9.3 Perform Reflections

Obj.: Reflect a figure in any given line.

Key Vocabulary

• **Line of reflection -** A ***reflection*** is a transformation that uses a line like a **mirror** to reflect an image. The mirror **line** is called the **line of reflection**.

• **Reflection -** A **reflection** uses a *line of reflection* to create a **mirror** image of the **original** figure.

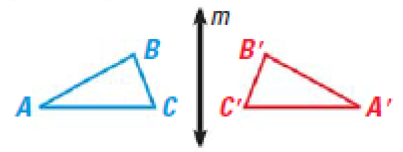
*******KEY CONCEPT*******

Coordinate Rules for Reflections

• If (a, b) is reflected in the ***x*-axis**, its image is the point **(a, -b)**.
• If (a, b) is reflected in the ***y*-axis**, its image is the point **(-a, b)**.
• If (a, b) is reflected in the line ***y = x***, its image is the point **(b, a)**.
• If (a, b) is reflected in the line ***y = -x***, its image is the point **(-b, -a)**.

Reflection Theorem **Reflect Thm.**

A reflection is an **isometry**.



▲ABC ≅ ▲A'B'C'

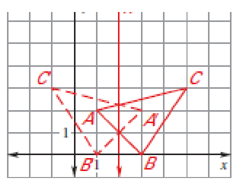
EXAMPLE 1 Graph reflections in horizontal and vertical lines

The vertices of $\triangle ABC$ are $A(1, 2)$, $B(3, 0)$, and $C(5, 3)$. Graph the reflection of $\triangle ABC$ described.

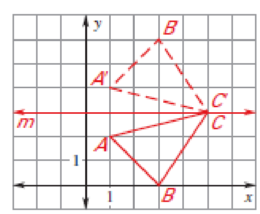
a. In the line n: $x = 2$ b. In the line m: $y = 3$

Solution

a. Point A is 1 unit left of n, so its reflection A' is 1 unit right of n at (3 , 2). Also, B' is 1 unit left of n at (1 , 0), and C' is 3 units left of n at (−1 , 3).



b. Point A is 1 unit below m, so A' is 1 unit above m at (1 , 4). Also, B' is 3 units above m at (3 , 6). Because point C is on line m, you know that C = C' .



EXAMPLE 2 Graph a reflection in *y* = *x*

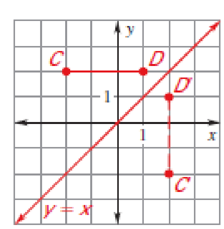
The endpoints of $\overline{CD}$ are C(−2, 2) and D(1, 2). Reflect the segment in the line y = x. Graph the segment and its image.

The product of the slopes of perpendicular lines is −1.

Solution

The slope of $y = x$ is 1 . The segment from C to its image, $\overline{CC'}$, is perpendicular to the line of reflection $y = x$, so the slope of $\overline{CC'}$ will be −1 (because $1(-1) =$ −1). From C, move 2 units right and 2 units down to $y = x$. From that point, move 2 units right and 2 units down to locate C'(2 , −2).

The slope of $\overline{DD'}$ will also be −1 . From D, move 0.5 units right and 0.5 units down to $y = x$. Then move 0.5 units right and 0.5 units down to locate D'(2 , 1).



EXAMPLE 3 Graph a reflection in $y = -x$

(cont. 9.3)

Reflect $\overline{CD}$ from Example 2 in the line $y = -x$. Graph $\overline{CD}$ and its image.

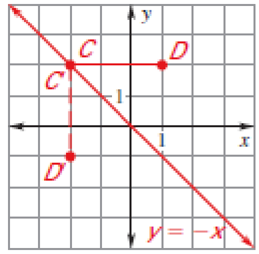
Solution

Use the coordinate rule for reflecting in the line $y = -x$.

$(a, b) \rightarrow (-b, -a)$

$C(-2, 2) \rightarrow C'($ <u>−2</u> , <u>2</u> $)$

$D(1, 2) \rightarrow D'($ <u>−2</u> , <u>−1</u> $)$

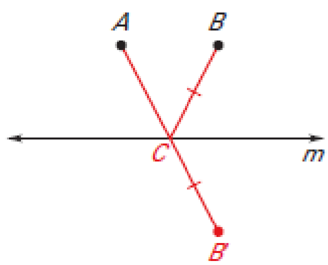


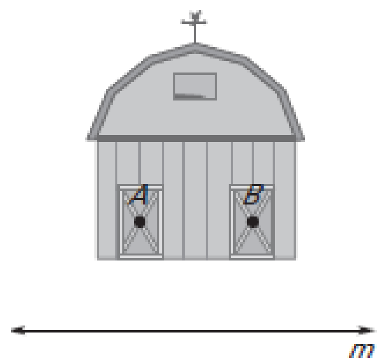
EXAMPLE 4 Find a minimum distance

Tools Workers are retrieving tools that they need for a project. One will enter the building at point A and the other at point B. Where should they park on driveway m to minimize the distance they will walk?

Solution

Reflect B in line m to obtain B'. Then draw $\overline{AB'}$. Label the <u>intersection</u> of $\overline{AB'}$ and m as C. Because AB' is the <u>shortest</u> distance between A and B' and $BC =$ <u>$B'C$</u>, park at point <u>C</u> to minimize the combined distance, $AC + BC$, they have to walk.

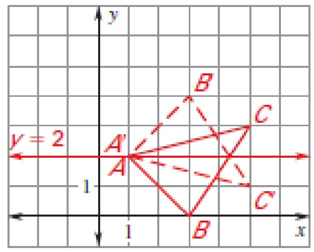




9.3 Cont.

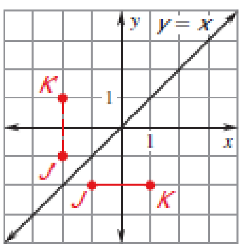
Checkpoint **Complete the following exercise.**

1. Graph the reflection of $\triangle ABC$ from Example 1 in the line $y = 2$.

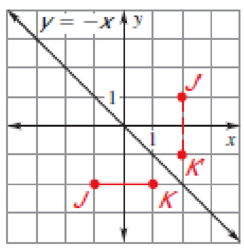


Checkpoint **The endpoints of $\overline{JK}$ are $J(-1, -2)$ and $K(1, -2)$. Reflect the segment in the given line. Graph the segment and its image.**

2. $y = x$

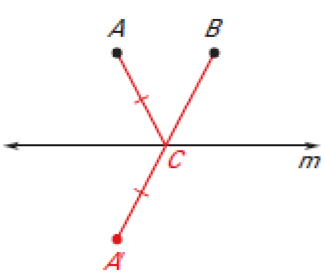


3. $y = -x$



Checkpoint **Complete the following exercise.**

4. In Example 4, reflect A in line m. What do you notice?



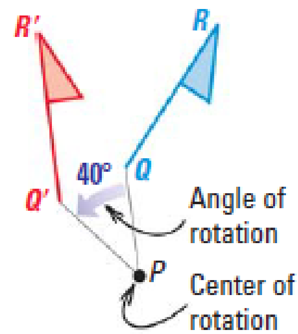
You obtain the same point C at which to park.

9.4 Perform Rotations

Obj.: Rotate figures about a point.

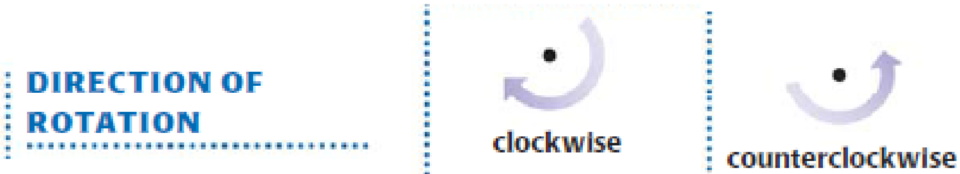
Key Vocabulary
- **Center of rotation -** A *rotation* is a transformation in which a figure is turned about a fixed point called the **center of** rotation.
- **Angle of rotation -** Rays drawn from the center of rotation to a point and its image form the angle **of rotation.**
- **Rotation -** A **rotation** turns a figure about a fixed point, called the *center of rotation*.

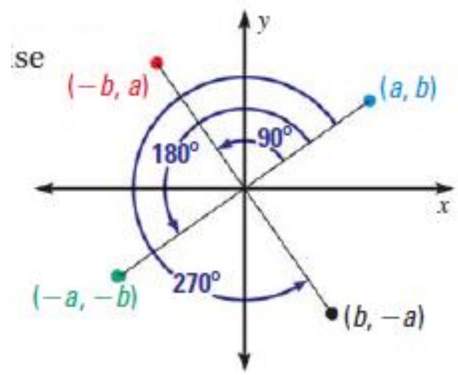
A rotation about a point *P* through an angle of $x°$ maps every point *Q* in the plane to a point *Q*' so that one of the following properties is true:



- If *Q* is not the center of rotation *P*, then *QP = Q'P* and *m QPQ' = x°*, or
- If *Q* is the center of rotation *P*, then the image of *Q* is Q'.

A 40° counterclockwise rotation is shown at the right.
Rotations can be *clockwise* or counterclockwise. In this chapter, all rotations are counterclockwise.





Coordinate Rules for Rotations about the Origin

When a point (*a*, *b*) is rotated counterclockwise about the origin, the following are true:

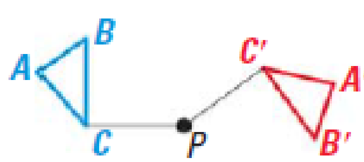
1. For a rotation of 90°, (*a*, *b*) → (-*b*, *a*).

2. For a rotation of 180°,(*a*, *b*) → (-*a*, -*b*).

3. For a rotation of 270°,(*a*, *b*) → (*b*, -*a*).

Rotation Theorem Rotat. Thm

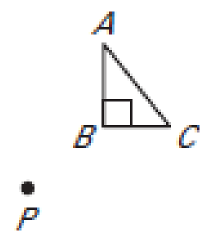
A rotation is an isometry.



ΔABC ≅ ΔA'B'C'

EXAMPLE 1 Draw a rotation

Draw a 150° rotation of ▲*ABC* about *P*.



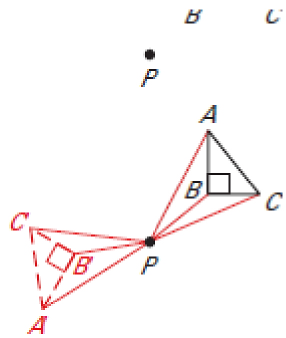
Solution

Step 1 Draw a segment from *A* to *P*.

Step 2 Draw a ray to form a 150° angle with $\overline{PA}$.

Step 3 Draw *A′* so that *PA′* = *PA*.

Step 4 Repeat Steps 1–3 for each vertex. Draw △*A′B′C′*.



EXAMPLE 2 Rotate a figure using the coordinate rules

Graph quadrilateral *KLMN* with vertices *K*(3, 2), *L*(4, 2), *M*(4, −3), and *N*(2, −1). Then rotate the quadrilateral 270° about the origin.

Solution

Graph *KLMN*. Use the coordinate rule for a 270° rotation to find the images of the vertices.

$(a, b) \to (b, -a)$

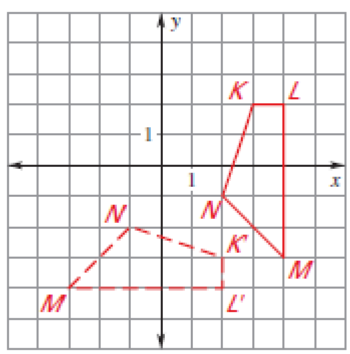
$K(3, 2) \to K'(\underline{\ 2\ }, \underline{\ -3\ })$

$L(4, 2) \to L'(\underline{\ 2\ }, \underline{\ -4\ })$

$M(4, -3) \to M'(\underline{\ -3\ }, \underline{\ -4\ })$

$N(2, -1) \to N'(\underline{\ -1\ }, \underline{\ -2\ })$

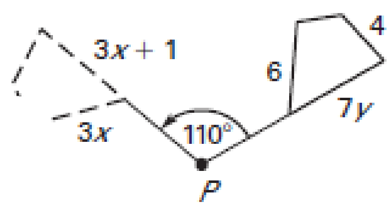
Graph the image *K′L′M′N′*.



EXAMPLE 3 Find side lengths in a rotation

The quadrilateral is rotated about P. Find the value of y.

Solution



By Theorem 9.3, the rotation is an <u>isometry</u>, so corresponding side lengths are <u>equal</u>. Then 3*x* = <u>6</u>, so *x* = <u>2</u>. Now set up an equation to solve for *y*.

$\underline{\ 7\ }y = \underline{\ 3x + 1\ }$ Corresponding lengths in an isometry are equal.

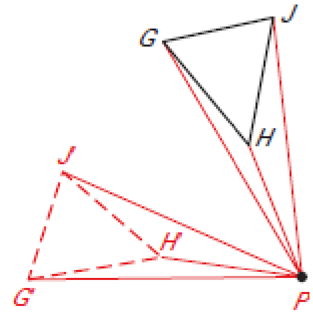
$\underline{\ 7\ }y = \underline{\ 3(2) + 1\ }$ Substitute <u>2</u> for *x*.

$y = \underline{\ 1\ }$ Solve for *y*.

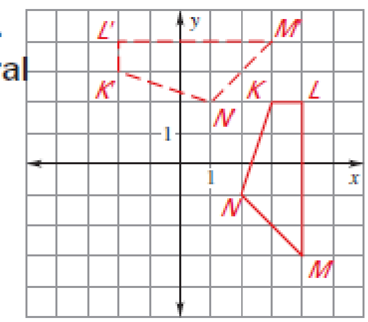
9.4 Cont.

Checkpoint **Complete the following exercise.**

1. Draw a 60° rotation of $\triangle GHJ$ about P.

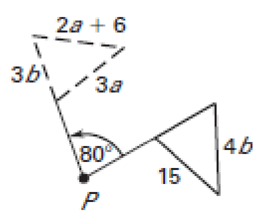


2. Graph $KLMN$ in Example 2. Then rotate the quadrilateral 90° about the origin.



4. The triangle is rotated about P. Find the value of b.

$b = 4$



9.5 Apply Compositions of Transformations

Obj.: <u>**Perform combinations of two or more transformations.**</u>

Key Vocabulary

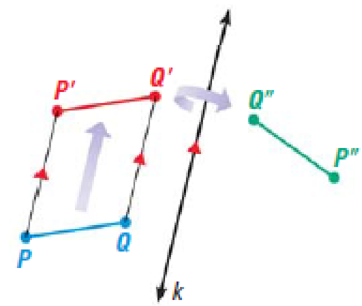
• **Glide reflection -** A <u>**translation**</u> followed by a <u>**reflection**</u> can be performed one after the other to produce a <u>***glide***</u> <u>***reflection***</u>.

• **Composition of transformations -** When <u>**two**</u> or more transformations are <u>**combined**</u> to form a <u>**single**</u> transformation, the result is a <u>**composition**</u> **of transformations**.

A <u>**glide**</u> <u>**reflection**</u> is a transformation in which every point <u>**P**</u> is mapped to a point <u>***P''***</u> by the following steps.

STEP 1 First, a <u>**translation**</u> maps *P* to *P*'.

STEP 2 Then, a <u>**reflection**</u> in a line *k* <u>**parallel**</u> to the direction of the <u>**translation**</u> maps *P*' to *P*''.



Composition Theorem <u>**Comp Thm**</u>

The <u>**composition**</u> of two (or more) isometries is an <u>**isometry**</u>.

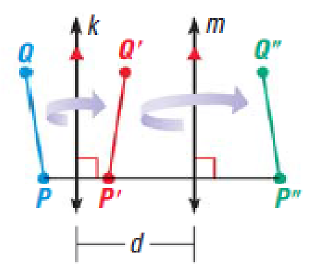
Reflections in Parallel Lines Theorem <u>**Reflect // lines Thm**</u>

If lines *k* and *m* are <u>**parallel**</u>, then a <u>**reflection**</u> in line *k* followed by a <u>**reflection**</u> in line *m* is the same as a <u>**translation**</u>.

If *P*'' is the image of *P*, then:

1. *PP*'' is <u>**perpendicular**</u> to *k* and *m*, and

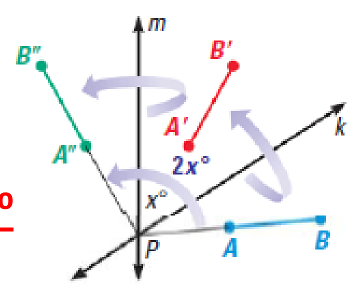
2. <u>***PP'' = 2d***</u>, where *d* is the distance between <u>***k* and *m***</u>.



Reflections in Intersecting Lines Theorem <u>**Reflect int. lines Thm**</u>

If lines *k* and *m* <u>**intersect**</u> at point *P*, then a <u>**reflection**</u> in *k* <u>**followed**</u> by a reflection in <u>***m***</u> is the same as a <u>**rotation**</u> about point *P*. The <u>**angle**</u> of rotation is <u>**$2x^o$**</u>, where <u>**x^o**</u> is the measure of the <u>**acute**</u> or right <u>**angle**</u> formed by <u>***k* and *m***</u>.

<u>***m∠BPB'' = $2x^o$***</u>



EXAMPLE 1 Find the image of a glide reflection

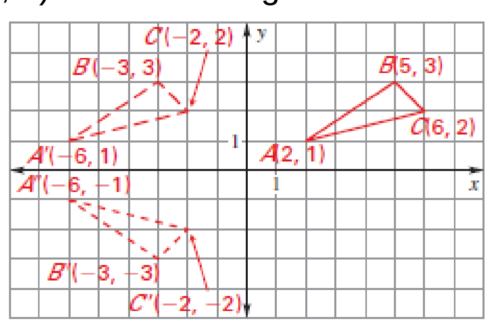
The vertices of ▲*ABC* are *A*(2, 1), *B*(5, 3), and *C*(6, 2). Find the image of ▲*ABC* after the glide reflection.

Translation: $(x, y) \rightarrow (x - 8, y)$

Reflection: in the *x*-axis

Solution

Begin by graphing △*ABC*. Then graph △*A*′*B*′*C*′ after a translation 8 units <u>left</u>. Finally, graph △*A*″*B*″*C*″ after a reflection in the *x*-axis.



The line of reflection must be parallel to the direction of the translation to be a glide reflection.

EXAMPLE 2 Find the image of a composition (cont. 9.5)

The endpoints of $\overline{CD}$ are $C(-1, 6)$ and $D(-1, 3)$. Graph the image of $\overline{CD}$ after the composition.

Reflection: in the *y*-axis

Rotation: 90° about the origin

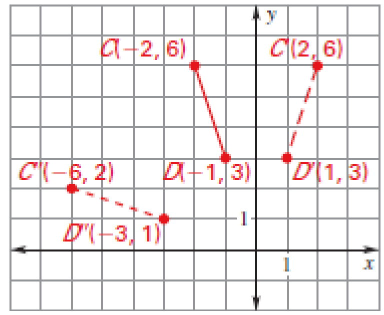
Solution

Unless you are told otherwise, do the transformations in the order given.

Step 1 Graph $\overline{CD}$.

Step 2 Reflect $\overline{CD}$ in the *y*-axis. $\overline{C'D'}$ has endpoints $C'($ 2 , 6 $)$ and $D'($ 1 , 3 $)$.

Step 3 Rotate $\overline{C'D'}$ 90° about the origin. $\overline{C''D''}$ has endpoints $C''($ −6 , 2 $)$ and $D''($ −3 , 1 $)$.



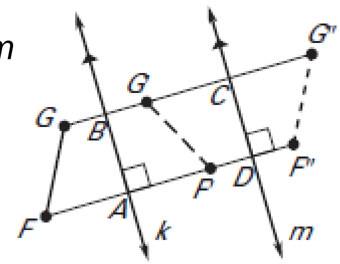
EXAMPLE 3 Use Reflect // lines Thm

In the diagram, a reflection in line *k* maps $\overline{GF}$ to $\overline{G'F'}$. A reflection in line *m* maps $\overline{G'F'}$ to $\overline{G''F''}$. Also, FA = 6 and *DF'' = 3.*

a. Name any segments congruent to each segment: $\overline{GF}$, $\overline{FA}$, and $\overline{GB}$.

b. Does AD = BC? Explain.

c. What is the length of $\overline{GG''}$. ?



Solution

a. $\overline{GF} \cong$ $\overline{G'F'}$, and $\overline{GF} \cong$ $\overline{G''F''}$. $\overline{FA} \cong$ $\overline{F'A}$.
$\overline{GB} \cong$ $\overline{G'B}$.

b. Yes, AD = BC because $\overline{GG''}$ and $\overline{FF''}$ are perpendicular to both *k* and *m*, so $\overline{BC}$ and $\overline{AD}$ are opposite sides of a rectangle.

c. By the properties of reflections, $F'A =$ 6 and $F'D =$ 3. Theorem 9.5 implies that $GG'' = FF'' =$ 2 • *AD*, so the length of GG'' is 2 (6 + 3), or 18 units.

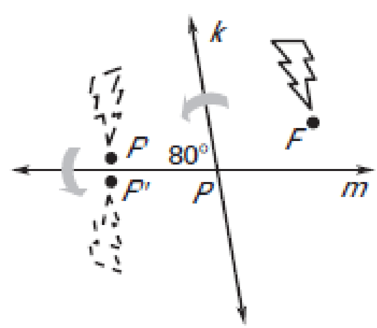
EXAMPLE 4 Use Reflect int. lines Thm

In the diagram, the figure is reflected in line *k*. The image is then reflected in line *m*. Describe a single transformation that maps F to F'.

Solution

The measure of the acute angle formed between lines *k* and *m* is 80°. So, by Theorem 9.6, a single transformation that maps *F* to *F''* is a 160° rotation about point *P*.

You can check that this is correct by tracing lines *k* and *m* and point *F*, then rotating the point 160°.



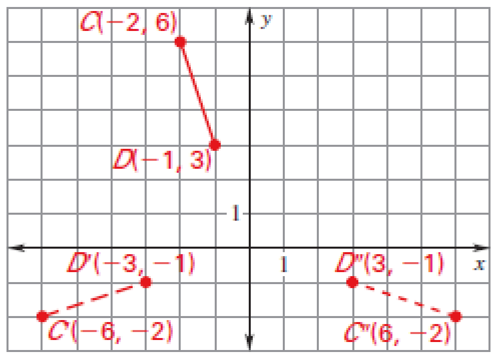
9.5 Cont.

Checkpoint **Complete the following exercises.**

1. Suppose $\triangle ABC$ in Example 1 is translated 5 units down, then reflected in the y-axis. What are the coordinates of the vertices of the image?

$A''(-2, -4)$, $B''(-5, -2)$, $C''(-6, -3)$

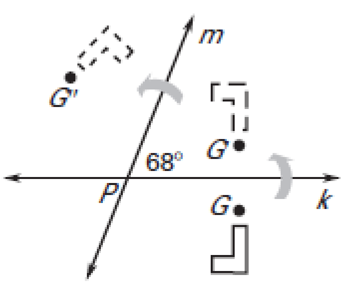
2. Graph $\overline{CD}$ from Example 2. Do the rotation first, followed by the reflection.



3. In Example 3, suppose you are given that $BC = 10$ and $G'F' = 6$. What is the perimeter of quadrilateral $GG''F''F$?

52 units

4. In the diagram below, the preimage is reflected in line k, then in line m. Describe a single transformation that maps G to G''.

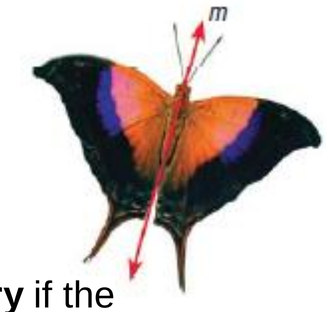


136° rotation about point P

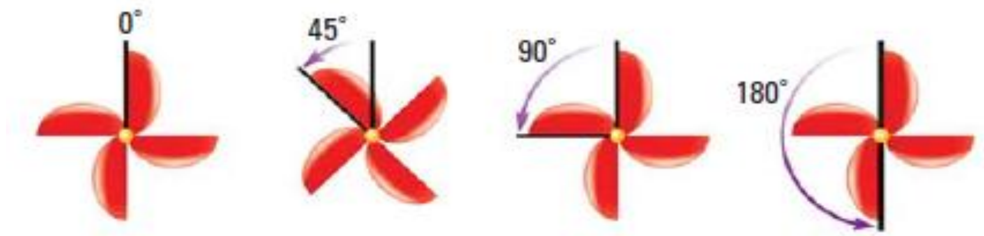
9.6 Identify Symmetry

Obj.: Identify line and rotational symmetries of a figure.

Key Vocabulary



- **Line symmetry -** A figure in the plane has **line** symmetry if the figure can be mapped onto itself by a reflection in a line.
- **Line of symmetry -** This line of reflection is a **line of symmetry**, such as line *m* at the right. A figure can have more than one line of symmetry.
- **Rotational symmetry -** A figure in a plane has rotational **symmetry** if the figure can be mapped onto itself by a rotation of 180^o or less about the center of the figure.
- **Center of Symmetry -** This point (the center of the figure) is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.



No Yes Yes

The figure above also has ***point*** *symmetry*, which is 180^o rotational symmetry.

EXAMPLE 1 Identify lines of symmetry

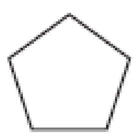
How many lines of symmetry does the figure have?

Solution

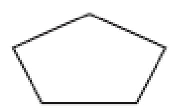
Notice that the lines of symmetry are also lines of reflection.

a.

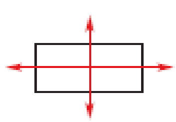
b.



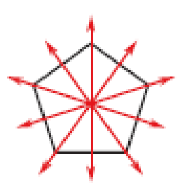
c.



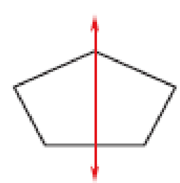
a. Two lines of symmetry



b. Five lines of symmetry



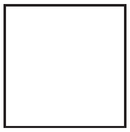
c. One line of symmetry

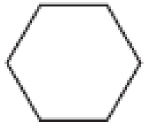


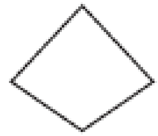
EXAMPLE 2 Identify rotational symmetry
(cont. 9.6)

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

a. Square **b. Regular hexagon** **c. Kite**

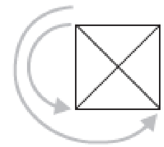






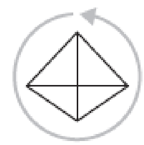
Solution

a. The square has rotational symmetry. The center is the intersection of the diagonals. Rotations of 90° or 180° about the center map the square onto itself.

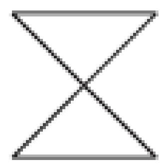


b. The regular hexagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 60°, 120°, or 180° about the center all map the hexagon onto itself.

c. The kite does not have rotational symmetry because no rotation of 180° or less maps the kite onto itself.

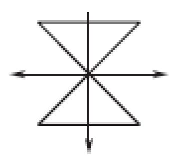


EXAMPLE 3 Identify symmetry
Identify the line symmetry and rotational symmetry of the equilateral figure at the right.

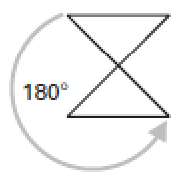


Solution

The figure has line symmetry. Two lines of symmetry can be drawn for the figure.



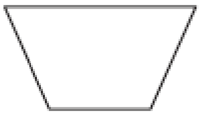
For a figure with s lines of symmetry, the smallest rotation that maps the figure onto itself has the measure $\frac{360°}{s}$. So, the figure has $\frac{360°}{2}$, or 180° rotational symmetry.



9.6 Cont.

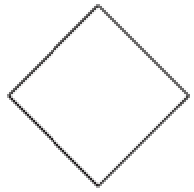
Checkpoint **How many lines of symmetry does the figure have?**

1.



1

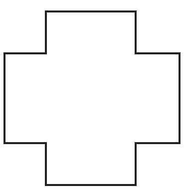
2.



4

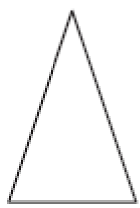
In Exercises 3 and 4, does the figure have rotational symmetry? If so, *describe* any rotations that map the figure onto itself.

3.



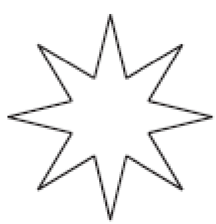
yes; 90° or 180° about the center

4.



no

5. Describe the lines of symmetry and rotational symmetry of the figure at the right.



8 lines of symmetry, 4 through the convex vertices and 4 through the concave vertices; 45°, 90°, 135°, or 180° about the center

9.7 Identify and Perform Dilations

Obj.: Use drawing tools to draw dilations.

Key Vocabulary

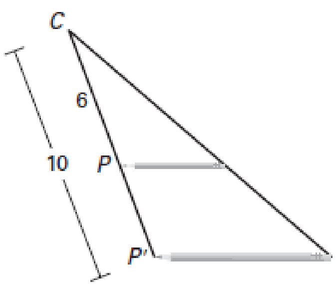
- **Dilation -** A **dilation** is a transformation that **stretches** or **shrinks** a figure to create a **similar** figure.
- **Reduction -** If **$0 < k < 1$**, the dilation is a **reduction**.
- **Enlargement -** If **$k > 1$**, the dilation is an **enlargement**.

EXAMPLE 1 Identify dilations

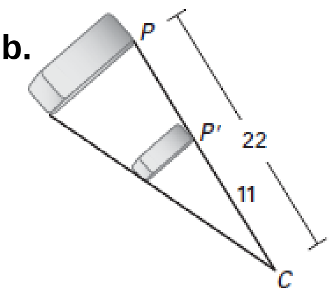
Find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*.

Solution

a.



b.



a. Because $\frac{CP'}{CP} = \frac{10}{6}$, the scale factor is $k = \frac{5}{3}$. The image P' is an enlargement.

b. Because $\frac{CP'}{CP} = \frac{11}{22}$, the scale factor is $k = \frac{1}{2}$. The image P' is a reduction.

EXAMPLE 2 Draw a dilation

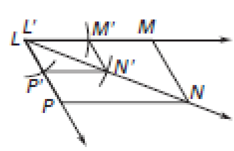
Draw and label ▱*LMNP*. Then construct a dilation of ▱*LMNP* with point *L* as the center of dilation and a scale factor of ½.

Solution

Step 1 Draw *LMNP*. Draw rays from *L* through vertices *M*, *N*, and *P*.

Step 2 Open the compass to the length of *LM*. Locate *M′* on $\overrightarrow{LM}$ so $LM' = \frac{1}{2}(LM)$. Locate *N′* and *P′* the same way.

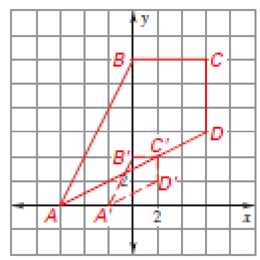
Step 3 Add a second label *L′* to point *L*. Draw the sides of *L′M′N′P′*.



EXAMPLE 4 Use scale factor multiplication in a dilation (cont. 9.7)

The vertices of quadrilateral ABCD are A(−3, 0), B(0, 6), C(3, 6), and D(3, 3). Find the image of ABCD after a dilation with its center at the origin and a scale factor of ⅓. Graph ABCD and its image.

Solution



EXAMPLE 5 Find the image of a composition

The vertices of ▲*KLM* are *K*(−3, 0), *L*(−2, 1), and *M*(−1, −1). Find the image of ▲KLM after the given composition.

Translation: (x, y) → (x + 4, y + 2)

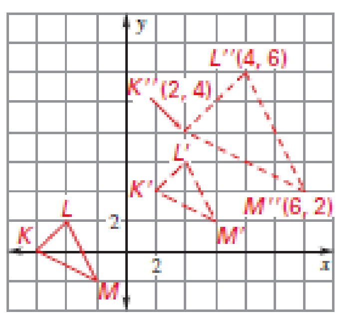
Dilation: centered at the origin with a scale factor of 2

Solution

Step 1 Graph the preimage △*KLM* in the coordinate plane.

Step 2 Translate △*KLM* 4 units to the _right_ and 2 units _up_. Label it △*K′L′M′*.

Step 3 Dilate △*K′L′M′* using the _origin_ as the center and a scale factor of 2 to find △*K″L″M″*.



9.6 Cont.

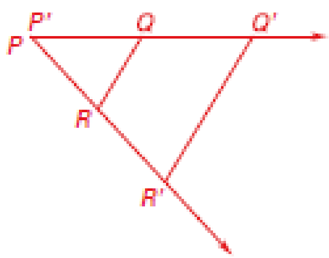
Checkpoint Complete the following exercise.

1. In a dilation, $CP' = 4$ and $CP = 20$. Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor.

   reduction; $\frac{1}{5}$

2. Draw and label $\triangle PQR$. Then construct a dilation of $\triangle PQR$ with P as the center of dilation and a scale factor of 2.

   Sample answer:



5. The vertices of $\triangle RST$ are $R(-4, 3)$, $S(-1, -2)$, and $T(2, 1)$. Use scalar multiplication to find the vertices of $\triangle R'S'T'$ after a dilation with its center at the origin and a scale factor of 2.

   '(8, 6), S'(2, 4), T'(4, 2)

6. A segment has the endpoints $C(-2, 2)$ and $D(2, 2)$. Find the image of CD after a 90° rotation about the origin followed by a dilation with its center at the origin and a scale factor of 2.

   C''(4, 4), D''(4, 4)