9. TEKNIK PENGINTEGRALAN

9.1 Integral Parsial

Formula Integral Parsial :

$$\int u\, dv = uv - \int v\, du$$

Cara : pilih *u* yang turunannya lebih sederhana

Contoh : Hitung $\int xe^x dx$

misal $u = x$, maka $du=dx$

$$dv = e^x\, dx \;\Rightarrow v = \int e^x\, dx = e^x$$

sehingga

$$\int xe^x dx = x\ e^x - \int e^x\, dx = x\ e^x - e^x + C$$

Integral parsial dapat dilakukan lebih dari satu kali

Contoh Hitung $\int x^2 \sin x\, dx = -x^2 \cos x + \int x \cos x dx$

Integral parsial

Jawab

(i) Misal $u = x^2 \longrightarrow$ du = 2xdx

dv = sinxdx $\longrightarrow$ V=-cosx

(ii) Misal u = x $\longrightarrow$ du = dx

dv = cosx dx $\longrightarrow$ v = sinx

$= -x^2 \cos x + (x \sin x - \int \sin x\, dx)$

$= -x^2 \cos x + x \sin x + \cos x + C$

Ada kemungkinan integran (f(x)) muncul lagi diruas kanan

Contoh Hitung $\int e^x \cos x dx$

Integral parsial

Jawab :

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

(i) Misal $u = e^x \longrightarrow du = e^x dx$

dv=cosxdx $\longrightarrow$ v=sinx

(ii) Misal $u = e^x \longrightarrow du = e^x dx$

dv = sinxdx $\longrightarrow$ v=-cosx

$$= e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) + C$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx) + C$$

Integral yang dicari ,bawa keruas kanan

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C$$

Soal latihan

Hitung

1. $\int_1^e \ln x\,dx$
2. $\int x \ln x\,dx$
3. $\int \ln(1+x^2)\,dx$
4. $\int \sin^{-1} x\,dx$
5. $\int \tan^{-1} x\,dx$
6. $\int x \tan^{-1} x\,dx$

9.2 Integral Fungsi Trigonometri

Bentuk : $\int \cos^n x\, dx \;\; \& \;\; \int \sin^n x\, dx$

* Untuk *n* ganjil, Tuliskan :

$$\sin^n x = \sin x \, \sin^{n-1} x \;\; \text{dan} \;\; \cos^n x = \cos x \cos^{n-1} x$$

dan gunakan identitas $\sin^2 x + \cos^2 x = 1$

* Untuk *n* genap, Tuliskan :

$$\sin^n x = \sin^2 x \sin^{n-2} x \;\; \text{dan} \;\; \cos^n x = \cos^2 x \cos^{n-2} x$$

dan gunakan identitas $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Contoh Hitung

1. $\int \sin^3 x\,dx$
2. $\int \sin^4 x\,dx$

Jawab

1. $\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1-\cos^2 x) d(\cos x) = \cos x - \frac{1}{3}\cos^3 x + C$

2. $\int \sin^4 x\,dx = \int \sin^2 x \sin^2 x\,dx = \int (\frac{1-\cos 2x}{2})(\frac{1-\cos 2x}{2})\,dx$

$$= \frac{1}{4}\int (1-2\cos 2x + \cos^2 2x)dx = \frac{1}{4}(\int dx - 2\int \cos 2x\,dx + \int \frac{1+\cos 4x}{2}\,dx)$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

- Bentuk $\int \sin^m x \cos^n x \, dx$

a). Untuk *n* atau *m* ganjil, keluarkan sin *x* atau cos *x* dan gunakan identitas $\sin^2 x + \cos^2 x = 1$

b). Untuk *m* dan *n* genap, tuliskan $\sin^m x$ dan $\cos^n x$ menjadi jumlah suku-suku dalam cosinus, gunakan identitas $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Contoh :

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = -\int \left(1 - \cos^2 x\right) \cos^2 x \, d(\cos x)$$

$$= -\int \left(\cos^2 x - \cos^4 x\right) x \, d(\cos x)$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos 2x}{2} \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{1}{4}\int (1+\cos^2 2x)dx \quad = \frac{1}{4}(\int 1+\frac{1+\cos 4x}{2} dx)$$

$$= \frac{3}{8}\int dx + \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{3}{8}x + \frac{1}{32}\sin 4x + C$$

Bentuk $\int \tan^m x \sec^n x dx$ dan $\int \cot^m x \csc^n x dx$

Gunakan identitas

$$\tan^2 x = \sec^2 x - 1 \ , \cot^2 x = \csc^2 x - 1$$

serta turunan tangen dan kotangen

$$d(\tan x) = \sec^2 x d \ , d(\cot x) = -\csc^2 x dx$$

Contoh

a. $\int \tan^4 x dx$

$$= \int \tan^2 x \tan^2 x dx = \int \tan^2 x (\sec^2 - 1) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$$= \int \tan^2 x d(\tan x) - \int (\sec^2 x - 1) dx$$

$$= \tfrac{1}{3} \tan^3 x - \tan x + x + C$$

b. $$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x dx$$

$$= \int \tan^2 x(1+\tan^2 x)d(\tan x)$$

$$= \int \tan^2 x + \tan^4 x \, dx$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$$

Soal Latihan

Hitung

1. $\int \sin^4 x \ \cos^5 x \ dx$

2. $\int_0^{\pi/4} \tan^4 t \sec^2 t \, dt$

3. $\int \sec^4 x \, dx$

4. $\int \cot^2 w \csc^4 w \, dw$

5. $\int \csc^3 x \ dx$

9.3 Substitusi Trigonometri

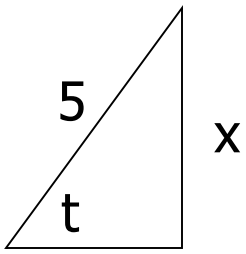
a. Integran memuat bentuk $\sqrt{a^2 - x^2}$,misal $x = a\sin t$

Contoh Hitung $\int \frac{\sqrt{25 - x^2}}{x^2} dx$

$$\int \frac{\sqrt{25 - x^2}}{x^2} dx = \int \frac{\sqrt{25 - 25\sin^2 t}\ \ 5\cos t\ dt}{25\sin^2 t}$$

Misal $x = 5\sin t$

dx = 5 cost dt



$$= \int \frac{\sqrt{25(1 - \sin^2 t)}}{5\sin^2 t} \cos t dt = \int \frac{\cos^2 t}{\sin^2 t} dt = \int \cot^2 t\ dt$$

$$= \int (\csc^2 t - 1) dt = -\cot t - t + c$$

$$= -\frac{\sqrt{25 - x^2}}{x} - \sin^{-1}(\frac{x}{5}) + C$$

b. Integran memuat bentuk $\sqrt{a^2+x^2}$,misal $x = a\tan t$

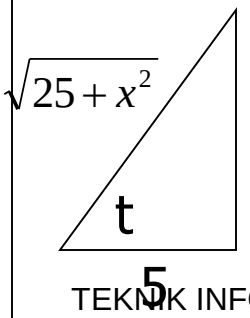
Contoh Hitung $\int \frac{1}{x^2\sqrt{25+x^2}}\,dx$

$$\int \frac{1}{x^2\sqrt{25+x^2}}\,dx = \int \frac{5\sec^2 t\ dt}{25\tan^2 t\sqrt{25+25\tan^2 t}}$$

Misal $x = 5\tan t$

$dx = 5\sec^2 t\ dt$

$\tan t = \frac{x}{5}$



$$= \frac{1}{25}\int \frac{\sec^2 t\,dt}{\tan^2 t\ \sec t} = \frac{1}{25}\int \frac{\cos t}{\sin^2 t}\,dt = \frac{1}{25}\int \frac{d(\sin(t))}{\sin^2 t}$$

$$= -\frac{1}{25\sin t} + C = -\frac{\sqrt{25+x^2}}{25x} + C$$

c. Integran memuat bentuk $\sqrt{x^2-a^2}$,misal $x = a\sec t$

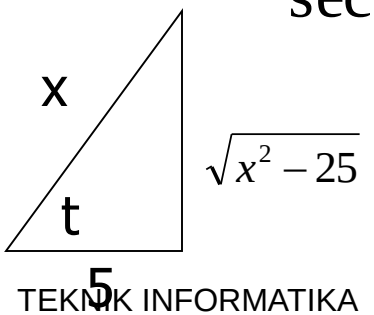
Contoh Hitung $\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

$$\int \frac{1}{x^2\sqrt{x^2-25}}\,dx = \int \frac{5\sec t\,\tan t\,dt}{25\sec^2 t\sqrt{25\sec^2 t-25}}$$

Misal $x = 5\sec t$

$$dx = 5\sec t\,\tan t\,dt$$

$$\sec t = \frac{x}{5}$$



$$= \frac{1}{25}\int \frac{\sec t\tan t\,dt}{\sec^2 t\tan t} = \frac{1}{25}\int \frac{\sec t}{\sec^2 t}\,dt = \frac{1}{25}\int \cos t\,dt$$

$$= \frac{1}{25}\sin t + C = \frac{\sqrt{x^2-25}}{25x} + C$$

Soal Latihan

Hitung

1. $\int \frac{x^2\,dx}{\sqrt{9-x^2}}dx$
2. $\int \frac{2x-3}{\sqrt{4-x^2}}\,dx$
3. $\int \frac{dx}{x^2\sqrt{4-x^2}}$
4. $\int \frac{dx}{x\sqrt{x^2+9}}$
5. $\int \frac{dx}{x^2\sqrt{x^2-16}}$
6. $\int \frac{dx}{\left(x^2+9\right)^{3/2}}$
7. $\int \frac{3x\ \ dx}{\sqrt{x^2+2x+5}}$
8. $\int \sqrt{5-4x-x^2}\ dx$
9. $\int \frac{2x+1}{x^2+2x+2}\,dx$

Substitusi Bentuk Akar

Integran memuat $\sqrt[n]{ax+b}$,misal $u=\sqrt[n]{ax+b}$

Contoh Hitung $\int \frac{dx}{2+2\sqrt{x}}$

Jawab :

$$\int \frac{dx}{2+2\sqrt{x}} = \int \frac{2u\,du}{2+2u} = \int \frac{u}{u+1}\,du$$

Misal $u=\sqrt{x} \longrightarrow u^2 = x$

Dengan turunan implisit

$2u\frac{du}{dx}=1 \longrightarrow$ dx=2udu

$$= \int \frac{u+1-1}{u+1}du = \int (1-\frac{1}{u+1})du$$

$$= u - \ln(u+1) + C$$

$$= \sqrt{x} - \ln\left(1+\sqrt{x}\right) + C$$

Soal Latihan

Hitung

1. $\int x \sqrt[3]{x+4}\ dx$
2. $\int \frac{x^2+2x}{\sqrt{x+1}}\ dx$
3. $\int \frac{\sqrt{t}}{t+1}\ dt$
4. $\int x\sqrt{x+1}\ dx$
5. $\int \frac{t}{\sqrt{3t+4}}\ dt$
6. $\int x(1-x)^{2/3}\ dx$

9.4 Integral Fungsi Rasional

- Integran berbentuk fungsi rasional : $f(x) = \dfrac{P(x)}{Q(x)}$, der (P)< der(Q)

- Ada 4 kasus dari pemfaktoran penyebut ($Q(x)$) yaitu :
  1. Faktor linear tidak berulang.
  2. Faktor linear berulang.
  3. Faktor kuadratik tidak berulang.
  4. Faktor kuadratik berulang.

- **Kasus 1 (linier tidak berulang)**

  Misal $Q(x) = (a_1 x + b_1)(a_2 x + b_2) \ldots (a_n x + b_n)$

  maka, $\dfrac{P(x)}{Q(x)} \equiv \dfrac{A_1}{a_1 x + b_1} + \dfrac{A_2}{a_2 x + b_2} + \ldots + \dfrac{A_n}{a_n x + b_n}$

  dengan $A_1, A_2, \ldots, A_n$ konstanta yang dicari.

Contoh Hitung $\int \frac{x+1}{x^2-9}dx$

Jawab

Faktorkan penyebut : $x^2-9=(x-3)(x+3)$

$$\frac{x+1}{x^2-9}=\frac{A}{(x+3)}+\frac{B}{(x-3)}=\frac{A(x-3)+B(x+3)}{(x-3)(x+3)}$$

$$\Leftrightarrow x+1=A(x-3)+B(x+3)\ =(A+B)x+(-3A+3B)$$

Samakan koefisien ruas kiri dan ruas kanan

A +B =1 | x3
-3A+3B=1 | x1

$\Rightarrow$

3A +3B=3
-3A+3B=1 +
6B=4 $\Rightarrow$ B=2/3 ,A=1/3

Sehingga

$$\int \frac{x+1}{x^2-9}dx=\int \frac{1/3}{(x+3)}dx+\int \frac{2/3}{(x-3)}dx\ =\frac{1}{3}\ln|x+3|+\frac{2}{3}\ln|x-3|+C$$

Kasus 2 Linear berulang

Misal $Q(x) = \left(a_i\, x + b_i \right)^p$

Maka

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_i x + b_i)} + \frac{A_2}{(a_i x + b_i)^2} + \ldots + \frac{A_{p-1}}{(a_i x + b_i)^{p-1}} + \frac{A_p}{(a_i x + b_i)^p}$$

dengan konstanta $A_1, A_2, \ldots, A_{p-1}, A_p$ akan dicari

Contoh Hitung $\int \frac{1}{(x+2)^2 (x-1)}\, dx$

Jawab

$$\frac{1}{(x+2)^2 (x-1)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)}$$

$$\frac{1}{(x+2)^2(x-1)} = \frac{A(x+2)(x-1)+B(x-1)+C(x+2)^2}{(x+2)^2(x-1)}$$

Penyebut ruas kiri = penyebut ruas kanan

$$1 = A(x+2)(x-1)+B(x-1)+C(x+2)^2$$

$$1 = (A+C)x^2+(A+B+4C)x+(4C-2A-B)$$

A+C=0
A+B+4C=0
-2A-B+4C=1

A+B+4C=0
-2A-B+4C=1 +
-A+8C=1

A+C=0
-A+8C=1 +
9C=1 ⟶ C=1/9

B=-1/3
A=-1/9

$$\int \frac{1}{(x+2)^2(x-1)}dx = \frac{-1}{9}\int \frac{1}{(x+2)}dx - \frac{1}{3}\int \frac{1}{(x+2)^2}dx + \frac{1}{9}\int \frac{1}{(x-1)}dx$$

$$= -\frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + \frac{1}{9}\ln|x-1| + C$$

Kasus 3 Kuadratik tak berulang

Misal

$$Q(x) = \left(a_1 x^2 + b_1 x + c_1 \right) \left(a_2 x^2 + b_2 x + c_2 \right) \ldots \left(a_n x^2 + b_n x + c_n \right)$$

Maka

$$\frac{P(x)}{Q(x)} \equiv \frac{A_1 x + B_1}{a_1 x^2 + b_1 x + c_1} + \frac{A_2 x + B_2}{a_2 x^2 + b_2 x + c_2} + \ldots + \frac{A_n x + B_n}{a_n x^2 + b_n x + c_n}$$

Dengan $A_1, A_2, \ldots, A_n$, dan $B_1, B_2, \ldots, B_n$ konstanta yang akan dicari

Contoh Hitung $\int \frac{dx}{x\left(x^2+1\right)}$

Jawab

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{\left(x^2+1\right)} \quad = \frac{A\left(x^2+1\right)+(Bx+c)x}{x\left(x^2+1\right)}$$

$$1 = A\left(x^2+1\right)+(Bx+c)x \longrightarrow 1 = (A+B)x^2 + cx + A$$

A+B=0
C=0
A=1 $\longrightarrow$ B=-1

$$\int \frac{1}{x\left(x^2+1\right)}dx = \int \frac{1}{x}dx - \int \frac{x}{\left(x^2+1\right)}dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

$$\int \frac{x}{\left(x^2+1\right)}dx = \int \frac{x}{x^2+1}\frac{d(x^2+1)}{2x}$$

$$= \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$$

Kasus 4 Kuadratik berulang

Misal $Q(x) = \left(a_i\, x^2 + b_i\, x + c_i \right)^p$

Maka

$$\frac{P(x)}{Q(x)} \equiv \frac{A_1 x + B_1}{\left(a_i x^2 + b_i x + c_i\right)} + \frac{A_2 x + B_2}{\left(a_i x^2 + b_i x + c_i\right)^2} + \ldots + \frac{A_{p-1} x + B_{p-1}}{\left(a_i x^2 + b_i x + c_i\right)^{p-1}} + \frac{A_p x + B_p}{\left(a_i x^2 + b_i x + c_i\right)^p}$$

Dimana $A_1, A_2, \ldots, A_{p-1}, A_p \; dan \; B_1, B_2, \ldots, B_{p-1}, B_p$ konstanta yang akan dicari

Contoh Hitung $\int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}\,dx$

Jawab :

$$\frac{6x^2-15x+22}{(x+3)(x^2+2)^2} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+2)} + \frac{Dx+E}{(x^2+2)^2}$$

$$= \frac{A(x^2+2)^2 + (Bx+C)(x^2+2)(x+3) + (Dx+E)(x+3)}{(x+3)(x^2+2)^2}$$

$$6x^2 - 15x + 22 = A(x^2+2)^2 + (Bx+C)(x^2+2)(x+3) + (Dx+E)(x+3)$$

$$6x^2 - 15x + 22 = (A+B)x^4 + (3B+C)x^3 + (4A+2B+3C+D)x^2 +$$

$$(6B+2C+3D+E)x + (4A+6C+3E)$$

Dengan menyamakan koefisien ruas kiri dan kanan diperoleh

A+B=0
3B+C=0
4A+2B+3C+D=1
6B+2C+3D+E=-15
4A+6C+3E=22

Dengan eliminasi : A=1,B=-1, C=3
D=-5, E=0

Sehingga

$$\int \frac{6x^2-15x+22}{(x+3)(x^2+2)^2}\,dx = \int \frac{1}{(x+3)}dx - \int \frac{x-3}{(x^2+2)}dx - 5\int \frac{x}{(x^2+2)^2}dx$$

$$= \int \frac{dx}{x+3} - \frac{1}{2}\int \frac{2x}{x^2+2}dx + 3\int \frac{dx}{x^2+2} - \frac{5}{2}\int \frac{2x}{(x^2+2)^2}\,dx$$

$$= \ln|x+3| - \frac{1}{2}\ln(x^2+2) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{5}{2(x^2+2)} + C.$$

Catatan jika $der(P(x)) \geq der(Q(x))$, bagi terlebih dahulu P(x) dengan Q(x), sehingga

$$\frac{P(x)}{Q(x)} = H(x) + \frac{S(x)}{Q(x)} \quad , der(S(x)) < der(Q(x))$$

Contoh Hitung

$$\int \frac{x^3 + 2x^2 + x - 4}{x^2 - 4} dx \longrightarrow$$ Der(P(x))=3>der(Q(x))=2

Bagi terlebih dahulu P(x) dengan Q(x)

$$\begin{array}{r|l} & x + 2 \\ \hline x^2 - 4 & x^3 + 2x^2 + x - 4 \\ & x^3 - 4x \quad - \\ \hline & 2x^2 + 5x - 4 \\ & 2x^2 - 8 \quad - \\ \hline & 5x + 4 \end{array}$$

$$\Rightarrow \quad \frac{x^3 + 2x^2 + x - 4}{x^2 - 4} = x + 2 + \frac{5x + 4}{x^2 - 4}$$

$$\frac{5x+4}{x^2-4} = \frac{5x+4}{(x-2)(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)}$$

$$= \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$5x+4 = A(x+2)+B(x-2) \quad \text{.....(*)}$$

Persamaan (*) berlaku untuk sembarang x, sehingga berlaku juga untuk Untuk x=2 dan x=-2

Untuk x = 2 ⟶ 5.2+4=A(2+2) ⟶ A=7/2

Untuk x = -2 ⟶ 5.(-2)+4=B(-2-2) ⟶ B=3/2

Dengan menggunakan hasil diatas :

$$\int \frac{x^3+2x^2+x-4}{x^2-4}\,dx = \int (x+2)dx + \frac{7}{2}\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}x^2 + 2x + \frac{7}{2}\ln|x-2| + \frac{3}{2}\ln|x+2| + C$$

Soal Latihan

Hitung

1. $\int \frac{2x-1}{x^2-7x-18}dx$
2. $\int \frac{1}{(x+5)^2(x-1)}dx$
3. $\int \frac{5x^2+3x-2}{x^3+2x^2}dx$
4. $\int \frac{dx}{x(x^2+1)^2}$
5. $\int_2^5 \frac{x^2+2x}{x^3+3x^2+4}dx$
6. $\int \frac{2x^2-3x-36}{(2x-1)(x^2+9)}dx$
7. $\int \frac{x^3+x^2}{x^2+5x+6}dx$