PYTHAGOREAN TRIANGLES OF EQUAL AREAS

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ABSTRACT. The main intent in this paper is to find triples of Rational Pythagorean Triangles (abbr. RPT) having equal areas. A new method of solving $a^2 + ab + b^2 = c^2$ is to set $a = y - 1$, $b = y + 1$, $y \in N - \{0,1\}$ and get Pell's equation $c^2 - 3y^2 = 1$. To solve $a^2 - ab + b^2 = c^2$, we set $a = \frac{1}{2}(y+1)$, $b = y - 1$, $y \geq 2$, $y \in N$ and get a corresponding Pell's equation. The infinite number of solutions in Pell's equation gives rise to an infinity of solutions to $a^2 \pm ab + b^2 = c^2$. From this fact the following theorems are proved.

Theorem 1 Let $c^2 = a^2 + ab + b^2$, $a + b > c > b > a > 0$, then the three RPT-s formed by (c,a), (c,b), $(a+b,c)$ have the same area $S = abc(b + a)$ and there are infinitely many such triples of RPT.

Theorem 2 Let $c^2 = a^2 - ab + b^2$, $b > c > a > 0$, then the three RPT-s formed by (b,c), (c,a), $(c,b-a)$ have the same area $S = abc(b-a)$ and there are infinitely many such triples of RPT.

1. DEFINITIONS AND PREVIOUS RESULTS. In one of his papers Bernstein [1] returned to the Grecian classical mathematics, and investigated primitive rational Pythagorean Triangles concerning mainly k-tuples of them having equal perimeters 2P.

In this paper we deal with rational integral right triangles having equal areas and give the following

<u>DEFINITION</u>   A rational triangle with sides a,b,c which are represented by a triple (a,b,c) of natural numbers will be called a Rational Pythagorean Triangle, if and only if

$$\left. \begin{array}{l} \text{there exists } (u,v), (u,v) \in N^2 - \{(0,0)\}, \\ u > v, \text{ such that} \\ a = u^2 - v^2, \ b = 2uv, \ c = u^2 + v^2, \\ a,b,c \in N - \{0\} = 1,2,\ldots \end{array} \right\} \qquad (1.1)$$

We abbreviate Rational Pythagorean Triangles by RPT and we write RPT(u,v) for RPT formed by (u,v). We also write S(u,v) for the area and P(u,v) for half perimeter of the RPT(u,v)

$$D = \text{RPT}(u,v) : S(u,v) = \tfrac{1}{2}ab = uv(u^2 - v^2) \qquad (1.2)$$

$$P(u,v) = \tfrac{1}{2}(a+b+c) = u(u+v)$$

The main intent in this paper is to find triplets of RPT-s having equal areas.

The first who asked this question was the great Diophantus [3] and Dickson [2] enlarged the topic.

Let D be a triangle with integral sides and $\hat{C} = 120°$ one of its angles. Then if c is the side opposite $\hat{C}$ and a,b the two adjacent sides of $\hat{C}$, we have, by $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$

$$\left. \begin{array}{l} a^2 + ab + b^2 = c^2 \\ a + b > c > b > a; \ a,b,c \in N - \{0\} \end{array} \right\} \qquad (1.3)$$

and if $\hat{C} = 60°$ we have

$$\left. \begin{array}{l} a^2 - ab + b^2 = c^2 \\ b > c > a > 0; \ a,b,c \in N - \{0\}. \end{array} \right\} \qquad (1.4)$$

The totality of solutions to $a^2 \pm ab + b^2 = c^2$ is given in parameter form by Hasse [6]. <u>The new idea in this paper rests in the fact that (1.3) and (1.4) are connected with the areas of the triangles</u>. In order to find a formula to derive explicitly the infinity of RPT-s of equal area, since we cannot use Hasse's [6] parametric form, we will give a new method to prove that the equations $a^2 \pm ab + b^2 = c^2$ have infinitely many solutions and state some of them explicitly.

The new method will bring us to the solution of a Pell's equation. The infinite number of solutions of Pell's equation [4] will give rise to an infinity of solutions to $a^2 \pm ab + b^2 = c^2$.

2. <u>PELL'S (EULER'S) EQUATION</u>: $u_n^2 - 3v_n^2 = 1; \ n = 0,1,\ldots$. In the sequel we shall permanently have to make use of Pell's equation

$$u_n^2 - 3v_n^2 = 1, \ n = 0,1,\ldots . \qquad (2.1)$$

This could be solved by continued fraction with $\sqrt{3} + 1 = [\overline{2,1}]$, but we use a simpler method since a solution of (2.1) is easily found. Neglecting $(u_0, v_0) = (1,0)$ we have

$$(u_1, v_1) = (2,1), \tag{2.2}$$

hence using [5]

$$u_n + v_n\sqrt{3} = (2 + \sqrt{3})^n, \quad n = 0,1,\ldots . \tag{2.3}$$

From (2.3) we deduce

$$\left.\begin{array}{l}
u_n = \displaystyle\sum_{i=0}^{[\frac{n}{2}]} \binom{n}{2i} 2^{n-2i} 3^i; \quad n = 0,1,\ldots \\[20pt]
v_n = \displaystyle\sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} 2^{n-1-2i} 3^i; \quad n = 1,2,\ldots
\end{array}\right\} \tag{2.4}$$

From (2.4) we obtain

$$\left.\begin{array}{l}
u_{2m} = \displaystyle\sum_{i=0}^{m} \binom{2m}{2i} 2^{2m-2i} 3^i; \quad m = 1,2,\ldots \\[20pt]
v_{2m} = \displaystyle\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^{2m-1-2i} 3^i; \quad m = 1,2,\ldots \\[12pt]
(u_0, v_0) = (1,0)
\end{array}\right\} \tag{2.5}$$

$$\left.\begin{array}{l}
u_{2m+1} = \displaystyle\sum_{i=0}^{m} \binom{2m+1}{2i} 2^{2m+1-2i} 3^i; \quad m = 0,1,\ldots \\[20pt]
v_{2m+1} = \displaystyle\sum_{i=0}^{m} \binom{2m+1}{2i+1} 2^{2m-2i} 3^i; \quad m = 0,1,\ldots
\end{array}\right\} \tag{2.6}$$

In u_{2m} all summands are even but the last which is 3^m; in v_{2m} all summands are even, hence

$$\left.\begin{array}{l}
u_{2m} = 2F + 1; \\
v_{2m} = 2G; \quad F,G \in N
\end{array}\right\} \tag{2.7}$$

In u_{2m+1} all summands are even; in v_{2m+1} all summands are even but the last which is 3^m; hence

$$\begin{array}{l}
u_{2m+1} = 2S; \\
v_{2m+1} = 2T + 1; \quad S,T \in N.
\end{array} \tag{2.7a}$$

We have the initial values

$$\left.\begin{array}{l}
(u_0, v_0) = (1,0); \quad (u_1, v_1) = (2,1); \quad (u_2, v_2) = (7,4) \\
(u_3, v_3) = (26,15); \quad (u_4, v_4) = (97,56); \quad (u_5, v_5) = (362,209) \\
(u_6, v_6) = (1351,780); \quad (u_7, v_7) = (5092,2911).
\end{array}\right\} \tag{2.8}$$

3. A SOLUTION OF (1.3) $a^2 + ab + b^2 = c^2$. Here we shall give infinitely many solutions of (1.3) in explicit form, though they won't constitute all solutions of (1.3). Let

$$a^2 + ab + b^2 = c^2; \quad a + b > c > b > a > 0,$$
$$a, b, c \in N - \{0\}. \quad\quad\quad\quad\quad\quad\quad\quad (3.1)$$

We set

$$a = y - 1; \quad b = y + 1; \quad y \in N - \{0\}. \quad\quad\quad (3.2)$$

Substituting the values of a,b from (3.2) in (3.1) we obtain

$$(y - 1)^2 + (y^2 - 1) + (y + 1)^2 = c^2$$

or $\quad\quad c^2 - 3y^2 = 1$.

Now we set $c = u_n$, $y = v_n$, $n = 1, 2, \ldots$ and we get $u_n^2 - 3v_n^2 = 1$ or $u_n^2 - 3v_n^2 = 1$ which is (2.1).

The infinite number of solutions u_n, v_n in Pell's equation (2.1), give rise to an infinity of solutions to $a^2 + ab + b^2 = c^2$.

THEOREM 1. Let $c^2 = a^2 + ab + b^2$, $a+b > c > b > a > 0$ then the three RPT-s formed by (c,a), (c,b), (a+b,c) have the same area $S = abc(b+a)$ and there are infinitely many such triples of RPT-s.

Proof. To get the example given by Diophantus [3, p. 172] as a particular case of our formulas we consider $(a,b,c) = 1$. Since we prefer $(a,b,c) = 1$, though this is not absolutely necessary, we have to set v_n even in (2.1), so that a,b are both odd. If $v_n = v_{2m}$ we obtain

$$a = v_{2m} - 1; \quad b = v_{2m} + 1; \quad c = u_{2m}; \quad m = 1, 2, \ldots \quad (3.3)$$

Using (1.2) and $a^2 + ab + b^2 = c^2$ it is easy to show that

$$S(c,a) = S(c,b) = S(a+b,c) = abc(b+a).$$
$$D_1 = RPT(c,a) = RPT(u_{2m}, v_{2m}-1);$$
$$D_2 = RPT(c,b) = RPT(u_{2m}, v_{2m}+1);$$
$$D_3 = RPT(a+b,c) = RPT(2v_{2m}, u_{2m}),$$

and following $S = abc(b+a)$ the common area is

$$S = 2 u_{2m} v_{2m} (v_{2m}^2 - 1). \quad\quad\quad\quad (3.4)$$

That there are infinitely many such triples of RPT-s follow from the infinity of solutions of (2.1).

Choosing m = 1, we have from (2.8)

$$(u_2, v_2) = (7, 4); \quad u_2 = 7, \quad v_2 = 4$$
$$c = 7; \quad a = 3; \quad b = 5 \quad\quad\quad\quad\quad (3.5)$$
$$S = 3 \cdot 5 \cdot 7 \cdot 8 = 840$$

and this is exactly the example given by Diophantus.

If we choose m = 2, we have

$$(u_4, v_4) = (97,56); \quad u_4 = 97; \quad v_4 = 56;$$

$$c = 97; \quad a = 55; \quad b = 57. \qquad\qquad\qquad (3.6)$$

$$S = 55 \cdot 57 \cdot 97 \cdot 112 = 34,058,640.$$

For RPT(97,55) using (1.2) we have

$$S(97,55) = 97 \cdot 55(97^2 - 55^2) =$$
$$= 55 \cdot 57 \cdot 97 \cdot 112 = S \text{ in } (3.6).$$

4. <u>SOLUTION OF (1.4)</u> $a^2 - ab + b^2 = c^2$. In this chapter we state more triples of triangles RPT having the same area. We shall first prove

Let

$$a^2 - ab + b^2 = c^2; \quad a,b,c \in N - \{0\}; \text{ then} \qquad (4.1)$$
$$b > c > a > 0; \quad (a,b,c) = 1$$

In (4.1) we set arbitrarily $b > a$; then we have

$$b^2 - c^2 = a(b-a) > 0; \quad b > c; \qquad\qquad\qquad (4.2)$$
$$c^2 - a^2 = b(b-a) > 0; \quad c > a;$$

Thus $b > c > a > 0$, as stated in (4.1). We shall now prove

<u>THEOREM 2</u>. Let $c^2 = a^2 - ab + b^2$, $b > c > a > 0$. Then the three RPT-s formed by (b,c), (c,a), $(c,b-a)$ have the same area S, viz.

$$S = abc(b-a) \qquad\qquad\qquad\qquad\qquad (4.3)$$

and there are infinitely many such triples RPT.

<u>Proof</u>. If we set in (1.4) $a \rightarrow -a$, we obtain from equation (1.3), the equation (4.1). But this is only an algebraic formality, since the RPT formed by $(c,-a)$ makes sense, though from $S = abc(a+b)$ we obtain (4.4) by substituting $-a$ for a. We also have

$$c^2 = (a-b)^2 + ab, \quad c > a-b, \ b-a. \qquad\qquad (4.4)$$

Now

$$D_1 = RPT(b,c); \quad S(b,c) = \frac{1}{2} \cdot 2bc(b^2 - c^2) = \frac{1}{2} xy$$
$$= \frac{1}{2} \cdot 2bca(b-a) = abc(b-a);$$

$$D_2 = RPT(c,a); \quad S(c,a) = \frac{1}{2} \cdot 2ca(c^2 - a^2) = \frac{1}{2} xy$$
$$= \frac{1}{2} \cdot 2cab(b-a) = abc(b-a);$$

$$D_3 = RPT(c,b-a); \quad S(c,b-a) = \frac{1}{2} \cdot 2c(b-a)[c^2 - (b-a)^2] = \frac{1}{2} xy$$
$$= c(b-a)ab = abc(b-a)$$

$$S = S(b,c) = S(c,a) = S(c,b-a) = abc(b-a).$$

We still have to prove that (4.1) has infinitely many solutions. We shall give two methods to find these solutions, though these may not be <u>all</u> infinitely many solutions of (4.1). This was done by Hasse [6] with algebraic number theory which we shall avoid here, giving simple methods to solve (4.1) in explicit form.

In (4.1) we set
$$b = a + v, \quad v = b - a > 0; \tag{4.5}$$
we obtain, setting $a = b - v$,
$$(b-v)^2 - b(b-v) + b^2 = c^2$$
$$b^2 - bv + v^2 - c^2 = 0$$
$$v^2 - bv - (c^2 - b^2) = 0. \tag{4.6}$$
From (4.6) we obtain
$$v = \frac{1}{2}(b \pm \sqrt{b^2 + 4(c^2 - b^2)}),$$
$$v = \frac{1}{2}(b \pm \sqrt{4c^2 - 3b^2}). \tag{4.7}$$
We have, since setting $4c^2 - 3b^2 = x^2$ contradicts our condition $(a,b,c) = 1$
$$4c^2 - 3b^2 = 1. \tag{4.8}$$
Thus we have arrived at Pell's equation, setting
$$2c = u_n, \quad b = v_n. \tag{4.9}$$
Thus u_n is even, and we must take $u_n = u_{2m+1}$ and from (2.7a) we obtain
$$\left. \begin{array}{l} b = v_{2m+1}; \quad c = \frac{1}{2} u_{2m+1}; \\ a = b - v = b - \frac{1}{2}(b \pm 1). \end{array} \right\} \tag{4.10}$$

Since we obtain two values for a, we may have obtained <u>six</u> RPT-s with equal area. We shall investigate this later. We first take $v = \frac{1}{2}(b+1)$
$$\left. \begin{array}{l} a = \frac{1}{2}(b-1) = \frac{1}{2}(v_{2m+1} - 1) \\ b = v_{2m+1}; \quad c = \frac{1}{2} u_{2m+1}; \\ b - a = \frac{1}{2}(v_{2m+1} + 1). \end{array} \right\} \tag{4.11}$$
Hence, from (4.3), or forming
$$D_1 = RPT(b,c), \quad D_2 = RPT(c,a), \quad D_3 = RPT(c,b-a)$$
$$S = \frac{1}{2}(v_{2m+1} - 1) \cdot v_{2m+1} \cdot \frac{1}{2} u_{2m+1} \cdot \frac{1}{2}(v_{2m+1} + 1)$$
$$S = \frac{1}{8} u_{2m+1} v_{2m+1} (v_{2m+1}^2 - 1). \tag{4.12}$$
We take, for an example $m = 1$,
$$(u_3, v_3) = (26, 15)$$
$$u_{2m+1} = 26; \quad v_{2m+1} = 15$$
$$S = \frac{1}{8} \cdot 26 \cdot 15(15^2 - 1) = 10,920.$$

The reader should note that since
$$v_{2m+1} = 2S + 1, \quad v_{2m+1}^2 - 1 \equiv 0 (\bmod 8).$$
Hence, in (4.12) S is an integer. We now take $v = \frac{1}{2}(b-1)$ and obtain from (4.10)

$$b = v_{2m+1}; \quad c = \frac{1}{2} u_{2m+1};$$

$$a = b-v = \frac{1}{2}(b+1) = \frac{1}{2}(v_{2m+1} + 1) \qquad\qquad (4.13)$$

$$b-a = \frac{1}{2}(v_{2m+1} - 1),$$

and from $D_4 = RPT(b,c)$, $D_5 = RPT(c,a)$, $D_6 = RPT(c,b-a)$. Previously, we had for $v = \frac{1}{2}(b+1)$,

$$a = \frac{1}{2}(v_{2m}-1); \quad b = v_{2m+1}, \quad c = \frac{1}{2} u_{2m+1}$$

$$b - a = \frac{1}{2}(v_{2m} + 1).$$

Comparing (4.11) with (4.13) we see that

$$D_2 = D_6$$

$$D_3 = D_5$$

$$D_1 = D_4$$

and thus get the same triple of RTS-s in both cases.

5. <u>SECOND SOLUTION OF $c^2 = a^2 - ab + b^2$</u>. We give a second method of finding infinitely many solutions of (4.1). We set here

$$a = \frac{1}{2}(y+1); \quad b = y-1; \quad y \geq 2; \quad y \in N. \qquad\qquad (5.1)$$

Substituting these values in (4.1), we obtain

$$[\frac{1}{2}(y+1)]^2 - \frac{1}{2}(y^2-1) + (y-1)^2 = c^2$$

$$y^2 + 2y+1 - 2y^2 + 2 + 4y^2 - 8y + 4 = 4c^2$$

$$3y^2 - 6y + 7 = 4c^2$$

$$3(y-1)^2 + 4 = 4c^2$$

$$4c^2 - 3(y-1)^2 = 4. \qquad\qquad (5.2)$$

Since from (2.7) y-1 has to be even we can cancel (5.2) by 4 and we obtain

$$y-1 = 2v_{2m}, \quad y = 2v_{2m} + 1; \qquad\qquad (5.3)$$

and with $c = u_{2m}$ we obtain

$$u_{2m}^2 - 3v_{2m}^2 = 1$$

$$a = \frac{1}{2}(2v_{2m} + 2),$$

$$a = v_{2m} + 1; \quad b = 2v_{2m}, \quad c = u_{2m} \qquad\qquad (5.4)$$

$$S = abc(b-a) = (v_{2m} + 1)2v_{2m} \cdot u_{2m}(v_{2m} - 1)$$

$$S = 2u_{2m}v_{2m}(v_{2m}^2 - 1). \qquad\qquad (5.5)$$

Comparing (5.5) with (4.12) we may think about the different expressions for area S; though one is expressed by u_{2m+1}, v_{2m+1}, the other by u_{2m},

v_{2m}, it is so because we are dealing here with different areas, depending on the values of a,b,c, which are different in each case. For an example for (5.5) we choose m = 2,

$$(u_4, v_4) = (97, 56),$$

$$S = 2 \cdot 97 \cdot 56 \cdot (56^2 - 1) = 34,058,640.$$

If we set

$$\left.\begin{array}{l} a = 2m-1, \quad b = 2v_{2m}, \quad c = u_n, \quad b-a = v_{2m}+1, \\[2mm] u_{2m}^2 - 3v_{2m}^2 = 1 \end{array}\right\} \qquad (5.6)$$

then $a^2 - ab + b^2 = c^2$, $S = 2u_{2m}v_{2m}(v_{2m}^2 - 1)$. But we obtain nothing new since RPT(c,a), RPT(c,b-a) are interchanged with RPT(c,b-a), RPT(c,a).

The author is asking whether two successive Fibonacci numbers could be solutions a,b of $a^2 \pm ab + b^2 = c^2$. The Fibonacci sequence, see [8] with $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, n = 1,2,... goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Now $(F_4, F_5) = (3,5)$ is a solution of the Diophantine example $3^2 + 3 \cdot 5 + 5^2 = 7^2$, and $(F_5, F_6) = (5,8)$ is a solution of

$$5^2 - 5 \cdot 8 + 8^2 = 7^2.$$

Here S = abc(b-a) = $5 \cdot 8 \cdot 7 \cdot 3$ = 840 whether there are more pairs of adjacent Fibonacci numbers serving as solutions of $a^2 - ab + b^2 = c^2$ could not be decided generally here.

6. **PERIMETERS AND AREAS**. As known, P, half the perimeter of a RPT is a divisor of its area S, since

$$P = u(u+v), \quad S = u \cdot v(u^2 - v^2) \qquad (6.1)$$

$$S \div P = v(u-v)$$

where (u,v) forms the RPT. The question arises whether there are other connections between these two elements. Here we could only prove

THEOREM 3. Let $a^2 - ab + b^2 = c^2$, b > c > a > 0, a,b,c $\in$ N - {0}. Then the sum of the three perimeters formed by a,b,c, viz. RPT(b,c), RPT(c,a), RPT(c,b-a) is the sum of two squares.

Proof. We have

$$D_1 = RPT(b,c): \quad 2P_1 = x_1 + y_1 + z_1 = b^2 - c^2 + 2bc + b^2 + c^2$$

$$2P_1 = 2b(b+c);$$

$$D_2 = RPT(c,a): \quad 2P_2 = x_2 + y_2 + z_2 = c^2 - a^2 + 2ca + c^2 + a^2$$

$$= 2c(c+a);$$

$$D_3 = RPT(c,b-a): \quad 2P_3 = x_3 + y_3 + z_3 = c^2 - (b-a)^2 + 2c(b-a) + (b-a)^2 + c^2$$

$$= 2c^2 + 2c(b-a) = 2c(c+b-a).$$

$$P_1 + P_2 + P_3 = b(b+c) + c(c+a) + c(c+b) - ac$$

$$= 2c^2 + 2bc + b^2 = (b+c)^2 + c^2$$

7. **ENTIRELY NEW RPT-s.** In a very little known paper Hillyer [7] has given a most surprising infinity of triples of Rational Pythagorean triangles in an explicit form, having equal areas. These are formed by

$$
\left.
\begin{aligned}
D_1 &= RPT(u,v) = RPT(a^2 + ab + b^2,\ b^2 - a^2) \\
D_2 &= RPT(u,v) = RPT(a^2 + ab + b^2,\ 2ab + a^2) \\
D_3 &= RPT(u,v) = RPT(2ab + b^2,\ a^2 + ab + b^2) \\
&\quad b > a > 0;\ b,a \in Q^+
\end{aligned}
\right\}
\qquad (7.1)
$$

As we see, Hillyer was not concerned about RPT-s; his Pythagorean tri-angles had just to have rational sides. We shall operate with RPT-s only and set

$$a,b \in N - \{0\};\quad b > a > 0. \qquad (7.2)$$

We first investigate, whether the condition

$$u > v > 0 \qquad (7.3)$$

is fulfilled for D_1, D_2, D_3. We have

D_1: $u-v = a^2 + ab + b^2 - (b^2 - a^2)$
$\qquad = 2b^2 + ab > 0;\ u > v > 0.$

D_2: $u-v = a^2 + ab + b^2 - (2ab + a^2)$
$\qquad = b^2 - ab = b(b-a) > 0,$

since $b > a$ hence $u > v > 0$

D_3: $u-v = 2ab + b^2 - (a^2 + ab + b^2)$
$\qquad = ab - a^2 = a(b-a) > 0,$

since $b-a > 0$, $u > v > 0$.

We shall now find the areas of D_1, D_2, D_3, and have

D_1:$S_1 = (a^2+ab+b^2)(b^2-a^2)[(a^2+ab+b^2)^2-(b^2-a^2)^2]$
$\qquad = (a^2+ab+b^2)(b^2-a^2)(a^2+ab+b^2+b^2-a^2) \cdot (a^2+ab+b^2-b^2+a^2)$
$\qquad = (a^2+ab+b^2)(b^2-a^2)(ab+2b^2)(ab+2a^2),$

$$S_1 = ab(a^2+ab+b^2)(b^2-a^2)(a+2b)(b+2a). \qquad (7.4)$$

D_2:$S_2 = (a^2+ab+b^2)(2ab+a^2)[(a^2+ab+b^2)^2-(2ab+a^2)^2]$
$\qquad = (a^2+ab+b^2)a(a+2b)(a^2+ab+b^2+2ab+a^2) \cdot (a^2+ab+b^2-2ab-a^2)$
$\qquad = (a^2+ab+b^2)a(a+2b)(2a^2+3ab+b^2) \cdot (b^2-ab).$

Now

$$2a^2 + 3ab + b^2 = (2a+b)(a+b),$$

hence

$$S_2 = ab(a^2 + ab + b^2)(b^2-a^2)(a+2b)(b+2a) \qquad (7.5)$$

D_3:$S_3 = (2ab+b^2)(a^2+ab+b^2)[(2ab+b^2)^2-(a^2+ab+b^2)^2]$
$\qquad = b(2a+b)(a^2+ab+b^2)(2b^2+3ab+b^2)(ab-a^2).$

Now

$$2b^2 + 3ab + b^2 = (2b-a)(b+a).$$

Hence

$$S_3 = ab(a^2 + ab + b^2)(b^2 - a^2)(a + 2b)(b + 2a). \tag{7.6}$$

Thus $S_1 = S_2 = S_3 = \sum$,

$$\sum = ab(a^2 + ab + b^2)(b^2 - a^2)(a + 2b)(b + 2a). \tag{7.7}$$

We now return to the equation of Diophantus from (1.3)

$$\left. \begin{array}{l} a^2 + ab + b^2 = c^2 \\ a+b > c > b > a > 0 \\ a,b,c \in N - \{0\}. \end{array} \right\} \tag{1.3}$$

We found that the three triples RPT(c,b), RPT(c,a), RPT(a+b,c) have equal areas, viz.

$$S = abc(b+a).$$

If in (7.7) we demand that $a^2 + ab + b^2 = c^2$, solvable in natural number $a+b > c > b > a > 0$ we obtain

$$\sum = abc^2(b^2 - a^2)(a + 2b)(b + 2a), \tag{7.8}$$

and from (7.8), and $S = abc(a+b)$,

$$\sum \div S = c(b-a)(a+2b)(b+2a). \tag{7.9}$$

Now the quotient $\sum \div S$ is a natural number, and many authors have asked and solved the question of the ratio of the areas of two RPT-s.

8.   <u>THE MAIN RESULT</u>.   We now form three RPT-s, having equal areas. They are entirely new and unknown.   We investigate

$$\left. \begin{array}{l} D_1 = RPT(u,v) = RPT(a^2 - ab + b^2,\ b^2 - a^2); \\ D_2 = RPT(u,v) = RPT(a^2 - ab + b^2,\ 2ab - b^2); \\ D_3 = RPT(u,v) = RPT(2ab - a^2,\ a^2 - ab + b^2) \\ 2a > b > a > 0;\ b,a \in Q^+ \end{array} \right\} \tag{8.1}$$

We first check for D_1, D_2, D_3,

$$u > v > 0,\ u-v > 0.$$

The reader note the condition

$$2a > b.$$

Later when we shall operate with RPT-s and the equation

$$a^2 - ab + b^2 = c^2$$

we shall see that solutions of this equation are possible with $2a > b$. We have, $(u,v) \in Q^+$)

$$\begin{array}{l} D_1:\quad u-v = a^2 - ab + b^2 - (b^2 - a^2) \\ \qquad\quad = 2a^2 - ab = a(2a - b) > 0 \\ \qquad v = b^2 - a^2 > 0. \end{array}$$

D_2: $u-v = a^2 - ab + b^2 - (2ab - b^2)$

$\quad\quad = a^2 - 3ab + 2b^2$

$\quad\quad = (2b-a)(b-a) > 0$

$\quad v = 2ab - b^2 = b(2a-b) > 0.$

D_3: $u-v = 2ab - a^2 - (a^2 - ab + b^2) = 3ab - 2a^2 - b^2$

$\quad\quad = (2a - b)(b - a) > 0.$

$\quad u = 2ab - a^2 = a(2b - a) > 0$

$\quad v = a^2 - ab + b^2 = (a - b)^2 + ab > 0.$

We shall now find the areas formed by D_1, D_2, D_3, and have

D_1:$S_1 = (a^2-ab+b^2)(b^2-a^2)[(a^2-ab+b^2)^2-(b^2-a^2)]$

$\quad = (a^2-ab+b^2)(b^2-a^2)(a^2-ab+b^2-b^2+a^2)\cdot(a^2-ab+b^2+b^2-a^2)$

$\quad = (a^2-ab+b^2)(b^2-a^2)(2a^2-ab)(2b^2-ab)$

$\quad = (a^2-ab+b^2)(b^2-a^2)a(2a-b)(2b-a).$

$$S_1 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \qquad (8.2)$$

D_2: $S_2 = (a^2-ab+b^2)(2ab-b^2)[(a^2-ab-b^2)-(2ab-a)^2]$

$\quad = (a^2-ab+b^2)b(2a-b)(a^2-ab+b^2-2ab+b^2)(ab+a^2)$

$\quad = (a^2-ab+b^2)b(2a-b)(a^2-3ab+2b^2)a(b+a)$

$\quad = (a^2-ab+b^2)b(2a-b)(2b-a)(b-a)a(b+a)$

$\quad = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a).$

$$S_2 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \qquad (8.3)$$

D_3: $S_3 = (2ab-a^2)(a^2-ab+b^2)[(2ab-a^2)^2-(a^2-ab+b^2)^2]$

$\quad = (2ab-a^2)(a^2-ab+b^2)(2ab-a^2-a^2+ab-b^2)\cdot(2ab-a^2+a^2-ab+b^2)$

$\quad = a(2b-a)(a^2-ab+b^2)(3ab-2a^2-b^2)\cdot(ab+b^2)$

$\quad = ab(a^2-ab+b^2)(2b-a)(2a-b)(b-a)\cdot(a+b)$

$\quad = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a).$

$$S_3 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \qquad (8.4)$$

Thus we have obtained the wanted result

$$S_1 = S_2 = S_3 = \sum{}'.$$

We now return to the equation

$$\left.\begin{aligned} a^2 - ab + b^2 &= c^2, \quad b > a > 0 \\ b > c > 0&; \ b,c,a \in N - \{0\} \end{aligned}\right\} \qquad (4.1)$$

and recall from (5.4) that there is a solution with

$$a = \frac{b}{2} + 1; \quad 2a > b \qquad (8.5)$$

as we needed. With equation (4.1), $\sum{}'$ takes the form

$$\sideset{}{'}\sum = abc^2(b^2 - a^2)(2a-b)(2b-a)$$
$$2a > b > c > a; \quad a,b,c \in N - \{0\}. \qquad \left.\right\} \qquad (8.6)$$

Now with $a^2 - ab + b^2 = c^2$, and (4.3), viz.

$$S = abc(b-a)$$

and obtain thus the quotient

$$\sideset{}{'}\sum \div S = c(b+a)(2a-b)(2b-a). \qquad (8.7)$$

As an example we shall take
$$a^2 - ab + b^2 = c^2$$
$$(a,b,c) = (8,15,13).$$

Here $2a = 16 > 15 = b$; we obtain

$$\sideset{}{'}\sum = 8 \cdot 15 \cdot 169 \cdot 161 \cdot 1 \cdot 22,$$

$$\sideset{}{'}\sum = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 23 = 71,831,760.$$

$$S = abc(b-a) = 8 \cdot 15 \cdot 13 \cdot 7$$
$$= 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 10,920.$$

$$\sideset{}{'}\sum \div S = 2 \cdot 11 \cdot 13 \cdot 23 = 6,578$$

$$71,831,760 \div 10,920 = 6,578.$$

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