9.5 Graphing Other Trigonometric Functions

Essential Question What are the characteristics of the graph of the tangent function?

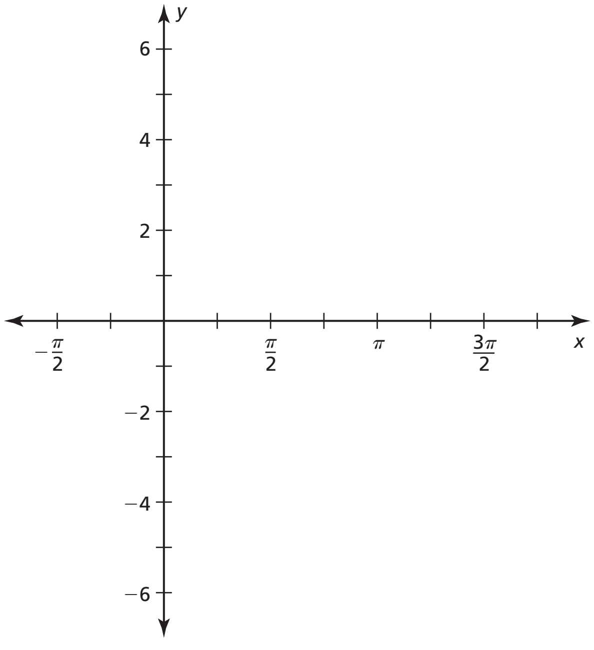
EXPLORATION 1 Graphing the Tangent Function

Work with a partner.

a. Complete the table for $y = \tan x$, where x is an angle measure in radians.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \tan x$									
x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$y = \tan x$									

b. The graph of $y = \tan x$ has vertical asymptotes at x-values where $\tan x$ is undefined. Plot the points (x, y) from part (a). Then use the asymptotes to sketch the graph of $y = \tan x$.



c. For the graph of $y = \tan x$, identify the asymptotes, the x-intercepts, and the intervals for which the function is increasing or decreasing over $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. Is the tangent function *even*, *odd*, or *neither*?

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to consider analogous problems and try special cases of the original problem in order to gain insight into its solution.

Communicate Your Answer

2. What are the characteristics of the graph of the tangent function?

3. Describe the asymptotes of the graph of $y = \cot x$ on the interval $-\frac{\pi}{2} < x < \frac{3\pi}{2}$.

9.5 Lesson

Core Vocabulary

Previous
asymptote
period
amplitude
x-intercept
transformations

What You Will Learn

- Explore characteristics of tangent and cotangent functions.
- Graph tangent and cotangent functions.
- Graph secant and cosecant functions.

Exploring Tangent and Cotangent Functions

The graphs of tangent and cotangent functions are related to the graphs of the parent functions $y = \tan x$ and $y = \cot x$, which are graphed below.

← x approaches $-\frac{\pi}{2}$ | x approaches $\frac{\pi}{2}$ →

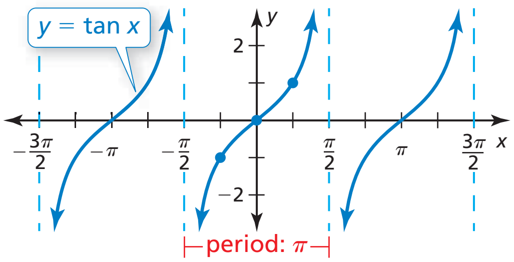
x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$y = \tan x$	Undef.	-1256	-14.10	-1	0	1	14.10	1256	Undef.

← $\tan x$ approaches $-\infty$ | $\tan x$ approaches ∞ →

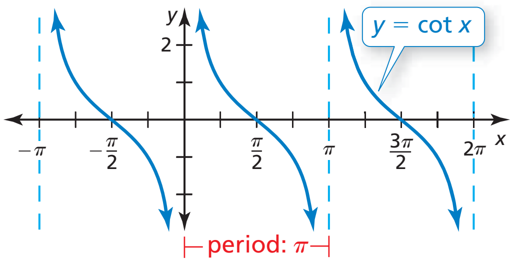
Because $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is undefined for x-values at which $\cos x = 0$, such as

$x = \pm\frac{\pi}{2} \approx \pm 1.571.$

The table indicates that the graph has asymptotes at these values. The table represents one cycle of the graph, so the period of the graph is π.



You can use a similar approach to graph $y = \cot x$. Because $\cot x = \frac{\cos x}{\sin x}$, $\cot x$ is undefined for x-values at which $\sin x = 0$, which are multiples of π. The graph has asymptotes at these values. The period of the graph is also π.



Core Concept

Characteristics of $y = \tan x$ and $y = \cot x$

The functions $y = \tan x$ and $y = \cot x$ have the following characteristics.

- The domain of $y = \tan x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \cot x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is all real numbers. So, the functions do not have maximum or minimum values, and the graphs do not have an amplitude.
- The period of each graph is π.
- The x-intercepts for $y = \tan x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cot x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

STUDY TIP

Odd multiples of $\frac{\pi}{2}$ are values such as these:

$\pm 1 \cdot \frac{\pi}{2} = \pm\frac{\pi}{2}$

$\pm 3 \cdot \frac{\pi}{2} = \pm\frac{3\pi}{2}$

$\pm 5 \cdot \frac{\pi}{2} = \pm\frac{5\pi}{2}$

Graphing Tangent and Cotangent Functions

The graphs of $y = a \tan bx$ and $y = a \cot bx$ represent transformations of their parent functions. The value of a indicates a vertical stretch $(a > 1)$ or a vertical shrink $(0 < a < 1)$. The value of b indicates a horizontal stretch $(0 < b < 1)$ or a horizontal shrink $(b > 1)$ and changes the period of the graph.

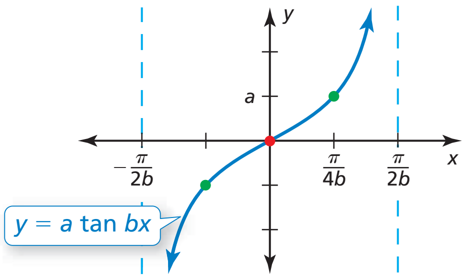
Core Concept

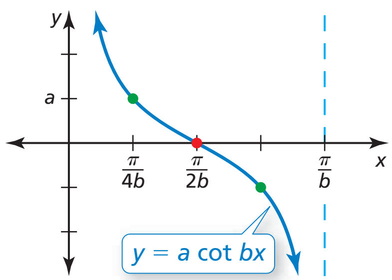
Period and Vertical Asymptotes of $y = a \tan bx$ and $y = a \cot bx$

The period and vertical asymptotes of the graphs of $y = a \tan bx$ and $y = a \cot bx$, where a and b are nonzero real numbers, are as follows.

- The period of the graph of each function is $\dfrac{\pi}{|b|}$.
- The vertical asymptotes for $y = a \tan bx$ are at odd multiples of $\dfrac{\pi}{2|b|}$.
- The vertical asymptotes for $y = a \cot bx$ are at multiples of $\dfrac{\pi}{|b|}$.

Each graph below shows five key x-values that you can use to sketch the graphs of $y = a \tan bx$ and $y = a \cot bx$ for $a > 0$ and $b > 0$. These are the x-intercept, the x-values where the asymptotes occur, and the x-values halfway between the x-intercept and the asymptotes. At each halfway point, the value of the function is either a or $-a$.





EXAMPLE 1 **Graphing a Tangent Function**

Graph one period of $g(x) = 2 \tan 3x$. Describe the graph of g as a transformation of the graph of $f(x) = \tan x$.

SOLUTION

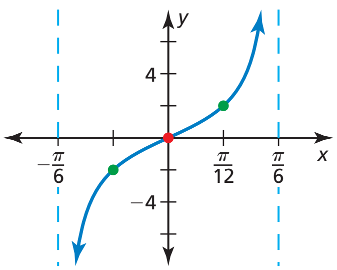
The function is of the form $g(x) = a \tan bx$ where $a = 2$ and $b = 3$. So, the period is $\dfrac{\pi}{|b|} = \dfrac{\pi}{3}$.

Intercept: $(0, 0)$

Asymptotes: $x = \dfrac{\pi}{2|b|} = \dfrac{\pi}{2(3)}$, or $x = \dfrac{\pi}{6}$; $\quad x = -\dfrac{\pi}{2|b|} = -\dfrac{\pi}{2(3)}$, or $x = -\dfrac{\pi}{6}$

Halfway points: $\left(\dfrac{\pi}{4b}, a\right) = \left(\dfrac{\pi}{4(3)}, 2\right) = \left(\dfrac{\pi}{12}, 2\right)$;

$$\left(-\dfrac{\pi}{4b}, -a\right) = \left(-\dfrac{\pi}{4(3)}, -2\right) = \left(-\dfrac{\pi}{12}, -2\right)$$



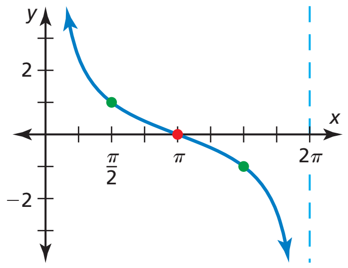
▶ The graph of g is a vertical stretch by a factor of 2 and a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.

EXAMPLE 2 Graphing a Cotangent Function

Graph one period of $g(x) = \cot \frac{1}{2}x$. Describe the graph of g as a transformation of the graph of $f(x) = \cot x$.

SOLUTION

The function is of the form $g(x) = a \cot bx$ where $a = 1$ and $b = \frac{1}{2}$. So, the period is $\frac{\pi}{|b|} = \frac{\pi}{\frac{1}{2}} = 2\pi$.



Intercept: $\left(\frac{\pi}{2b}, 0\right) = \left(\frac{\pi}{2\left(\frac{1}{2}\right)}, 0\right) = (\pi, 0)$

Asymptotes: $x = 0$; $x = \frac{\pi}{|b|} = \frac{\pi}{\frac{1}{2}}$, or $x = 2\pi$

Halfway points: $\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4\left(\frac{1}{2}\right)}, 1\right) = \left(\frac{\pi}{2}, 1\right)$; $\left(\frac{3\pi}{4b}, -a\right) = \left(\frac{3\pi}{4\left(\frac{1}{2}\right)}, -1\right) = \left(\frac{3\pi}{2}, -1\right)$

▶ The graph of g is a horizontal stretch by a factor of 2 of the graph of f.

Monitoring Progress

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Graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function.

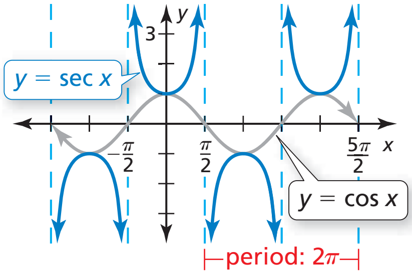
1. $g(x) = \tan 2x$ **2.** $g(x) = \frac{1}{3} \cot x$ **3.** $g(x) = 2 \cot 4x$ **4.** $g(x) = 5 \tan \pi x$

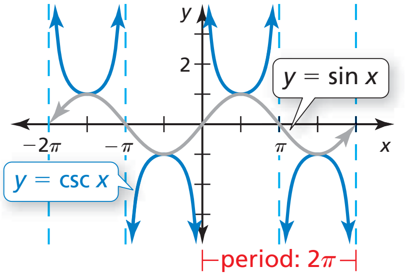
STUDY TIP

Because $\sec x = \frac{1}{\cos x}$, $\sec x$ is undefined for x-values at which $\cos x = 0$. The graph of $y = \sec x$ has vertical asymptotes at these x-values. You can use similar reasoning to understand the vertical asymptotes of the graph of $y = \csc x$.

Graphing Secant and Cosecant Functions

The graphs of secant and cosecant functions are related to the graphs of the parent functions $y = \sec x$ and $y = \csc x$, which are shown below.





Core Concept

Characteristics of $y = \sec x$ and $y = \csc x$

The functions $y = \sec x$ and $y = \csc x$ have the following characteristics.

- The domain of $y = \sec x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \csc x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is $y \leq -1$ and $y \geq 1$. So, the graphs do not have an amplitude.
- The period of each graph is 2π.

To graph $y = a \sec bx$ or $y = a \csc bx$, first graph the function $y = a \cos bx$ or $y = a \sin bx$, respectively. Then use the asymptotes and several points to sketch a graph of the function. Notice that the value of b represents a horizontal stretch or shrink by a factor of $\frac{1}{b}$, so the period of $y = a \sec bx$ and $y = a \csc bx$ is $\frac{2\pi}{|b|}$.

EXAMPLE 3 Graphing a Secant Function

Graph one period of $g(x) = 2 \sec x$. Describe the graph of g as a transformation of the graph of $f(x) = \sec x$.

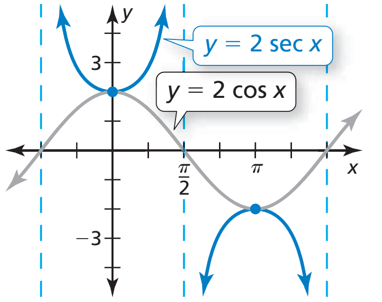
SOLUTION

Step 1 Graph the function $y = 2 \cos x$.

The period is $\frac{2\pi}{1} = 2\pi$.

Step 2 Graph asymptotes of g. Because the asymptotes of g occur when $2 \cos x = 0$, graph $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$.

Step 3 Plot points on g, such as $(0, 2)$ and $(\pi, -2)$. Then use the asymptotes to sketch the curve.



▶ The graph of g is a vertical stretch by a factor of 2 of the graph of f.

EXAMPLE 4 Graphing a Cosecant Function

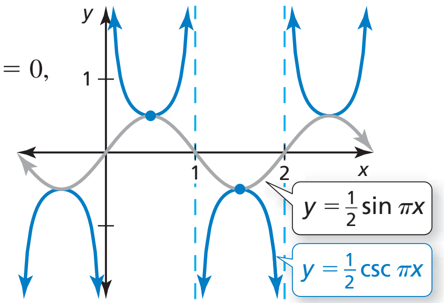
Graph one period of $g(x) = \frac{1}{2} \csc \pi x$. Describe the graph of g as a transformation of the graph of $f(x) = \csc x$.

SOLUTION

Step 1 Graph the function $y = \frac{1}{2} \sin \pi x$. The period is $\frac{2\pi}{\pi} = 2$.

Step 2 Graph asymptotes of g. Because the asymptotes of g occur when $\frac{1}{2} \sin \pi x = 0$, graph $x = 0$, $x = 1$, and $x = 2$.

Step 3 Plot points on g, such as $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{3}{2}, -\frac{1}{2}\right)$. Then use the asymptotes to sketch the curve.



▶ The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ and a horizontal shrink by a factor of $\frac{1}{\pi}$ of the graph of f.

LOOKING FOR A PATTERN

In Examples 3 and 4, notice that the plotted points are on both graphs. Also, these points represent a local maximum on one graph and a local minimum on the other graph.

Monitoring Progress

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Graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function.

5. $g(x) = \csc 3x$ **6.** $g(x) = \frac{1}{2} \sec x$ **7.** $g(x) = 2 \csc 2x$ **8.** $g(x) = 2 \sec \pi x$

9.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain why the graphs of the tangent, cotangent, secant, and cosecant functions do not have an amplitude.

2. **COMPLETE THE SENTENCE** The ________ and ________ functions are undefined for x-values at which $\sin x = 0$.

3. **COMPLETE THE SENTENCE** The period of the function $y = \sec x$ is ______, and the period of $y = \cot x$ is ______.

4. **WRITING** Explain how to graph a function of the form $y = a \sec bx$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function. *(See Examples 1 and 2.)*

5. $g(x) = 2 \tan x$
6. $g(x) = 3 \tan x$
7. $g(x) = \cot 3x$
8. $g(x) = \cot 2x$
9. $g(x) = 3 \cot \frac{1}{4}x$
10. $g(x) = 4 \cot \frac{1}{2}x$
11. $g(x) = \frac{1}{2} \tan \pi x$
12. $g(x) = \frac{1}{3} \tan 2\pi x$

13. **ERROR ANALYSIS** Describe and correct the error in finding the period of the function $y = \cot 3x$.

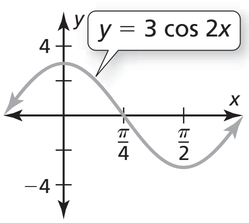
✗ Period: $\frac{2\pi}{|b|} = \frac{2\pi}{3}$

14. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of $f(x) = \tan x$ represented by $g(x) = 2 \tan 5x$.

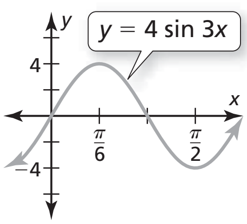
✗ A vertical stretch by a factor of 5 and a horizontal shrink by a factor of $\frac{1}{2}$.

15. **ANALYZING RELATIONSHIPS** Use the given graph to graph each function.

a. $f(x) = 3 \sec 2x$



b. $f(x) = 4 \csc 3x$



16. **USING EQUATIONS** Which of the following are asymptotes of the graph of $y = 3 \tan 4x$?

Ⓐ $x = \frac{\pi}{8}$ Ⓑ $x = \frac{\pi}{4}$

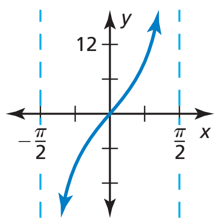
Ⓒ $x = 0$ Ⓓ $x = -\frac{5\pi}{8}$

In Exercises 17–24, graph one period of the function. Describe the graph of g as a transformation of the graph of its parent function. *(See Examples 3 and 4.)*

17. $g(x) = 3 \csc x$
18. $g(x) = 2 \csc x$
19. $g(x) = \sec 4x$
20. $g(x) = \sec 3x$
21. $g(x) = \frac{1}{2} \sec \pi x$
22. $g(x) = \frac{1}{4} \sec 2\pi x$
23. $g(x) = \csc \frac{\pi}{2}x$
24. $g(x) = \csc \frac{\pi}{4}x$

ATTENDING TO PRECISION **In Exercises 25–28, use the graph to write a function of the form $y = a \tan bx$.**

25.



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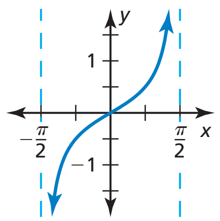
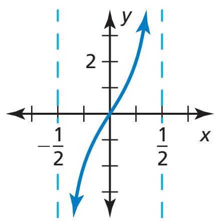
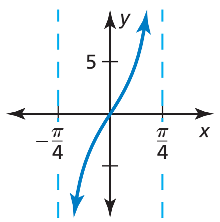


27.



28.

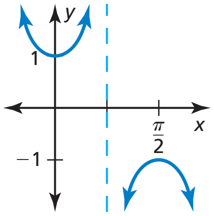


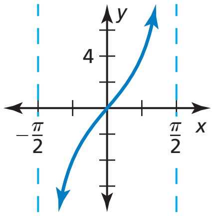
USING STRUCTURE In Exercises 29–34, match the equation with the correct graph. Explain your reasoning.

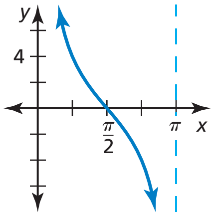
29. $g(x) = 4 \tan x$ **30.** $g(x) = 4 \cot x$

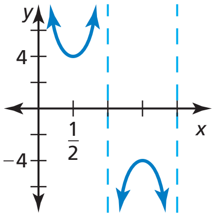
31. $g(x) = 4 \csc \pi x$ **32.** $g(x) = 4 \sec \pi x$

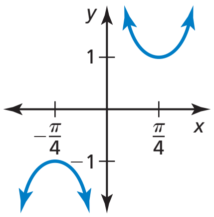
33. $g(x) = \sec 2x$ **34.** $g(x) = \csc 2x$

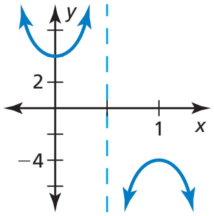
A.


B.


C.


D.


E.


F.


35. WRITING Explain why there is more than one tangent function whose graph passes through the origin and has asymptotes at $x = -\pi$ and $x = \pi$.

36. USING EQUATIONS Graph one period of each function. Describe the transformation of the graph of its parent function.

a. $g(x) = \sec x + 3$ **b.** $g(x) = \csc x - 2$

c. $g(x) = \cot(x - \pi)$ **d.** $g(x) = -\tan x$

WRITING EQUATIONS In Exercises 37–40, write a rule for *g* that represents the indicated transformation of the graph of *f*.

37. $f(x) = \cot 2x$; translation 3 units up and $\frac{\pi}{2}$ units left

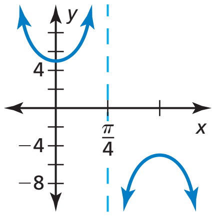
38. $f(x) = 2 \tan x$; translation π units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$

39. $f(x) = 5 \sec (x - \pi)$; translation 2 units down, followed by a reflection in the x-axis

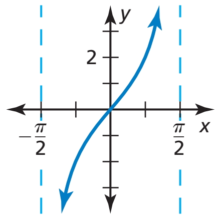
40. $f(x) = 4 \csc x$; vertical stretch by a factor of 2 and a reflection in the x-axis

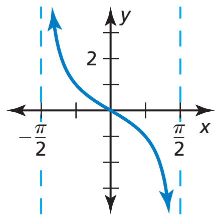
41. MULTIPLE REPRESENTATIONS Which function has a greater local maximum value? Which has a greater local minimum value? Explain.

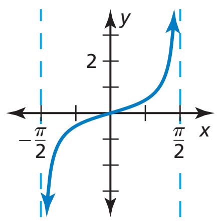
A. $f(x) = \frac{1}{4} \csc \pi x$

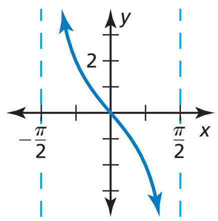
B.


42. ANALYZING RELATIONSHIPS Order the functions from the least average rate of change to the greatest average rate of change over the interval $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

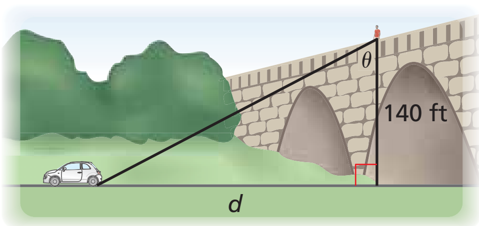
A.


B.


C.


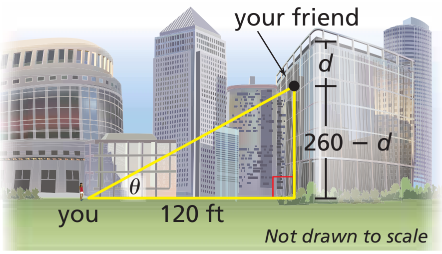
D.


43. REASONING You are standing on a bridge 140 feet above the ground. You look down at a car traveling away from the underpass. The distance d (in feet) the car is from the base of the bridge can be modeled by $d = 140 \tan \theta$. Graph the function. Describe what happens to θ as d increases.



44. USING TOOLS You use a video camera to pan up the Statue of Liberty. The height h (in feet) of the part of the Statue of Liberty that can be seen through your video camera after time t (in seconds) can be modeled by $h = 100 \tan \frac{\pi}{36}t$. Graph the function using a graphing calculator. What viewing window did you use? Explain.

45. MODELING WITH MATHEMATICS You are standing 120 feet from the base of a 260-foot building. You watch your friend go down the side of the building in a glass elevator.



a. Write an equation that gives the distance d (in feet) your friend is from the top of the building as a function of the angle of elevation θ.

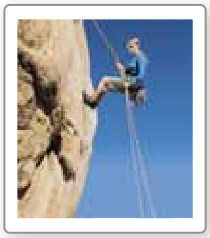
b. Graph the function found in part (a). Explain how the graph relates to this situation.

46. MODELING WITH MATHEMATICS You are standing 300 feet from the base of a 200-foot cliff. Your friend is rappelling down the cliff.

a. Write an equation that gives the distance d (in feet) your friend is from the top of the cliff as a function of the angle of elevation θ.

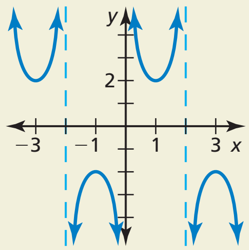
b. Graph the function found in part (a).

c. Use a graphing calculator to determine the angle of elevation when your friend has rappelled halfway down the cliff.



47. MAKING AN ARGUMENT Your friend states that it is not possible to write a cosecant function that has the same graph as $y = \sec x$. Is your friend correct? Explain your reasoning.

48. HOW DO YOU SEE IT? Use the graph to answer each question.



a. What is the period of the graph?

b. What is the range of the function?

c. Is the function of the form $f(x) = a \csc bx$ or $f(x) = a \sec bx$? Explain.

49. ABSTRACT REASONING Rewrite $a \sec bx$ in terms of $\cos bx$. Use your results to explain the relationship between the local maximums and minimums of the cosine and secant functions.

50. THOUGHT PROVOKING A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a *trigonometric identity*. Use a graphing calculator to graph the function

$$y = \frac{1}{2}\left(\tan \frac{x}{2} + \cot \frac{x}{2}\right).$$

Use your graph to write a trigonometric identity involving this function. Explain your reasoning.

51. CRITICAL THINKING Find a tangent function whose graph intersects the graph of $y = 2 + 2 \sin x$ only at minimum points of the sine function.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write a cubic function whose graph passes through the given points. *(Section 4.9)*

52. $(-1, 0), (1, 0), (3, 0), (0, 3)$

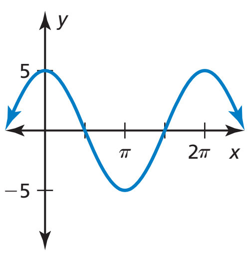
53. $(-2, 0), (1, 0), (3, 0), (0, -6)$

54. $(-1, 0), (2, 0), (3, 0), (1, -2)$

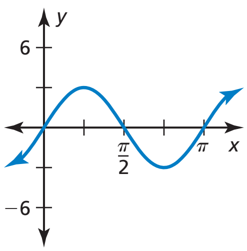
55. $(-3, 0), (-1, 0), (3, 0), (-2, 1)$

Find the amplitude and period of the graph of the function. *(Section 9.4)*

56.



57.



58.