

A stochastic process $W = (W_t,\ t \in \mathbb{R}_+)$ is called the **Wiener process** (or the **standard Brownian motion**) if the following conditions hold:

1. $W_0 = 0$.
2. Sample paths of the process W, that is, the maps $t \to W_t(\omega)$ are continuous functions.
3. The process W has the Gaussian (i.e. normal) distribution with the expected value $\mathbb{E}_{\mathbb{P}}(W_t) = 0$ for all $t \geq 0$ and the covariance

$$Cov\,(W_s, W_t) = \min\,(s, t), \quad s, t \geq 0.$$

Wiener Process: Equivalent Definition

Definition (Wiener Process: Equivalent Definition)

A stochastic process $W = (W_t,\ t \in \mathbb{R}_+)$ on Ω is called the **Wiener process** if the following conditions hold:

1. $W_0 = 0$.
2. Sample paths of W are continuous functions.
3. For any $0 \leq s < t$, $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.
4. For any $0 \leq t_1 < t_2 < \cdots < t_n$,

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are mutually independent.

Existence of the Wiener Process

- The existence of a stochastic process satisfying the definition of a Wiener process is not obvious.
- The following theorem was first rigorously established by Norbert Wiener in his paper published in 1923.

Theorem (Wiener's Theorem)

There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process W defined on this space, such that conditions 1)-3) of the definition of the Wiener process are met.

- It is known that almost all sample paths of the Wiener process are continuous functions of the time parameter, but they are non-differentiable everywhere. This striking feature makes the Wiener process rather difficult to analyse.

Wiener Process: Sample Paths

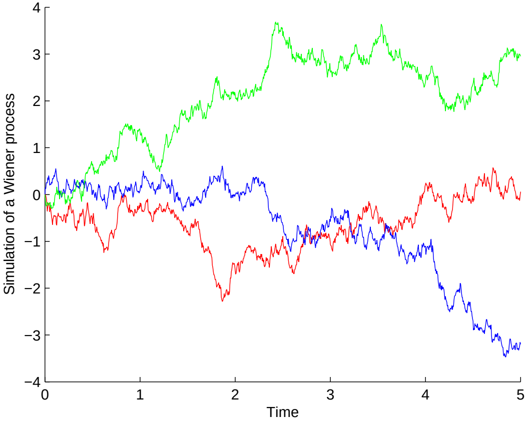

Figure: Three sample paths of a Wiener process with $\Delta t = 0.005$

Gaussian Distribution

Remark (Gaussian Distribution)

- *We say that X has the* **Gaussian (normal) distribution** *with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ if its pdf equals*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.$$

  We write $X \sim N(\mu, \sigma^2)$.

- *One can show that*

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1.$$

- *We have*

$$\mathbb{E}_{\mathbb{P}}(X) = \mu \quad \text{and} \quad Var(X) = \sigma^2.$$

Standard Normal Distribution

Remark (Standard Normal Distribution)

- *If we set* $\mu = 0$ *and* $\sigma^2 = 1$ *then we obtain the* **standard normal distribution** $N(0, 1)$ *with the following pdf*

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } x \in \mathbb{R}.$$

- *The cdf of the probability distribution* $N(0, 1)$ *equals*

$$N(x) = \int_{-\infty}^{x} n(u)\, du = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \quad \text{for } x \in \mathbb{R}.$$

- *The values of* $N(x)$ *can be found in the* **cumulative standard normal table** *(also known as the* **Z table***).*
- *If* $X \sim N(\mu, \sigma^2)$ *then* $Z := \frac{X-\mu}{\sigma} \sim N(0, 1)$.

Marginal Distributions of the Wiener Process

- Let $N(\mu, \sigma^2)$ denote the Gaussian (normal) distribution with mean μ and variance σ^2.
- For any $t > 0$, $W_t \sim N(0, t)$ and thus $(\sqrt{t})^{-1} W_t \sim N(0, 1)$.
- The random variable W_t has the pdf $p(x, t)$ given by

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \text{for } x \in \mathbb{R}.$$

- Hence for any real numbers $a \leq b$

$$\begin{aligned} \mathbb{P}(W_t \in [a, b]) &= \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx = \int_{\frac{a}{\sqrt{t}}}^{\frac{b}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \\ &= \int_{\frac{a}{\sqrt{t}}}^{\frac{b}{\sqrt{t}}} n(x) \, dx = N\left(\frac{b}{\sqrt{t}}\right) - N\left(\frac{a}{\sqrt{t}}\right). \end{aligned}$$

Proposition (8.1)

The Wiener process W is a Markov process in the following sense: for every $n \geq 1$, any sequence of times $0 < t_1 < \ldots < t_n < t$ and any real numbers $x_1, \ldots, x_n$, the following holds for all $x \in \mathbb{R}$

$$\mathbb{P}\left(W_t \leq x \middle| W_{t_1} = x_1, \ldots, W_{t_n} = x_n \right) = \mathbb{P}\left(W_t \leq x \middle| W_{t_n} = x_n \right).$$

Moreover, for all $s < t$ and $x, y \in \mathbb{R}$ we have

$$\mathbb{P}\left(W_t \leq y \middle| W_s = x \right) = \int_{-\infty}^{y} p(t-s, z-x)\, dz$$

where

$$p(t-s, z-x) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(z-x)^2}{2(t-s)} \right)$$

is the transition probability density function of the Wiener process.

Proposition (8.2)

Let W be the Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the process W is a martingale with respect to its natural filtration $\mathcal{F}_t = \mathcal{F}_t^W$, that is, the filtration generated by W.

Proof of Proposition 8.2.

For all $0 \leq s < t$, using the independence of increments of the Wiener process W, we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(W_t \,|\, \mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}\big((W_t - W_s) + W_s \,|\, \mathcal{F}_s\big) \\ &= \mathbb{E}_{\mathbb{P}}(W_t - W_s) + W_s \\ &= W_s. \end{aligned}$$

We conclude that W is a martingale with respect to its natural filtration. □

PART 2

THE BLACK-SCHOLES MARKET MODEL

Stock Price Process

- We note that the values of the Wiener process W can be negative and thus it cannot be used to directly model the movements of the stock price.
- Following Samuelson (1965) and Black and Scholes (1973), we postulate that the stock price process S is governed under the risk-neutral probability measure $\widetilde{\mathbb{P}}$ by the following **stochastic differential equation** (**SDE**)

$$dS_t = r\,S_t\,dt + \sigma S_t\,dW_t \tag{1}$$

  with a constant initial value $S_0 > 0$.
- The term $\sigma S_t\,dW_t$ is aimed to give a plausible description of the uncertainty of the stock price.
- The **volatility** parameter $\sigma > 0$ is used to control the size of random fluctuations of the stock price.

Stochastic Differential Equation

- Sample path of the Wiener process W are not differentiable so that equation (1) cannot be represented as

$$dS_t = r\, S_t\, dt + \sigma S_t W_t'\, dt.$$

- It should be understood as the **stochastic integral equation**

$$S_t = S_0 + \int_0^t rS_u\, du + \int_0^t \sigma S_u\, dW_u$$

  where the second integral is the Itō stochastic integral.
- The Itō stochastic integration theory, which extends the classic integrals and underpins financial modelling in continuous time, is beyond the scope of this course.

- It turns out that stochastic differential equation (1) can be solved explicitly yielding the unique solution

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{1}{2}\sigma^2\right)t\right). \tag{2}$$

  The process S is called the **geometric Brownian motion**.
- Note that S_t has the **lognormal** distribution for every $t > 0$.
- It can be shown that S is a Markov process. Note, however, that S is not a process of independent increments.
- We assume that the continuously compounded interest rate r is constant. Hence the **savings account** equals

$$B_t = B_0 e^{rt}, \quad t \geq 0,$$

  where $B_0 = 1$. Hence $dB_t = rB_t\,dt$ for $t \geq 0$.

Sample Paths of Stock Price

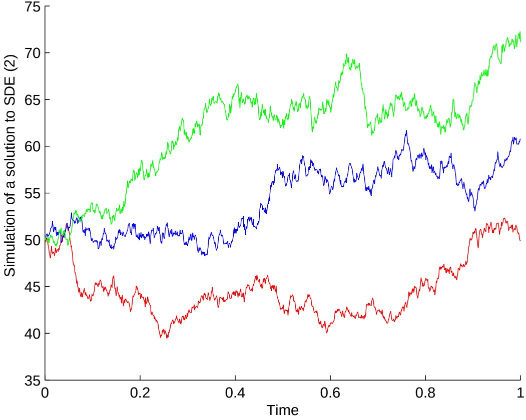

Figure: Three sample paths of the stock price with $r = 10\%$, $\sigma = 0.2$ and $\Delta t = 0.001$

The Black-Scholes Model $\mathcal{M} = (B, S)$

Assumptions of the Black-Scholes market model $\mathcal{M} = (B, S)$:

- There are no arbitrage opportunities in the class of trading strategies.
- It is possible to borrow or lend any amount of cash at a constant interest rate $r \geq 0$.
- The stock price dynamics are governed by a geometric Brownian motion.
- It is possible to purchase any amount of a stock and short-selling is allowed.
- The market is frictionless: there are no transaction costs (or any other costs).
- The underlying stock does not pay any dividends.

Discounted Stock Price (MATH3975)

As in a multi-period market model, the discounted stock price $\hat{S}$ is a martingale.

Proposition (8.3)

The discounted stock price, that is, the process $\hat{S}$ given by the formula

$$\hat{S}_t = \frac{S_t}{B_t} = e^{-rt} S_t$$

is a martingale with respect to its natural filtration under $\widetilde{\mathbb{P}}$, that is, for every $0 \leq s \leq t$,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\ u \leq s\big) = \hat{S}_s.$$

Proof of Proposition 8.3.

- We observe that equality (2) yields

$$\hat{S}_t = S_0\, e^{\sigma W_t - \frac{1}{2}\sigma^2 t} = \hat{S}_s\, e^{\sigma (W_t - W_s) - \frac{1}{2}\sigma^2 (t-s)}. \tag{3}$$

- Hence if we know $\hat{S}_t$ then we also know the value of W_t and vice versa. This immediately implies that $\mathbb{F}^{\hat{S}} = \mathbb{F}^W$.
- Therefore, the following conditional expectations coincide

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(X \,\big|\, \hat{S}_u,\, u \leq s\big) = \mathbb{E}_{\widetilde{\mathbb{P}}}\big(\, X \big|\, W_u,\, u \leq s\big) \tag{4}$$

for any integrable random variable X

□

Proof of Proposition 8.3 (Continued).

- We obtain the following chain of equalities

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\ u \leq s\big) \\
&= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\hat{S}_s\, e^{\sigma\left(W_t - W_s - \frac{1}{2}\sigma^2(t-s)\right)} \,\Big|\, \hat{S}_u,\ u \leq s\right) && \text{(from (3))} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,\Big|\, \hat{S}_u,\ u \leq s\right) && \text{(conditioning)} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,\Big|\, W_u,\ u \leq s\right) && \text{(from (4))} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)}\right). && \text{(independence)}
\end{aligned}
$$

- It remains to compute the expected value above.

□

Proof of Proposition 8.3 (Continued).

- Recall also that $W_t - W_s = \sqrt{t-s}\, Z$ where $Z \sim N(0,1)$, and thus

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\ u \le s\big) = \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{\sigma\sqrt{t-s}Z}\big).$$

- Let us finally observe that if $Z \sim N(0,1)$ then for any real a

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{aZ}\big) = e^{a^2/2}.$$

- By setting $a = \sigma\sqrt{t-s}$, we finally obtain

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\ u \le s\big) = \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, e^{\frac{1}{2}\sigma^2(t-s)} = \hat{S}_s$$

which shows that $\hat{S}$ is indeed a martingale.

□

PART 3

THE BLACK-SCHOLES CALL OPTION PRICING FORMULA

Call Option

- Recall that the **European call option** written on the stock is a traded security, which pays at its maturity T the random amount
$$C_T = (S_T - K)^+$$
where $x^+ = \max(x, 0)$ and $K > 0$ is a fixed strike.
- We take for granted that for $t \leq T$ the price $C_t(x)$ of the call option when $S_t = x$ is given by the **risk-neutral pricing formula**
$$C_t(x) = e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left((S_T - K)^+ \,\middle|\, S_t = x\right).$$
- This formula can be supported by the replication principle. However, this argument requires the knowledge of the Itō **stochastic integration** theory with respect to the Brownian motion, as was developed by Kiyoshi Itō (1944).

The Black-Scholes Call Pricing Formula

- The following call option pricing result was established in the seminal paper by Black and Scholes (1973).

Theorem (8.1)

The arbitrage price of the call option at time $t \leq T$ equals

$$C_t(S_t) = S_t N\big(d_+(S_t, T-t)\big) - Ke^{-r(T-t)} N\big(d_-(S_t, T-t)\big)$$

where

$$d_\pm(S_t, T-t) = \frac{\ln \frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and N is the standard normal cumulative distribution function.

Proof of Theorem 8.1.

- Our goal is to compute the conditional expectation

$$C_t(x) = e^{-r(T-t)} \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left((S_T - K)^+ \middle| \, S_t = x \right).$$

- We can represent the stock price S_T as follows

$$S_T = S_t \, e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)}.$$

- As in the proof of Proposition 8.3, we write

$$W_T - W_t = \sqrt{T-t} Z$$

where Z has the standard Gaussian probability distribution, that is, $Z \sim N(0,1)$.

□

Proof of Theorem 8.1 (Continued).

- Using the independence of increments of the Wiener process W, we obtain, for a generic value $x > 0$ of the stock price S_t at time t

$$C_t(x) = e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\left(S_t\,e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(W_T-W_t)} - K\right)^+ \,\Big|\, S_t = x\right)$$

$$= e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(x\,e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}Z} - K\right)^+$$

$$= e^{-r(T-t)}\int_{-\infty}^{\infty}\left(x\,e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right)^+ n(z)\,dz$$

- We denote here by n the pdf of Z, that is, the standard normal probability density function.

□

Proof of Theorem 8.1 (Continued).

- It is clear that the function under the integral sign is non-zero if and only if the following inequality holds

$$x\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K \geq 0.$$

- This in turn is equivalent to the following inequality

$$z \geq \frac{\ln \frac{K}{x} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = -d_-(x, T-t).$$

- Let us denote

$$d_+ = d_+(x, T-t), \quad d_- = d_-(x, T-t).$$

□

Proof of Theorem 8.1 (Continued).

$$
\begin{aligned}
C_t(x) &= e^{-r(T-t)} \int_{-d_-}^{\infty} \left(x\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K \right) n(z)\, dz \\
&= x\, e^{-\frac{1}{2}\sigma^2(T-t)} \int_{-d_-}^{\infty} e^{\sigma\sqrt{T-t}z} n(z)\, dz - Ke^{-r(T-t)} \int_{-d_-}^{\infty} n(z)\, dz \\
&= x\, e^{-\frac{1}{2}\sigma^2(T-t)} \int_{-d_-}^{\infty} e^{\sigma\sqrt{T-t}z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, dz - Ke^{-r(T-t)} N(d_-) \\
&= x \int_{-d_-}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(z-\sigma\sqrt{T-t}\right)^2}\, dz - Ke^{-r(T-t)} N(d_-) \\
&= x \int_{-d_- -\sigma\sqrt{T-t}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du - Ke^{-r(T-t)} N(d_-) \\
&= x \int_{-d_+}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du - Ke^{-r(T-t)} N(d_-) \\
&= xN(d_+) - Ke^{-r(T-t)} N(d_-).
\end{aligned}
$$

Put-Call Parity

- The price of the put option can be computed from the usual put-call parity

$$C_t - P_t = S_t - Ke^{-r(T-t)} = S_t - KB(t, T).$$

- Specifically, the put option price equals

$$P_t = Ke^{-r(T-t)}N\big(-d_-(S_t, T-t)\big) - S_t N\big(-d_+(S_t, T-t)\big).$$

- It is worth noting that $C_t > 0$ and $P_t > 0$.
- It can also be checked that the prices of call and put options are increasing functions of the volatility parameter σ (if all other quantities are fixed). Hence the options become more expensive when the underlying stock becomes more risky.
- The price of a call (put) option is an increasing (decreasing) function of the interest rate r.

Example: Call Option

Example (8.1)

- Suppose that the current stock price equals \$31, the stock price volatility is $\sigma = 10\%$ per annum, and the risk-free rate is $r = 5\%$ per annum with continuous compounding.
- Consider a call option on the stock S, with strike price \$30 and with 3 months to expiry. We may assume that $t = 0$ and $T = 0.25$. We obtain $d_+(S_0, T) = 0.93$ and thus

$$d_-(S_0, T) = d_+(S_0, T) - \sigma\sqrt{T} = 0.88.$$

- The Black-Scholes call option pricing formula yields (approximately)

$$C_0 = 31N(0.93) - 30e^{-0.05/4}N(0.88) = 25.42 - 23.9 = 1.52$$

since $N(0.93) \approx 0.82$ and $N(0.88) \approx 0.81$.

Replicating Strategy

Example (8.1 Continued)

- Let $C_t = \phi_t^0 B_t + \phi_t^1 S_t$. The hedge ratio for the call option is known to be given by the formula

$$\phi_t^1 = N(d_+(S_t, T - t)).$$

- Hence the replicating portfolio at time $t = 0$ is given by

$$\phi_0^0 = -23.9, \quad \phi_0^1 = N(d_+(S_0, T)) = 0.82.$$

- This means that to hedge a short position in the call option, which was sold at the arbitrage price $C_0 = \$1.52$, the writer needs to buy at time 0 the number $\delta = 0.82$ shares of stock.
- The purchase of shares requires an additional borrowing of 23.9 units of cash.

Elasticity of the Call Price

Example (8.1 Continued)

- The **elasticity** at time 0 of the call option price with respect to the stock price equals

$$\eta_0^c := \frac{\partial C}{\partial S}\left(\frac{C_0}{S_0}\right)^{-1} = \frac{N\big(d_+(S_0, T)\big)S_0}{C_0} = 16.72.$$

- Suppose that the stock price rises immediately from \$31 to \$31.2, yielding a return rate of 0.65% flat.
- Then the option price will move by approximately 16.5 cents from \$1.52 to \$1.685, giving a return rate of 10.86% flat.
- The option has nearly 17 times the return rate of the stock; this also means that it will drop 17 times as fast.

Example: Put Option

Example (8.1 Continued)

- We now assume that an option is a put. The price of the put option at time 0 equals

$$P_0 = 30e^{-0.05/4}N(-0.88) - 31N(-0.93) = 5.73 - 5.58 = 0.15$$

- The hedge ratio corresponding to a short position in the put option equals approximately -0.18 (since $N(-0.93) \approx 0.18$).
- Therefore, to hedge the exposure an investor needs to short 0.18 shares of stock for one put option. The proceeds from the option and share-selling transactions, which amount to \$5.73, should be invested in risk-free bonds.
- Notice that the elasticity of the put option is several times larger than the elasticity of the call option. For instance, if the stock price rises immediately from \$31 to \$31.2 then the price of the put option will drop to less than 12 cents.

PART 4

THE BLACK-SCHOLES PDE

The Black-Scholes PDE

Proposition (8.4)

Consider a path-independent contingent claim $X = g(S_T)$. *Let the price of the contingent claim at* t *given the current stock price* $S_t = s$ *be denoted by* $v(s,t)$. *Then* $v(s,t)$ *is the solution of the* **Black-Scholes partial differential equation**

$$\frac{\partial}{\partial t}v(s,t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2}v(s,t) + rs\frac{\partial}{\partial s}v(s,t) - rv(s,t) = 0$$

with the terminal condition $v(s,T) = g(s)$.

Proof of Proposition 8.4.

The statement is an immediate consequence of the risk-neutral valuation formula and the classic Feynman-Kac formula. □

Sensitivities of the Call Price

- It can be checked that arbitrage prices of call and put options satisfy this PDE.
- We denote by $c(s, \tau)$ the function $c : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ such that $C_t = c(S_t, T - t)$. Then

$$\begin{aligned}
c_s &= N(d_+) = \delta > 0, \\
c_{ss} &= \frac{n(d_+)}{s\sigma\sqrt{\tau}} = \gamma > 0, \\
c_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_+) + Kre^{-r\tau} N(d_-) = \theta > 0, \\
c_\sigma &= s\sqrt{\tau} n(d_+) = \lambda > 0, \\
c_r &= \tau K e^{-r\tau} N(d_-) = \rho > 0, \\
c_K &= -e^{-r\tau} N(d_-) < 0,
\end{aligned}$$

where $d_+ = d_+(s, \tau)$, $d_- = d_-(s, \tau)$ and n stands for the standard Gaussian probability density function.

Sensitivities of the Put Price

- We denote by $p(s,\tau)$ the function $p : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ such that $P_t = p(S_t, T - t)$. Then

$$
\begin{aligned}
p_s &= N(d_+) - 1 = -N(-d_+) = \delta < 0, \\
p_{ss} &= \frac{n(d_+)}{s\sigma\sqrt{\tau}} = \gamma > 0, \\
p_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_+) + Kre^{-r\tau}(N(d_-) - 1) = \theta, \\
p_\sigma &= s\sqrt{\tau} n(d_+) = \lambda > 0, \\
p_r &= \tau K e^{-r\tau}(N(d_-) - 1) = \rho < 0, \\
p_K &= e^{-r\tau}(1 - N(d_-)) > 0.
\end{aligned}
$$

where $d_+ = d_+(s,\tau)$, $d_- = d_-(s,\tau)$ and n stands for the standard Gaussian probability density function.

PART 5

RANDOM WALK APPROXIMATIONS

Random Walk Approximations

- Our final goal is to examine a judicious approximation of the Black-Scholes model by a sequence of CRR models.
- In the first step, we will first examine an approximation of the Wiener process by a sequence of symmetric random walks.
- In the next step, we will use this result in order to show how to approximate the Black-Scholes stock price process by a sequence of the CRR stock price models.
- We will also recognise that the proposed approximation of the stock price leads to the Jarrow-Rudd parametrisation of the CRR model in terms of the short term rate r and the stock price volatility σ.

Symmetric Random Walk

Definition (Symmetric Random Walk)

A process $Y = (Y_k,\ k = 0, 1, \dots)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the **symmetric random walk** starting at zero if $Y_0 = 0$ and $Y_k = \sum_{i=1}^{k} X_i$ where the random variables $X_1, X_2, \dots$ are independent with the following common probability distribution

$$\mathbb{P}(X_i = 1) = 0.5 = \mathbb{P}(X_i = -1).$$

The **scaled random walk** Y^h is obtained from Y as follows: we fix $h = \sqrt{\Delta t}$ and for every $k = 0, 1, \dots$ we set

$$Y^h_{k\Delta t} = \sqrt{\Delta t}\, Y_k = \sum_{i=1}^{k} \sqrt{\Delta t} X_i$$

Of course, for $h = \sqrt{\Delta t} = 1$ we obtain $Y^h_{k\Delta t} = Y^1_k = Y_k$.

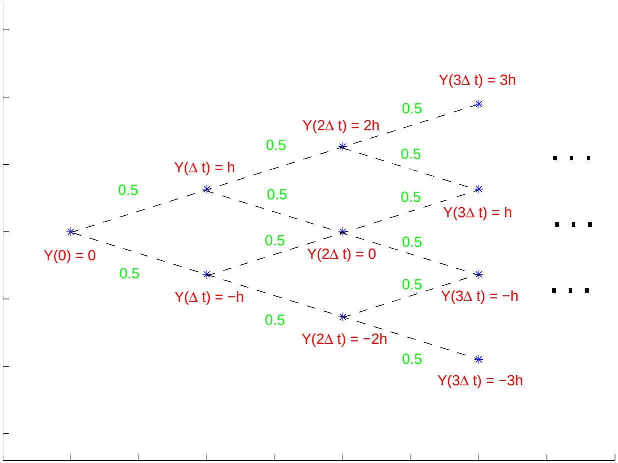

Figure: Representation of the scaled random walk Y^h

Approximation of the Wiener Process

- The following result is an easy consequence of the classic Central Limit Theorem (CLT) for sequences of independent and identically distributed (i.i.d.) random variables.

Theorem (8.2)

Let Y_t^h for $t = 0, \Delta t, \dots,$ be a random walk starting at 0 with increments $\pm h = \pm\sqrt{\Delta t}$. If

$$\mathbb{P}\left(Y_{t+\Delta t}^h = y + h \,\middle|\, Y_t^h = y\right) = \mathbb{P}\left(Y_{t+\Delta t}^h = y - h \,\middle|\, Y_t^h = y\right) = 0.5$$

then, for any fixed $t \geq 0$, the limit $\lim_{h\to 0} Y_t^h$ exists in the sense of probability distribution. Specifically, $\lim_{h\to 0} Y_t^h \sim W_t$ where W is the Wiener process and $\sim$ denotes the equality of probability distributions. In other words, $\lim_{h\to 0} Y_t^h \sim N(0, t)$.

Proof of Theorem 8.2.

- We fix $t > 0$ and we set $k = t/\Delta t$. Hence if $\Delta t \to 0$ then $k \to \infty$. We recall that $h = \sqrt{\Delta t}$ and

$$Y^h_{k\Delta t} = \sum_{i=1}^{k} \sqrt{\Delta t} X_i.$$

- Since $\mathbb{E}_{\mathbb{P}}(X_i) = 0$ and $Var\,(X_i) = \mathbb{E}_{\mathbb{P}}(X_i^2) = 1$, we obtain

$$\mathbb{E}_{\mathbb{P}}(Y^h_{k\Delta t}) = \sqrt{\Delta t} \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}}(X_i) = 0$$

$$Var\,(Y^h_{k\Delta t}) = \sum_{i=1}^{k} \Delta t \, Var\,(X_i) = \sum_{i=1}^{k} \Delta t = k\Delta t = t.$$

- Hence the statement follows from the (slightly extended) CLT.

□

Central Limit Theorem (CLT)

Theorem (Central Limit Theorem)

Assume that $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean μ and variance $\sigma^2 > 0$. Then for all real x

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le x \right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = N(x).$$

Note that if we denote $Y_n = \sum_{i=1}^{n} X_i$ then

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{Y_n - \mathbb{E}_{\mathbb{P}}(Y_n)}{\sqrt{Var\,(Y_n)}}.$$

Approximation of the Wiener Process

The sequence of random walks Y^h approximates the Wiener process W when $h = \sqrt{\Delta t} \to 0$ meaning that:

- For any fixed $t \geq 0$, the convergence $\lim_{h \to 0} Y_t^h \sim W_t$ holds, where $\sim$ denotes the equality of probability distributions on $\mathbb{R}$. This follows from Theorem 8.2.
- For any fixed n and any dates $0 \leq t_1 < t_2 < \cdots < t_n$, we have

$$\lim_{h \to 0} (Y_{t_1}^h, \ldots, Y_{t_n}^h) \sim (W_{t_1}, \ldots, W_{t_n})$$

  where $\sim$ denotes the equality of probability distributions on the space $\mathbb{R}^n$. This can be proven by extending Theorem 8.2.
- The sequence of linear versions of the random walk processes Y^h converge to a continuous time process W in the sense of the weak convergence of stochastic processes on the space of continuous functions (this is due to Donsker (1951)).

Approximation of the Stock Price

- Recall that the JR parameterisation for the CRR binomial model postulates that

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \text{ and } d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}},$$

  whereas under the CRR convention we set $u = e^{\sigma\sqrt{\Delta t}} = 1/d$.
- We will show that it corresponds to a particular approximation of the stock price process S
- Recall that for all $0 \leq s \leq t$

$$\begin{aligned} S_t &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t} \\ &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)s + \sigma W_s} \, e^{\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W_t - W_s)} \qquad (5) \\ &= S_s e^{\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W_t - W_s)}. \end{aligned}$$

Approximation of the Stock Price

- Let us set $t - s = \Delta t$ and let us replace the Wiener process W by the random walk Y^h in equation (5). Then

$$W_{t+\Delta t} - W_t \approx Y^h_{t+\Delta t} - Y^h_t = \pm h = \pm\sqrt{\Delta t}.$$

- Consequently, we obtain the following approximation

$$S_{t+\Delta t} \approx \begin{cases} S_t e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} & \text{if the price increases,} \\ S_t e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} & \text{if the price decreases.} \end{cases}$$

- More explicitly, for $k = 0, 1, \dots$

$$S^h_{k\Delta t} = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)k\Delta t + \sigma Y^h_{k\Delta t}}.$$

If we denote $t = k\Delta t$ then

$$S^h_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Y^h_t}.$$

Jarrow-Rudd Parametrisation

- We observe that this approximation of the stock price process leads to the Jarrow-Rudd parameterisation

$$u = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

- The convergence of the sequence of random walks Y^h to the Wiener process W (Donsker's Theorem) implies that the sequence S^h of CRR stock price models converges to the Black-Scholes stock price model S.
- The convergence of S^h to the stock price process S justifies the claim that the JR parametrisation is more suitable than the CRR method.
- This is especially important when dealing with valuation and hedging of path-dependent and American contingent claims.

THE END