9-6

1. Plan

Objectives
1 To use a composition of reflections
2 To identify glide reflections

Examples
1 Recognizing the Transformation
2 Composition of Reflections Across Parallel Lines
3 Composition of Reflections in Intersecting Lines
4 Finding a Glide Reflection Image
5 Classifying Isometries

Professional Development

Math Background

The four distinct isometry types can be divided into two sets: the direct, or sense-preserving, set that contains translations and rotations; and the opposite, or sense-reversing, set that contains reflections and glide reflections. The theorems in this lesson summarize the abstract algebra group properties of these isometries.

More Math Background: p. 468D

Lesson Planning and Resources

See p. 468E for a list of the resources that support this lesson.

PowerPoint

Bell Ringer Practice

Check Skills You'll Need
For intervention, direct students to:

Drawing Reflection Images
Lesson 9-2: Example 2
Extra Skills, Word Problems, Proof Practice, Ch. 9

Finding a Translation Image
Lesson 9-1: Example 3
Extra Skills, Word Problems, Proof Practice, Ch. 9

9-6 Compositions of Reflections

What You'll Learn
- To use a composition of reflections
- To identify glide reflections

. . . And Why
To classify isometries, as in Example 5

Check Skills You'll Need

GO for Help Lessons 9-1 and 9-2

Given points $R(-1, 1)$, $S(-4, 3)$, and $T(-2, 5)$, draw $\triangle RST$ and its reflection image in each line. 1–3. See back of book.

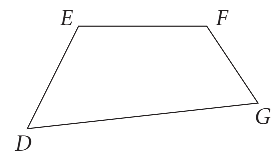
1. the y-axis **2.** the x-axis **3.** $y = 1$

Draw $\triangle RST$ described above and its translation image for each translation. 4–6. See back of book.

4. $(x, y) \rightarrow (x, y - 3)$ **5.** $(x, y) \rightarrow (x + 4, y)$

6. $(x, y) \rightarrow (x + 2, y - 5)$

7. Copy the figure at the right. Draw the image of the figure for a reflection across $\overleftrightarrow{DG}$. See back of book.



New Vocabulary • glide reflection

1 Compositions of Reflections

If two figures are congruent, there is a transformation that maps one onto the other. If no reflection is involved, then the figures are either translation or rotation images of each other.

1 EXAMPLE Recognizing the Transformation

The two figures are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain.

The orientations of these congruent figures do not appear to be opposite, so one is a translation image or a rotation image of the other. Clearly, it's not a translation image, so it must be a rotation image.

Quick Check 1 The two figures are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain.
Neither; the figures do not have the same orientation.

Any translation or rotation can be expressed as the composition of two reflections.

Key Concepts

Theorem 9-1
A translation or rotation is a composition of two reflections.

The examples that illustrate Theorems 9-2 and 9-3 suggest a proof of Theorem 9-1 (how to find two reflections for a given translation or rotation).

Differentiated Instruction Solutions for All Learners

Special Needs L1
For Example 4, have students trace $\triangle TEX$ and illustrate the translation. Students then fold the paper to find its reflection.

learning style: tactile

Below Level L2
Before students read the theorems in this lesson, have them try compositions of reflections using geometry software or paper and pencil.

learning style: visual

Theorems 9-2 and 9-3 together form the converse of Theorem 9-1.

Key Concepts

Theorem 9-2

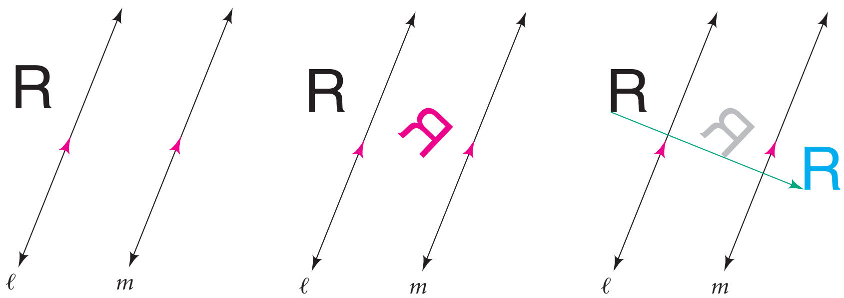
A composition of reflections across two parallel lines is a translation.

Theorem 9-3

A composition of reflections across two intersecting lines is a rotation.

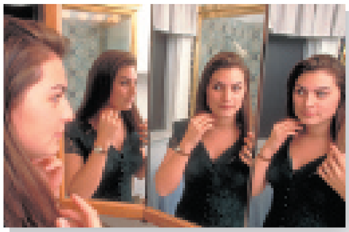
2 EXAMPLE Composition of Reflections Across Parallel Lines

Find the image of R for a reflection across line ℓ followed by a reflection across line m. Describe the resulting translation.



Reflect in ℓ. Reflect in m.

R is translated the distance and direction shown by the green arrow. The arrow is perpendicular to lines ℓ and m with length equal to twice the distance from ℓ to m.



Real-World Connection

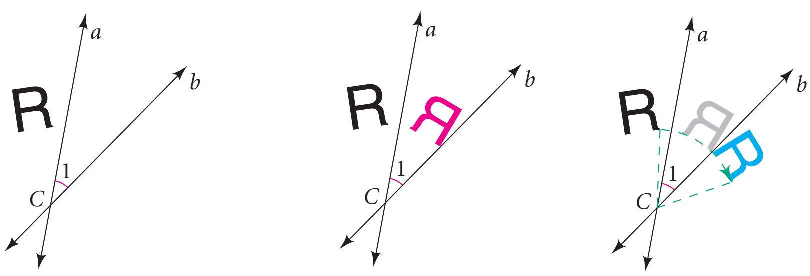
Each mirror shows a reverse image. But bend the mirrors like this and you get compositions of reflections.

Quick Check

2 Draw lines ℓ and m as shown above. Draw R between ℓ and m. Find the image of R for a reflection across line ℓ and then across line m. Describe the resulting translation. **See back of book.**

3 EXAMPLE Composition of Reflections in Intersecting Lines

Lines a and b intersect in point C and form acute $\angle 1$ with measure 35. Find the image of R for a reflection across line a and then a reflection across line b. Describe the resulting rotation.



Reflect in a. Reflect in b.

R rotates clockwise through the angle shown by the green arrow. The center of rotation is C and the measure of the angle is twice $m\angle 1$, or 70.

Quick Check

3 Repeat Example 3, but begin with R in a different position. **See back of book.**

Advanced Learners L4

After Example 2, have students find the image of R reflected across line ℓ and then across line m, when $\ell \perp m$, and develop a theorem for this composition of reflections.

learning style: verbal

English Language Learners ELL

Watch for students who confuse the vocabulary terms. In Example 1, make sure students understand the transformation of *5* is not a translation and the transformation of *Hi* is not a reflection.

learning style: verbal

2. Teach

Guided Instruction

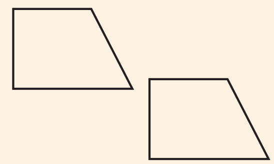
2 EXAMPLE Error Prevention

Students may think that each R should look like a translation of the original R. Have them use paper folding to see why the orientation of the second R must be different from the first and third Rs.

PowerPoint

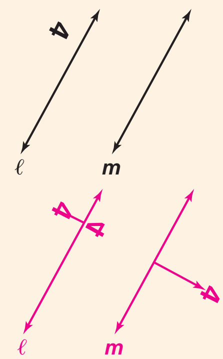
Additional Examples

1 Judging by appearances, is one figure a translation image or a rotation image of the other? Explain.

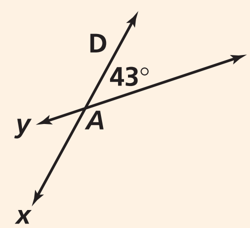


translation; congruent with same orientation

2 Find the image of the figure for a reflection across line ℓ and then across line m.



3 The letter D is reflected across line x and then across line y. Describe the resulting rotation.



***D* rotates 86° clockwise about the center of rotation *A*.**

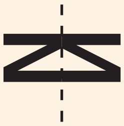
Guided Instruction

PowerPoint

Additional Examples

4 $\triangle ABC$ has vertices $A(-4, 5)$, $B(6, 2)$, and $C(0, 0)$. Find the image of $\triangle ABC$ for a glide reflection where the translation is $(x, y) \rightarrow (x, y + 2)$ and the reflection line is $x = 1$. **$A'(6, 7)$, $B'(-4, 4)$, and $C'(2, 2)$**

5 Tell whether the orientations are the same or opposite. Then classify the isometry.



opposite; reflection in vertical line

Resources

- Daily Notetaking Guide 9-6 L3
- Daily Notetaking Guide 9-6—Adapted Instruction L1

Closure

Name four isometries. Then choose two, and explain which composition of transformations results in each. **Glide reflection, reflection, rotation, translation; sample: Glide reflection is the composition of a translation and a reflection in a line ∥ to the translation vector; rotation is the composition of two reflections.**

2 Glide Reflections

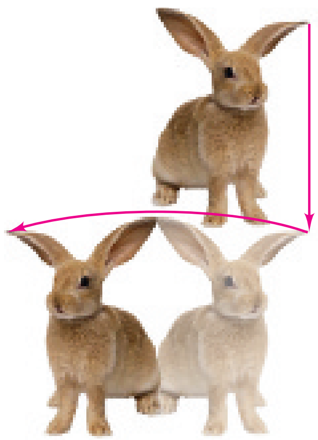
Two plane figures A and B can be congruent with opposite orientations. Reflect A and you get a figure A′ that has the same orientation as B. Thus, B is a translation or rotation image of A′. By Theorem 9-1, two reflections map A′ to B. The net result is that three reflections map A to B.

This is summarized in what is sometimes called the Fundamental Theorem of Isometries.

Key Concepts

Theorem 9-4 **Fundamental Theorem of Isometries**

In a plane, one of two congruent figures can be mapped onto the other by a composition of at most three reflections.



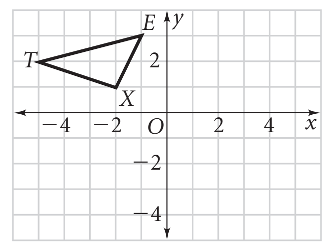
Real-World Connection

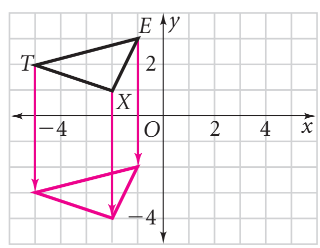
A computer can translate an image and then reflect it, or vice versa. The two rabbit images are glide reflection images of each other.

If two figures are congruent and have opposite orientations (but are not simply reflections of each other), then there is a slide and a reflection that will map one onto the other. A **glide reflection** is the composition of a glide (translation) and a reflection across a line parallel to the direction of translation.

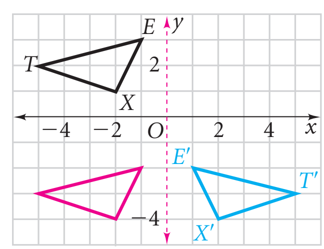
4 EXAMPLE Finding a Glide Reflection Image

Coordinate Geometry Find the image of $\triangle TEX$ for a glide reflection where the translation is $(x, y) \rightarrow (x, y - 5)$ and the reflection line is $x = 0$.



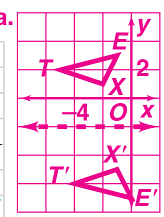


Translate $\triangle TEX$.



Reflect the image in $x = 0$.

4a.



Quick Check

4 Use $\triangle TEX$ from Example 4 above.

a. Find the image of $\triangle TEX$ under a glide reflection where the translation is $(x, y) \rightarrow (x + 1, y)$ and the reflection line is $y = -2$. **See above.**

b. **Critical Thinking** Would the result of part (a) be the same if you reflected $\triangle TEX$ first, and then translated it? Explain. **Yes; if you reflected it and then moved it right, the result would be the same.**

You can map one of any two congruent figures onto the other by a single reflection, translation, rotation, or glide reflection. Thus, you are able to classify any isometry.

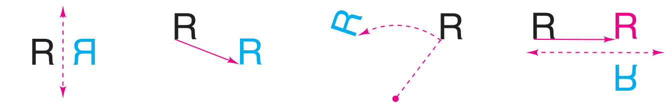
![Key Concepts]

Key Concepts

Theorem 9-5 **Isometry Classification Theorem**

There are only four isometries. They are the following.

Reflection | Translation | Rotation | Glide reflection



5 EXAMPLE Classifying Isometries

Each figure is an isometry image of the figure at the left. Tell whether their orientations are the same or opposite. Then classify the isometry.

a. opposite; a reflection

b. opposite; a glide reflection

c. same; a translation

d. same; a rotation

Quick Check **5** Classify the isometry. **rotation**

EXERCISES

For more exercises, see *Extra Skill, Word Problem, and Proof Practice.*

Practice and Problem Solving

A **Practice by Example**

Example 1 (page 506)

GO for Help

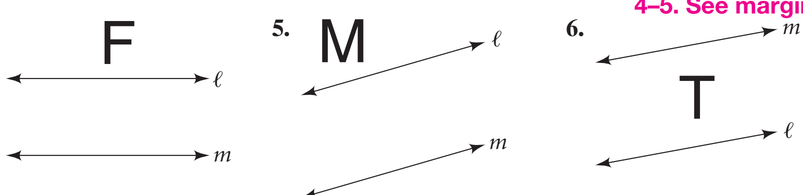
The two figures in each pair are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain. **1–3. See margin.**

1. **2.** **3.**

Example 2 (page 507)

Find the image of each letter for a reflection across line ℓ and then a reflection across line m. Describe the resulting translation or rotation. **4–5. See margin.**

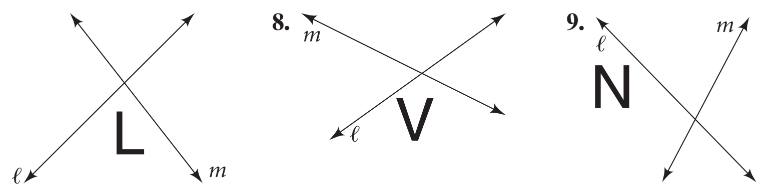
4. **5.** **6.**



6–9. See back of book.

Example 3 (page 507)

7. **8.** **9.**

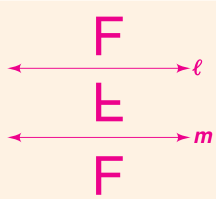


1. rotation

2. translation

3. Neither; the figures do not have the same orientation.

4.



F is translated down twice the distance between ℓ and m.

3. Practice

Assignment Guide

1 **A B** 1-9, 25, 26, 31-34

2 **A B** 10-24, 27-30, 35-44

C Challenge 45-54

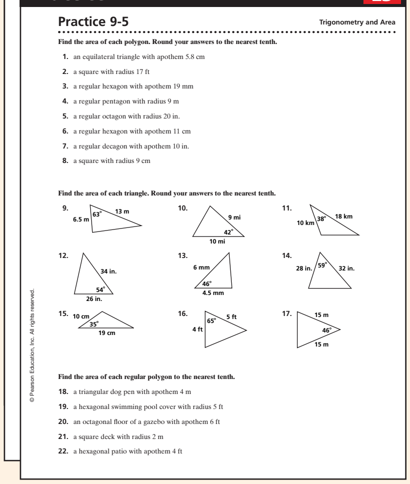
Test Prep 55-58
Mixed Review 59-67

Homework Quick Check
To check students' understanding of key skills and concepts, go over Exercises 8, 23, 24, 27, 36.

Exercises 4–6 Before students begin, ask: *How can you tell that the composition of reflections will result in a translation?* **Lines ℓ and m are parallel.**

Differentiated Instruction Resources

GPS **Guided Problem Solving** L3
Enrichment L4
Reteaching L2
Adapted Practice L1
Practice L3


Practice 9-5 Trigonometry and Area

Find the area of each polygon. Round your answers to the nearest tenth.

1. an equilateral triangle with apothem 5.8 cm
2. a square with radius 17 ft
3. a regular hexagon with apothem 19 mm
4. a regular pentagon with radius 9 m
5. a regular octagon with radius 20 in.
6. a regular hexagon with apothem 11 cm
7. a regular decagon with apothem 10 in.
8. a square with radius 9 cm

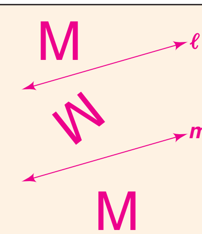
Find the area of each triangle. Round your answers to the nearest tenth.

9.–17.

Find the area of each regular polygon to the nearest tenth.

18. a triangular dog pen with apothem 4 m
19. a hexagonal swimming pool cover with radius 5 ft
20. an octagonal floor of a gazebo with apothem 6 ft
21. a square deck with radius 2 m
22. a hexagonal patio with apothem 4 ft

5.



M is translated across line *m* twice the distance between ℓ and *m*.

Connection to Algebra

Exercises 12, 13 Help students discover that the image of (x, y) is (y, x) in the reflection line $y = x$ and is $(-y, -x)$ in the reflection line $y = -x$.

Tactile Learners

Exercises 16–23 Suggest that students trace each original figure and manipulate their tracings to help them understand how the transformations were made.

Exercise 26 If students select answer choice A, they most likely read $x = -2$ incorrectly as $y = -2$.

Auditory Learners

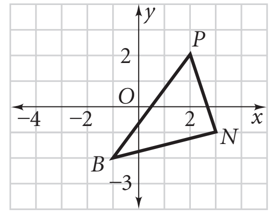
Exercise 27 Ask several volunteers to read their explanations to the rest of the class and answer questions about the math terminology they used.

Exercises 31–34 A kaleidoscope produces repeated reflections in intersecting mirrors. Consequently, the images are reflected isosceles triangles.

Exercise 45 Encourage students to provide several descriptions to help them realize that the glide and reflection are not unique, although the lines of reflection must be parallel.

Example 4 (page 508)

Find the glide reflection image of △*PNB* for the given translation and reflection line. 10–15. See back of book.



10. $(x, y) \rightarrow (x + 2, y); y = 3$

11. $(x, y) \rightarrow (x, y - 3); x = 0$

12. $(x, y) \rightarrow (x + 2, y + 2); y = x$

13. $(x, y) \rightarrow (x - 1, y + 1); y = -x$

14. $(x, y) \rightarrow (x, y - 1); x = 2$

15. $(x, y) \rightarrow (x - 2, y - 2); y = x$

Example 5 (page 509)

Each figure is an isometry image of the figure at the left. Tell whether their orientations are the same or opposite. Then classify the isometry.

16.  opp.; reflection

17.  17. opp.; glide reflection

18.  same; translation

19.  same; rotation

20.  same; rotation

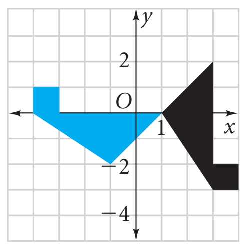
21.  same; translation

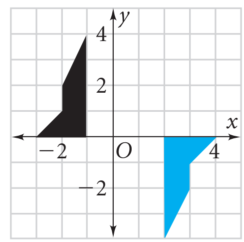
22.  opp.; reflection

23.  23. opp.; glide reflection

B Apply Your Skills

The two figures are congruent. Name the isometry that maps one onto the other.

24. 

25. 

24. glide reflection; $(x, y) \rightarrow (x - 2, y - 2)$, refl. in $y = x - 1$

25. rotation; 180° about the pt. $\left(0, \frac{1}{2}\right)$

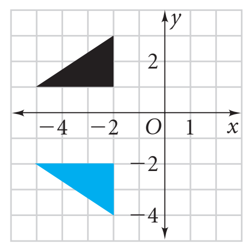
26. Multiple Choice Which transformation maps the black triangle onto the blue triangle? C

(A) a translation $(x, y) \rightarrow (x, y - 3)$ followed by a reflection across $x = -2$

(B) a rotation of 180° about the origin

(C) a reflection across $y = -\frac{1}{2}$

(D) a reflection across the *y*-axis followed by a 180° rotation about the origin



27. Writing Reflections and glide reflections are *odd isometries*, while translations and rotations are *even isometries*. Use what you learned in this lesson to explain why these categories make sense. See left.

27. Odd isometries can be expressed as the composition of an odd number of reflections. Even isometries are the composition of an even number of reflections.

28. Open-Ended Draw △*ABC*. Then, describe a reflection, a translation, a rotation, and a glide reflection, and draw the image of △*ABC* for each transformation. Check students' work.

29. For center of rotation *P*, does an $x°$ rotation followed by a $y°$ rotation give the same image as a $y°$ rotation followed by an $x°$ rotation? Explain. See margin.

30. Does an $x°$ rotation about a point *P* followed by a reflection in a line ℓ give the same image as a reflection in ℓ followed by an $x°$ rotation about *P*? Explain. No; explanations may vary.

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29. Yes; a rotation of $x°$ followed by a rotation of $y°$ is equivalent to a rotation of $(x + y)°$.

46. If $\overline{XY}$ is reflected in line ℓ, then ℓ is the ⊥ bis. of $\overline{XX'}$ and $\overline{YY'}$, so $\overline{XX'} \parallel \overline{YY'}$ and $XX'YY'$ is an isosc. trap. Therefore $\overline{XY} \cong \overline{X'Y'}$.

47. $\overline{XX'} \parallel \overline{YY'}$ and $\overline{XX'} \cong \overline{YY'}$, so $XX'Y'Y$ is a ▱. Therefore, $\overline{XY} \cong \overline{X'Y'}$.

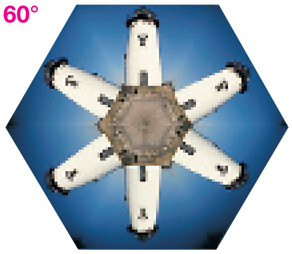
48. If $\overline{XY}$ is rotated $x°$ about pt. *R*, then $\overline{RX} \cong \overline{RX'}$ and $\overline{RY} \cong \overline{RY'}$. Also, $m\angle XRY + m\angle YRX' = m\angle YRX' + m\angle X'RY' = x°$, so $\angle XRY \cong \angle X'RY'$. So $\triangle XRY \cong \triangle X'RY'$ by SAS and $\overline{XY} \cong \overline{X'Y'}$ by CPCTC.

GO for Help

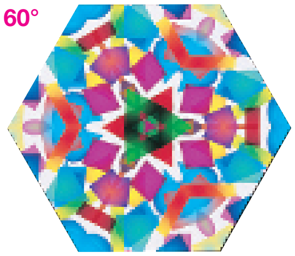
To learn more about kaleidoscopes, see p. 513.

Kaleidoscopes **The vibrant images of a kaleidoscope are produced by compositions of reflections in intersecting mirrors. Determine the angle between the mirrors in each kaleidoscope image.**

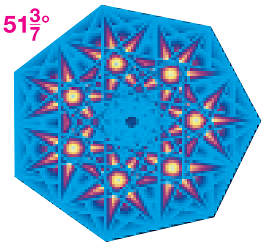
31. 60°



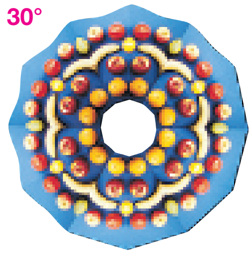
32. 60°



33. $51\frac{3}{7}$°

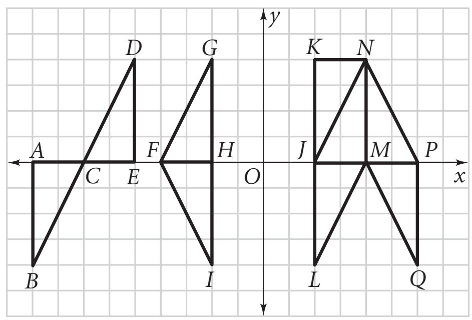


34. 30°



Identify each mapping as a reflection, translation, rotation, or glide reflection. Find the reflection line, translation rule, center and angle of rotation, or glide translation and reflection line.

35–44. See left.



35. $\triangle ABC \rightarrow \triangle EDC$

36. $\triangle EDC \rightarrow \triangle PQM$ (GPS)

37. $\triangle MNJ \rightarrow \triangle EDC$

38. $\triangle HIF \rightarrow \triangle HGF$

39. $\triangle PQM \rightarrow \triangle JLM$

40. $\triangle MNP \rightarrow \triangle EDC$

41. $\triangle JLM \rightarrow \triangle MNJ$

42. $\triangle PQM \rightarrow \triangle KJN$

43. $\triangle KJN \rightarrow \triangle ABC$

44. $\triangle HGF \rightarrow \triangle KJN$

35. rotation; center *C*, ∠ of rotation 180°
36. glide reflection; $(x, y) \rightarrow (x + 11, y)$, $y = 0$
37. translation; $(x, y) \rightarrow (x - 9, y)$
38. reflection; $y = 0$
39. reflection; $x = 4$
40. reflection; $x = -\frac{1}{2}$
41. rotation; center (3, 0), ∠ of rotation 180°
42. glide reflection; $(x, y) \rightarrow (x, y + 4)$, $x = 4$
43. translation; $(x, y) \rightarrow (x - 11, y - 4)$
44. rotation; center (0, 2), ∠ of rotation 180°

C Challenge

45. Describe a glide and a reflection that maps the red R to the blue R. See left.

45. Sample: Translate the red R so that one point moves to its corresponding point on the blue R. Then reflect across a line passing through that point.

For the given transformation mapping $\overline{XY}$ to $\overline{X'Y'}$, give a convincing argument why $\overline{XY} \cong \overline{X'Y'}$. 46–48. See margin p. 510.

46. a reflection **47.** a translation **48.** a rotation

49. The definition states that a glide reflection is the composition of a translation and a reflection. Explain why these can occur in either order. See margin.

50. For lines of reflection *r* and *s*, does a reflection in *r* followed by a reflection in *s* give the same image as a reflection in *s* followed by a reflection in *r*? Explain. See margin.

$P \rightarrow P'(3, -1)$ for the given translation and reflection line. Find the coordinates of *P*.

51. $(x, y) \rightarrow (x - 3, y)$; $y = 2$ (6, 5)

52. $(x, y) \rightarrow (x, y - 3)$; $y = 2$ (1, 2)

53. $(x, y) \rightarrow (x - 3, y - 3)$; $y = x$ (2, 6)

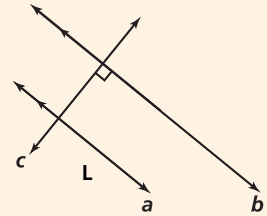
54. $(x, y) \rightarrow (x + 4, y - 4)$; $y = -x$ (−3, 1)

4. Assess & Reteach

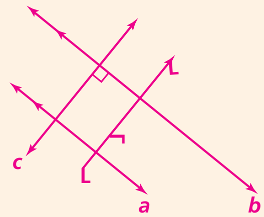
PowerPoint

Lesson Quiz

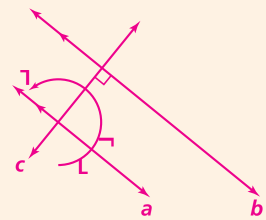
Use the diagram for Exercises 1–3.



1. Find the image of L for a reflection across line *a* and then across line *b*.



2. Find the image of L for a reflection across line *a* and then across line *c*.



3. Describe the rotation in Exercise 2. 180° rotation with center of rotation at the intersection of lines *a* and *c*

4. $\overline{PQ}$ has endpoints *P*(4, 15) and *Q*(−6, 10). Find the image of $\overline{PQ}$ for a glide reflection where the translation is $(x, y) \rightarrow (x, y - 8)$ and the reflection line is $x = 0$. Check that students' images have endpoints *P*′(-4, 7) and *Q*′(6, 2).

5. Name the four types of isometries. glide reflection, reflection, rotation, translation

49. Answers may vary. Sample: since a reflection moves a pt. in the direction ⊥ to the translation, the order does not matter.

50. No; explanations may vary. Sample: If (1, 1) is reflected over the line $y = x$ and then the *x*-axis, the image is (1, −1). If the reflections are reversed, the image is (−1, 1).

Alternative Assessment

Have each student use a right scalene triangle preimage to show
- that two reflections result in either a translation or a rotation, and
- that three reflections result in either a reflection or a glide reflection.

Students may use compass, straightedge, ruler, and protractor.

Test Prep

Resources

For additional practice with a variety of test item formats:
- Standardized Test Prep, p. 527
- Test-Taking Strategies, p. 522
- Test-Taking Strategies with Transparencies

Test Prep

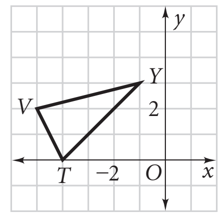
Multiple Choice

55. Find the image of $P(11, -5)$ for the translation $(x, y) \rightarrow (x - 12, y - 6)$ followed by a reflection in $x = 0$. **A**
   A. $(1, -11)$ **B.** $(-1, 11)$ **C.** $(1, 11)$ **D.** $(-1, -11)$

56. A reflection in the y-axis followed by a reflection in the x-axis does NOT give the same result as which of the following transformations? **H**
   F. a reflection in the x-axis followed by a reflection in the y-axis
   G. a rotation of 180°
   H. a rotation of 90° followed by a reflection in the x-axis
   J. a reflection in the line $y = x$ followed by a reflection in the line $y = -x$

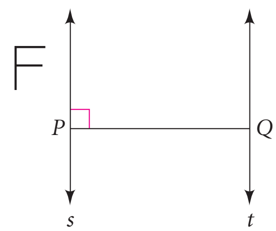
Short Response

57. Find the image of $\triangle VTY$ for the given glide reflection. Show all your steps.
   translation: $(x, y) \rightarrow (x + 3, y - 3)$
   reflection line: $y = -x$ **See margin.**



Extended Response

58. Copy the diagram with $s \parallel t$.
   a. Draw the image of F for a composition of two reflections. Reflect first in line s and then in line t. **a–b. See margin.**
   b. Explain why the resulting image is the same image as found by translating F in a direction parallel to $\overline{PQ}$ through a distance $2 \cdot PQ$.



Mixed Review

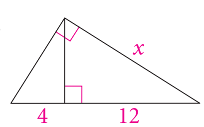
GO for Help

Lesson 9-5 **Coordinate Geometry** A figure has a vertex at $(-2, 7)$. If the figure has the given type of symmetry, state the coordinates of another vertex of the figure.

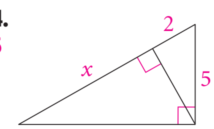
59. line symmetry about the x-axis **(−2, −7)**
60. line symmetry about the y-axis **(2, 7)**
61. point symmetry about the origin **(2, −7)**
62. line symmetry about the line $y = x$ **(7, −2)**

Lesson 7-4 **Algebra** Find the value of x.

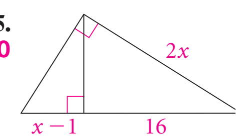
63. **$8\sqrt{3}$**



64. **10.5**



65. **10**



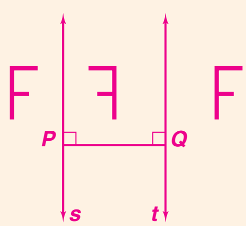
Lesson 5-4 Identify the two statements that contradict each other.

66. I. $\triangle ABC$ is right. **I and II**
   II. $\triangle ABC$ is equiangular.
   III. $\triangle ABC$ is isosceles.

67. I. In right $\triangle ABC$, $m\angle B = 90$. **I and III**
   II. In right $\triangle ABC$, $m\angle A = 80$.
   III. In right $\triangle ABC$, $m\angle C = 90$.

57. [2] $V(-5, 2)$, $T(-4, 0)$, $Y(-1, 3)$ glided give $V'(-2, -1)$, $T'(-1, -3)$, $Y'(2, 0)$. These vertices reflected over $y = -x$ give $V''(1, 2)$, $T''(3, 1)$, $Y''(0, -2)$.
   [1] incorrect method or answer

58. [4] a.



b. Suppose pt. A in F is x units from s. Thus, A reflected across s gives A', x units right of s. A' is then $PQ - x$ units left of t. Thus, A' reflected across t gives A'', $PQ - x$ units right of t. Thus, the total distance travelled is $PQ - x + PQ + x = 2PQ$.

[3] correct composition, vague explanation
[2] part (a) only
[1] composition is partially correct