9.6 Modeling with Trigonometric Functions

Essential Question What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

EXPLORATION 1 Modeling Electric Currents

Work with a partner. Find a sine function that models the electric current shown in each oscilloscope screen. State the amplitude and period of the graph.

> **MODELING WITH MATHEMATICS**
> To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.

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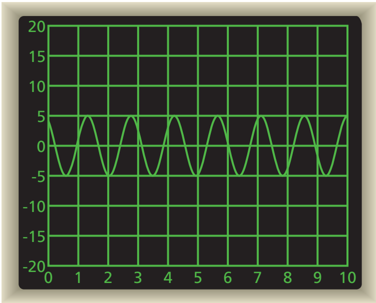

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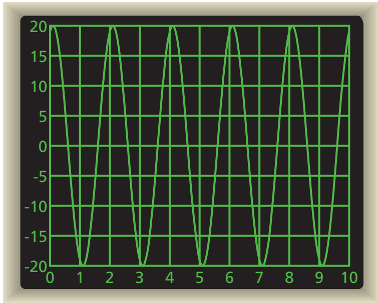

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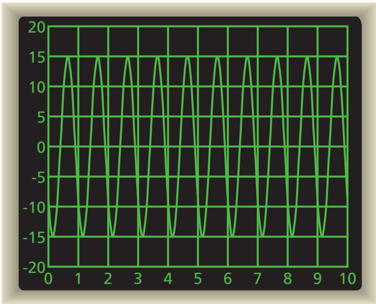

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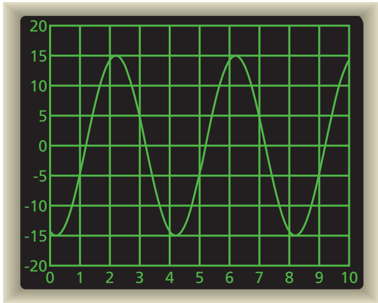

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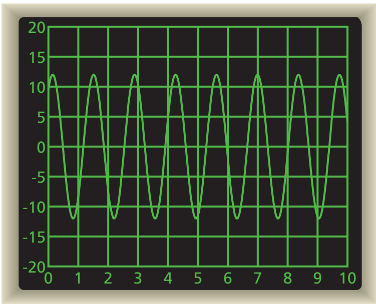

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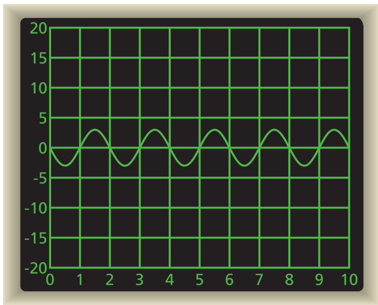


Communicate Your Answer

2. What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

3. Use the Internet or some other reference to find examples of real-life situations that can be modeled by trigonometric functions.

9.6 Lesson

Core Vocabulary

frequency, *p. 506*
sinusoid, *p. 507*

Previous
amplitude
period
midline

What You Will Learn

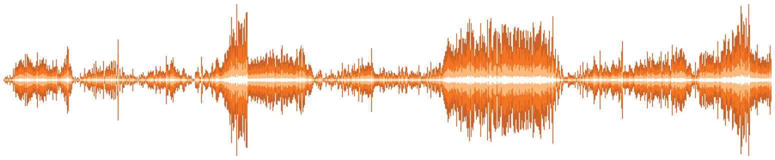
- Interpret and use frequency.
- Write trigonometric functions.
- Use technology to find trigonometric models.

Frequency

The periodic nature of trigonometric functions makes them useful for modeling *oscillating* motions or repeating patterns that occur in real life. Some examples are sound waves, the motion of a pendulum, and seasons of the year. In such applications, the reciprocal of the period is called the **frequency**, which gives the number of cycles per unit of time.

EXAMPLE 1 Using Frequency

A sound consisting of a single frequency is called a *pure tone*. An audiometer produces pure tones to test a person's auditory functions. An audiometer produces a pure tone with a frequency f of 2000 hertz (cycles per second). The maximum pressure P produced from the pure tone is 2 millipascals. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds).



SOLUTION

Step 1 Find the values of a and b in the model $P = a \sin bt$. The maximum pressure is 2, so $a = 2$. Use the frequency f to find b.

$$\text{frequency} = \frac{1}{\text{period}}$$ Write relationship involving frequency and period.

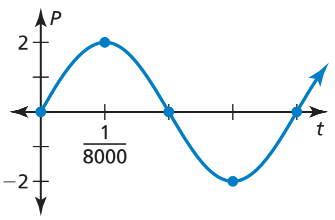
$$2000 = \frac{b}{2\pi}$$ Substitute.

$$4000\pi = b$$ Multiply each side by 2π.

The pressure P as a function of time t is given by $P = 2 \sin 4000\pi t$.

Step 2 Graph the model. The amplitude is $a = 2$ and the period is

$$\frac{1}{f} = \frac{1}{2000}.$$



The key points are:

Intercepts: $(0, 0)$; $\left(\frac{1}{2} \cdot \frac{1}{2000}, 0\right) = \left(\frac{1}{4000}, 0\right)$; $\left(\frac{1}{2000}, 0\right)$

Maximum: $\left(\frac{1}{4} \cdot \frac{1}{2000}, 2\right) = \left(\frac{1}{8000}, 2\right)$

Minimum: $\left(\frac{3}{4} \cdot \frac{1}{2000}, -2\right) = \left(\frac{3}{8000}, -2\right)$

▶ The graph of $P = 2 \sin 4000\pi t$ is shown at the left.

Monitoring Progress

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1. WHAT IF? In Example 1, how would the function change when the audiometer produced a pure tone with a frequency of 1000 hertz?

Writing Trigonometric Functions

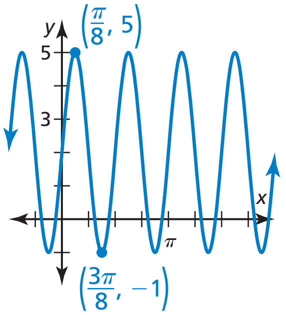
Graphs of sine and cosine functions are called **sinusoids**. One method to write a sine or cosine function that models a sinusoid is to find the values of a, b, h, and k for

$$y = a \sin b(x - h) + k \qquad \text{or} \qquad y = a \cos b(x - h) + k$$

where $|a|$ is the amplitude, $\frac{2\pi}{b}$ is the period ($b > 0$), h is the horizontal shift, and k is the vertical shift.

EXAMPLE 2 Writing a Trigonometric Function

Write a function for the sinusoid shown.



SOLUTION

Step 1 Find the maximum and minimum values. From the graph, the maximum value is 5 and the minimum value is -1.

Step 2 Identify the vertical shift, k. The value of k is the mean of the maximum and minimum values.

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{5 + (-1)}{2} = \frac{4}{2} = 2$$

Step 3 Decide whether the graph should be modeled by a sine or cosine function. Because the graph crosses the midline $y = 2$ on the y-axis, the graph is a sine curve with no horizontal shift. So, $h = 0$.

Step 4 Find the amplitude and period. The period is

$$\frac{\pi}{2} = \frac{2\pi}{b} \quad \Rightarrow \quad b = 4.$$

The amplitude is

$$|a| = \frac{(\text{maximum value}) - (\text{minimum value})}{2} = \frac{5 - (-1)}{2} = \frac{6}{2} = 3.$$

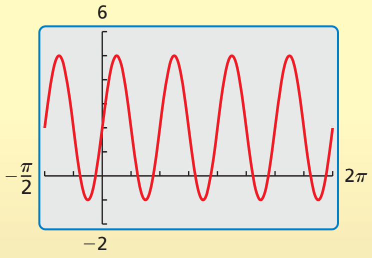
The graph is not a reflection, so $a > 0$. Therefore, $a = 3$.

▶ The function is $y = 3 \sin 4x + 2$. Check this by graphing the function on a graphing calculator.

STUDY TIP

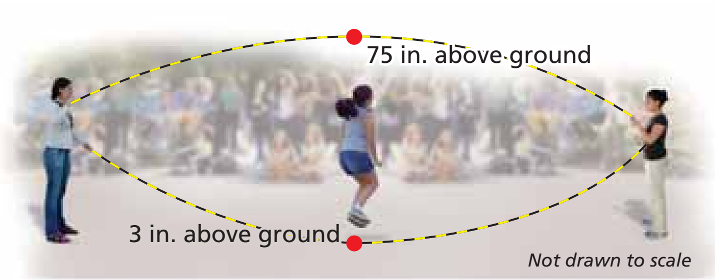
Because the graph repeats every $\frac{\pi}{2}$ units, the period is $\frac{\pi}{2}$.

Check



EXAMPLE 3 Modeling Circular Motion

Two people swing jump ropes, as shown in the diagram. The highest point of the middle of each rope is 75 inches above the ground, and the lowest point is 3 inches. The rope makes 2 revolutions per second. Write a model for the height h (in inches) of a rope as a function of the time t (in seconds) given that the rope is at its lowest point when $t = 0$.



SOLUTION

A rope oscillates between 3 inches and 75 inches above the ground. So, a sine or cosine function may be an appropriate model for the height over time.

Step 1 Identify the maximum and minimum values. The maximum height of a rope is 75 inches. The minimum height is 3 inches.

Step 2 Identify the vertical shift, k.

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2} = \frac{75 + 3}{2} = 39$$

Step 3 Decide whether the height should be modeled by a sine or cosine function. When $t = 0$, the height is at its minimum. So, use a cosine function whose graph is a reflection in the x-axis with no horizontal shift ($h = 0$).

Step 4 Find the amplitude and period.

The amplitude is $|a| = \dfrac{(\text{maximum value}) - (\text{minimum value})}{2} = \dfrac{75 - 3}{2} = 36.$

Because the graph is a reflection in the x-axis, $a < 0$. So, $a = -36$. Because a rope is rotating at a rate of 2 revolutions per second, one revolution is completed in 0.5 second. So, the period is $\dfrac{2\pi}{b} = 0.5$, and $b = 4\pi$.

▶ A model for the height of a rope is $h(t) = -36 \cos 4\pi t + 39$.

Check

Use the *table* feature of a graphing calculator to check your model.

X	Y1	
0	3	
.25	75	
.5	3	2 revolutions
.75	75	
1	3	
1.25	75	
1.5	3	

X=0

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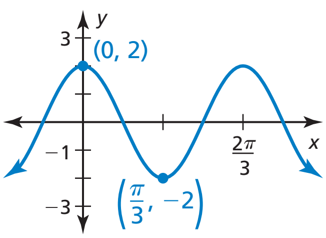
Write a function for the sinusoid.

2.



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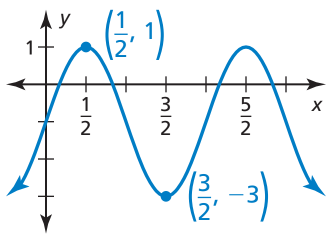
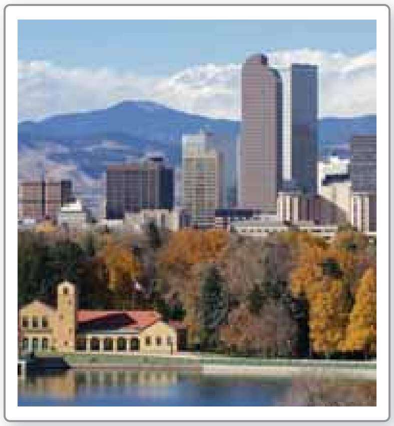


4. WHAT IF? Describe how the model in Example 3 changes when the lowest point of a rope is 5 inches above the ground and the highest point is 70 inches above the ground.

Using Technology to Find Trigonometric Models

Another way to model sinusoids is to use a graphing calculator that has a sinusoidal regression feature.

EXAMPLE 4 Using Sinusoidal Regression



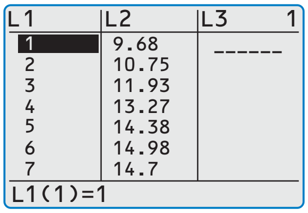
The table shows the numbers N of hours of daylight in Denver, Colorado, on the 15th day of each month, where $t = 1$ represents January. Write a model that gives N as a function of t and interpret the period of its graph.

t	1	2	3	4	5	6
N	9.68	10.75	11.93	13.27	14.38	14.98

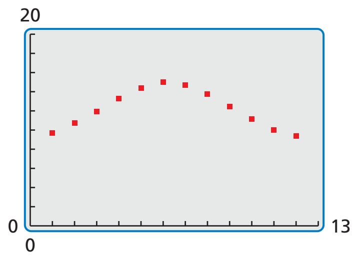
t	7	8	9	10	11	12
N	14.70	13.73	12.45	11.17	9.98	9.38

SOLUTION

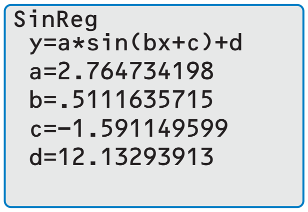
Step 1 Enter the data in a graphing calculator.



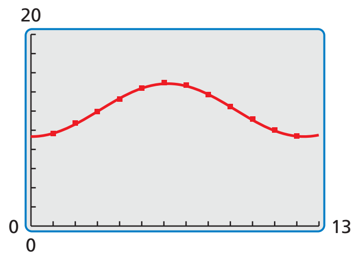
Step 2 Make a scatter plot.



Step 3 The scatter plot appears sinusoidal. So, perform a sinusoidal regression.



Step 4 Graph the data and the model in the same viewing window.



STUDY TIP

Notice that the *sinusoidal regression* feature finds a model of the form $y = a \sin(bx + c) + d$. This function has a period of $\frac{2\pi}{b}$ because it can be written as $y = a \sin b\left(x + \frac{c}{b}\right) + d$.

▶ The model appears to be a good fit. So, a model for the data is $N = 2.76 \sin(0.511t - 1.59) + 12.1$. The period, $\frac{2\pi}{0.511} \approx 12$, makes sense because there are 12 months in a year and you would expect this pattern to continue in following years.

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5. The table shows the average daily temperature T (in degrees Fahrenheit) for a city each month, where $m = 1$ represents January. Write a model that gives T as a function of m and interpret the period of its graph.

m	1	2	3	4	5	6	7	8	9	10	11	12
T	29	32	39	48	59	68	74	72	65	54	45	35

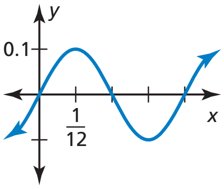
9.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Graphs of sine and cosine functions are called ___________.

2. **WRITING** Describe how to find the frequency of the function whose graph is shown.



Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the frequency of the function.

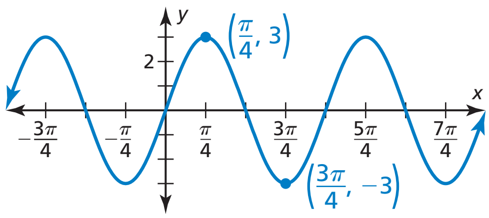
3. $y = \sin x$
4. $y = \sin 3x$
5. $y = \cos 4x + 2$
6. $y = -\cos 2x$
7. $y = \sin 3\pi x$
8. $y = \cos \dfrac{\pi x}{4}$
9. $y = \dfrac{1}{2} \cos 0.75x - 8$
10. $y = 3 \sin 0.2x + 6$

11. **MODELING WITH MATHEMATICS** The lowest frequency of sounds that can be heard by humans is 20 hertz. The maximum pressure P produced from a sound with a frequency of 20 hertz is 0.02 millipascal. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds). *(See Example 1.)*

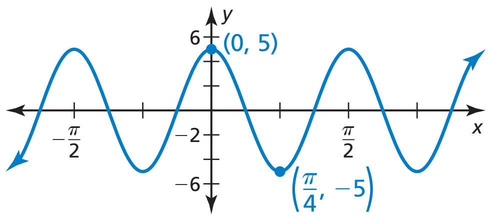
12. **MODELING WITH MATHEMATICS** A middle-A tuning fork vibrates with a frequency f of 440 hertz (cycles per second). You strike a middle-A tuning fork with a force that produces a maximum pressure of 5 pascals. Write and graph a sine model that gives the pressure P as a function of the time t (in seconds).

In Exercises 13–16, write a function for the sinusoid. *(See Example 2.)*

13.



14.



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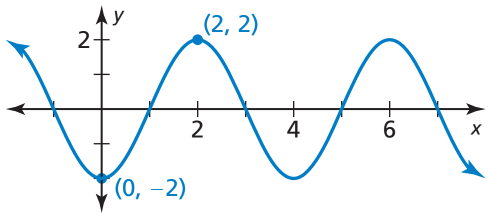
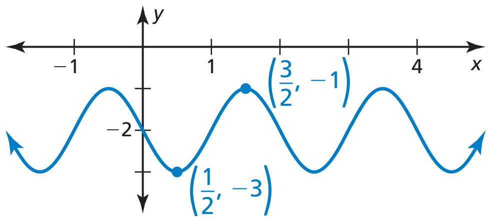


16.



17. ERROR ANALYSIS Describe and correct the error in finding the amplitude of a sinusoid with a maximum point at (2, 10) and a minimum point at (4, −6).

$$|a| = \frac{(\text{maximum value}) + (\text{minimum value})}{2}$$
$$= \frac{10 - 6}{2}$$
$$= 2$$

18. ERROR ANALYSIS Describe and correct the error in finding the vertical shift of a sinusoid with a maximum point at (3, −2) and a minimum point at (7, −8).

$$k = \frac{(\text{maximum value}) + (\text{minimum value})}{2}$$
$$= \frac{7 + 3}{2}$$
$$= 5$$

19. MODELING WITH MATHEMATICS One of the largest sewing machines in the world has a *flywheel* (which turns as the machine sews) that is 5 feet in diameter. The highest point of the handle at the edge of the flywheel is 9 feet above the ground, and the lowest point is 4 feet. The wheel makes a complete turn every 2 seconds. Write a model for the height h (in feet) of the handle as a function of the time t (in seconds) given that the handle is at its lowest point when $t = 0$. *(See Example 3.)*

20. MODELING WITH MATHEMATICS The Great Laxey Wheel, located on the Isle of Man, is the largest working water wheel in the world. The highest point of a bucket on the wheel is 70.5 feet above the viewing platform, and the lowest point is 2 feet below the viewing platform. The wheel makes a complete turn every 24 seconds. Write a model for the height h (in feet) of the bucket as a function of time t (in seconds) given that the bucket is at its lowest point when $t = 0$.

USING TOOLS In Exercises 21 and 22, the time t is measured in months, where $t = 1$ represents January. Write a model that gives the average monthly high temperature D as a function of t and interpret the period of the graph. *(See Example 4.)*

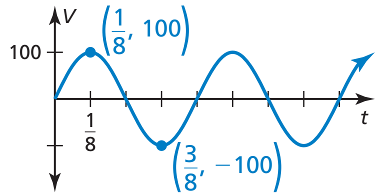
21.

Air Temperatures in Apple Valley, CA						
t	1	2	3	4	5	6
D	60	63	69	75	85	94
t	7	8	9	10	11	12
D	99	99	93	81	69	60

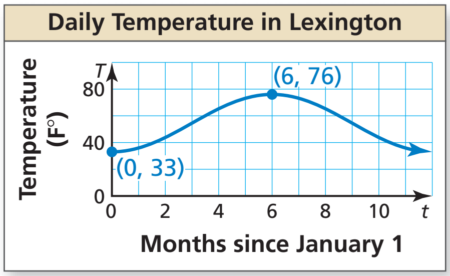
22.

Water Temperatures at Miami Beach, FL						
t	1	2	3	4	5	6
D	71	73	75	78	81	85
t	7	8	9	10	11	12
D	86	85	84	81	76	73

23. MODELING WITH MATHEMATICS A circuit has an alternating voltage of 100 volts that peaks every 0.5 second. Write a sinusoidal model for the voltage V as a function of the time t (in seconds).



24. MULTIPLE REPRESENTATIONS The graph shows the average daily temperature of Lexington, Kentucky. The average daily temperature of Louisville, Kentucky, is modeled by $y = -22 \cos \frac{\pi}{6}t + 57$, where y is the temperature (in degrees Fahrenheit) and t is the number of months since January 1. Which city has the greater average daily temperature? Explain.



25. USING TOOLS The table shows the numbers of employees N (in thousands) at a sporting goods company each year for 11 years. The time t is measured in years, with $t = 1$ representing the first year.

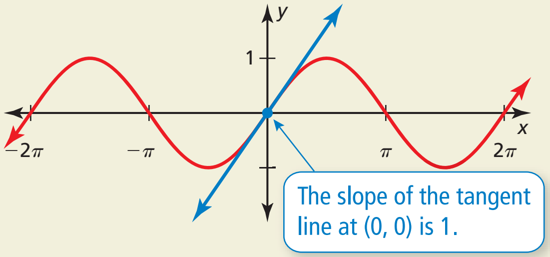
t	1	2	3	4	5	6
N	20.8	22.7	24.6	23.2	20	17.5

t	7	8	9	10	11	12
N	16.7	17.8	21	22	24.1	

a. Use sinusoidal regression to find a model that gives N as a function of t.

b. Predict the number of employees at the company in the 12th year.

26. THOUGHT PROVOKING The figure shows a tangent line drawn to the graph of the function $y = \sin x$. At several points on the graph, draw a tangent line to the graph and estimate its slope. Then plot the points (x, m), where m is the slope of the tangent line. What can you conclude?



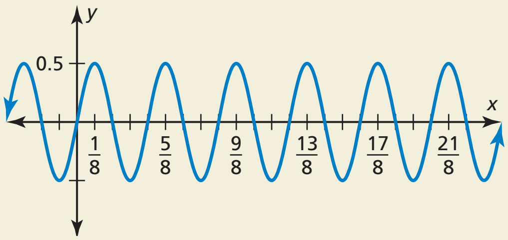
27. REASONING Determine whether you would use a sine or cosine function to model each sinusoid with the y-intercept described. Explain your reasoning.

a. The y-intercept occurs at the maximum value of the function.

b. The y-intercept occurs at the minimum value of the function.

c. The y-intercept occurs halfway between the maximum and minimum values of the function.

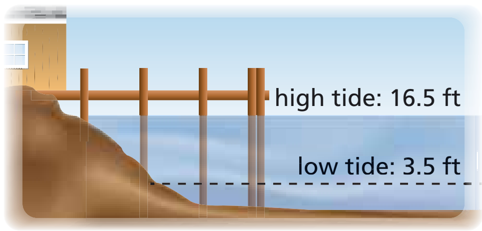
28. HOW DO YOU SEE IT? What is the frequency of the function whose graph is shown? Explain.



29. USING STRUCTURE During one cycle, a sinusoid has a minimum at $\left(\frac{\pi}{2}, 3\right)$ and a maximum at $\left(\frac{\pi}{4}, 8\right)$. Write a sine function *and* a cosine function for the sinusoid. Use a graphing calculator to verify that your answers are correct.

30. MAKING AN ARGUMENT Your friend claims that a function with a frequency of 2 has a greater period than a function with a frequency of $\frac{1}{2}$. Is your friend correct? Explain your reasoning.

31. PROBLEM SOLVING The low tide at a port is 3.5 feet and occurs at midnight. After 6 hours, the port is at high tide, which is 16.5 feet.



a. Write a sinusoidal model that gives the tide depth d (in feet) as a function of the time t (in hours). Let $t = 0$ represent midnight.

b. Find all the times when low and high tides occur in a 24-hour period.

c. Explain how the graph of the function you wrote in part (a) is related to a graph that shows the tide depth d at the port t hours after 3:00 A.M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Section 5.2)*

32. $\dfrac{17}{\sqrt{2}}$ **33.** $\dfrac{3}{\sqrt{6} - 2}$ **34.** $\dfrac{8}{\sqrt{10} + 3}$ **35.** $\dfrac{13}{\sqrt{3} + \sqrt{11}}$

Expand the logarithmic expression. *(Section 6.5)*

36. $\log_8 \dfrac{x}{7}$ **37.** $\ln 2x$ **38.** $\log_3 5x^3$ **39.** $\ln \dfrac{4x^6}{y}$