9.6 Solving Nonlinear Systems of Equations

Essential Question How can you solve a system of two equations when one is linear and the other is quadratic?

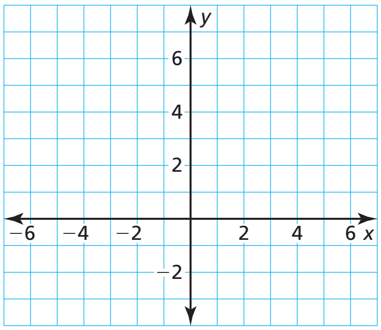
EXPLORATION 1 Solving a System of Equations

Work with a partner. Solve the system of equations by graphing each equation and finding the points of intersection.

System of Equations

$y = x + 2$ Linear

$y = x^2 + 2x$ Quadratic



EXPLORATION 2 Analyzing Systems of Equations

Work with a partner. Match each system of equations with its graph. Then solve the system of equations.

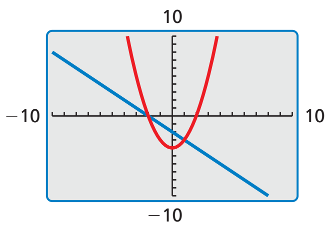
a. $y = x^2 - 4$
$y = -x - 2$

b. $y = x^2 - 2x + 2$
$y = 2x - 2$

c. $y = x^2 + 1$
$y = x - 1$

d. $y = x^2 - x - 6$
$y = 2x - 2$

A.



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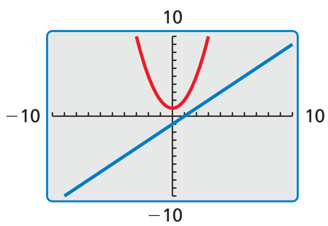

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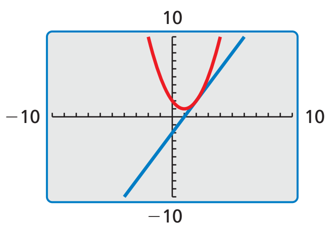

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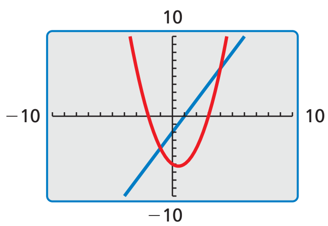


MAKING SENSE OF PROBLEMS

To be proficient in math, you need to analyze givens, relationships, and goals.

Communicate Your Answer

3. How can you solve a system of two equations when one is linear and the other is quadratic?

4. Write a system of equations (one linear and one quadratic) that has (a) no solutions, (b) one solution, and (c) two solutions. Your systems should be different from those in Explorations 1 and 2.

9.6 Lesson

Core Vocabulary

system of nonlinear equations, *p. 526*

Previous

system of linear equations

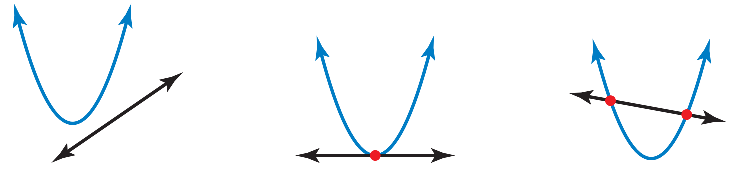
What You Will Learn

- Solve systems of nonlinear equations by graphing.
- Solve systems of nonlinear equations algebraically.
- Approximate solutions of nonlinear systems and equations.

Solving Nonlinear Systems by Graphing

The methods for solving systems of linear equations can also be used to solve *systems of nonlinear equations*. A **system of nonlinear equations** is a system in which at least one of the equations is nonlinear.

When a nonlinear system consists of a linear equation and a quadratic equation, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions, as shown.



No solutions **One solution** **Two solutions**

EXAMPLE 1 Solving a Nonlinear System by Graphing

Solve the system by graphing.

$y = 2x^2 + 5x - 1$ Equation 1

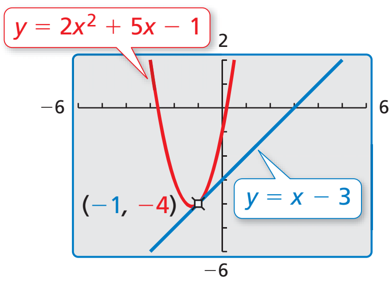
$y = x - 3$ Equation 2

SOLUTION

Step 1 Graph each equation.

Step 2 Estimate the point of intersection. The graphs appear to intersect at $(-1, -4)$.

Step 3 Check the point from Step 2 by substituting the coordinates into each of the original equations.



Equation 1

$y = 2x^2 + 5x - 1$

$-4 \stackrel{?}{=} 2(-1)^2 + 5(-1) - 1$

$-4 = -4$ ✓

Equation 2

$y = x - 3$

$-4 \stackrel{?}{=} -1 - 3$

$-4 = -4$ ✓

▶ The solution is $(-1, -4)$.

Monitoring Progress

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Solve the system by graphing.

1. $y = x^2 + 4x - 4$
$y = 2x - 5$

2. $y = -x + 6$
$y = -2x^2 - x + 3$

3. $y = 3x - 15$
$y = \frac{1}{2}x^2 - 2x - 7$

REMEMBER

The algebraic procedures that you use to solve nonlinear systems are similar to the procedures that you used to solve linear systems in Sections 5.2 and 5.3.

Solving Nonlinear Systems Algebraically

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system by substitution.

$y = x^2 + x - 1$ Equation 1

$y = -2x + 3$ Equation 2

SOLUTION

Step 1 The equations are already solved for y.

Step 2 Substitute $-2x + 3$ for y in Equation 1 and solve for x.

$-2x + 3 = x^2 + x - 1$ Substitute $-2x + 3$ for y in Equation 1.

$3 = x^2 + 3x - 1$ Add $2x$ to each side.

$0 = x^2 + 3x - 4$ Subtract 3 from each side.

$0 = (x + 4)(x - 1)$ Factor the polynomial.

$x + 4 = 0$ or $x - 1 = 0$ Zero-Product Property

$x = -4$ or $x = 1$ Solve for x.

Step 3 Substitute -4 and 1 for x in Equation 2 and solve for y.

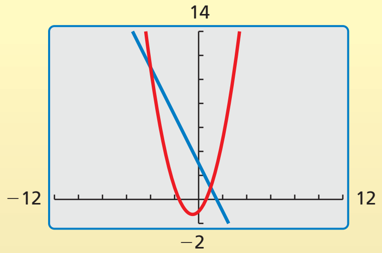
$y = -2(-4) + 3$ Substitute for x in Equation 2. $y = -2(1) + 3$

$= 11$ Simplify. $= 1$

▶ So, the solutions are $(-4, 11)$ and $(1, 1)$.

Check

Use a graphing calculator to check your answer. Notice that the graphs have two points of intersection at $(-4, 11)$ and $(1, 1)$.



EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system by elimination.

$y = x^2 - 3x - 2$ Equation 1

$y = -3x - 8$ Equation 2

SOLUTION

Step 1 Because the coefficients of the y-terms are the same, you do not need to multiply either equation by a constant.

Step 2 Subtract Equation 2 from Equation 1.

$y = x^2 - 3x - 2$ Equation 1

$y = \quad -3x - 8$ Equation 2

$0 = x^2 \quad + 6$ Subtract the equations.

Step 3 Solve for x.

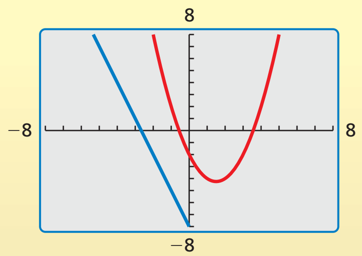
$0 = x^2 + 6$ Resulting equation from Step 2

$-6 = x^2$ Subtract 6 from each side.

▶ The square of a real number cannot be negative. So, the system has no real solutions.

Check

Use a graphing calculator to check your answer. The graphs do not intersect.



Monitoring Progress

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Solve the system by substitution.

4. $y = x^2 + 9$
$y = 9$

5. $y = -5x$
$y = x^2 - 3x - 3$

6. $y = -3x^2 + 2x + 1$
$y = 5 - 3x$

Solve the system by elimination.

7. $y = x^2 + x$
$y = x + 5$

8. $y = 9x^2 + 8x - 6$
$y = 5x - 4$

9. $y = 2x + 5$
$y = -3x^2 + x - 4$

Approximating Solutions

When you cannot find the exact solution(s) of a system of equations, you can analyze output values to approximate the solution(s).

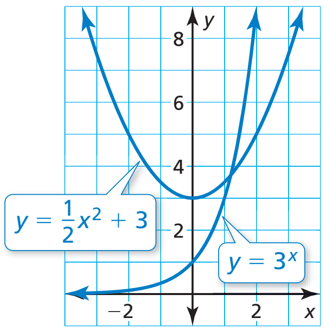
EXAMPLE 4 Approximating Solutions of a Nonlinear System

Approximate the solution(s) of the system to the nearest thousandth.

$y = \frac{1}{2}x^2 + 3$ — Equation 1

$y = 3^x$ — Equation 2

SOLUTION



Sketch a graph of the system. You can see that the system has one solution between $x = 1$ and $x = 2$.

Substitute 3^x for y in Equation 1 and rewrite the equation.

$3^x = \frac{1}{2}x^2 + 3$ — Substitute 3^x for y in Equation 1.

$3^x - \frac{1}{2}x^2 - 3 = 0$ — Rewrite the equation.

Because you do not know how to solve this equation algebraically, let $f(x) = 3^x - \frac{1}{2}x^2 - 3$. Then evaluate the function for x-values between 1 and 2.

$f(1.1) \approx -0.26$
$f(1.2) \approx 0.02$
Because $f(1.1) < 0$ and $f(1.2) > 0$, the zero is between 1.1 and 1.2.

REMEMBER
The function values that are closest to 0 correspond to x-values that best approximate the zeros of the function.

$f(1.2)$ is closer to 0 than $f(1.1)$, so decrease your guess and evaluate $f(1.19)$.

$f(1.19) \approx -0.012$ — Because $f(1.19) < 0$ and $f(1.2) > 0$, the zero is between 1.19 and 1.2. So, increase guess.

$f(1.191) \approx -0.009$ — Result is negative. Increase guess.

$f(1.192) \approx -0.006$ — Result is negative. Increase guess.

$f(1.193) \approx -0.003$ — Result is negative. Increase guess.

$f(1.194) \approx -0.0002$ — Result is negative. Increase guess.

$f(1.195) \approx 0.003$ — Result is positive.

Because $f(1.194)$ is closest to 0, $x \approx 1.194$.

Substitute $x = 1.194$ into one of the original equations and solve for y.

$$y = \frac{1}{2}x^2 + 3 = \frac{1}{2}(1.194)^2 + 3 \approx 3.713$$

▶ So, the solution of the system is about (1.194, 3.713).

REMEMBER

When entering the equations, be sure to use an appropriate viewing window that shows all the points of intersection. For this system, an appropriate viewing window is $-4 \le x \le 4$ and $-4 \le y \le 4$.

Recall from Section 5.5 that you can use systems of equations to solve equations with variables on both sides.

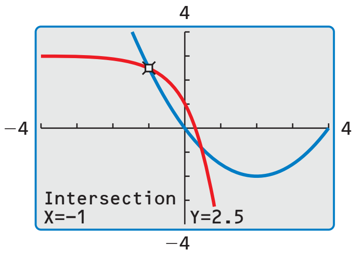
EXAMPLE 5 Approximating Solutions of an Equation

Solve $-2(4)^x + 3 = 0.5x^2 - 2x$.

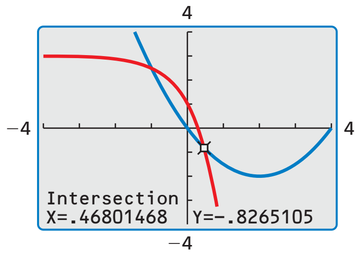
SOLUTION

You do not know how to solve this equation algebraically. So, use each side of the equation to write the system $y = -2(4)^x + 3$ and $y = 0.5x^2 - 2x$.

Method 1 Use a graphing calculator to graph the system. Then use the *intersect* feature to find the coordinates of each point of intersection.



One point of intersection is $(-1, 2.5)$.



The other point of intersection is about $(0.47, -0.83)$.

▶ So, the solutions of the equation are $x = -1$ and $x \approx 0.47$.

Method 2 Use the *table* feature to create a table of values for the equations. Find the x-values for which the corresponding y-values are approximately equal.

X	Y1	Y2
-1.03	2.5204	2.5905
-1.02	2.5137	2.5602
-1.01	2.5069	2.5301
-1	2.5	2.5
-.99	2.493	2.4701
-.98	2.4859	2.4402
-.97	2.4788	2.4105
X=-1		

When $x = -1$, the corresponding y-values are 2.5.

X	Y1	Y2
.44	-.6808	-.7832
.45	-.7321	-.7988
.46	-.7842	-.8142
.47	-.8371	-.8296
.48	-.8906	-.8448
.49	-.9449	-.86
.50	-1	-.875
X=.47		

When $x = 0.47$, the corresponding y-values are approximately -0.83.

▶ So, the solutions of the equation are $x = -1$ and $x \approx 0.47$.

STUDY TIP

You can use the differences between the corresponding y-values to determine the best approximation of a solution.

Monitoring Progress

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Use the method in Example 4 to approximate the solution(s) of the system to the nearest thousandth.

10. $y = 4^x$
$y = x^2 + x + 3$

11. $y = 4x^2 - 1$
$y = -2(3)^x + 4$

12. $y = x^2 + 3x$
$y = -x^2 + x + 10$

Solve the equation. Round your solution(s) to the nearest hundredth.

13. $3^x - 1 = x^2 - 2x + 5$

14. $4x^2 + x = -2\left(\frac{1}{2}\right)^x + 5$

9.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Describe how to use substitution to solve a system of nonlinear equations.

2. **WRITING** How is solving a system of nonlinear equations similar to solving a system of linear equations? How is it different?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the system of equations with its graph. Then solve the system.

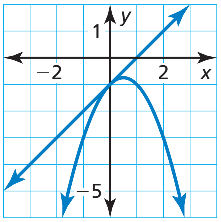
3. $y = x^2 - 2x + 1$
   $y = x + 1$

4. $y = x^2 + 3x + 2$
   $y = -x - 3$

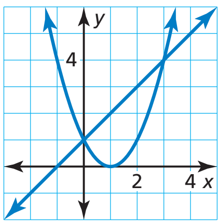
5. $y = x - 1$
   $y = -x^2 + x - 1$

6. $y = -x + 3$
   $y = -x^2 - 2x + 5$

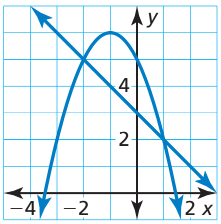
A.



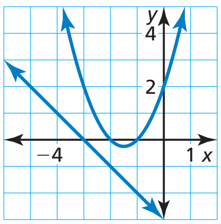
B.



C.



D.



In Exercises 7–12, solve the system by graphing. *(See Example 1.)*

7. $y = 3x^2 - 2x + 1$
   $y = x + 7$

8. $y = x^2 + 2x + 5$
   $y = -2x - 5$

9. $y = -2x^2 - 4x$
   $y = 2$

10. $y = \frac{1}{2}x^2 - 3x + 4$
    $y = x - 2$

11. $y = \frac{1}{3}x^2 + 2x - 3$
    $y = 2x$

12. $y = 4x^2 + 5x - 7$
    $y = -3x + 5$

In Exercises 13–18, solve the system by substitution. *(See Example 2.)*

13. $y = x - 5$
    $y = x^2 + 4x - 5$

14. $y = -3x^2$
    $y = 6x + 3$

15. $y = -x + 7$
    $y = -x^2 - 2x - 1$

16. $y = -x^2 + 7$
    $y = 2x + 4$

17. $y - 5 = -x^2$
    $y = 5$

18. $y = 2x^2 + 3x - 4$
    $y - 4x = 2$

In Exercises 19–26, solve the system by elimination. *(See Example 3.)*

19. $y = x^2 - 5x - 7$
    $y = -5x + 9$

20. $y = -3x^2 + x + 2$
    $y = x + 4$

21. $y = -x^2 - 2x + 2$
    $y = 4x + 2$

22. $y = -2x^2 + x - 3$
    $y = 2x - 2$

23. $y = 2x - 1$
    $y = x^2$

24. $y = x^2 + x + 1$
    $y = -x - 2$

25. $y + 2x = 0$
    $y = x^2 + 4x - 6$

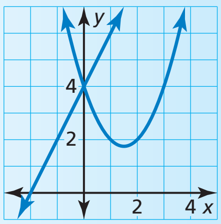
26. $y = 2x - 7$
    $y + 5x = x^2 - 2$

27. **ERROR ANALYSIS** Describe and correct the error in solving the system of equations by graphing.

✗

$y = x^2 - 3x + 4$
$y = 2x + 4$

The only solution of the system is (0, 4).



28. ERROR ANALYSIS Describe and correct the error in solving for one of the variables in the system.

✗

$y = 3x^2 - 6x + 4$

$y = 4$

$y = 3(4)^2 - 6(4) + 4$ Substitute.

$y = 28$ Simplify.

In Exercises 29–32, use the table to describe the locations of the zeros of the quadratic function *f*.

29.

x	−4	−3	−2	−1	0	1
$f(x)$	−2	2	4	4	2	−2

30.

x	−1	0	1	2	3	4
$f(x)$	11	5	1	−1	−1	1

31.

x	−4	−3	−2	−1	0	1
$f(x)$	3	−1	−1	3	11	23

32.

x	1	2	3	4	5	6
$f(x)$	−25	−9	1	5	3	−5

In Exercises 33–38, use the method in Example 4 to approximate the solution(s) of the system to the nearest thousandth. *(See Example 4.)*

33. $y = x^2 + 2x + 3$
$y = 3^x$

34. $y = 2^x + 5$
$y = x^2 - 3x + 1$

35. $y = 2(4)^x - 1$
$y = 3x^2 + 8x$

36. $y = -x^2 - 4x - 4$
$y = -5^x - 2$

37. $y = -x^2 - x + 5$
$y = 2x^2 + 6x - 3$

38. $y = 2x^2 + x - 8$
$y = x^2 - 5$

In Exercises 39–46, solve the equation. Round your solution(s) to the nearest hundredth. *(See Example 5.)*

39. $3x + 1 = x^2 + 7x - 1$

40. $-x^2 + 2x = -2x + 5$

41. $x^2 - 6x + 4 = -x^2 - 2x$

42. $2x^2 + 8x + 10 = -x^2 - 2x + 5$

43. $-4\left(\frac{1}{2}\right)^x = -x^2 - 5$

44. $1.5(2)^x - 3 = -x^2 + 4x$

45. $8^{x-2} + 3 = 2\left(\frac{3}{2}\right)^x$

46. $-0.5(4)^x = 5^x - 6$

47. COMPARING METHODS Solve the system in Exercise 37 using substitution. Compare the exact solutions to the approximated solutions.

48. COMPARING METHODS Solve the system in Exercise 38 using elimination. Compare the exact solutions to the approximated solutions.

49. MODELING WITH MATHEMATICS The attendances y for two movies can be modeled by the following equations, where x is the number of days since the movies opened.

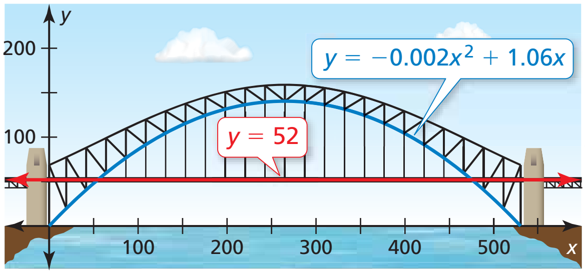
$y = -x^2 + 35x + 100$ Movie A

$y = -5x + 275$ Movie B

When is the attendance for each movie the same?

50. MODELING WITH MATHEMATICS You and a friend are driving boats on the same lake. Your path can be modeled by the equation $y = -x^2 - 4x - 1$, and your friend's path can be modeled by the equation $y = 2x + 8$. Do your paths cross each other? If so, what are the coordinates of the point(s) where the paths meet?

51. MODELING WITH MATHEMATICS The arch of a bridge can be modeled by $y = -0.002x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. The road can be modeled by the equation $y = 52$. To the nearest meter, how far from the left pylons are the two points where the road intersects the arch of the bridge?



52. MAKING AN ARGUMENT Your friend says that a system of equations consisting of a linear equation and a quadratic equation can have zero, one, two, or infinitely many solutions. Is your friend correct? Explain.

COMPARING METHODS **In Exercises 53 and 54, solve the system of equations by (a) graphing, (b) substitution, and (c) elimination. Which method do you prefer? Explain your reasoning.**

53. $y = 4x + 3$
$y = x^2 + 4x - 1$

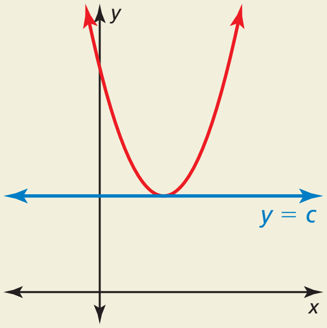
54. $y = x^2 - 5$
$y = -x + 7$

55. **MODELING WITH MATHEMATICS** The function $y = -x^2 + 65x + 256$ models the number y of subscribers to a website, where x is the number of days since the website launched. The number of subscribers to a competitor's website can be modeled by a linear function. The websites have the same number of subscribers on Days 1 and 34.

a. Write a linear function that models the number of subscribers to the competitor's website.

b. Solve the system to verify the function from part (a).

56. **HOW DO YOU SEE IT?** The diagram shows the graphs of two equations in a system that has one solution.



a. How many solutions will the system have when you change the linear equation to $y = c + 2$?

b. How many solutions will the system have when you change the linear equation to $y = c - 2$?

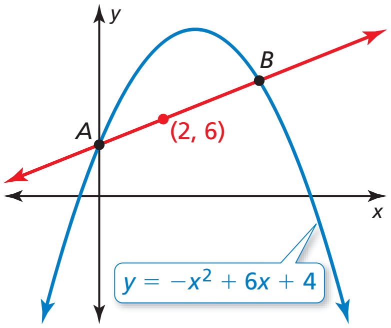
57. **WRITING** A system of equations consists of a quadratic equation whose graph opens up and a quadratic equation whose graph opens down. Describe the possible numbers of solutions of the system. Sketch examples to justify your answer.

58. **PROBLEM SOLVING** The population of a country is 2 million people and increases by 3% each year. The country's food supply is sufficient to feed 3 million people and increases at a constant rate that feeds 0.25 million additional people each year.

a. When will the country first experience a food shortage?

b. The country doubles the rate at which its food supply increases. Will food shortages still occur? If so, in what year?

59. **ANALYZING GRAPHS** Use the graphs of the linear and quadratic functions.



a. Find the coordinates of point A.

b. Find the coordinates of point B.

60. **THOUGHT PROVOKING** Is it possible for a system of two quadratic equations to have exactly three solutions? exactly four solutions? Explain your reasoning. (*Hint*: Rotations of the graphs of quadratic equations still represent quadratic equations.)

61. **PROBLEM SOLVING** Solve the system of three equations shown.

$y = 2x - 8$
$y = x^2 - 4x - 3$
$y = -3(2)^x$

62. **PROBLEM SOLVING** Find the point(s) of intersection, if any, of the line $y = -x - 1$ and the circle $x^2 + y^2 = 41$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the system of linear inequalities. *(Section 5.7)*

63. $y > 2x$
$y > -x + 4$

64. $y \geq 4x + 1$
$y \leq 7$

65. $y - 3 \leq -2x$
$y + 5 < 3x$

66. $x + y > -6$
$2y \leq 3x + 4$

Graph the function. Describe the domain and range. *(Section 8.3)*

67. $y = 3x^2 + 2$

68. $y = -x^2 - 6x$

69. $y = -2x^2 + 12x - 7$

70. $y = 5x^2 + 10x - 3$