A brief introduction to dynamics (and stability)

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The Linear Oscillator

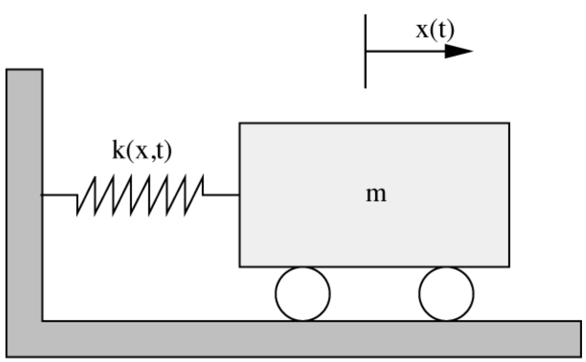
In mechanics we are primarily interested in the time evolution of systems governed by ode's:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) \quad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

And especially how the system responds to changes in a control parameter:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu, t) \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in \mathbb{R},$$

We start with the undamped, unforced linear oscillator:



$$\omega_n = \sqrt{k/m}$$

$$\ddot{x} + \omega_n^2 x = 0,$$

The solution can be written in various ways:

$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}.$$

$$x(t) = C\cos(\omega_n t) + D\sin(\omega_n t).$$

$$x(t) = \bar{A}\cos(\omega_n t + \phi),$$

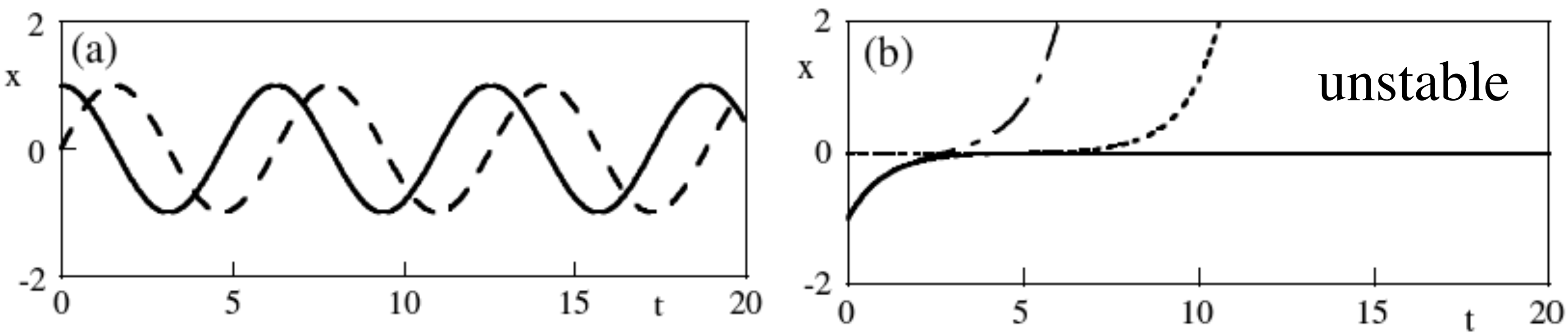
and more specifically, given two initial conditions:

$$x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t).$$

In terms of state space the system can be written:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

A conventional plot of the solution is the time series:



position and velocity tend to be π/2 out of phase.

We can also consider the conservation of total energy, where A is related to the initial conditions:

$$x^2(t) + (\dot{x}(t)/\omega)^2 = \bar{A}^2,$$

However, suppose the spring has a negative stiffness:

$$\ddot{x} - \omega_n^2 x = 0.$$

In this case the solution takes the form:

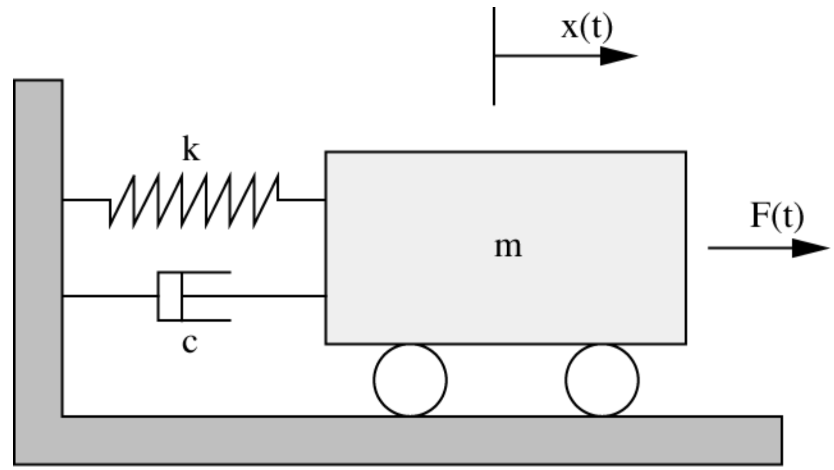
$$x(t) = ae^{\omega_n t} + be^{-\omega_n t},$$

and using the initial conditions:

$$x(t) = x(0)\cosh \omega_n t + (\dot{x}(0)/\omega_n)\sinh \omega_n t.$$

These solutions grow with time, for almost any initial conditions

Damping



Most real systems have some form of energy dissipation

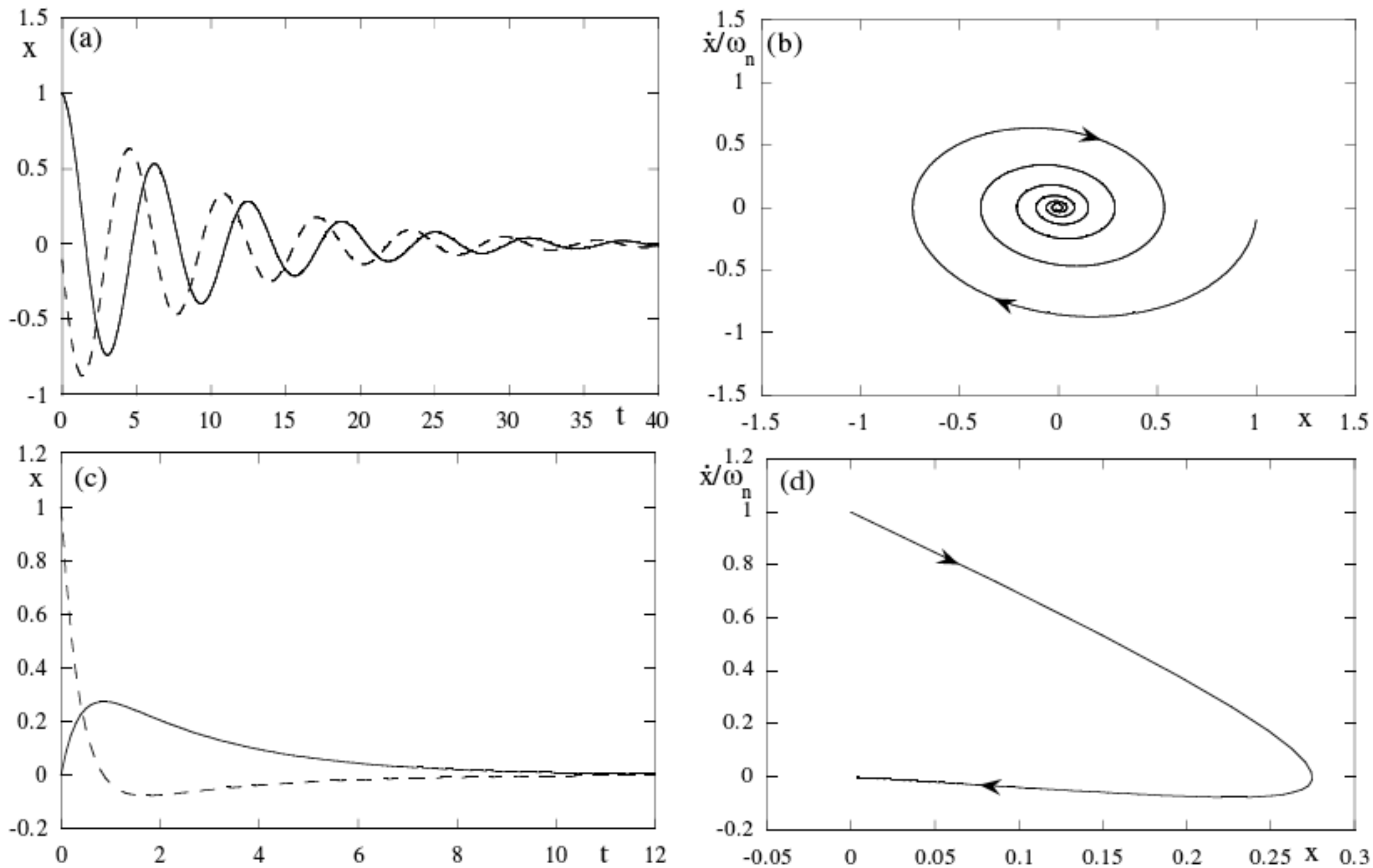
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0) + \zeta\omega_n\dot{x}(0)}{\omega_d}\sin\omega_d t + x(0)\cos\omega_d t\right),$$

For most structural/mechanical systems $\zeta \sim 0.1$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}.$$

The top two figures illustrate a typical underdamped response (the rhs show a phase portrait - position vs. velocity); The lower figures show a typical overdamped response:



and again the response can be written in terms of exponential functions:

$$x(t) = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$A = \frac{\dot{x}(0) + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$B = \frac{-\dot{x}(0) - (\zeta - \sqrt{\zeta^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2 - 1}}.$$

The state space form now becomes:

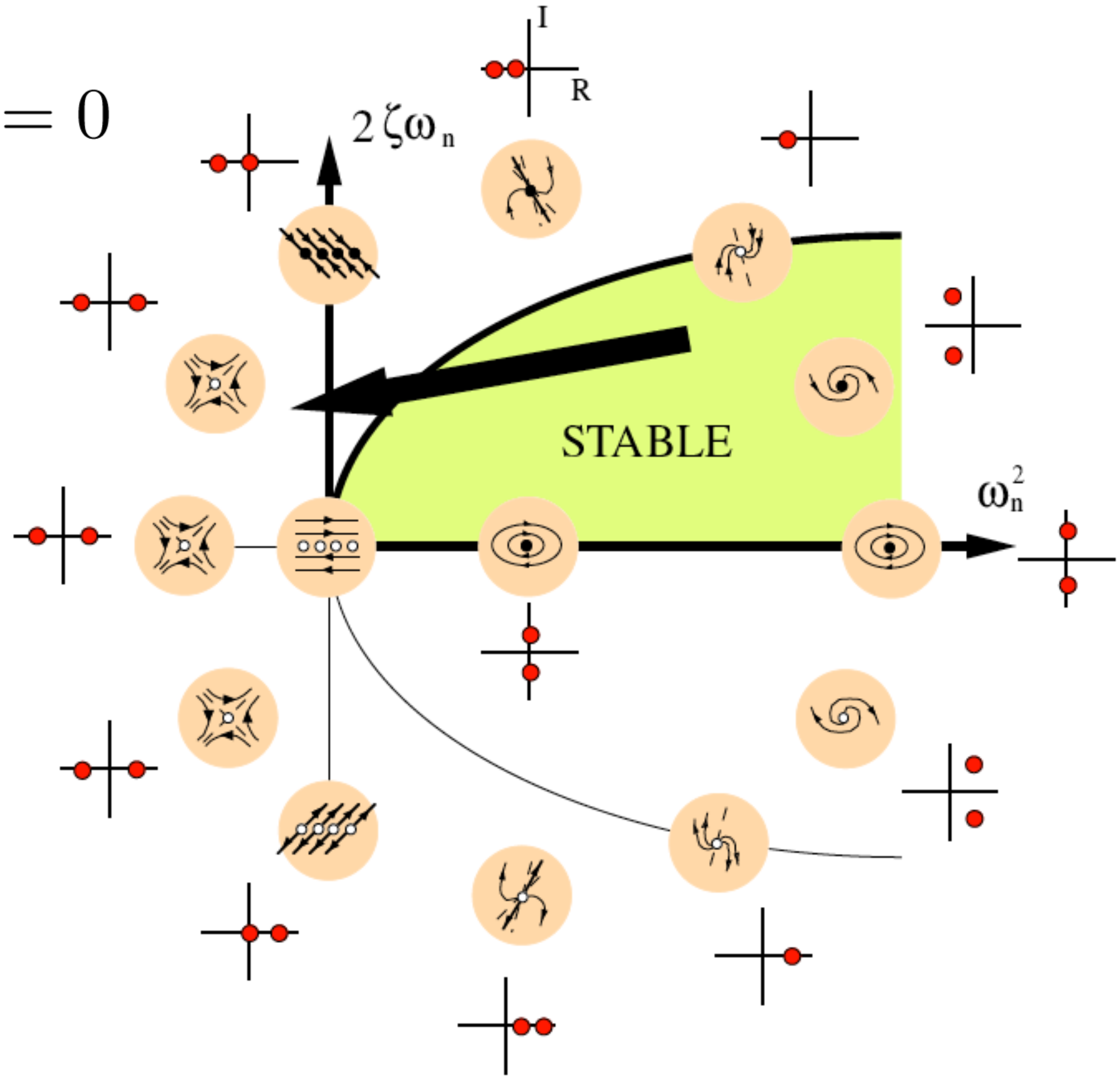
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and there are many possible types of response:

The linear oscillator is capable of exhibiting a wide variety of behavior depending on the 'stiffness' and 'damping' parameters.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

structural stiffness influenced by axial loading

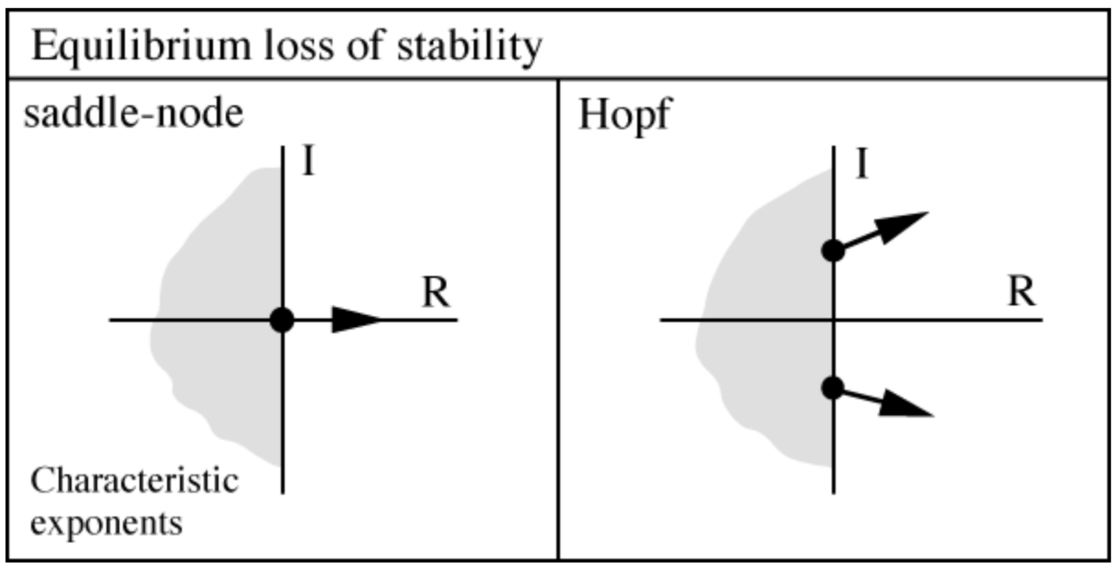


The root structure of a linear oscillator

This leads naturally to a description of stability:

- (i) If the system is stable, then $Re\{\lambda_i\} \leq 0, i = 1, 2..., n$.
- (ii) If either $Re\{\lambda_i\} < 0, i = 1, 2..., n$; or if $Re\{\lambda_i\} \leq 0, i = 1, 2..., n$ and there is no repeated eigenvalue; then the system is uniformly stable
- (iii) The system is asymptotically stable if and only if $Re\{\lambda_i\} < 0, i = 1, 2..., n$ (and then it is also uniformly stable, by (ii))
- If $Re\{\lambda_i\} > 0$ for any i, the solution is unstable.

In terms of critical eigenvalues we see that equilibrium may lose its stability by a loss of stiffness or a loss of damping. We will re-visit these ***bifurcations*** next:



Bifurcations

The Saddle-Node Bifurcation

The saddle-node is best described via first order ode:

$$\dot{x} = \mu - x^2,$$

However, in mechanics we can conveniently incorporate this into a second-order ode:

$$\ddot{x} + 0.1\dot{x} + x^2 - \mu = 0.$$

Equilibrium is given by:

$$x_e = \pm\sqrt{\mu}.$$

Stability is investigated by considering small perturbations:
$x = x_e + \delta,$

Plugging this back in the ode we obtain:

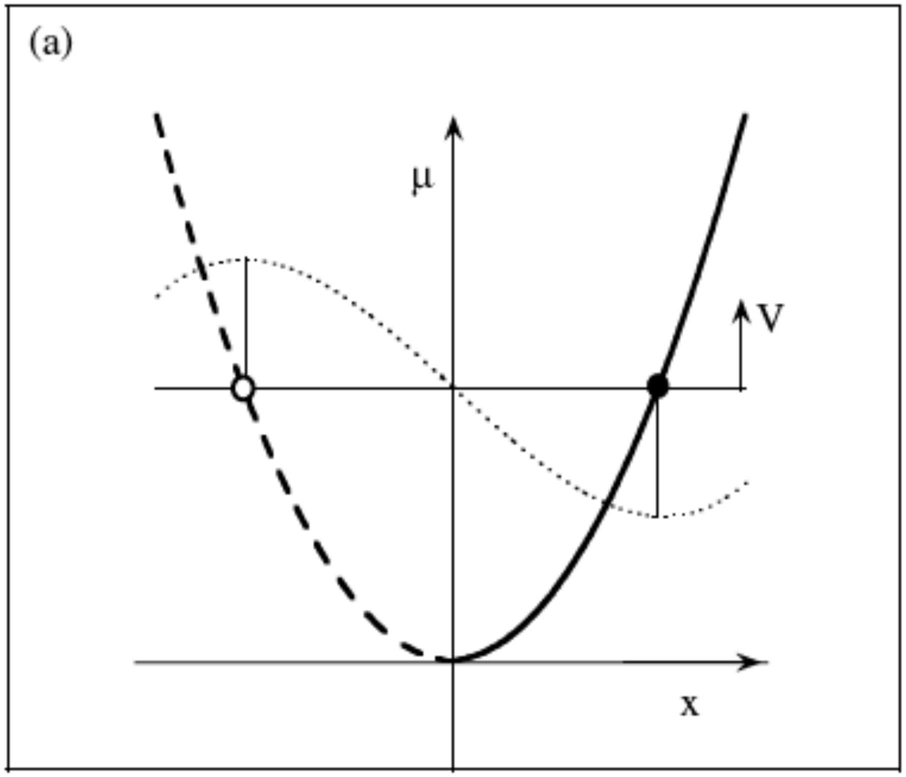
$$\ddot{\delta} + 0.1\dot{\delta} + x_e^2 + 2x_e\delta + \delta^2 - \mu = 0.$$

and since δ is small:

$$\ddot{\delta} + 0.1\dot{\delta} + 2x_e\delta = 0.$$

and thus the form of the stiffness depends on which equilibrium branch we are on:

$$\ddot{\delta} + 0.1\dot{\delta} + \pm 2\sqrt{\mu}\delta = 0.$$

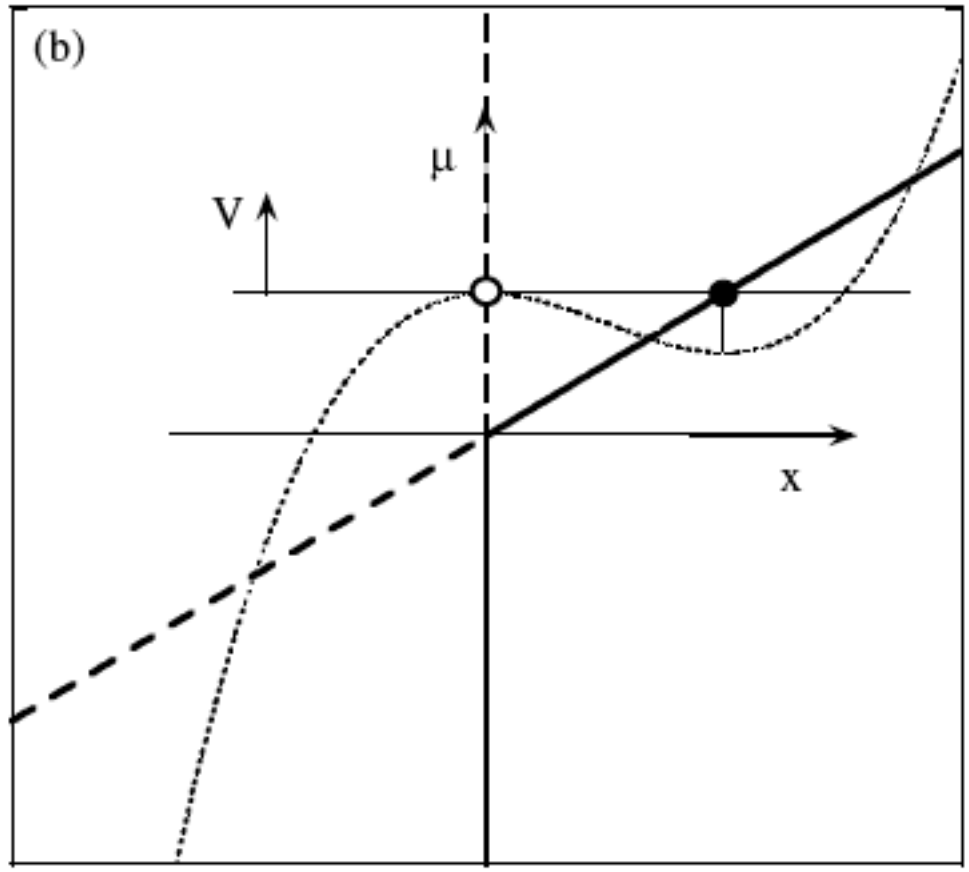


Note: in the figure, V (and the dotted line) shows potential energy.

The transcritical bifurcation

$$\ddot{x} + 0.1\dot{x} + x^2 - \mu x = 0.$$

Here, there is a difference in behavior depending on the sign of the deflection



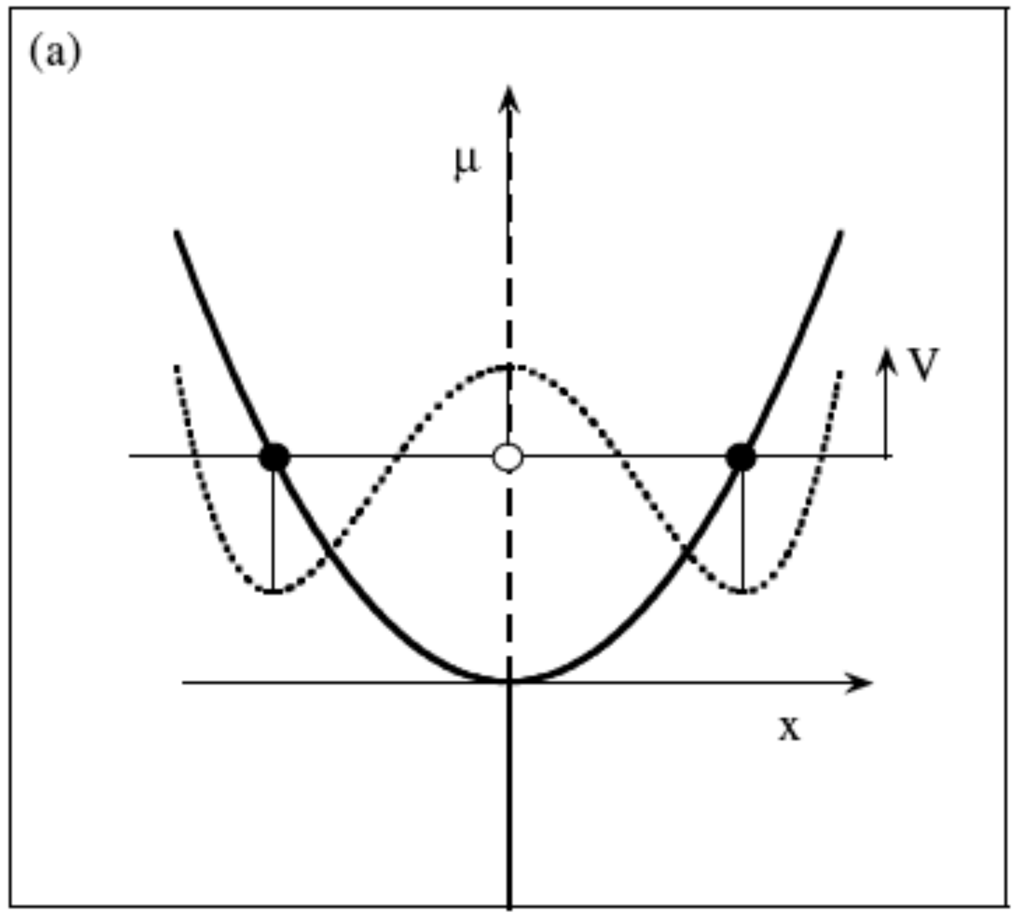
An example of this type of behavior is the Timoshenko (right-angled) frame.

The super-critical pitchfork bifurcation

$$\ddot{x} + 0.1\dot{x} + x^3 - \mu x = 0.$$

Many systems possess some kind of symmetry.

The classic example is the axially-loaded beam (Euler).

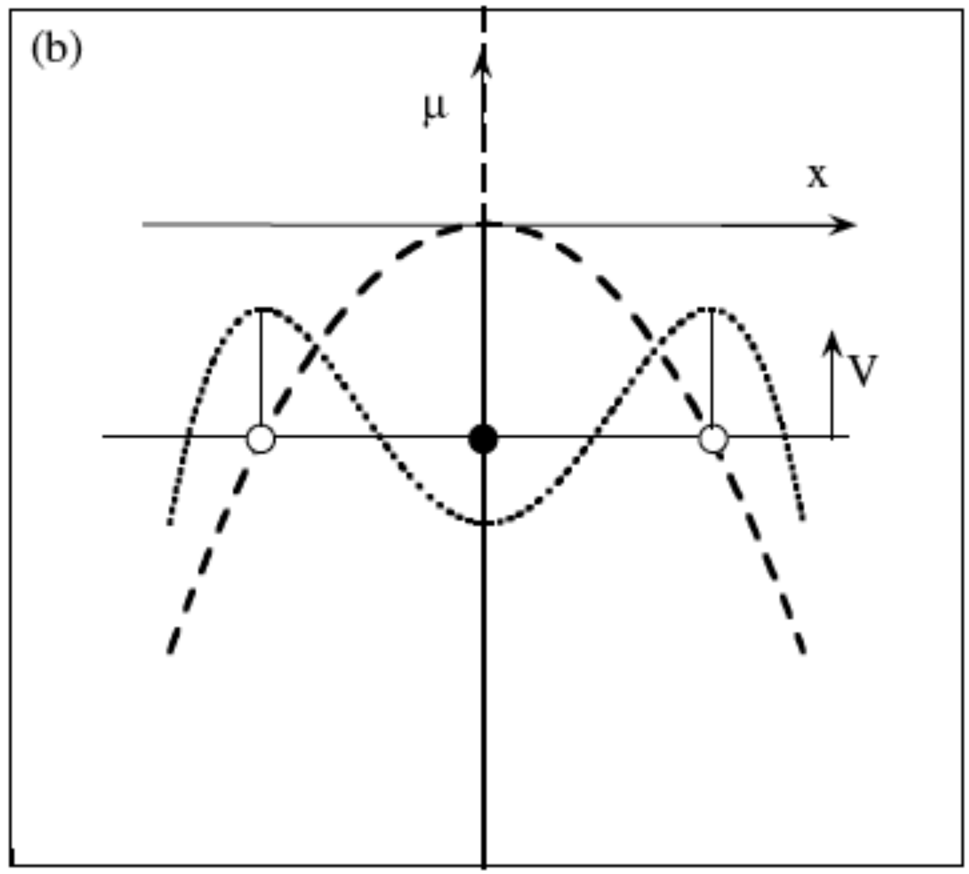


The sub-critical pitchfork bifurcation

$$\ddot{x} + 0.1\dot{x} + x^3 + \mu x = 0.$$

Sub-critical results in a sudden dynamic jump - catastrophic response.

Although this behavior is not as common as the super-critical form in mechanics.



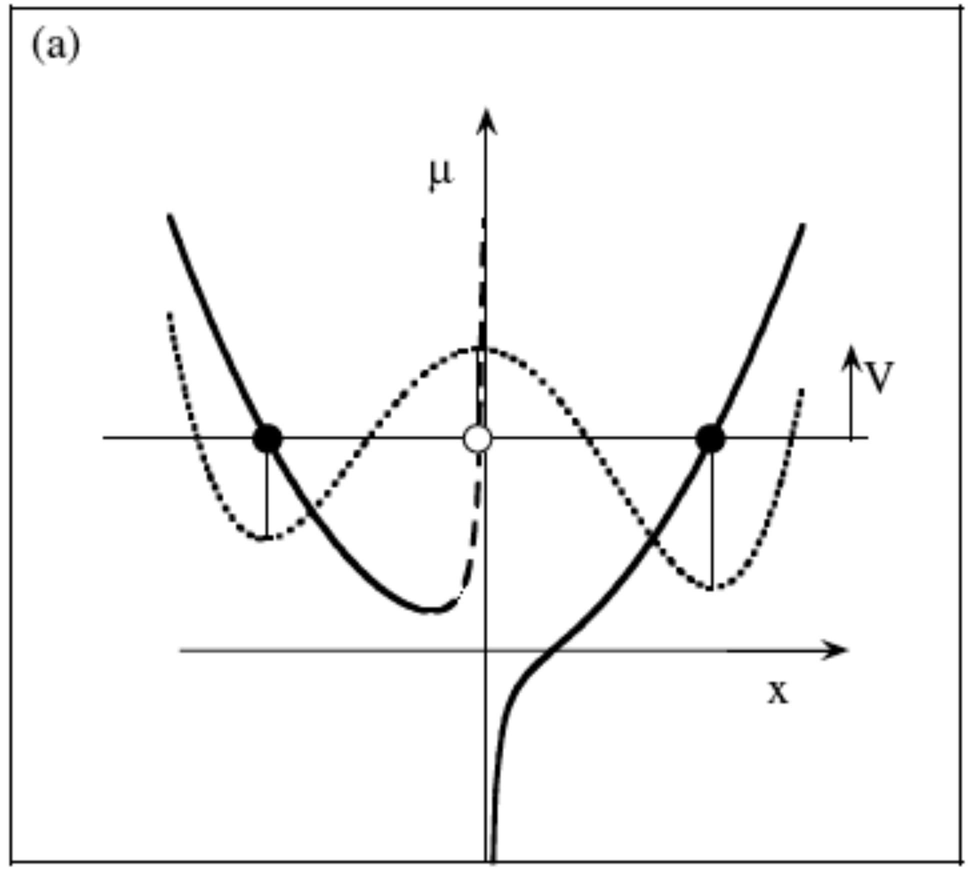
Initial Imperfections

$$\ddot{x} + 0.1\dot{x} + x^3 - \mu x + \epsilon = 0,$$

symmetry breaking

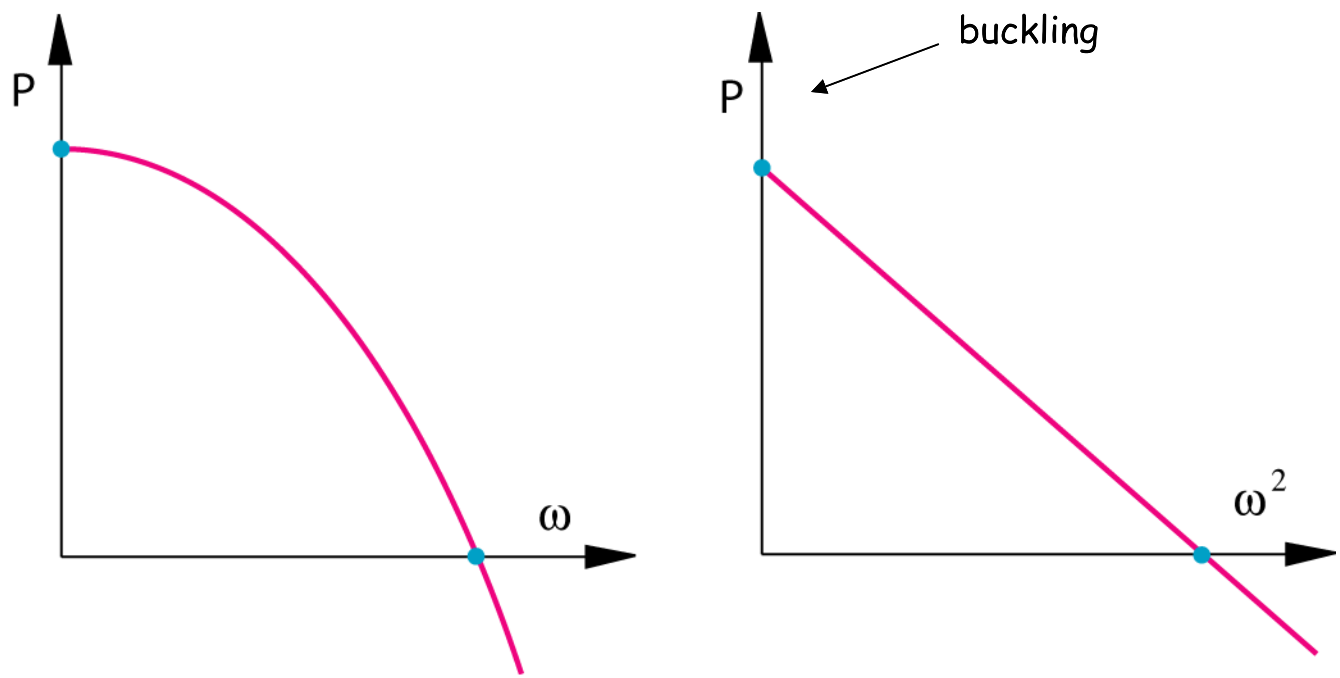
Symmetrical systems are somewhat pathological, and a new parameter ε is often incorporated.

For example, a strut will usually have a preferred direction of buckling.



Let's focus on the relation between dynamics and buckling:

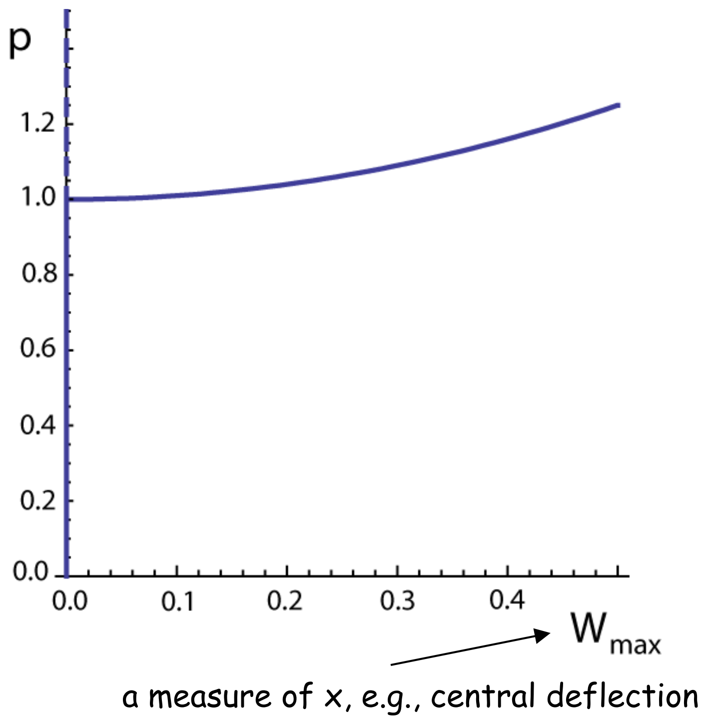
Assuming the buckling mode and vibration mode are identical, it can easily be shown that there is an especially simple relation between the axial load and the natural frequency:

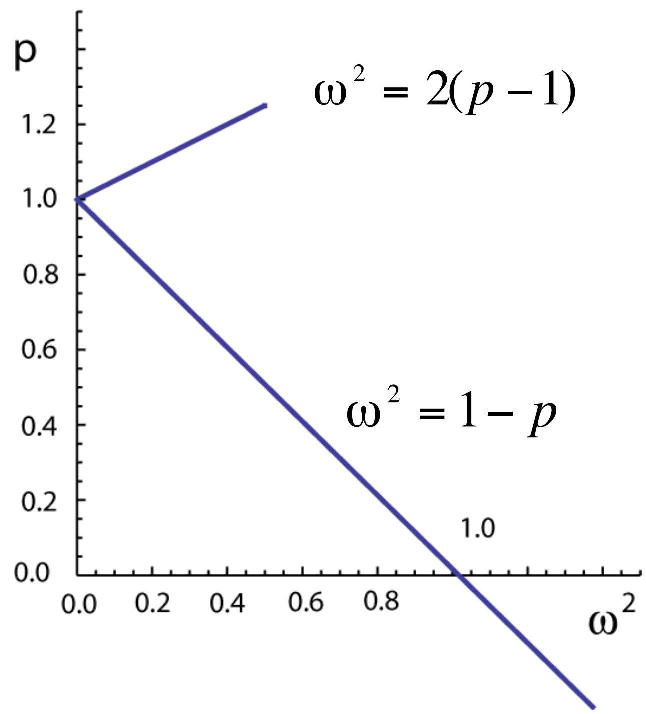


In fact, this relation is linear if we plot the axial load versus the square of the natural frequency. The relation is almost linear when the mode shapes are not quite the same. Potentially useful for prediction based on extrapolation.

As a starting point, consider an oscillator with a stiffness that depends on axial load: $$m\ddot{x} + c\dot{x} + k(1 - p)x = 0$$

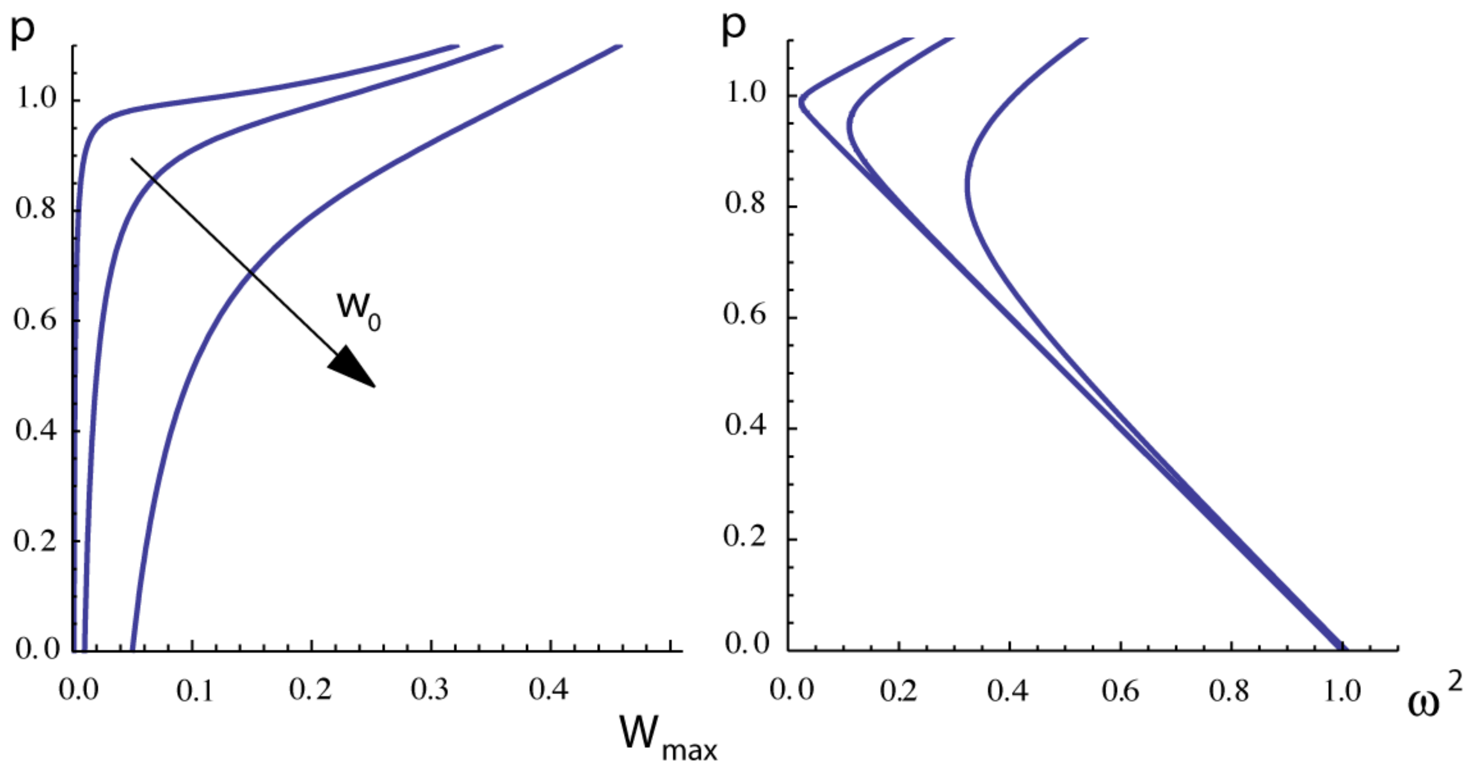
Clearly, in this case there is a linear relation between the axial load and the square of the natural frequency.





But it can also be shown that there is another simple relation for initial postbuckling.

We can add the effect of small (but inevitable) imperfections:



This is the super-critical pitchfork bifurcation.

A link model demonstration

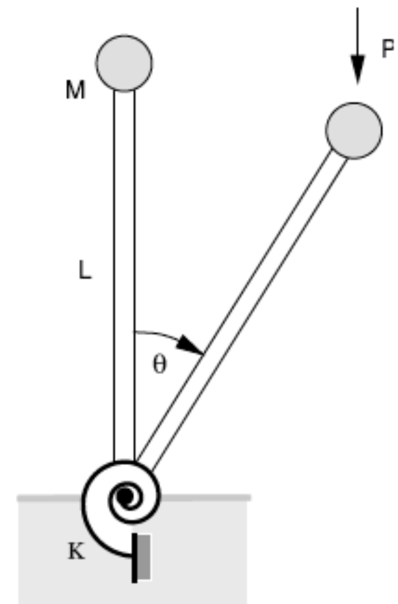
$$V = U - V_P = \frac{1}{2}K\theta^2 - PL(1-\cos\theta).$$

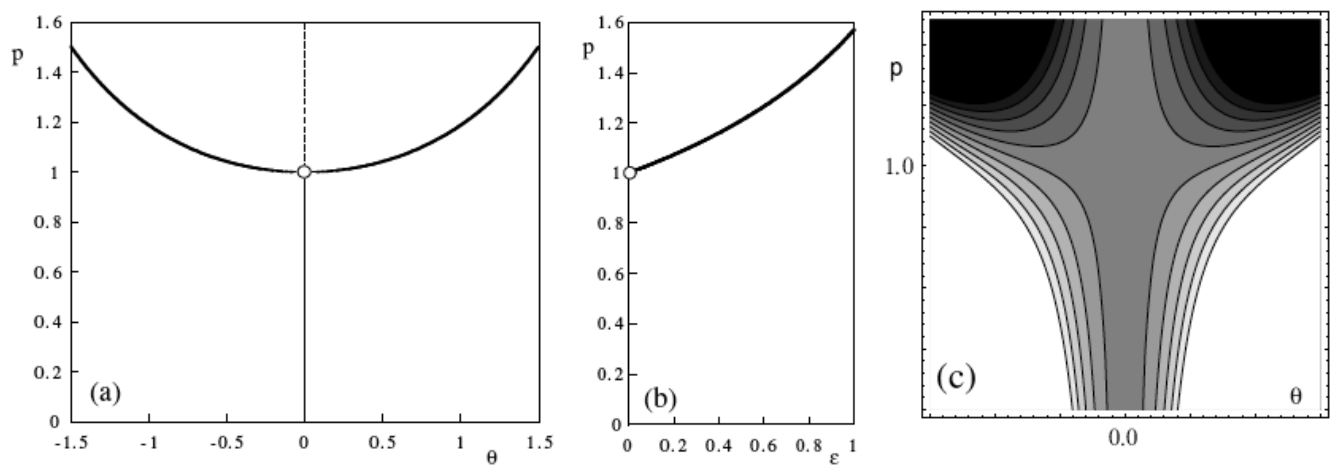
$$T = \frac{1}{2}ML^2\dot{\theta}^2,$$

$$ML^2\ddot{\theta} + K\theta - PL\sin\theta = 0,$$

$$\omega_n^2 = K/ML^2, \qquad p = PL/K,$$

$$\ddot{\theta} + \omega_n^2(\theta - p\sin\theta) = 0.$$





Linearization

Let's consider the behavior of small perturbations in the vicinity of equilibrium:

$$\theta = \theta_e + \delta,$$

and placing this in the nonlinear ode we get:

$$\ddot{\delta} + \omega_n^2[(\theta_e + \delta) - p\sin(\theta_e + \delta)] = 0,$$

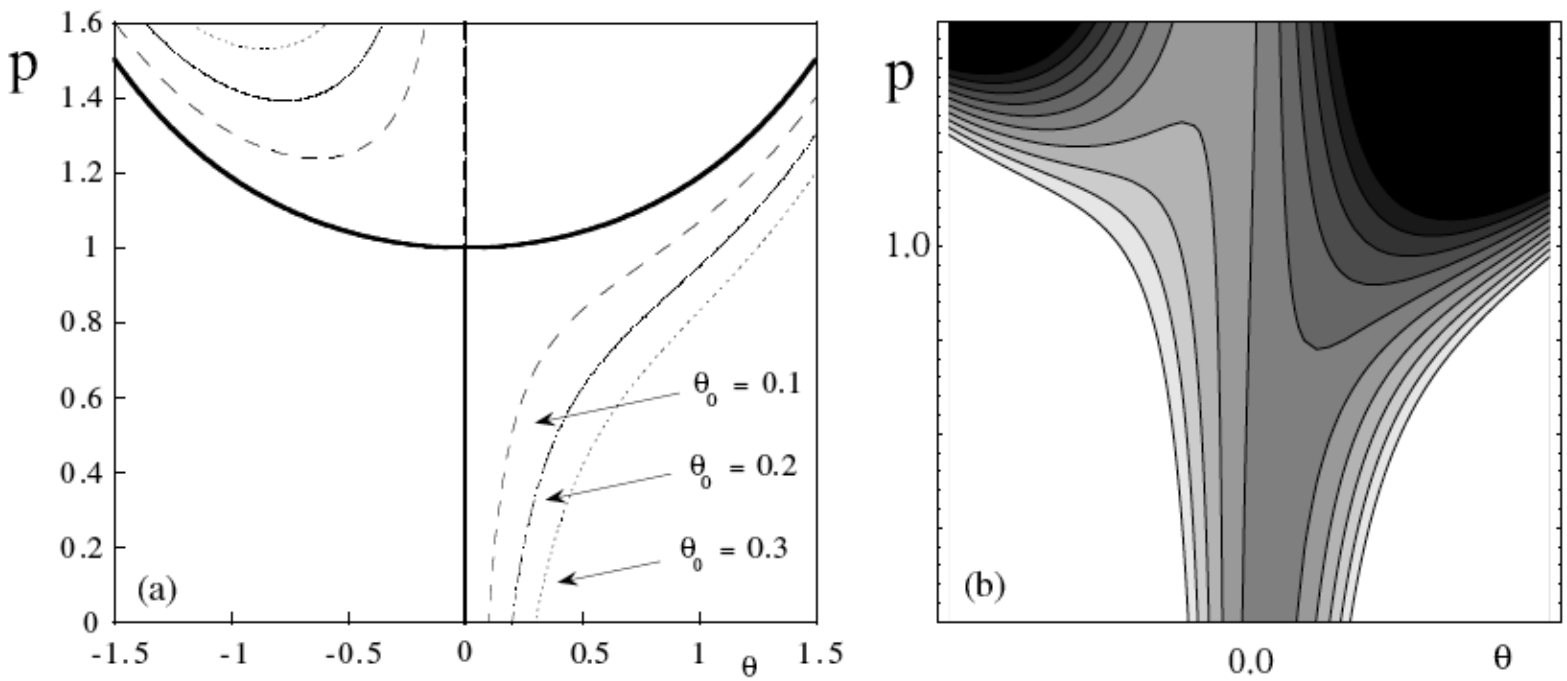
and since δ is small, and $\theta_e - p\sin\theta_e = 0$ have

$$\ddot{\delta} + \omega_n^2[1 - p\cos\theta_e]\delta = 0,$$

which supplies a linear equation about each equilibrium, and thus stability can be established.

Incorporating a small initial angle breaks the symmetry:

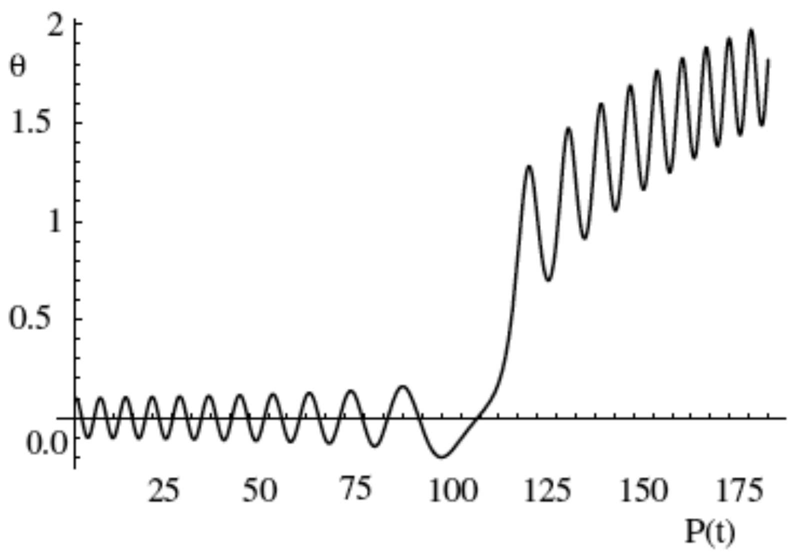
$$V = \frac{1}{2}K(\theta - \theta_0)^2 - PL(\cos\theta_0 - \cos\theta), \qquad p = \frac{(\theta - \theta_0)}{\sin\theta}.$$

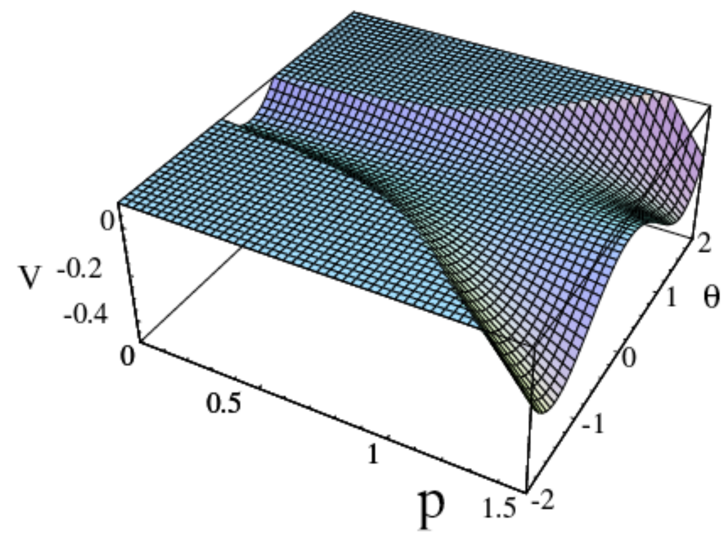


Stability information is still obtained using either the second variation of potential energy or the nature of small oscillations

We can also 'evolve' or 'degrade' the stiffness of the system as a linear function of time:

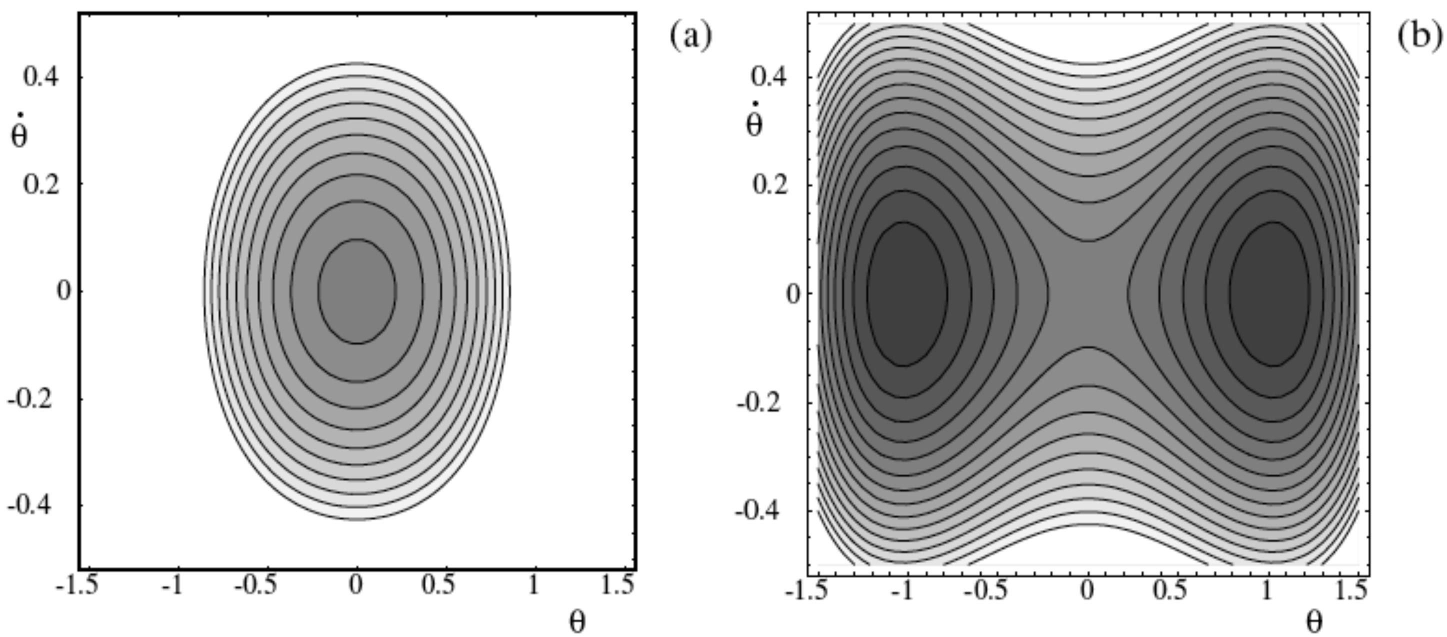
$$\ddot{\theta} + \theta - 0.01t \sin\theta = 0,$$





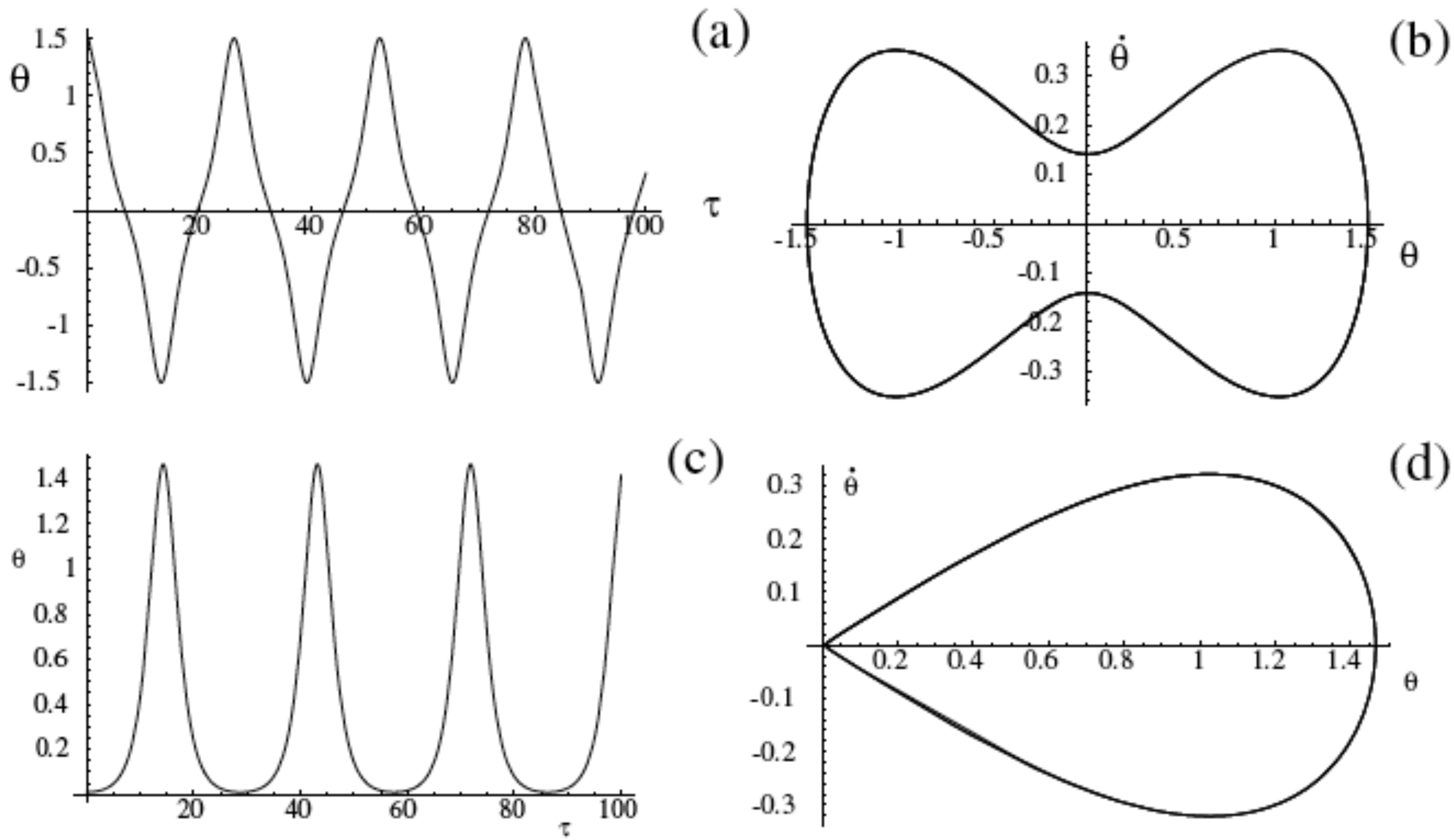
For undamped, unforced systems we can take advantage of the conservation of total energy (a first integral)

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \qquad\qquad \dot{\theta} = \pm\sqrt{C - \theta^2 - 2p\cos\theta},$$



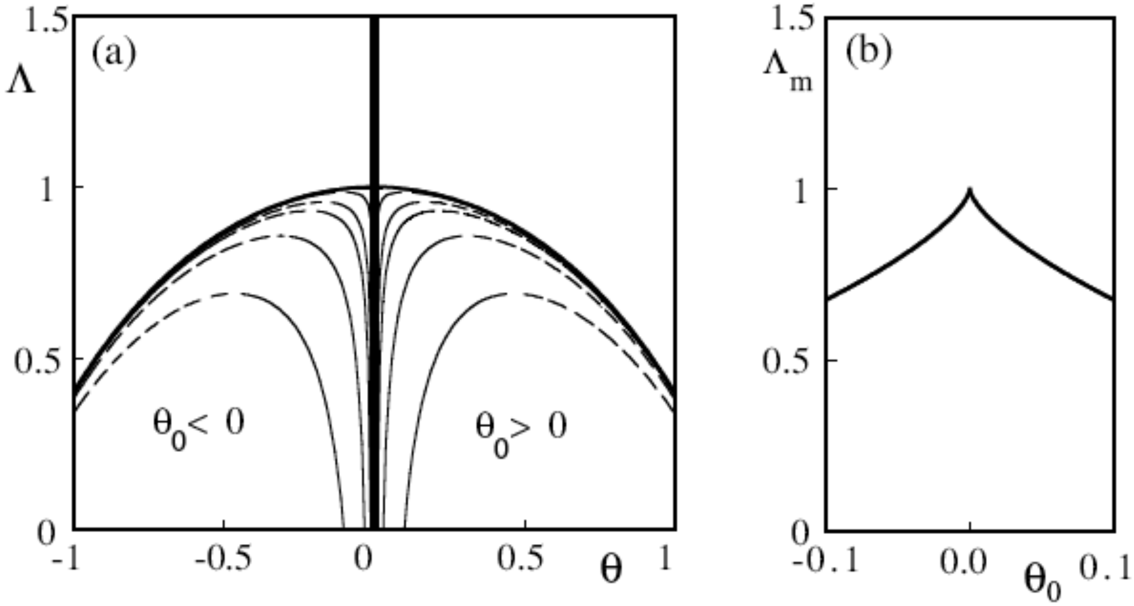
Contourplots display phase trajectories and depend on initial conditions

Here are some time series and phase projections for ‘large’ initial conditions:

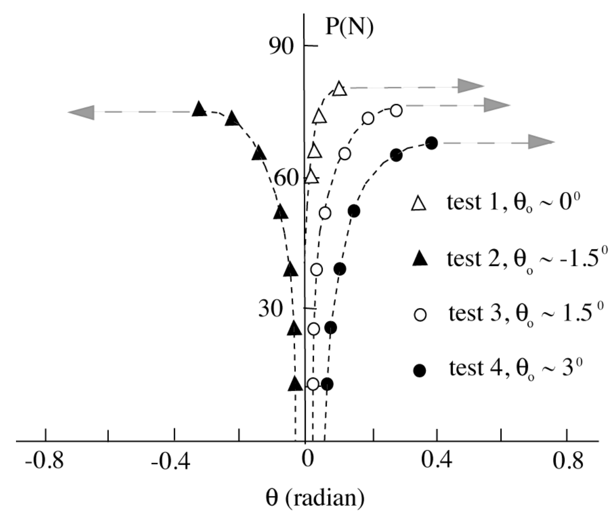


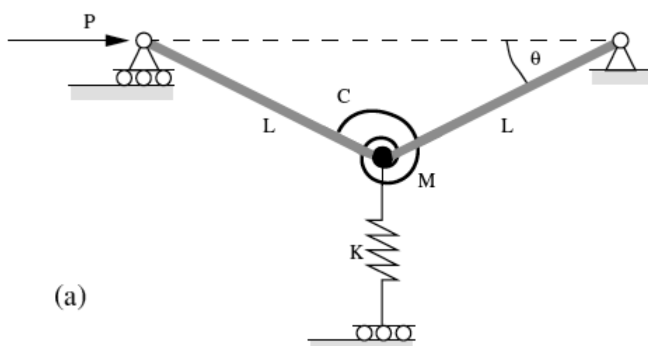
We shall come back to large (nonlinear) oscillations later.

Imperfection Sensitivity



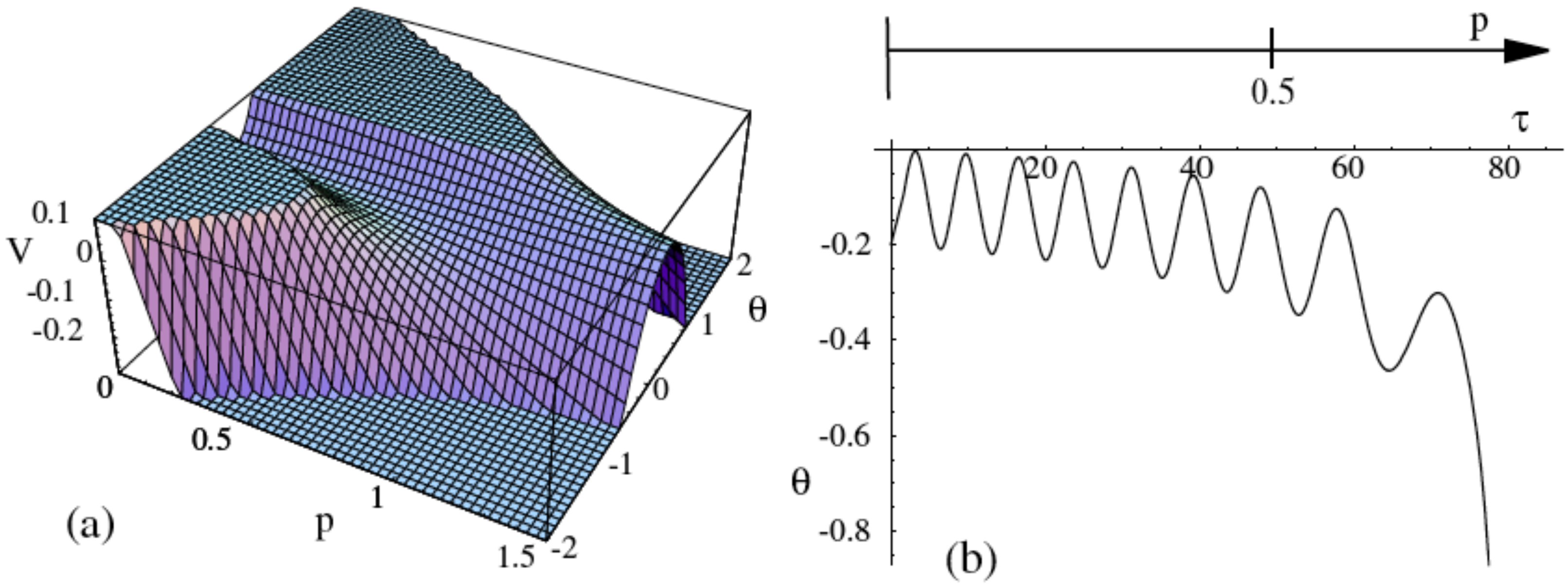
experimental data





The important point here is that the initial imperfection reduces the load-carrying capacity of the structure.

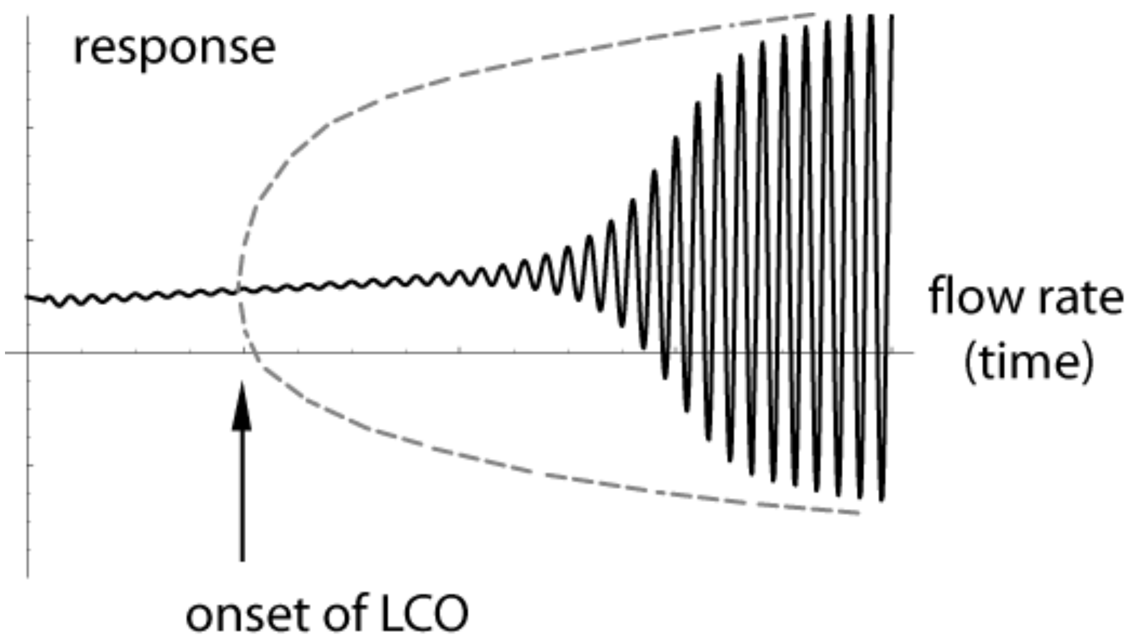
And an evolution on the potential energy surface:



$$V = \frac{1}{2}KL^2(\sin\theta - \sin\theta_0)^2 - 2PL(\cos\theta_0 - \cos\theta)$$

The Hopf Bifurcation

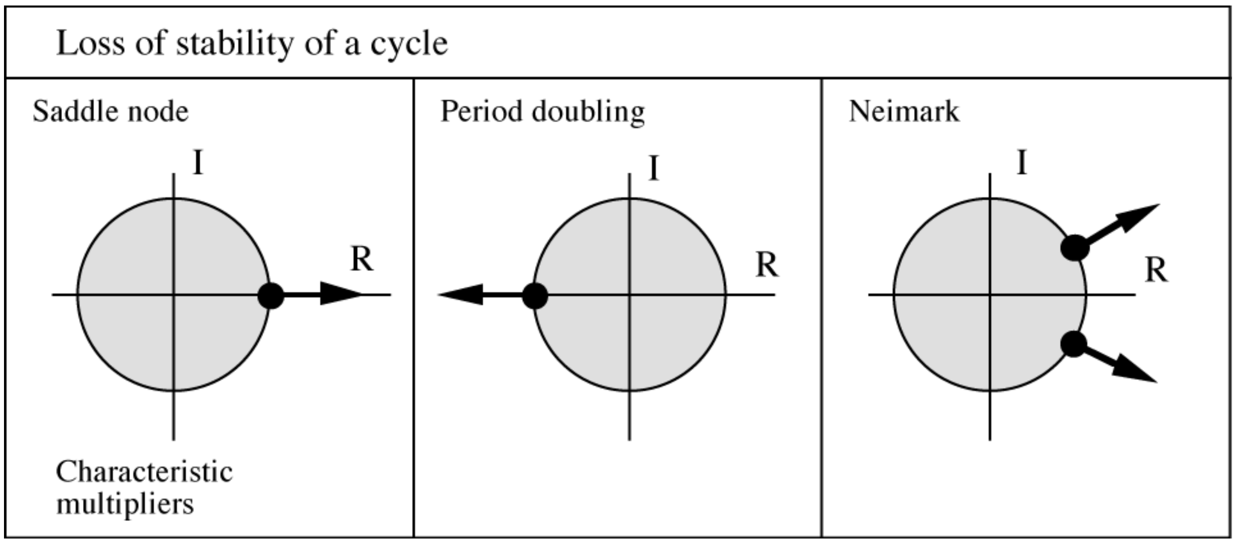
We can think of the previous bifurcations as 'static' in the sense that there is a qualitative change in behavior (e.g., as the potential energy ceases to exhibit a minimum). However, there are applications in which damping effectively becomes negative and the system 'flutters':



The post-critical response is often limited due to the presence of nonlinearities - limit cycle oscillations (LCO)

Bifurcations of Maps

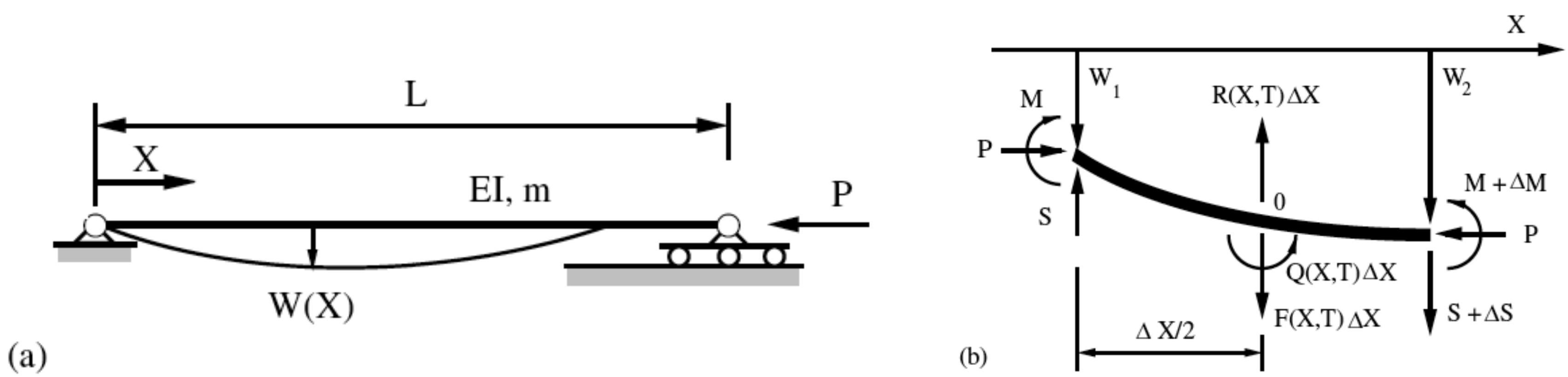
$$\mathbf{x}_{i+1} = F(\mathbf{x}_i, \mu).$$



The reason this becomes important is in the study of periodic attractors via the Poincare section in forced oscillations.

Continuous elastic systems

In the context of *distributed* systems we can consider a slender beam with an axial load. The behavior of this infinite dimensional system is a pde:



$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = F(x,t).$$

With this type of linear pde it is easy to separate variables:

$$w(x,t) = W(x)Y(t).$$

$$EI\frac{d^4W}{dx^4}Y + P\frac{d^2W}{dx^2}Y = -mW\frac{d^2Y}{dt^2}.$$

with each side equal to a constant (that we' ll call ω^2):

$$\frac{m}{Y}\frac{d^2Y}{dt^2} = -\frac{EI}{W}\frac{d^4W}{dx^4} - \frac{P}{W}\frac{d^2W}{dx^2} = -\omega^2$$

We' ll focus on the temporal and spatial parts separately:

The Temporal Solution

$$w(x,t) = \sum_{n=1}^{\infty} Y(t) \sin \frac{n\pi x}{L}.$$

[simply supported bc' s]

$$Y_n(t) = A_n e^{i\omega_n t},$$

$$\sum_{n=1}^{\infty} \left[\left[EI \frac{n^2\pi^2}{L^2} - P \right] \frac{n^2\pi^2}{L^2} - m\omega_n^2 \right] A_n \sin \frac{n\pi x}{L} e^{i\omega_n t} = 0.$$

$$\omega_n^2 = \frac{n^4 EI \pi^4}{mL^4} \left[1 - \frac{PL^2}{n^2 EI \pi^2} \right].$$

In nondimensional terms:

$$p_n = n^2 EI\pi^2/L^2, \qquad \bar{\omega}_n^2 = n^4 EI\pi^4/mL^4,$$

we recover the simple relation between axial load and natural frequency:

$$\omega_n = \pm\bar{\omega}_n\sqrt{1-\bar{p}},$$

And we see various possibilities as time evolves:

- If $\bar{p} = 0$ then

$$Y_n(t) = A_n \cos \bar{\omega}_n t + B_n \sin \bar{\omega}_n t$$

- If $\bar{p} < 1$ then

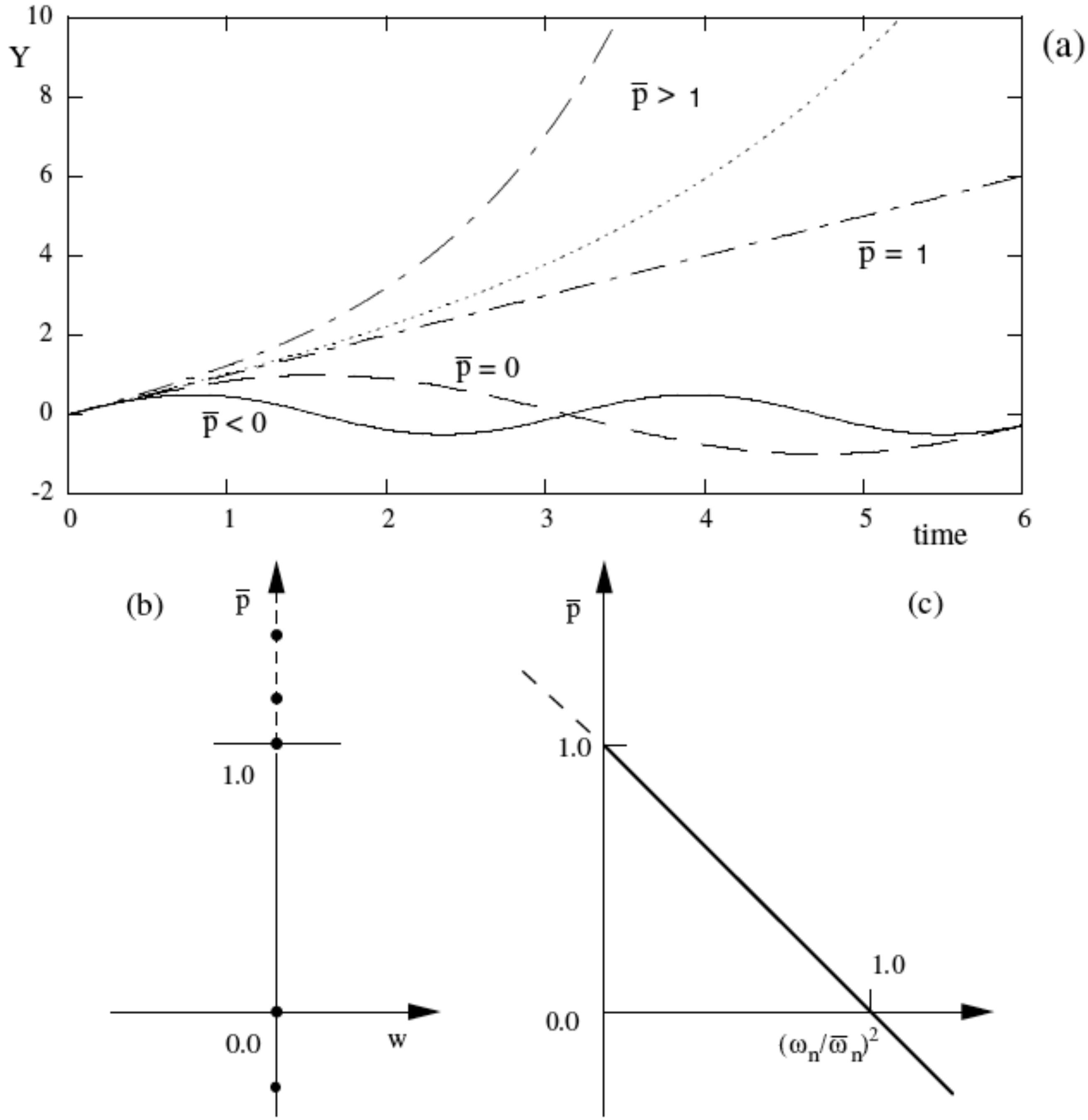
$$Y_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t,$$

- If $\bar{p} = 1$ then the solution can be written as

$$Y_n(t) = A_n + B_n t,$$

- If $\bar{p} > 1$ then

$$Y_n(t) = A_n \cosh \omega_n t + B_n \sinh \omega_n t,$$


(a)
Y
10
8
6
4
2
0
-2
0
1
2
3
4
5
6
time
$\overline{p} > 1$
$\overline{p} = 1$
$\overline{p} = 0$
$\overline{p} < 0$
(b)
$\overline{p}$
1.0
0.0
w
(c)
$\overline{p}$
1.0
1.0
0.0
$(\omega_n/\overline{\omega}_n)^2$

The Spatial Solution

$$EI\frac{d^4W(x)}{dx^4} + P\frac{d^2W(x)}{dx^2} - m\omega^2 W(x) = 0.$$

$$W(\bar{x}l) = c_1 \sinh M\bar{x} + c_2 \cosh M\bar{x} + c_3 \sin N\bar{x} + c_4 \cos N\bar{x},$$

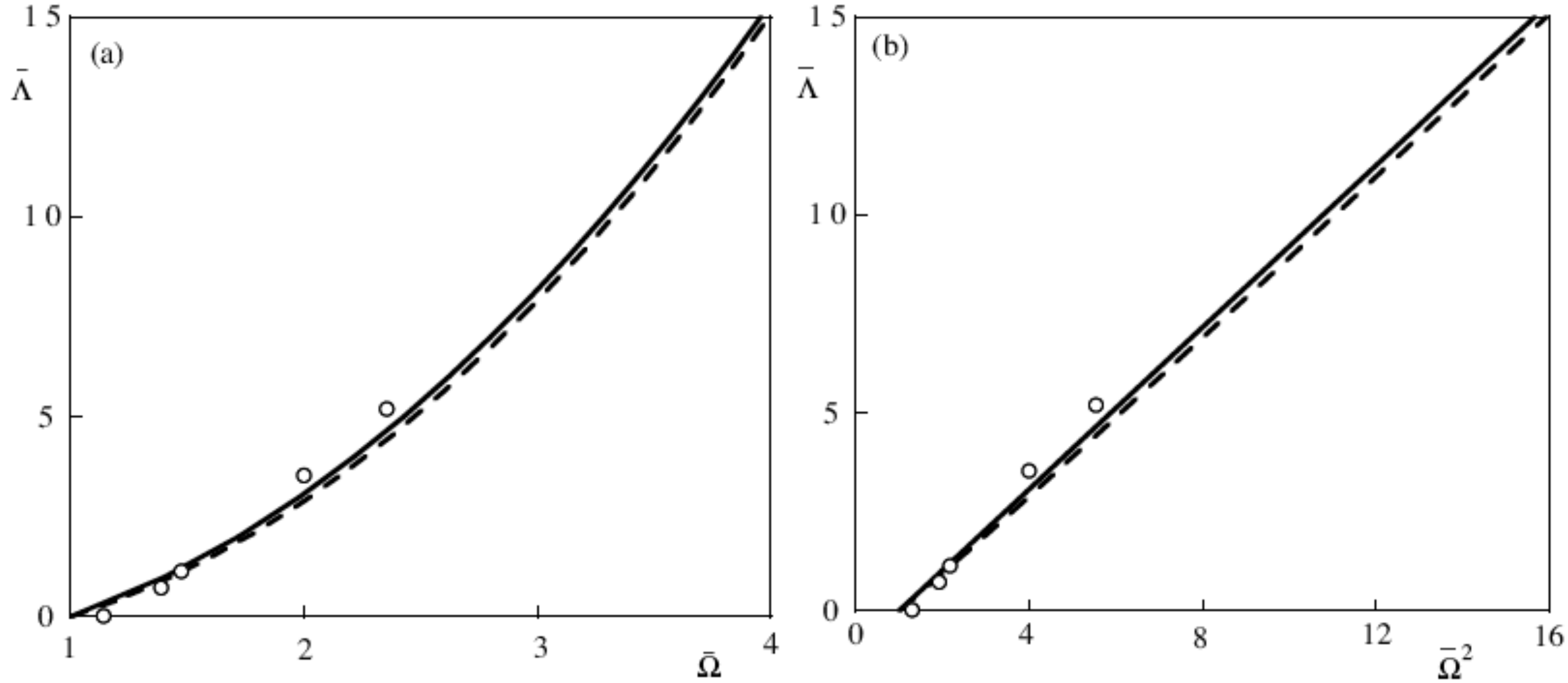
$$M = \sqrt{-\Lambda + \sqrt{\Lambda^2 + \Omega^2}}$$
$$N = \sqrt{\Lambda + \sqrt{\Lambda^2 + \Omega^2}}$$

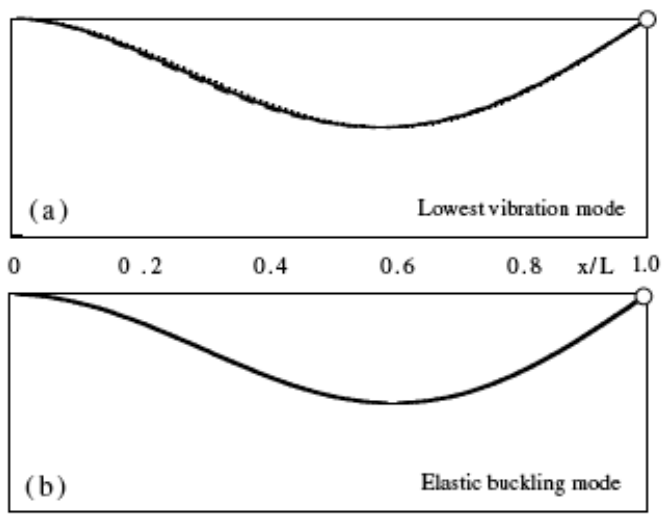
assuming now that the
bc's are clamped-pinned

$$\Lambda = PL^2/(2EI) \quad \Omega^2 = m\omega^2 L^4/(EI).$$

$$M \cosh M \sin N - N \sinh M \cos N = 0.$$

characteristic equation

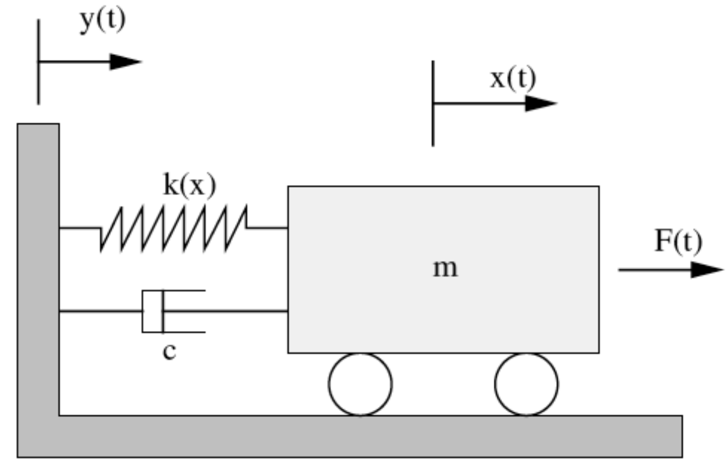

(a)
(b)
15
10
5
0
$\bar{\Lambda}$
1
2
3
4
$\bar{\Omega}$
0
4
8
12
16
$\bar{\Omega}^2$


(a)
Lowest vibration mode
0
0.2
0.4
0.6
0.8
x/L
1.0
(b)
Elastic buckling mode


LEXAN

Forcing (harmonic)

The presence of a harmonic external forcing term changes the long term behavior from point to periodic attractors as the phase space goes from 2 to 3.



$$F(t) = F_0 \sin \omega t,$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega t.$$

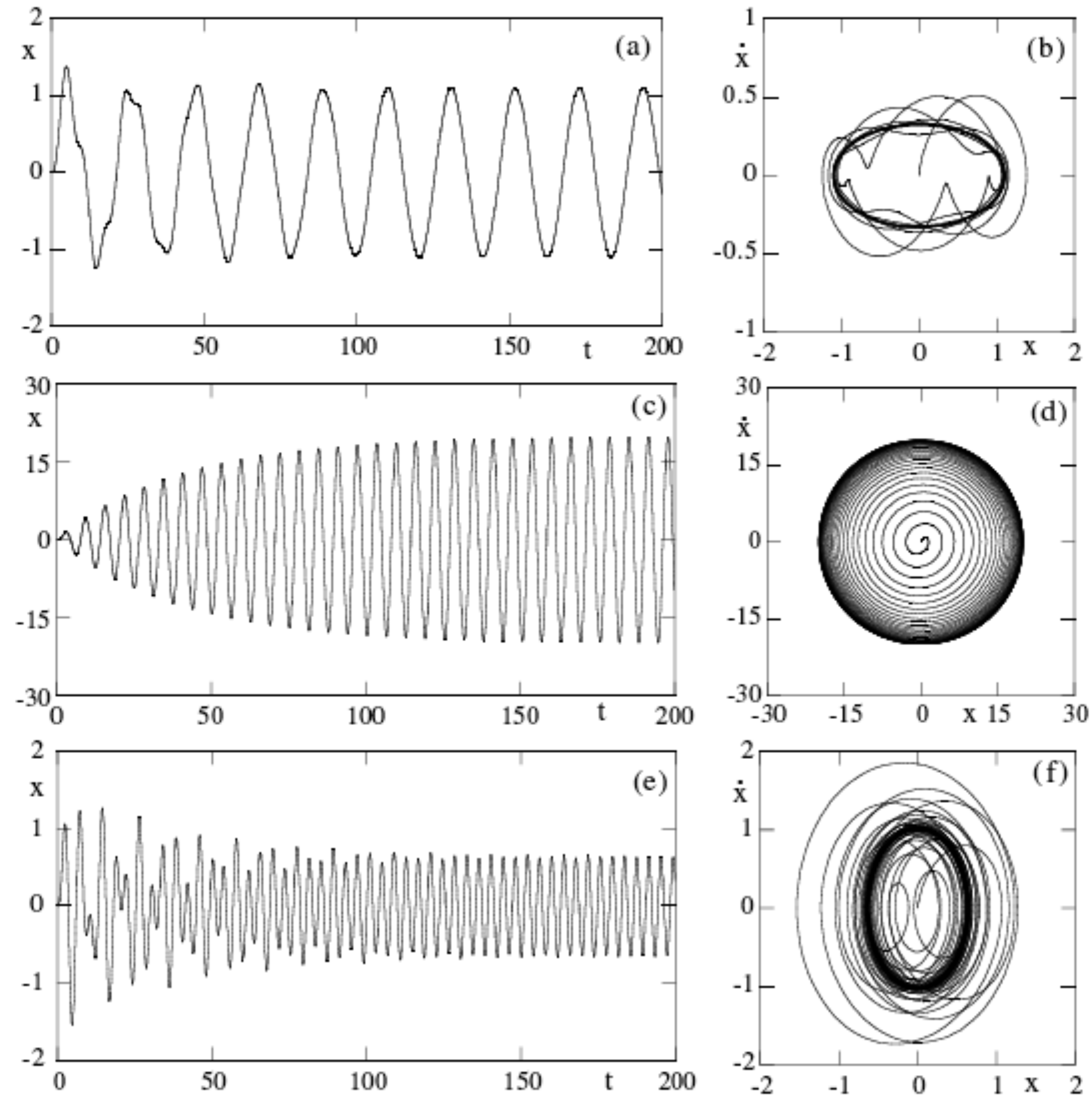
$$x(t) = \frac{f_0}{k} \frac{\sin(\omega t - \phi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}} + X_1 e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\omega_n t + \phi_1),$$

$\omega = 0.3$

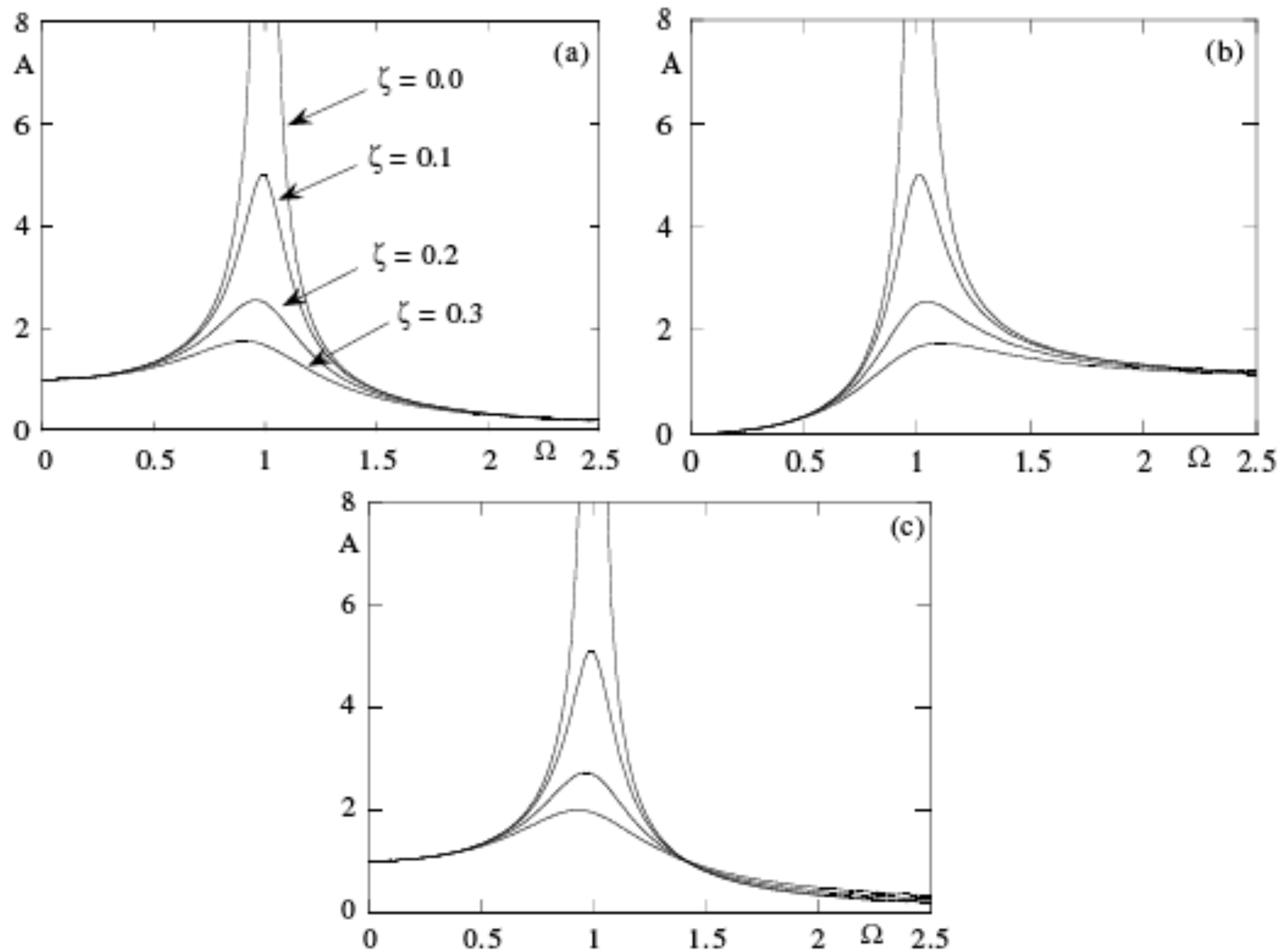
Some typical responses when starting from the rest state.

$\omega = 1.0$

$\omega = 1.6$



Focusing attention on the steady state response, we can plot direct (mass-excited) resonance, mass eccentricity, and base excitation.



Much of the preceding theory can be demonstrated with a (small) mass that rolls on a (potential-energy) surface: