A BRIEF TOUR OF VECTOR CALCULUS

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Contents

0. Prelude

This is an ongoing notes project to capture the essence of the subject of vector calculus by providing a variety of examples and visualizations, but also to present the main ideas of vector calculus in conceptual a framework that is adequate for the needs of mathematics, physics, and engineering majors.

The essential prerequisites are

- comfort with college level algebra, analytic geometry and trigonometry,
- calculus knowledge including exposure to multivariable functions, partial derivatives and multiple integrals,
- the material of my notes on *Vector Algebra, and the Equations of Lines and Planes in 3-Space* or equivalent, and
- the material related to polar, cylindrical and spherical frames in my notes on *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*, particularly for the optional sections.

Definitions and results are sometimes stated in terms of functions of n variables for an arbitrary number $n \geq 2$, but examples focus on $n = 2$ and $n = 3$. Since it is hard to really *see* what is happening for $n \geq 4$, pictures often show up for examples using $n = 2$ or 3, and the hope is that you internalize some intuition from these pictures and examples. If you have trouble understanding a statement that uses arbitrary n, just read it with $n = 2$ or $n = 3$ and try to understand the underlying geometry in these cases.

A warning: my notational conventions sometimes differ from other common sources, so use caution when comparing to other resources on vector calculus. In particular, my spherical coordinate system is not the one in most common use, but is an intuitive convention nonetheless, as explained in *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*, where they are constructed so as to adapt geographers' conventions for latitude and longitude.

These notes exist primarily to prop up the problems; the exposition and ideas herein are foremost to introduce enough language for the reader to then approach and tackle the various problems provided. Some of the problems are quite standard and are meant to drive home a particular concept, while others encourage you to fill in details omitted in examples, and a few problems are meant to give a flavor of a more advanced but mathematical subject, such as the study of differential equations, or topology. There's also a handful of computational problems meant to satisfy those who merely enjoy the meditative art of symbol pushing, but the bulk of the questions ought to provoke some serious thought about how the objects of vector calculus interact with each other and with mathematical models of the real world.

In addition to clarifying notations and terminology, footnotes are often used to exposit on more advanced directions, and hint at how the subject matter broadens and connects to contemporary mathematical thinking and research.

Finally, please forgive any typos and the clunkiness of formatting; these and the companion/prerequisite notes are work in progress that were primarily hastily assembled in the midst of my terminal years teaching at University of Massachusetts Amherst as a PhD student. I welcome suggestions as I work to improve these.

1. Directional Derivatives, the Gradient and the Del Operator

§ 1.1. Conceptual Review: Directional Derivatives and the Gradient

Recall that partial derivatives are defined by computing a difference quotient in which only one variable is perturbed. This has a geometric interpretation as slicing the graph of the function along a plane parallel to the directions of the coordinate of concern x_i and the coordinate of the dependent variable whose value is $f(x_1, \ldots, x_i, \ldots, x_n)$ (so this plane is thus normal to the coordinate directions of the variables held constant), and then measuring the rate of change of the function along the curve of intersection, as a function of the variable x_i. In two variables, the slicing for the two partial derivatives corresponds to a picture like that of figure 1.

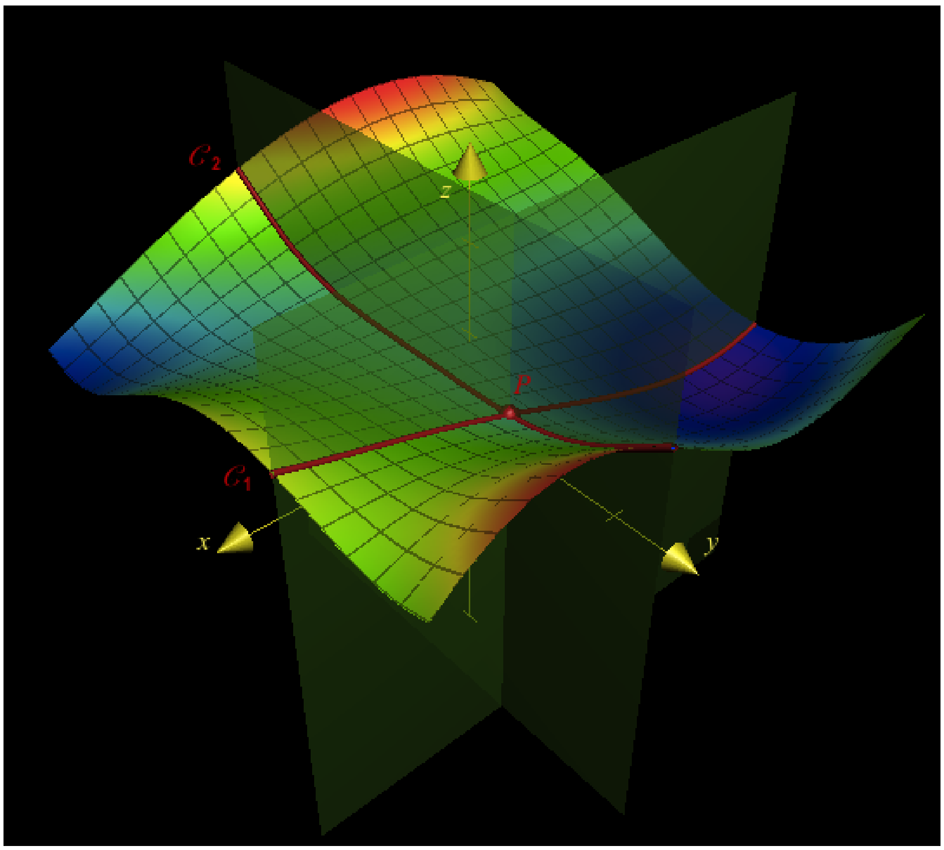


Figure 1. Curves $\mathcal{C}_1$ and $\mathcal{C}_2$ on the graph of a function along planes of constant y and x respectively.

This geometric picture suggests that we need not be confined to only know the rate of change of the function along coordinate directions, for we could slice the graph along any plane containing the direction of the dependent variable. This gives rise to the notion of a directional derivative, defined so as to allow us to measure the rate of change of a function f in any of the possible directions we might choose as we leave a point of the function's domain.

Fix a connected domain $\mathcal{D} \subset \mathbb{R}^n$, and let $f : \mathcal{D} \to \mathbb{R}$ be a multivariate function of n variables, and $\mathbf{r} = \langle x_1, \ldots x_n \rangle$ an arbitrary position vector for a point P of $\mathcal{D}$. We'll often conflate the idea of $\mathcal{D}$ as a set of points with the conception of it as a set of positions, and thus will unapologetically write things such as $\mathbf{r} \in \mathcal{D}$ to mean that the point $(x_1, \ldots, x_n)$ with position $\mathbf{r}$ is an element of $\mathcal{D}$. Definitions will often be stated in the general context of arbitrary n, but examples and pictures will be specialized to low dimensions.

Observe that a "direction" at a point $\mathbf{r} \in \mathcal{D}$ may be specified by giving a unit vector, i.e. a vector $\hat{\mathbf{u}}$ of length 1. In two dimensions, the set of unit vectors, and thus, of directions, is a circle, while in three dimensions it is the surface of a sphere. The set of unit vectors in $\mathbb{R}^n$ geometrically describes the origin centered $(n-1)$-dimensional sphere in $\mathbb{R}^n$:

$$\mathbb{S}^{n-1} = \{\mathbf{r} \in \mathbb{R}^n : \|\mathbf{r}\| = 1\}\,.$$

Definition. Given a unit vector $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$ and a function $f : \mathcal{D} \to \mathbb{R}$ of n variables, the directional derivative of f in the direction of $\hat{\mathbf{u}}$ at a point $\mathbf{r}_0 \in \mathcal{D}$ is

$$D_{\hat{\mathbf{u}}} f(\mathbf{r}_0) = \lim_{h\to 0} \frac{f(\mathbf{r}_0 + h\hat{\mathbf{u}}) - f(\mathbf{r}_0)}{h}\,.$$

Again, in two dimensions we can actually see and interpret this limit in terms of the familiar notion of a slope of a tangent line. This, together with our discussion of the gradient in two dimensions, will set the stage for understanding tangent space objects later.

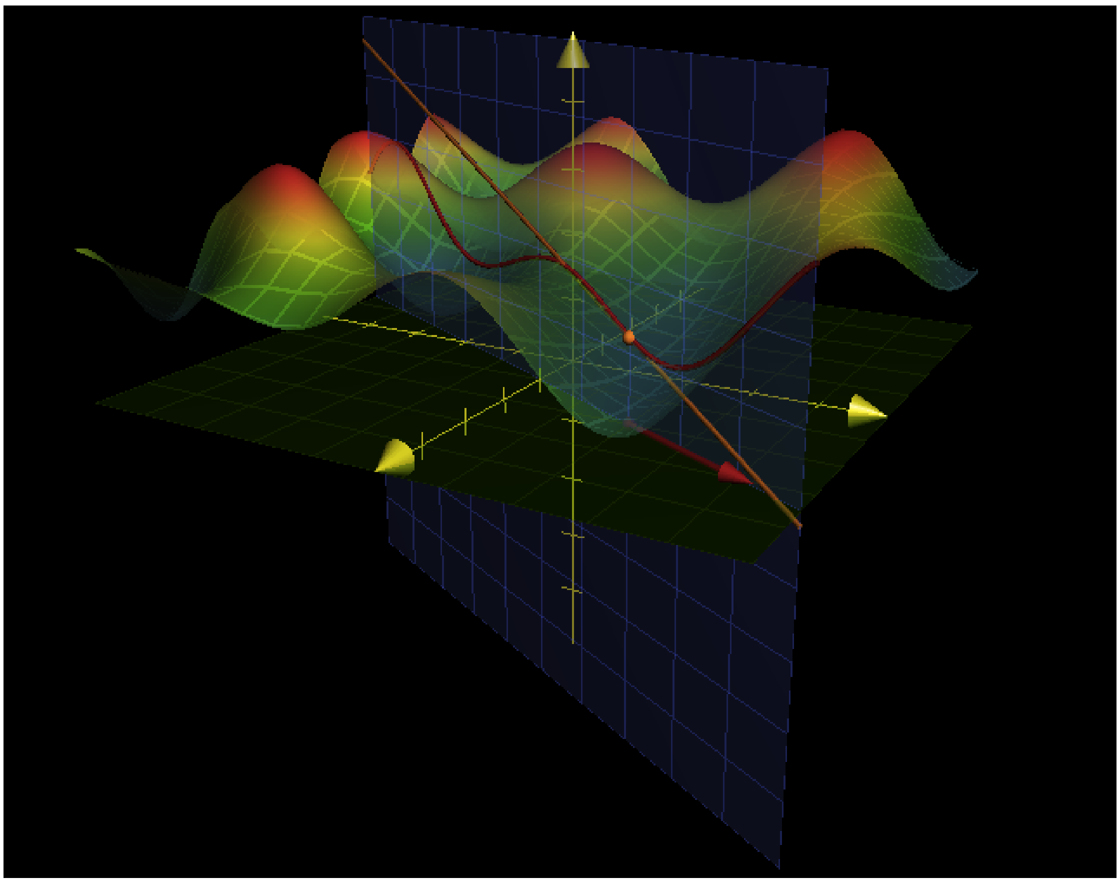


FIGURE 2. The directional derivative computes a slope to a curve of intersection of a vertical plane slicing the graph surface in the direction specified by a unit vector.

Recall that the graph $\mathcal{G}_f$ of a two-variable function $f(x, y)$ is the locus of points $(x, y, z) \in \mathbb{R}^3$ satisfying $z = f(x, y)$. For f continuously differentiable, this is a smooth[1] surface over $\mathcal{D}$. Fix a point $\mathbf{r}_0 \in \mathcal{D}$ at which we are interested in the directional derivative in the direction of a given $\hat{\mathbf{u}} \in \mathbb{S}^1$. Note that $\hat{\mathbf{u}}$ and $\hat{\mathbf{k}}$ determine a plane $\Pi_{\hat{\mathbf{u}},\mathbf{r}_0}$ containing the point $\mathbf{r}_0 + f(\mathbf{r}_0)\hat{\mathbf{k}} = \langle x_0, y_0, f(x_0, y_0)\rangle$, and this plane slices the surface $\mathcal{G}_f$ along some curve. In the plane $\Pi_{\hat{\mathbf{u}},\mathbf{r}_0}$, the variation of the curve as one displaces from $\mathbf{r}_0$ in the direction of $\pm\hat{\mathbf{u}}$ is purely in the z direction, and so it is natural to try to study the rate of change of z as one moves along the $\hat{\mathbf{u}}$ direction by a small displacement $h\hat{\mathbf{u}}$. One sees easily that the directional derivative formula above is precisely the appropriate limit of a difference quotient to capture this rate of change. Observe also that the usual partial derivatives are just directional derivatives along the coordinate directions, e.g. for $\mathbb{R}^3$ with standard rectangular coordinates (x, y, z), one has:

$$D_{\hat{\imath}} = \frac{\partial}{\partial x}\,, \quad D_{\hat{\jmath}} = \frac{\partial}{\partial y}\,, \quad D_{\hat{\mathbf{k}}} = \frac{\partial}{\partial z}\,.$$

Proposition 1.1. *The directional derivative of f in the direction of $\hat{\mathbf{u}}$ at a point $\mathbf{r}_0 \in \mathcal{D}$ may be calculated as*

$$D_{\hat{\mathbf{u}}} f(\mathbf{r}_0) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(\mathbf{r}_0)\,,$$

where u_i are the components of $\hat{\mathbf{u}}$ in rectangular coordinates on $\mathbb{R}^n$ and x_i are the n variables of f giving the rectangular coordinates of a general vector argument $\mathbf{r}$.

Proof. This is a straightforward consequence of the multivariate chain rule – see (1). □

Several natural questions arise immediately:

- For a fixed point $\mathbf{r}_0 \in \mathcal{D}$, can one readily determine for what $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$ the directional derivative $D_{\hat{\mathbf{u}}} f(\mathbf{r}_0)$ is largest? That is, how do we determine the direction leaving $\mathbf{r}_0$ that maximizes the rate of change of the function f?
- What can be said about directions in which the directional derivative vanishes?
- Is there a coordinate free and geometric way to understand the directional derivative operator, other than its defining limit formula? That is, as the formula above to calculate it is not coordinate independent, can we instead describe the directional derivative operator in a geometric way that doesn't invoke rectangular coordinates, or some other arbitrary choice? After all, the directional derivative "frees us" from considering only the way f changes along coordinate directions, and its limit definition suggests that it lives independently of coordinates.

Observe that the expression in the theorem is reminiscent of the formula for a dot product in terms of the components of two vectors in rectangular coordinates. Indeed, we shall realize it as such–we will momentarily define the gradient of f at $\mathbf{r}_0$ to be a vector which fulfills the necessary role to allow us to view this computation as being a dot product. But then we are left to ponder whether the expression would be so nice in another coordinate system. How should one compute a directional derivative of a two-variable function given in terms of polar variables, or a three variable function expressed in spherical coordinates? This amounts to asking about coordinate transformations of the gradient of f. As we shall see, there is a "coordinate-free" story, but the computations one does most often occur in a particular coordinate system, and thus we must understand the coordinate dependence of our methods as well.

Definition (The gradient at a point)**.** For $f : \mathcal{D} \to \mathbb{R}$ a multivariate function differentiable at the point $P(x_1, \ldots, x_n)$, the gradient of f at P is the unique vector $\nabla f(P)$ such that for any $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$, the directional derivative of f at P satisfies

$$D_{\hat{\mathbf{u}}} f(P) = \hat{\mathbf{u}} \cdot \nabla f(P) = \mathrm{comp}_{\hat{\mathbf{u}}} \nabla f(P)\,.$$

In rectangular coordinates $P(x_1, \ldots, x_n)$, the gradient can be expressed as

$$\nabla f(P) = \left\langle \frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \right\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)\hat{\mathbf{e}}_i\,,$$

where $(\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_n)$ is the usual rectangular orthonormal basis for $\mathbb{R}^n$.

[1] *Smooth* has a technical definition which involves grades; the smoothness of the graph surface $\mathcal{G}_f = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = f(x, y)\}$ for continuously differentiable f is called 1-smoothness, and the function f is said to be of class $\mathcal{C}^1(\mathcal{D}, \mathbb{R})$. If f has continuous partials of all orders less than or equal to k for some natural number k, then we say f is of class $\mathcal{C}^k(\mathcal{D}, \mathbb{R})$ and its graph is a k-smooth surface. In differential topology, "smooth" without a specified integer usually means k-smooth for all k, in which case the function f would be said to be of class $\mathcal{C}^\infty(\mathcal{D}, \mathbb{R})$.

Note that the way we defined $\nabla f(P)$, we did not need the coordinates (as the directional derivative is defined using limits and vector addition, without reference to coordinates), however we immediately have a convenient and memorable expression for the gradient at a point in terms of the partial derivatives with respect to the rectangular coordinate variables. But, if we exploit the geometry of the dot product, we can arrive at a second definition of the gradient at a point, in terms of giving an optimal answer to the question of how to choose a direction leaving the point P to change f most rapidly.

Let φ be the angle between $\nabla f(P)$ and $\hat{\mathbf{u}}$. Then we can rewrite the directional derivative as

$$D_{\hat{\mathbf{u}}} f(P) = \|\nabla f(P)\| \cos\varphi\,,$$

since $\|\hat{\mathbf{u}}\| = 1$ by definition. We see from this formula that the directional derivative is maximized by choosing $\hat{\mathbf{u}}$ in the same direction as $\nabla f(P)$, and for this choice, the directional derivative has value $\|\nabla f(P)\|$. Thus we have the alternative definition:

Definition (The gradient as the vector of steepest ascent)**.** For $f : \mathcal{D} \to \mathbb{R}$ a multivariate function differentiable at the point $P(x_1, \dots, x_n)$, the gradient of f at P is the unique vector $\nabla f(P)$ such that $D_{\hat{\mathbf{u}}} f(P)$ is maximized by choosing $\hat{\mathbf{u}} = \nabla f(P)/\|\nabla f(P)\|$, and

$$D_{\hat{\mathbf{u}}} f(P) = \|\nabla f(P)\|$$

gives the maximum rate of change of f at P. Observe that the minimum value of $D_{\hat{\mathbf{u}}} f(P)$ occurs for $\hat{\mathbf{u}} = -\nabla f(P)/\|\nabla f(P)\|$, and the minimum rate of change is $-\|\nabla f(P)\|$.

This gives a fruitful geometric intuition for the directional derivative in the direction of $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$: given that $\nabla f(P)$ represents the optimal direction and rate of increase of f at P, $D_{\hat{\mathbf{u}}} f(P)$ is just the scalar projection of this steepest ascent vector onto the direction $\hat{\mathbf{u}}$. That is, the rate of change of the function f in any direction $\hat{\mathbf{u}}$ is just the scalar projection of a single vector, the gradient at P, which encodes the maximum rate of change at P and the direction in which it occurs, onto the direction $\hat{\mathbf{u}}$.

Continuing from this observation, we can now give a subtly different coordinate-free interpretation of the directional derivative operator. Initially, we considered a fixed direction specified by a vector $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$, and a fixed point $P \in \mathcal{D}$, from which we obtained a quantity $D_{\hat{\mathbf{u}}} f(P)$ measuring the rate of change of f in the direction of $\hat{\mathbf{u}}$. We now change perspectives, by fixing f and P but allowing $\hat{\mathbf{u}} \in \mathbb{S}^{n-1}$ to vary over the whole sphere. That is, we now study the directional derivative operator on f at P as a map from the sphere $\mathbb{S}^{n-1}$ to $\mathbb{R}$.

Fix $f : \mathcal{D} \to \mathbb{R}$ and $P \in \mathcal{D}$ a point where at least one directional derivative of f is nonzero (and thus, P is *non-critical*). Imagine $\mathbb{S}^{n-1}$ as the unit sphere centered at P, capturing all of the directions, called escape vectors, that we might choose to leave from P. Then we have a map

$$\begin{aligned} D_{\bullet} f(P) : \mathbb{S}^{n-1} &\to \mathbb{R}\,, \\ \hat{\mathbf{u}} &\mapsto D_{\hat{\mathbf{u}}} f(P) = \hat{\mathbf{u}} \cdot \nabla f(P)\,, \end{aligned}$$

which captures the rate at which f changes for a choice of escape vector. Since $\mathbb{S}^{n-1}$ is a compact space, a version of the extreme value theorem applies: this function must have an absolute maximum, and the gradient direction $\hat{\mathbf{u}} = \nabla f(P)/\|\nabla f(P)\|$ is the escape vector which gives us this maximum. One might worry that there could be multiple escape vectors $\hat{\mathbf{u}}$ giving the same absolute maximum value for $D_{\hat{\mathbf{u}}} f(P)$, but since P is noncritical, the formula $D_{\hat{\mathbf{u}}} f(P) = \hat{\mathbf{u}} \cdot \nabla f(P) = \|\nabla f(P)\| \cos\varphi$ guarantees that there are in fact only two critical points for this map. Indeed, this map is in a strict sense a *minimal Morse function* for a sphere, that is, it is a function on the sphere with each critical point non-degenerate (the Hessian determinants are nonzero) and the minimal number of critical points[2] (one maximum and one minimum); as a map of $\mathbb{S}^{n-1}$, the directional derivative gives a "height function" (up to a constant factor of $\|\nabla f(P)\|$) relative to an axis in the direction of the gradient $\nabla f(P)$. We can regard $\nabla f(P)/\|\nabla f(P)\|$ as the "north pole" of this sphere, and the equator of this sphere relative to this induced height function is precisely the set of unit tangent directions to the *level set* containing P, as will be understood from the discussion below in section 1.2.

§ 1.2. The Gradient as a Vector Field

Having defined the gradient of a function at a point, we now study the gradient as a map of the domain of the function. If $f : \mathcal{D} \to \mathbb{R}$ is differentiable at all points of $\mathcal{D}$, then we can define $\nabla f(P)$ for each $P \in \mathcal{D}$. Thus, we have a map

$$\nabla f : \mathcal{D} \to \mathbb{R}^n$$
$$P \mapsto \nabla f(P)$$

sending points in $\mathcal{D}$ to vectors in $\mathbb{R}^n$. Such a map, assigning vectors to points of a geometric set, is called a *vector field* on that set. So in our case, we can view the operation of taking the gradient of f as giving a vector field on the domain $\mathcal{D}$ of f. If f is not everywhere differentiable, then we can define the vector field ∇f only on the subset of the domain where the partials of f exist.

Note that if we regard $\mathcal{D}$ as a set of vectors, we may think of vector fields as maps from vectors to vectors, though the domain vectors have a life as position vectors, while the image vectors may have different interpretations depending on context (e.g., we may consider force fields, where the vectors assigned describe force on a point mass or charge, or we may have a velocity field for wind, so the image vectors are velocities of particles at a point and a given moment of time). In this sense, vector fields generalize the idea of *vector-valued functions*, to allow vectors as *inputs* as well as outputs. Let us now define vector fields on domains in $\mathbb{R}^n$ formally:

Definition. Given a set $\mathcal{D} \subseteq \mathbb{R}^n$ and a vector space $\mathscr{V}$, a *vector field on* $\mathcal{D}$ is a map $\mathbf{F} : \mathcal{D} \to \mathscr{V}$ assigning to each point $\mathbf{r} \in \mathcal{D}$ a vector $\mathbf{F}(\mathbf{r}) \in \mathscr{V}$. In the context of classical vector calculus[3], $\mathscr{V}$ is taken to be $\mathbb{R}^n$.

Before we study many other examples of vector fields, let us return to our study of gradient vector fields, as we will use it to arrive at other constructions eventually. Note that the preceding definitions of the gradient of f at a point tell us that there is a geometric interpretation of the gradient vector field ∇f: it is the vector field whose vectors at any point P specify the direction in which f most rapidly increases, and with the vector lengths giving the maximum rates of change at the points of $\mathcal{D}$. There are some nice geometric consequences of this interpretation, in particular involving level sets, tangent spaces, and local extrema of functions.

Recall that a level set of a function $f : \mathcal{D} \to \mathbb{R}$ is a set of all points of $\mathcal{D}$ on which f has a given constant value. Letting $t = f(\mathbf{r})$, we can define, for any constant $t = t_0$, a level set

$$f^{-1}(t_0) = \{\mathbf{r} \in \mathcal{D} : f(\mathbf{r}) = t_0\}.$$

Note that f^{-1} here does not mean "inverse" but rather, "pre-image". That is, a level set of f is the subset of its domain $\mathcal{D}$ which is the pre-image of a constant value. If f is continuously differentiable

[2]Morse functions are a formal analogue of height functions for surfaces, and in some sense are generic among smooth functions. They are of great use to differential topologists, who study spaces called *smooth manifolds* up to *diffeomorphisms*, which are smooth bijective maps that are smoothly invertible. Put differently, a differential topologist is interested in classifying the types of smooth manifolds up to smooth and reversible deformations. One of the powerful results of Morse theory is that any compact manifold which admits a Morse function with just two critical points must be topologically a sphere. More generally, the kinds of critical points of a Morse function, as classified by the signs of the eigenvalues of their Hessians, encode a lot of topological information about a space, and lead to Big Ideas like handle decompositions and Morse Homology.

[3]In modern differential geometry, vector fields are often given as global differential operators, called *tangent vector fields* on $\mathcal{D}$: instead of assigning vectors from a fixed vector space $\mathscr{V}$, one would look at spaces $\mathcal{T}_P\mathbb{R}^n$ of differential operators at P, for each $P \in \mathcal{D} \subseteq \mathbb{R}^n$. The spaces $\mathcal{T}_P\mathbb{R}^n$ are called *tangent spaces to* $\mathbb{R}^n$ *at* P, and can be interpreted as being spaces of generalized directional derivative operators, with coordinate form $a_1(P)\frac{\partial}{\partial x_1}\big|_P + \cdots + a_n(P)\frac{\partial}{\partial x_n}\big|_P$. The tangent spaces $\mathcal{T}_P\mathbb{R}^n$ are each isomorphic to $\mathbb{R}^n$, and one can give a classical version of these modern fields for domains $\mathcal{D} \subseteq \mathbb{R}^n$. The modern approach has the advantage that it generalizes to spaces more general than $\mathbb{R}^n$, called *differentiable manifolds*, where there is not an immediately clear notion of what "attaching an arrow" would mean. Nevertheless, a differentiable manifold M admits tangent spaces $\mathcal{T}_pM$ of differentiable operators for points $p \in M$, and one can define vector fields and a suite of other calculus objects associated to M.

and $\mathcal{D}$ is an n-dimensional subspace of $\mathbb{R}^n$, then the level sets have dimension at most $n-1$ (this is the difference of the dimension of the domain and the dimension of the codomain). For example, a differentiable two-variable function has level sets which are generally curves, while a differentiable three variable function has level sets which are generally surfaces.

Example 1.1. Let $f(\mathbf{r}) = \mathbf{r}\cdot\mathbf{r}$. The value of f at a point $\mathbf{r}\in\mathbb{R}^n$ is the square of the distance of $\mathbf{r}$ from the origin $\mathbf{0}\in\mathbb{R}^n$. The level sets are spheres; in 2D the level sets are the "1-sphere" $\mathbb{S}^1$, i.e., the circle, while in $\mathbb{R}^3$ they are the familiar "2-sphere" $\mathbb{S}^2$, which is the surface of what non-mathematicians think of when they hear the word sphere. A quick calculation shows that the gradient of f is $2\mathbf{r}$, which is a radial vector field, pointing away from the level sets outwards (towards more distant level sets). This is depicted for 2 and 3 dimensions below in figure (3).

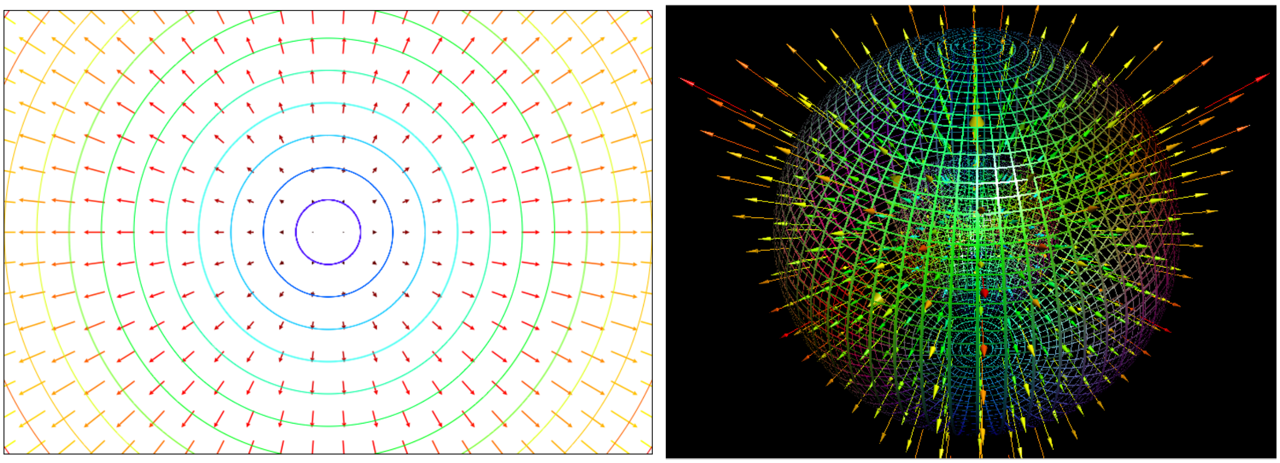


FIGURE 3. The level sets and gradients of the square distance function $f(\mathbf{r}) = \mathbf{r}\cdot\mathbf{r}$ in 2 and 3 dimensions.

Note that the gradient in the above examples is in a sense perpendicular to the level sets themselves (namely, it is perpendicular to the tangent spaces at any point of a level set). It turns out this is not merely because the level sets of the previous example were circles and spheres, while the gradients in the previous examples were radial. More generally, we should expect the directional derivative to vanish along directions tangent to level sets, since the value of the function doesn't change along a level set. Another intuition is that since the gradient tells us how to move away from P to most steeply ascend through values of f, we expect that the gradient should "point as much as possible away from level sets". One can show explicitly using the chain rule that in fact, the gradient is *always* orthogonal to the level sets (in the sense that it is perpendicular to any tangent vector):

Proposition 1.2. *For a given level set $\mathcal{S} = f^{-1}(k)$ of a differentiable function $f:\mathcal{D}\to\mathbb{R}$, and any point $P\in\mathcal{S}$, let $\mathbf{r}_0$ be the position of P and*

$$\mathcal{T}_P\mathcal{S} = \{\mathbf{v}\in\mathbb{R}^n : \mathbf{v} = \dot{\boldsymbol{\gamma}}(t_0) \text{ for } \boldsymbol{\gamma}: I\to\mathcal{S} \text{ a curve in } \mathcal{S} \text{ with } \boldsymbol{\gamma}(t_0) = \mathbf{r}_0\}$$

be the tangent vector space to $\mathcal{S}$ at P, i.e. the set of tangent vectors at the point P to curves in $\mathcal{S}$ through P. Then

$$\nabla f(P)\cdot\mathbf{v} = 0 \text{ for all } \mathbf{v}\in\mathcal{T}_P\mathcal{S}\,.$$

Thus the gradient of f along a level set $\mathcal{S}$ is a *normal vector field* to $\mathcal{S}$. We'll explore tangent and normal vectors in greater detail below in section 2.4.

Example 1.2. The following example goes through the solution to Exercise 3.3 from the notes on partial derivatives.

While exploring an exoplanet (alone and un-armed–what were you thinking‽) you've slid part way down a strangely smooth, deep hole. The alien terrain you are on is modeled locally (in a neighborhood around you spanning several dozen square kilometers) by the height function

$$z = f(x, y) = \ln\sqrt{16x^2 + 9y^2}\,,$$

where the height z is given in kilometers. Let $\hat{\imath}$ point eastward and $\hat{\jmath}$ point northward. Your current position is one eighth kilometers east, and one sixth kilometers south, relative to the origin of the (x, y) coordinate system given. You want to climb out of this strange crater to get away from the rumbling in the darkness below you.

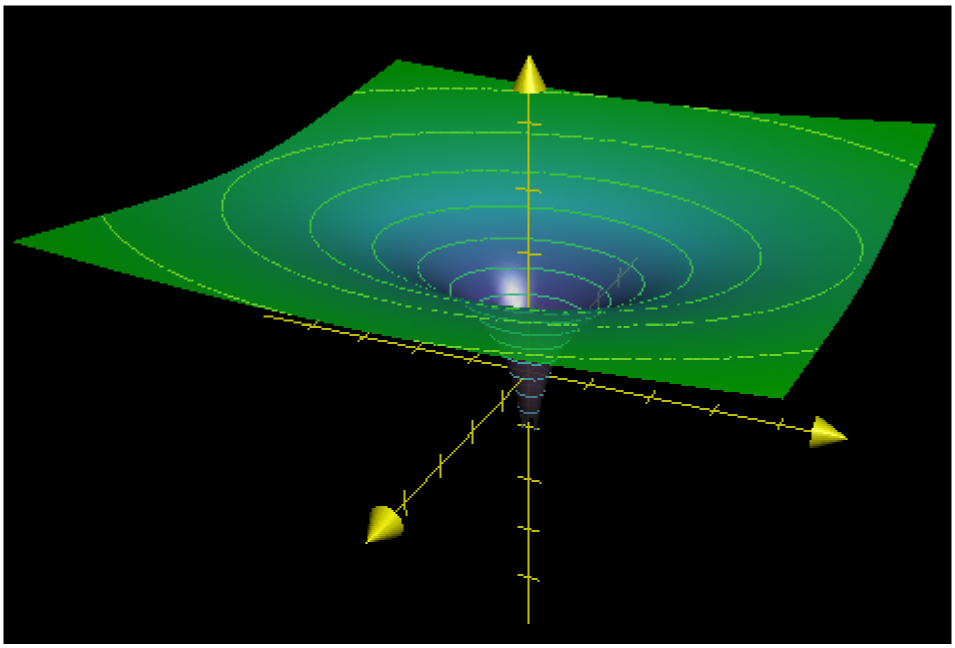


FIGURE 4. The graph of the surface $z = \ln\sqrt{16x^2 + 9y^2}$.

(a) Find your current height relative to the $z = 0$ plane.

(b) Show that the level curves $z = k$ for constants k are ellipses, and explicitly determine the semi-major and semi-minor axis lengths in terms of the level constant k.

(c) In what direction(s) should you initially travel if you wish to stay at the current altitude?

(d) What happens if you travel in the direction of the vector $-(1/8)\hat{\imath} + (1/6)\hat{\jmath}$? Should you try this?

(e) In what direction should you travel if you wish to climb up (and hopefully out) as quickly as possible? Justify your choice mathematically.

(f) For each of the directions described in parts (c), (d), and (e), explicitly calculate the rate of change of your altitude along those directions.

Solutions:

(a) From $z = f(x, y) = \ln\sqrt{16x^2 + 9y^2}$, your current height relative to the plane $z = 0$ is

$$z = f(1/8, -1/6) = \ln\sqrt{16\left(\tfrac{1}{8}\right)^2 + 9\left(-\tfrac{1}{6}\right)^2} = \ln\sqrt{\tfrac{16}{64} + \tfrac{9}{36}} = -\tfrac{1}{2}\ln 2 \approx -0.3466\,.$$

Thus you are about 35 meters (a bit shy of 115 feet) below the plane $z = 0$.

(b) Rewrite f as $f(x,y) = \frac{1}{2}\ln(16x^2+9y^2)$. Then if $z = f(x,y) = k$ for a constant k, we have

$$\begin{aligned} k = \frac{1}{2}\ln(16x^2+9y^2) &\implies 2k = \ln(16x^2+9y^2) \\ &\implies e^{2k} = 16x^2+9y^2 \\ &\implies 1 = \frac{x^2}{e^{2k}/16} + \frac{y^2}{e^{2k}/9} = \left(\frac{x}{e^k/4}\right)^2 + \left(\frac{y}{e^k/3}\right)^2 \end{aligned}$$

Thus the level curves are ellipses with semi-major axis length $\frac{1}{3}e^k$ and semi-minor axis length $\frac{1}{4}e^k$. See figure 5 for a visualization of the contours.

(c) To stay at the current altitude, you should initially choose a direction tangent to the level curve through your position. To calculate such directions, you can exploit that the gradient at a position $\mathbf{r}$ is perpendicular to the level curve through $\mathbf{r}$. The gradient of f is

$$\nabla f(x,y) = \left(\frac{16x}{16x^2+9y^2}\right)\hat{\mathbf{\imath}} + \left(\frac{9y}{16x^2+9y^2}\right)\hat{\mathbf{\jmath}}\,,$$

which gives a gradient at the starting position of $(1/8, -1/6)$ as

$$\nabla f(1/8, 1/6) = 4\hat{\mathbf{\imath}} - 3\hat{\mathbf{\jmath}}\,.$$

The perpendicular directions in which you could initially head to stay at the current altitude are

$$\pm 3\hat{\mathbf{\imath}} \pm 4\hat{\mathbf{\jmath}}\,.$$

(d) You can compute the directional derivative in the direction of the vector $-(1/8)\hat{\mathbf{\imath}} + (1/6)\hat{\mathbf{\jmath}}$ to see what is happening to your altitude. Let

$$\hat{\mathbf{u}} = \frac{-(1/8)\hat{\mathbf{\imath}} + (1/6)\hat{\mathbf{\jmath}}}{\|-(1/8)\hat{\mathbf{\imath}} + (1/6)\hat{\mathbf{\jmath}}\|} = -\frac{3}{5}\hat{\mathbf{\imath}} + \frac{4}{5}\hat{\mathbf{\jmath}}\,.$$

Then

$$\begin{aligned} D_{\hat{\mathbf{u}}}f(1/8,-1/6) &= \nabla f(1/8,-1/6)\cdot\hat{\mathbf{u}} \\ &= (4\hat{\mathbf{\imath}} - 3\hat{\mathbf{\jmath}})\cdot\left(-\frac{3}{5}\hat{\mathbf{\imath}} + \frac{4}{5}\hat{\mathbf{\jmath}}\right) \\ &= \frac{-12-12}{5} = -\frac{24}{5}\,. \end{aligned}$$

Thus, if you head in this direction, you are descending at an initial rate of nearly 5 meters downward per meter forward. Indeed noting that this vector is in the exact opposite direction as your initial position, it heads straight for the origin, which is where the hole is indefinitely deep. So you should *not* head this way if you hope to live very long.

(e) To climb out as quickly as possible, assuming you can maintain stamina, you should seek the route of steepest ascent, which is a route along the gradient direction. Starting from $(1/8,-1/6)$, you should then initially travel in the direction of $\nabla f(1/8,-1/6)/\|\nabla f(1/8,-1/6)\| = \frac{4}{5}\hat{\mathbf{\imath}} - \frac{3}{5}\hat{\mathbf{\jmath}}$. Note that this direction is *not* the radial direction as one might initially suspect; this discrepancy of directions is sensible given that the level curves are not circles, but ellipses. In fact, we can calculate the cosine angle between the direction of steepest ascent and the radial direction easily: just dot the corresponding unit vectors:

$$\hat{\mathbf{u}}_r(1/8,-1/6)\cdot\frac{\nabla f(1/8,-1/6)}{\|\nabla f(1/8,-1/6)\|} = \left(\frac{3}{5}\hat{\mathbf{\imath}} - \frac{4}{5}\hat{\mathbf{\jmath}}\right)\cdot\left(\frac{4}{5}\hat{\mathbf{\imath}} - \frac{3}{5}\hat{\mathbf{\jmath}}\right) = 24/25\,,$$

whence these directions make an angle of $\arccos(24/25) \approx 0.2838$ radians, or $16.26°$.

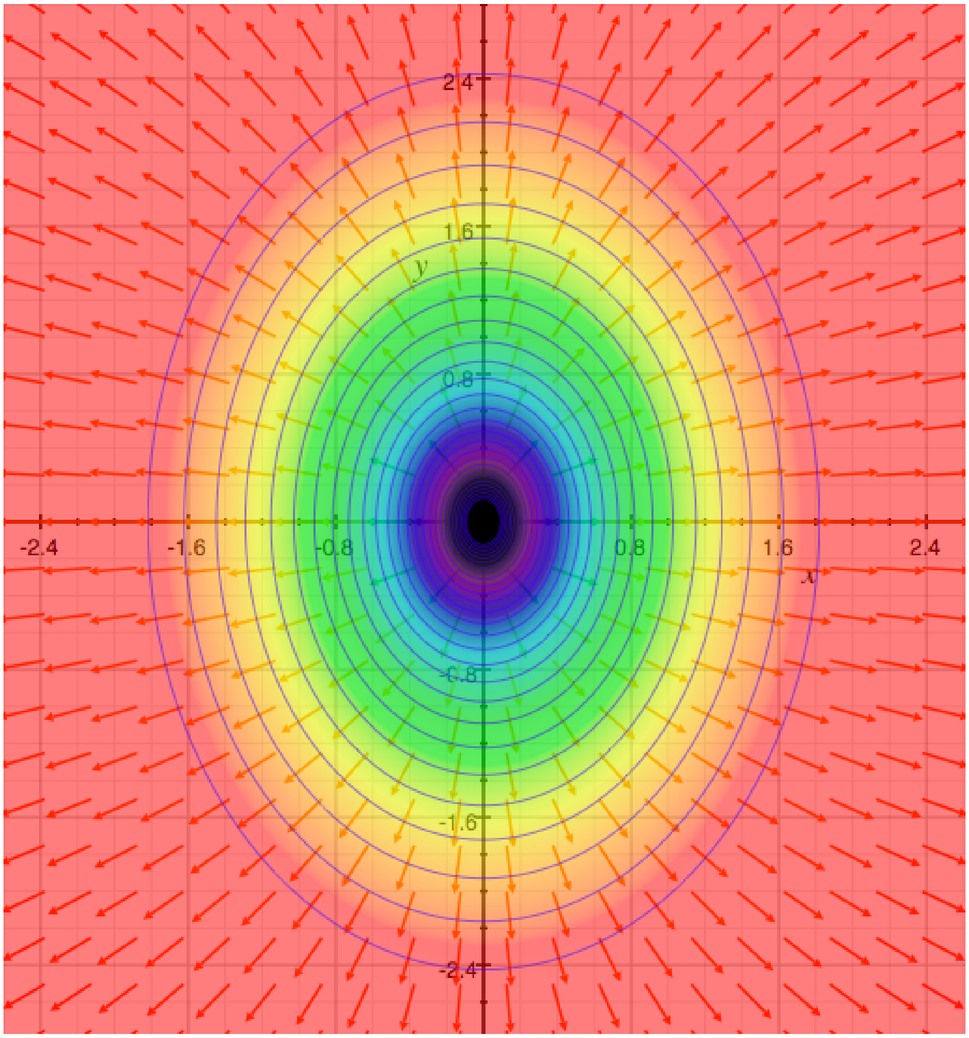


FIGURE 5. A color map of the altitude $z = \ln\sqrt{16x^2 + 9y^2}$, showing also the elliptical contours for z, and the gradient vector field (with vectors scaled down for clarity).

(f) For (c) the rate of change in the altitude is 0, as is easily verified by computing a directional derivative. It better be zero of course– if you wish to remain at the current altitude, then height function should not change initially in the direction chosen. For part (d) the rate of change was computed above as $-24/5$. For part (e), the rate of change in the gradient direction is $\|\nabla f(1/8, -1/6)\| = \|4\hat{\imath} - 3\hat{\jmath}\| = 5$. Note these rates represent (kilo)meters of incline or decline relative to a horizontal (kilo)meter displacement along a vector $\hat{\mathbf{u}} \in \mathbb{S}^1 \subset \mathbb{R}^2$.

In the preceding example, we used that the gradient of a bivariate function determines the direction of steepest ascent on the graph surface and that the gradient is perpendicular to level curves. Pushing this idea further, we can use that the gradient is normal to level sets $\mathcal{S}$ to determine an equation of the *affine tangent space* to a hypersurface $\mathcal{S} \subset \mathcal{D} \subset \mathbb{R}^n$ given as the level set of some function $f : \mathcal{D} \to \mathbb{R}$. Without loss of generality, we can assume such a hypersurface is given as the level zero set of an appropriate function. First, we define the affine tangent space:

Definition. Let $\mathcal{S}$ be a hypersurface given as the zero set $f^{-1}(0)$ of a differentiable function $f : \mathcal{D} \to \mathbb{R}$. Let $\mathbf{r}_0$ be the position of a point $P \in \mathcal{S}$. Then the *affine tangent space to* $\mathcal{S}$ *at* P is the set $\mathcal{AT}_P\mathcal{S}$ of all points of $\mathbb{R}^n$ that can be reached by displacing from $\mathbf{r}_0$ by a vector $\mathbf{v} \in T_P\mathcal{S}$.

Equivalently, it is the set of points swept out by all possible velocity vectors $\mathbf{v} = \dot{\boldsymbol{\gamma}}(t_0)$ to curves $\boldsymbol{\gamma} : I \to \mathcal{S}$ that pass through P when $t = t_0 \in I$, when these velocity vectors are placed at P:

$$\mathcal{AT}_P\mathcal{S} = \{\mathbf{r} \in \mathbb{R}^n : \mathbf{r} = \mathbf{r}_0 + \mathbf{v} \text{ for } \mathbf{v} \in \mathcal{T}_P\mathcal{S}\}\,.$$

Since the gradient is normal to tangent vectors to the hypersurface, and any point $\mathbf{r}$ of $\mathcal{AT}_P\mathcal{S}$ is displaced from $\mathbf{r}_0$ by a tangent vector $\mathbf{v}$, we know that $\nabla f(\mathbf{r}_0)$ is perpendicular to $\mathbf{v} = \mathbf{r} - \mathbf{r}_0$. Thus, we have the following proposition giving the equation of $\mathcal{AT}_P\mathcal{S}$:

Proposition 1.3. *The affine tangent space $\mathcal{AT}_P\mathcal{S}$ to $\mathcal{S}$ at P is given as the locus of points $\mathbf{r} \in \mathbb{R}^n$ satisfying the equation*

$$\nabla f(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0\,.$$

In rectangular coordinates, this yields a scalar equation

$$\sum_{i=1}^{n} a_i x_i = d\,,$$

where $a_i = \frac{\partial f}{\partial x_i}(\mathbf{r}_0)$ and $d = \nabla f(\mathbf{r}_0) \cdot \mathbf{r}_0$.

Thus, $\mathcal{AT}_P\mathcal{S}$ is genuinely a hyperplane tangent to the hypersurface $\mathcal{S}$ at P.

Example 1.3. Let $\mathcal{S}$ be the radius R sphere centered at $\mathbf{0}$ in $\mathbb{R}^3$. Find the tangent plane equation at the point with position $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$.

Solution: Let $f(\mathbf{r}) = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$. Then the sphere $\mathcal{S}$ is just the level set $f^{-1}\{R^2\} = \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\|^2 = R^2\} = \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\| = R\}$. Let $P(x_0, y_0, z_0)$ be the given point on $\mathcal{S}$, and $\mathbf{r}_0$ its position. Observe that $\nabla f(\mathbf{r}_0) = 2\mathbf{r}_0$. Then by the above proposition:

$$\begin{aligned}\mathcal{AT}_P\mathcal{S} &= \{\mathbf{r} \in \mathbb{R}^3 : \nabla f(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0\} \\ &= \{\mathbf{r} \in \mathbb{R}^3 : 2\mathbf{r}_0 \cdot (\mathbf{r} - \mathbf{r}_0) = 0\} \\ &= \{\mathbf{r} \in \mathbb{R}^3 : \mathbf{r}_0 \cdot \mathbf{r} = \|\mathbf{r}_0\|^2 = R^2\} \\ &= \{(x, y, z) : x_0 x + y_0 y + z_0 z = R^2\}\,.\end{aligned}$$

Thus an affine tangent plane to the sphere at P is a plane through P whose normal is given in coordinates by the position vector $\mathbf{r}_0$ of P itself! This is what one should expect; it is a result of classical geometry that the tangent plane to a sphere at a point is orthogonal to the radial line segment from the sphere's center to the point of tangency.

§ 1.3. The Gradient Flow and Critical Points

We can now discuss the relation of the gradient vector field of a differentiable function $f : \mathcal{D} \to \mathbb{R}$ to the local extrema of such functions. Let us first consider two-variable functions, as the picture we wish to paint is both simple and clear when there are only two variables.

If $f : \mathcal{D} \to \mathbb{R}$ is a differentiable two-variable function with graph a surface

$$\mathcal{S} = \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathcal{D}\}\,,$$

then at each point P of its domain, we have a vector $\nabla f(P)$ which points us in the direction in which the graph's slope is steepest. Imagining the graph as a mountain, the gradient $\nabla f(P)$ is pointing a hiker in the direction that allows her to climb away from her current location most efficiently. So, suppose (x, y) is moved a little distance $\boldsymbol{\Delta}\mathbf{r}$ along this direction (and so our hiker climbs up the mountain a little, initially parallel to a tangent vector whose xy-plane projection is the gradient). The hiker arrives at a new point, where there is a (potentially) new gradient direction pointing her in the direction of steepest ascent. We can imagine her repeatedly traveling along little displacements, with the (x, y) position displaced parallel to the gradient at each step. The smaller the steps, the more closely her motion follows the directions of the gradient vector field. One can take a limit, and find that there is some curve $\boldsymbol{\gamma}(t) \subset \mathcal{D}$ leaving P and traveling some ways in $\mathbb{R}^2$

such that the tangent vectors to $\boldsymbol{\gamma}(t)$ are always in the gradient direction. In fact, we can choose a parametrization

$$\boldsymbol{\gamma} : [a, b] \to \mathcal{D}$$

such that $\boldsymbol{\gamma}(a) = \mathbf{r}_0$ is the position of P, and $\dot{\boldsymbol{\gamma}}(t) = \nabla f\big(\boldsymbol{\gamma}(t)\big)$ for any $t \in [a, b]$.

Of course, one can try to extend this curve as far as possible by taking as many steps as possible along gradient directions, until one finds a point where the gradient gives no direction (that is, the gradient either vanishes or doesn't exist). One can also try to travel down the mountain, and so one can talk about extending this curve backwards. By extending as far as one can in either direction, one obtains a maximal path in $\mathcal{D}$ through P which is always tangent to the gradient vector field. Such a path is called a *field-line* or *integral curve* for the gradient vector field. Since f was presumed differentiable throughout $\mathcal{D}$, there is such a field-line through every point $P \in \mathcal{D}$ except those where $\nabla f(P) = \mathbf{0}$. At these points, multiple field-lines converge. Observe that the field-lines are necessarily perpendicular to the level curves.

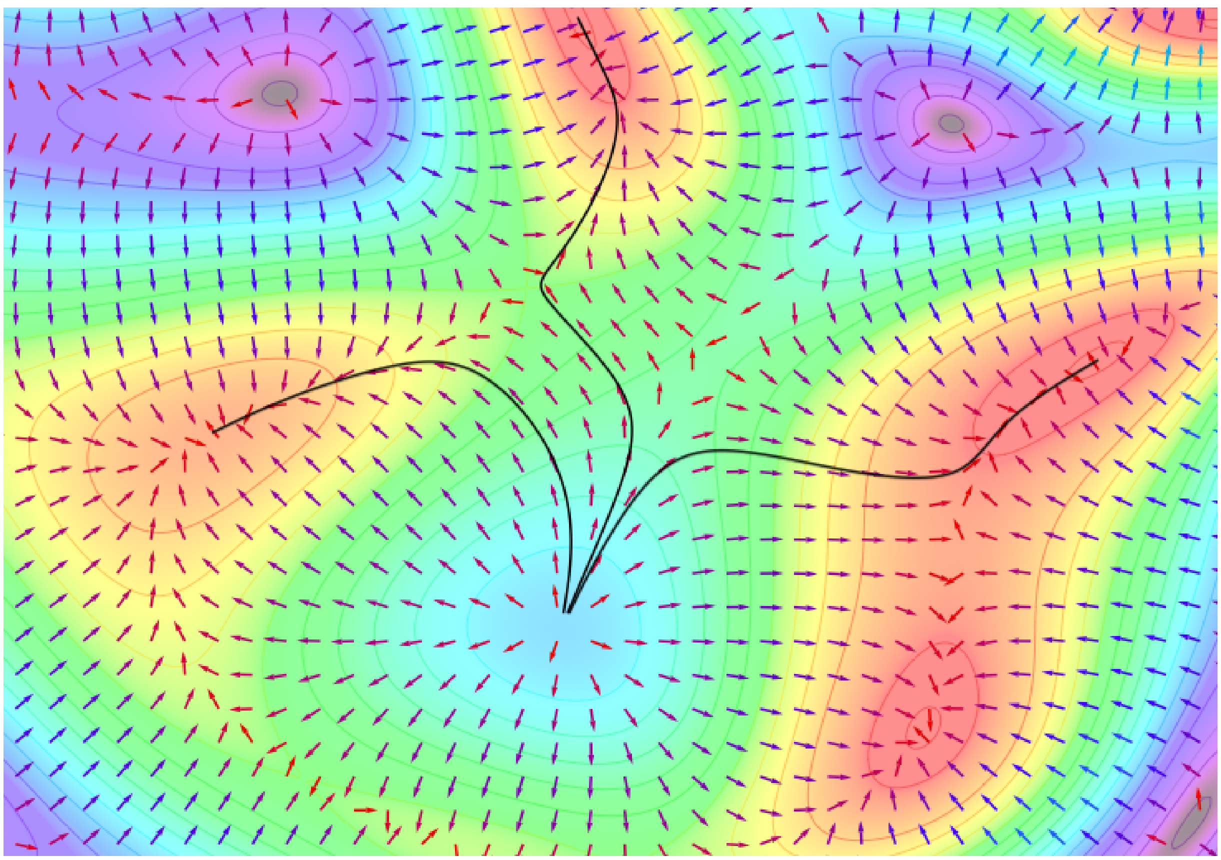


Figure 6. Three integral curves of the gradient for some bivariate function are illustrated along with a heat map, contours, and the gradient vector field itself (rescaled for clarity). Note that although these curves all originate near each other in a region near a local minimum, they each tend towards different local maxima. To reach a summit, just follow the gradient vectors from where-ever you stand! But be careful: note one curve narrowly misses a saddle point (look for the sharp rightward bend)–at a saddle critical point, it is ambiguous how to best proceed upwards.

Returning to our hiker's journey, we let her (x, y) position follow a field-line, leading her up the mountain. If her journey comes to an end, it is because her field-line has terminated in a point P with $\nabla f(P) = \mathbf{0}$. Such a point will be called a *critical point* of f. Note that at such a point, the tangent plane is necessarily parallel to the xy-plane, i.e. it is horizontal. If she is lucky, she has found a summit, though it is possible she has instead found a mountain pass (also called a saddle).

Note that if she wants to find her way to one of the lowest points of terrain, she can follow the field-lines backwards until she hits a critical point, hoping again it is not a saddle.

We can now formalize this idea for general multivariable functions. We will define critical points, and a map called *the gradient flow* which is defined on $\mathcal{D}$ and allows us to imagine pushing or flowing the domain towards certain critical points, and away from the local minima.

Definition. Let $f : \mathcal{D} \to \mathbb{R}$ be an n-variable function. The critical points of f are the points of $\mathcal{D}$ where the partial derivatives all vanish, or where any partial derivative fails to exist:

$$\mathrm{crit}(f) = \{\mathbf{r} \in \mathcal{D} : \nabla f(\mathbf{r}) = \mathbf{0} \text{ or } \nabla f(\mathbf{r}) \text{ does not exist}\}.$$

The numbers $f(\mathbf{r}_0)$ for $\mathbf{r}_0 \in \mathrm{crit}(f)$ are called the *critical values* of the function f. A critical value $f(\mathbf{r}_0)$ is a local maximum value if there exists a neighborhood $N \in \mathcal{D}$ of $\mathbf{r}_0$ such that $f(\mathbf{r}_0) \geq f(\mathbf{r})$ for all $\mathbf{r} \in N$. A critical value is a local minimum if there exists a neighborhood $N \in \mathcal{D}$ of $\mathbf{r}_0$ such that $f(\mathbf{r}_0) \leq f(\mathbf{r})$ for all $\mathbf{r} \in N$.

A point $\mathbf{r}$ where $\nabla f(\mathbf{r})$ exists and is nonzero is called a *regular point*.

Definition. The *gradient flow* of a function $f : \mathcal{D} \to \mathbb{R}$ of n variables is the map given by

$$\Phi(t, \mathbf{r}) = \boldsymbol{\gamma}_{\mathbf{r}}(t),$$

where $\boldsymbol{\gamma}_{\mathbf{r}}(t)$ is the *field-line* through $\mathbf{r}$ such that $\boldsymbol{\gamma}_{\mathbf{r}}(0) = \mathbf{r}$ and $\dot{\boldsymbol{\gamma}}_{\mathbf{r}}(t) = \nabla f(\boldsymbol{\gamma}_{\mathbf{r}}(t))$ for all t for which $\boldsymbol{\gamma}_{\mathbf{r}}(t)$ is defined.

Observe that $\Phi(0, \mathbf{r}) = \mathbf{r}$. As t increases, Φ maps the domain $\mathcal{D}$ onto itself such that any point $\mathbf{r}$ is moved along its field-line by a time step of t. If $\mathbf{r}$ is non-critical and the field-line $\boldsymbol{\gamma}_{\mathbf{r}}(t)$ terminates in a critical point, then $\mathbf{r}$ will move towards this critical point for t large enough.

One can define field-lines/integral curves for more general vector fields, which we will briefly discuss in section 2.5. For now, we offer a brief discussion of the field-lines of ∇f for f a function of two variables. From the definition we can extract a differential condition to be satisfied for the trajectory of a point (x, y) under the gradient flow. Writing $\boldsymbol{\gamma}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$, and $\dot{x}(t) = \mathrm{d}x/\mathrm{d}t$, $\dot{y}(t) = \mathrm{d}y/\mathrm{d}t$, the velocity vector for the gradient field-line $\boldsymbol{\gamma}(t)$ is $\dot{\boldsymbol{\gamma}}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}}$, and imposing the condition $\dot{\boldsymbol{\gamma}}(t) = \nabla f(\boldsymbol{\gamma}(t))$, we arrive at the *autonomous system of differential equations*[4]:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \partial_x(f)(x(t), y(t)) \\ \partial_y(f)(x(t), y(t)) \end{pmatrix},$$

where[5] $\partial_x(f) = \partial f/\partial x$ and $\partial_y(f) = \partial f/\partial y$.

These are often nonlinear differential equations and generally difficult or impossible to solve explicitly. Nevertheless, one can still use the idea of gradient flow to prove things, or gather a useful understanding of the behavior of functions and the geometry of their graphs. One observation we can make in this two-dimensional setting is that while it may be quite difficult to explicitly describe the flow with equations, there is occasionally some hope of understanding the field-lines as curves described implicitly. Since the gradient $\nabla f(P)$ of a two-variable function at the point P is a vector in $\mathbb{R}^2$, it specifies a *slope* for a tangent line to any curve through P that is tangent to the gradient field at P. Thus, we deduce that when y is implicitly a function of x along a field-line of ∇f through P, it must satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\hat{\boldsymbol{\jmath}} \cdot \nabla f(x, y)}{\hat{\boldsymbol{\imath}} \cdot \nabla f(x, y)} = \left.\frac{\partial_y(f)}{\partial_x(f)}\right|_{(x,y)}.$$

Similarly, one can describe the derivative $\frac{\mathrm{d}x}{\mathrm{d}y}$. Though these first order differential equations may be solvable in some instances where the system approach is fruitless, it is still often the case that

[4]A first order system of differential equations is a set of equations relating variables to their first derivatives with respect to some common parameter t, and to each other. These can often be given in the form $\dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t)$ where $\mathbf{F}$ is a vector-valued function dependent on position and the variable t, usually thought of as time. One can think of $\mathbf{F}$ as a *time-dependent vector field.* If $\mathbf{F}$ is time-independent, meaning $\partial\mathbf{F}/\partial t = \mathbf{0}$, then the system is called *autonomous.*

[5]In this and future sections, we will frequently abbreviate partial derivative operators like $\frac{\partial}{\partial x}$ as ∂_x, except in certain definitions and propositions, or in any context in which it might impact the clarity of the notation.

one cannot obtain closed form analytic solutions. We'll explore some qualitative ways one can still grasp at the gradient flow through examples.

Finding the equations of the gradient field-lines for a two-variable function thus amounts to solving first order differential equations involving x and y, either as a system or to obtain implicit equations. Similarly, finding integral curves of general vector fields (in any number of dimensions) amounts to solving differential equations. This connection will be explored briefly again in §2.2.

Example 1.4. Consider the function $f(x,y) = \sqrt{36 - x^2 - y^2}$. It should be easily recognized that the graph of $z = f(x,y)$ will be the "northern" hemisphere of a sphere of radius 6 placed with center at $\mathbf{0} \in \mathbb{R}^3$. The gradient is

$$\nabla f(x,y) = -\frac{x}{\sqrt{36 - x^2 - y^2}}\hat{\mathbf{\imath}} - \frac{y}{\sqrt{36 - x^2 - y^2}}\,.$$

This is a radial vector field pointing inwards, and the magnitude decreases as one approaches the origin. Indeed, we can rewrite this vector field as

$$\nabla f(x,y) = -\frac{1}{z}(x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}) = -\frac{\mathbf{r}}{\sqrt{36 - \mathbf{r}\cdot\mathbf{r}}}\,, \quad \mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}\,.$$

Note that there is exactly one isolated critical point, which is at the origin, corresponding to the maximum value $f(0,0) = 6$ (the "north pole"). The gradient is also undefined at the boundary, where the tangent planes to the surface are all vertical.

From the above gradient calculation, the gradient field-lines satisfy the differential equation

$$\dot{\mathbf{r}} = -\frac{\mathbf{r}}{\sqrt{36 - \mathbf{r}\cdot\mathbf{r}}}\,.$$

Since the right-hand side is in the opposite direction of $\mathbf{r}$, we know that $\dot{\mathbf{r}}$ and $\mathbf{r}$ are parallel, and thus the gradient field-lines are line segments heading toward the origin–the gradient flow instructs a hiker on this dome to head for the north pole along a direct trajectory, which on the sphere is an arc of a great circle, the shadow of which on the xy-plane is the line segment connecting the hiker's position to the origin. Note also that since the gradient is undefined at the equator, the differential equation is singular there.

Dotting both sides of the equation with $\mathbf{r}$ and doubling, we obtain

$$2\mathbf{r}\cdot\dot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{r}\|^2) = -\frac{2\mathbf{r}\cdot\mathbf{r}}{\sqrt{36 - \mathbf{r}\cdot\mathbf{r}}} = -\frac{2\|\mathbf{r}\|^2}{\sqrt{36 - \|\mathbf{r}\|^2}}\,.$$

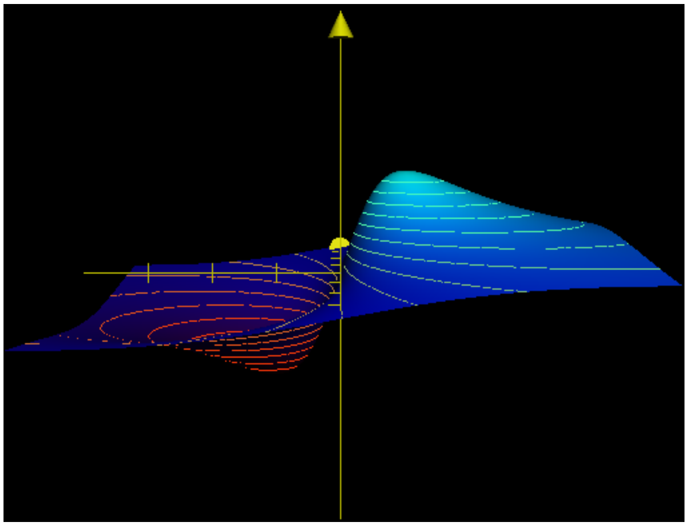
This gives us a differential equation for the square of the distance of the point $\mathbf{r}$ from $\mathbf{0}$, but from it we can also get an equation for the distance itself:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{r}\| = -\frac{\|\mathbf{r}\|}{\sqrt{36 - \|\mathbf{r}\|^2}}\,.$$

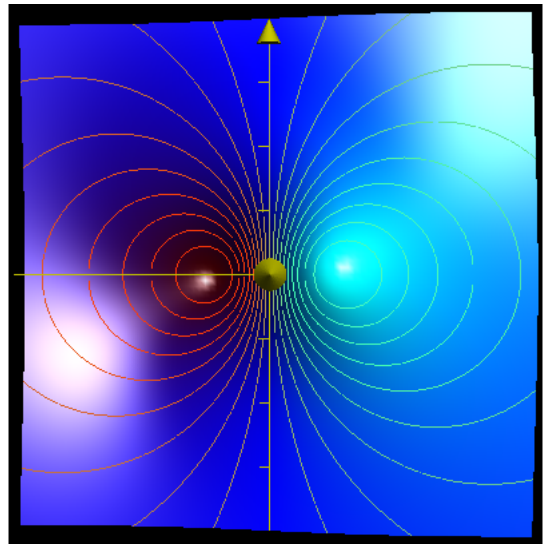
From either equation, we see that the rate of change of the the distance from $\mathbf{0}$ is negative, and approaches 0 from below as $\mathbf{r} \to \mathbf{0}$. We can actually solve either equation by separation and integration to get *t as a function of $\|\mathbf{r}\|^2$ or $\|\mathbf{r}\|$*. Since the derivatives are non-positive, we know that the position function is monotonic decreasing, and we can in principle invert to get position as a function of t. Unfortunately, inverting the resulting functions explicitly is not feasible, but it is interesting to note that you can still understand the flow: from our above analysis we know that points flow towards the origin, and if we know the initial distance from $\mathbf{0}$, and we know how far we want a point to travel towards $\mathbf{0}$ along the field-line carrying it to the origin, we can determine how long it will take for the flow to take it there. This will be explored in (11) in the problems below.

Example 1.5. Let

$$f(x,y) = \frac{2x}{x^2 + y^2 + 1}\,.$$



(A)



(B)

FIGURE 7. (A) – A view of the surface of the graph of $z = f(x, y)$ from just above the negative y-axis. (B) – A view of the surface of the graph of $z = f(x, y)$ from above, showing the contours as a family of circles

We'll describe the level sets and the extrema, and then we'll study the gradient flow.

Let $z = f(x, y)$. The level set corresponding to $z = z_0$ a constant is the set of points (x, y) satisfying

$$z_0 = f(x, y) = \frac{2x}{x^2 + y^2 + 1} \implies (x^2 + y^2 + 1)z_0 = 2x\,.$$

If $z_0 = 0$, then $x = 0$, and the y-axis is the level set. Let's consider when $z_0 \neq 0$. By rearranging and completing the square, one gets

$$\left(x - \frac{1}{z_0}\right)^2 + y^2 = \frac{1}{z_0^2} - 1\,.$$

From this, it is apparent that the level sets for $z - 0 \neq 0$ are circles with centers $(1/z_0, 0)$ and radii $1/z_0^2 - 1$, and $-1 \leq z_0 \leq 1$. Observe that our description of the level curves implies that the range of f is $[-1, 1]$, with extrema at $(\pm 1, 0)$.

The gradient of f is

$$\nabla f(x, y) = \left(\frac{2}{x^2 + y^2 + 1} - \frac{4x^2}{(x^2 + y^2 + 1)^2}\right)\hat{\mathbf{\imath}} - \frac{4xy}{(x^2 + y^2 + 1)^2}\hat{\mathbf{\jmath}}\,.$$

The trajectories $\boldsymbol{\gamma}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}}$ under the gradient flow satisfy the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} \frac{2}{x^2+y^2+1} - \frac{4x^2}{(x^2+y^2+1)^2} \\ -\frac{4xy}{(x^2+y^2+1)^2} \end{bmatrix}.$$

This is both highly nonlinear and algebraically intimidating to solve, so we'll instead look to understand the trajectories as implicit curves.

The $\hat{\mathbf{\imath}}$-component of $\nabla f(x, y)$ can be rewritten as $z/x - z^2$, and the $\hat{\mathbf{\jmath}}$-component can be rewritten as $-yz^2/x$. Thus the gradient field-lines satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{yz^2}{xz^2 - z} = \frac{2xy}{x^2 - y^2 - 1}\,.$$

Rather than explicitly and forcefully solving this differential equation, we will first look back to the level curves, and then study the geometry of the surface $z = f(x, y)$ to better understand the flow.

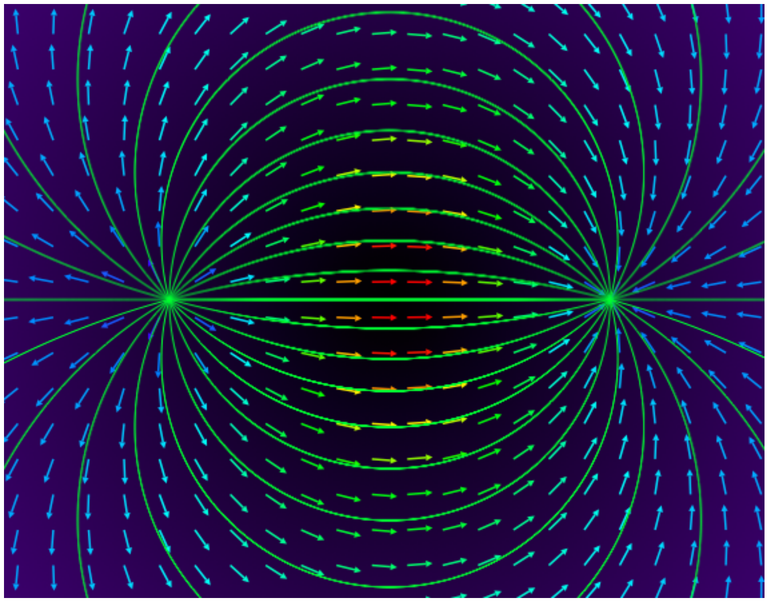


FIGURE 8. The gradient vector field together with some field-lines.

As shown in figure (7 B), the level curves are circles emanating from the minimum, growing in radius with centers moving outwards along the negative x axis, until we reach a limiting case which is a line (the y-axis), after which they are circles with centers on the positive x-axis, moving inwards, radii decreasing, enclosing the maximum at $(1, 0)$. The gradient field-lines are perpendicular to the level curves, and they too form a family of circles. These circles comprise the two kinds of families of *Apollonian circles*, pictured in figure (9), which are named after the Greek geometer Apollonius of Perga who discovered them. Note that each family also contains a "degenerate" circle, which is a line; in the case of the gradient flow, it will be the x axis, which contains three different field-lines.

To see that the field-lines and level curves really are this pair of Appolonian circle families, we need to understand the geometry of the level curve family, and see that a curve which is orthogonal to all of the level curves it meets must in fact be itself an arc of a circle with center on the y axis, and passing through both critical points. This is developed in (12) of the problems below.

The primary result of problem 12 is that $z(\mathbf{r})$ is an algebraic transformation of the ratio of the distances from $\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$ to the two critical points:

$$z = \tanh \ln \frac{\|\mathbf{r} + \hat{\imath}\|}{\|\mathbf{r} - \hat{\imath}\|} = \frac{\|\mathbf{r} + \hat{\imath}\|^2 - \|\mathbf{r} - \hat{\imath}\|^2}{\|\mathbf{r} + \hat{\imath}\|^2 + \|\mathbf{r} - \hat{\imath}\|^2}\,.$$

The level sets are thus the circles where this ratio is constant. Let τ be the inverse hyperbolic tangent of the corresponding level, i.e., τ satisfies

$$e^\tau = \frac{\|\mathbf{r} + \hat{\imath}\|}{\|\mathbf{r} - \hat{\imath}\|}\,.$$

The other family of circles is the sets of $\mathbf{r}$ such that the angle between the displacement vectors $\mathbf{r} - \hat{\imath}$ and $\mathbf{r} + \hat{\imath}$ is a constant σ, and so along any gradient field-line the angle σ is constant. The flow is given by letting τ act as the time parameter:

$$\Phi_\tau(\mathbf{r}(\tau_0, \sigma_0)) = \mathbf{r}(\tau_0 + \tau, \sigma_0)\,,$$

where $\mathbf{r}(\tau, \sigma)$ is given by

$$\mathbf{r}(\tau, \sigma) = \frac{\sinh \tau}{\cosh \tau - \cos \sigma}\,\hat{\imath} + \frac{\sin \sigma}{\cosh \tau - \cos \sigma}\,\hat{\jmath}\,.$$

Problem (12) guides you through the details of showing this.

The field-lines come in several families: there are circular arcs arching from $(-1, 0)$, where $\tau = -\infty$, to $(1, 0)$, where $\tau = \infty$, both above and below the x-axis. These fill out most of the plane, in

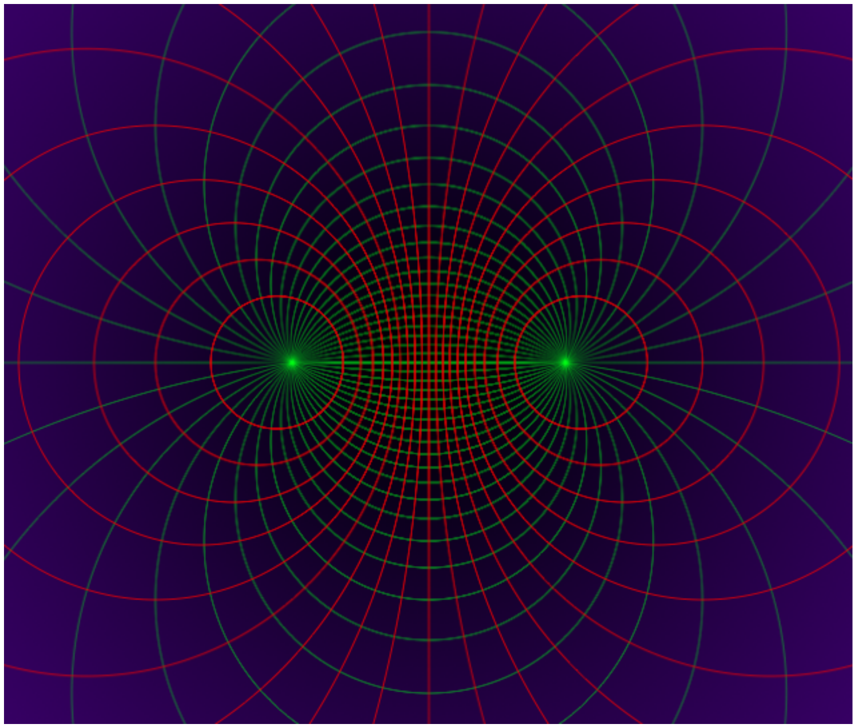


Figure 9. The two families of Apollonian circles constituting the families of level curves and gradient field-lines. The red circles are the level curves $(x - 1/z_0)^2 + y^2 = 1/z_0^2 - 1$, and the green circles are *pairs* of gradient field-lines; each green circle decomposes into two arcs, one above the x-axis, and one below, which are both field-lines, with the flow carrying points away from the minimum at $(-1, 0)$ and eventually towards the maximum at $(1, 0)$.

the family of circles given by setting $\sigma =$ a constant. i.e., circles centered on the y-axis and passing through $(\pm 1, 0)$. Thus, the true circles within this Apollonian family each split into two field-lines, one above the x-axis, and one below. The degenerate case is the x-axis itself ($\sigma = 0$), which splits into three field-lines. There is the straight line segment from $(-1, 0)$ to $(1, 0)$ which behaves much like the other field-lines, in the limit as $\tau \to \infty$ it reaches the critical point at $(1, 0)$ that gives the maximum, and in ancient time as $\tau \to -\infty$ it flows back to the critical point $(-1, 0)$ which gives the minimum. Then there are the two rays $\{\mathbf{r} \in \mathbb{R}^2 : 1 \leq \mathbf{r} \cdot \hat{\mathbf{\imath}} < \infty\}$ and $\{\mathbf{r} \in \mathbb{R}^2 : -\infty < \mathbf{r} \cdot \hat{\mathbf{\imath}} \leq -1\}$. The former approaches the critical point at $(1, 0)$, while the latter flows away from the critical point at $(-1, 0)$, as if to "wander off to infinity".

The pair of numbers (τ, σ) defines an orthogonal curvilinear coordinate system called *bipolar coordinates*, in this case, with its two focal points at $(\pm 1, 0)$. In the next section, we'll explore how to express gradients in terms of other coordinate systems, though we leave bipolar coordinates to the exercises.

§ 1.4. The Del Operator and the Gradient in Other Coordinates*

(Note: this is an optional section concerned with constructing formulae for the gradient and the del operator in polar and spherical coordinates, and outlining a general procedure to recover the operator in other coordinates using only differential calculus and linear algebra. This is not part of the current curriculum, but is a useful skill, especially if the reader plans to explore the later optional sections, and any examples using spherical coordinates.)

We've now seen that from a differentiable function of multiple variables $f : \mathcal{D} \to \mathbb{R}^n$ we can produce a vector field on $\mathcal{D}$ via the gradient, and moreover, when restricted to level sets $\mathcal{S} \subset \mathcal{D}$, we obtain a normal vector field. We have a way of concretely describing the gradient vector field in components when working in rectangular coordinates, using the partial derivatives $\partial_{x_1} f, \ldots, \partial_{x_n} f$. We are interested in two objectives: understanding the assignment of vector fields to functions via an operator, and understanding how to express this in other coordinates. We can think of the first objective as a step towards a coordinate free understanding of multiple constructions involving vector-valued derivatives as well as derivatives of vector fields. These perspectives will be developed in later sections.

To meet our objectives, we once again subtly adapt our perspective. Whereas in the last section we fixed f and examined the idea of the gradient as a vector field, sending points P to vectors $\nabla f(P)$, we now consider the idea of a map from the space of differentiable functions on $\mathcal{D} \subseteq \mathbb{R}^n$ to vector fields on $\mathcal{D}$. Let $\mathcal{C}^1(\mathcal{D}, \mathbb{R})$ be the space of continuously differentiable functions[6] on $\mathcal{D}$, and let $\mathfrak{V}(\mathcal{D}, \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued vector fields on $\mathcal{D}$. Then we have a map

$$\begin{aligned} \nabla : \mathcal{C}^1(\mathcal{D}, \mathbb{R}) &\to \mathfrak{V}(\mathcal{D}, \mathbb{R}^n) \\ f &\mapsto \nabla f\,. \end{aligned}$$

Writing vectors in rectangular coordinates, we know this map can be expressed as

$$f \mapsto \nabla f = \sum_{i=1}^{n} (\partial_{x_i} f)\mathbf{e}_i\,.$$

This motivates the following definition:

Definition. The operator ∇, called the "del operator," "nabla", or the "gradient operator" is the partial differential operator that sends a function $f \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$ to its gradient $\nabla f \in \mathfrak{V}(\mathcal{D}, \mathbb{R}^n)$.

The rectangular coordinate expression of the del operator is

$$\nabla = \sum_{i=1}^{n} \mathbf{e}_i \partial_{x_i} = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle.$$

The second part of the definition is really a notational convention, albeit one of great convenience. The map sending a function to its gradient then can be interpreted as "multiplying" ∇ on the right by the scalar function f, thus distributing f to the components, where the partials then act on f. This of course is a coordinate dependent expression for this operator, and so one might wonder what happens when we try to change coordinates. It will not suffice to merely replace the basis vectors $\mathbf{e}_i$, for we must also pair them with the appropriate partial differential operators, which can have a general form of a function of the new coordinates, times partial derivatives with respect to the new coordinates.

Perhaps the first natural choice of example is to express the two-dimensional del operator in the *polar frame* $(\hat{\mathbf{u}}_r, \hat{\mathbf{u}}_\theta)$. Recall that the polar frame is given by

$$\hat{\mathbf{u}}_r = \frac{x\hat{\mathbf{\i}} + y\hat{\mathbf{\j}}}{\sqrt{x^2+y^2}} = \cos(\theta)\,\hat{\mathbf{\i}} + \sin(\theta)\,\hat{\mathbf{\j}} \qquad \hat{\mathbf{u}}_\theta = \partial_\theta \hat{\mathbf{u}}_r = -\sin(\theta)\,\hat{\mathbf{\i}} + \cos(\theta)\,\hat{\mathbf{\j}} = \frac{-y\hat{\mathbf{\i}} + x\hat{\mathbf{\j}}}{\sqrt{x^2+y^2}}\,.$$

[6] Strictly speaking, we can define the del operator on a larger class of merely *differentiable functions*, as there exist functions which are differentiable but not necessarily continuously differentiable. But a function having continuous first partials guarantees that it is differentiable, so this is a safe and large class of functions to use as the domain of our operator for the purposes of this class.

Proposition 1.4. *The 2-dimensional del operator expressed in the polar frame is*

$$\nabla = \hat{\mathbf{u}}_r \frac{\partial}{\partial r} + \hat{\mathbf{u}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\,.$$

The actual calculation to show this is left as an exercise (see problem 20 of the notes *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*). We will however describe a procedure to convert del into a general curvilinear coordinate system, and apply this procedure to demonstrate del in a three-dimensional spherical coordinate system.

The general procedure to convert del to curvilinear coordinate frames has three steps:

(*i*) Express the standard basis of rectangular coordinates in the curvilinear coordinate frame[7]. That is, if the standard basis of rectangular coordinates is $(\mathbf{e}_1, \dots \mathbf{e}_n)$, and the new coordinates $(y_1, \dots y_n)$ give rise to a frame $(\hat{\mathbf{u}}_1, \dots \hat{\mathbf{u}}_n)$ where in general $\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(y_1, \dots, y_n)$ are functions of the new variables, you want to find functions $a_{ij}(y_1, \dots y_n)$ for each $\mathbf{e}_i$, such that

$$\mathbf{e}_i = \sum_{j=1}^{n} a_{ji}(y_1, \dots y_n)\hat{\mathbf{u}}_j(y_1, \dots y_n)\,.$$

Note that since $\mathbf{e}_i$ is constant, the partial derivative of $\mathbf{e}_i$ with respect to any y_k is $\mathbf{0}$, whence for each $i = 1, \dots, n$ and any $k = 1, \dots, n$

$$\partial_{y_k}\mathbf{e}_i = \sum_{j=1}^{n}\Big(\partial_{y_k}(a_{ji})\hat{\mathbf{u}}_j + a_{ji}\partial_{y_k}(\hat{\mathbf{u}}_j)\Big) = \mathbf{0}\,.$$

This gives a differential criteria one can use to check if the linear algebra was done correctly.

(*ii*) Apply the chain rule to express ∂_{x_i} in terms of the new coordinate functions and the partial differential operators with respect to them:

$$\partial_{x_i} = \sum_{k=1}^{n} b_{ik}(y_1, \dots, y_n)\partial_{y_k}\,,$$

where $b_{ik} = \partial_{x_i}(y_k)$ is expressed as a function of the variables $y_1, \dots, y_n$.

(*iii*) Assembling steps (*i*) and (*ii*), the del operator can be written as

$$\begin{aligned}\nabla = \sum_{i=1}^{n}\mathbf{e}_i\partial_{x_i} &= \sum_{i=1}^{n}\left[\Big(\sum_{j=1}^{n} a_{ji}\mathbf{u}_j\Big)\Big(\sum_{k=1}^{n}\partial_{x_i}(y_k)\partial_{y_k}\Big)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}\partial_{x_i}(y_k)\mathbf{u}_j\partial_{y_k}\;.\end{aligned}$$

Note that we can regroup the sums:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}\partial_{x_i}(y_k)\mathbf{u}_j\partial_{y_k} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ji}b_{ik}\hat{\mathbf{u}}_j\partial_{y_k} = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ji}b_{ik}\right)\hat{\mathbf{u}}_j\partial_{y_k} = \sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk}\hat{\mathbf{u}}_j\partial_{y_k}\,,$$

where $c_{jk} = \sum_{i=1}^{n} a_{ji}b_{ik}$. Letting $\mathrm{A} = (a_{ji})$ be the matrix for the change of frame $(\hat{\mathbf{u}}_1, \dots \hat{\mathbf{u}}_n) \to (\mathbf{e}_1, \dots \mathbf{e}_n)$ and $\mathrm{B} = (b_{ik})$ the matrix for the change of derivatives, we see that the coefficients c_{jk} form a matrix $\mathrm{C} = \mathrm{AB}$. Thus, the whole change can be computed using a product of matrices with entries given as functions of the variables $(y_1, \dots y_n)$, and the (j,k)-th entry of the result is the scale factor for the term of del involving the j-th frame vector $\hat{\mathbf{u}}_j$ and the partial derivative operator ∂_{y_k}.

Do not be intimidated! In practice many of the terms above might be zero or might cancel with other terms, and in low dimensions there are fewer terms to work with. We next partially

[7]A coordinate frame can be thought of as a collection of vector fields adapted to the coordinates. At each point of space, a frame element gives the tangent direction to the curve created by continuously changing a corresponding coordinate variable. The perspective of frames as vector fields is described in greater detail in section 2.5. Frames for polar, cylindrical, and spherical coordinates are described in the notes *Curvature, Natural Frames, and Acceleration for Plane and Space Curves.*

demonstrate this process by calculating the del operator in the spherical coordinate system defined in *Curvature, Natural Frames, and Acceleration for Plane and Space Curves* (see pages 11-14), leaving the details of the calculations to the exercises.

The transformation from the rectangular coordinates $(x, y, z)_{\mathscr{R}}$ on $\mathbb{R}^3$ to these spherical coordinates $(\varrho, \theta, \varphi)_{\mathscr{S}}$ is given as

$$x = \varrho\cos\theta\cos\varphi\,, \quad y = \varrho\sin\theta\cos\varphi\,, \quad z = \varrho\sin\varphi\,,$$

where $\varrho \in [0, \infty)$, $\theta \in (-\pi, \pi]$, and $\varphi \in [-\pi/2, \pi/2]$. The transformation of the spherical frame $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$ back to the rectangular frame $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$ is given by the equations

$$\begin{aligned}
\hat{\mathbf{u}}_\varrho &= \cos(\theta)\cos(\varphi)\,\hat{\mathbf{\imath}} + \sin(\theta)\cos(\varphi)\,\hat{\mathbf{\jmath}} + \sin(\varphi)\,\hat{\mathbf{k}}\,,\\
\hat{\mathbf{u}}_\theta &= -\sin(\theta)\,\hat{\mathbf{\imath}} + \cos(\theta)\,\hat{\mathbf{\jmath}}\,,\\
\hat{\mathbf{u}}_\varphi &= -\cos(\theta)\sin(\varphi)\,\hat{\mathbf{\imath}} - \sin(\theta)\sin(\varphi)\,\hat{\mathbf{\jmath}} + \cos(\varphi)\,\hat{\mathbf{k}}\,.
\end{aligned}$$

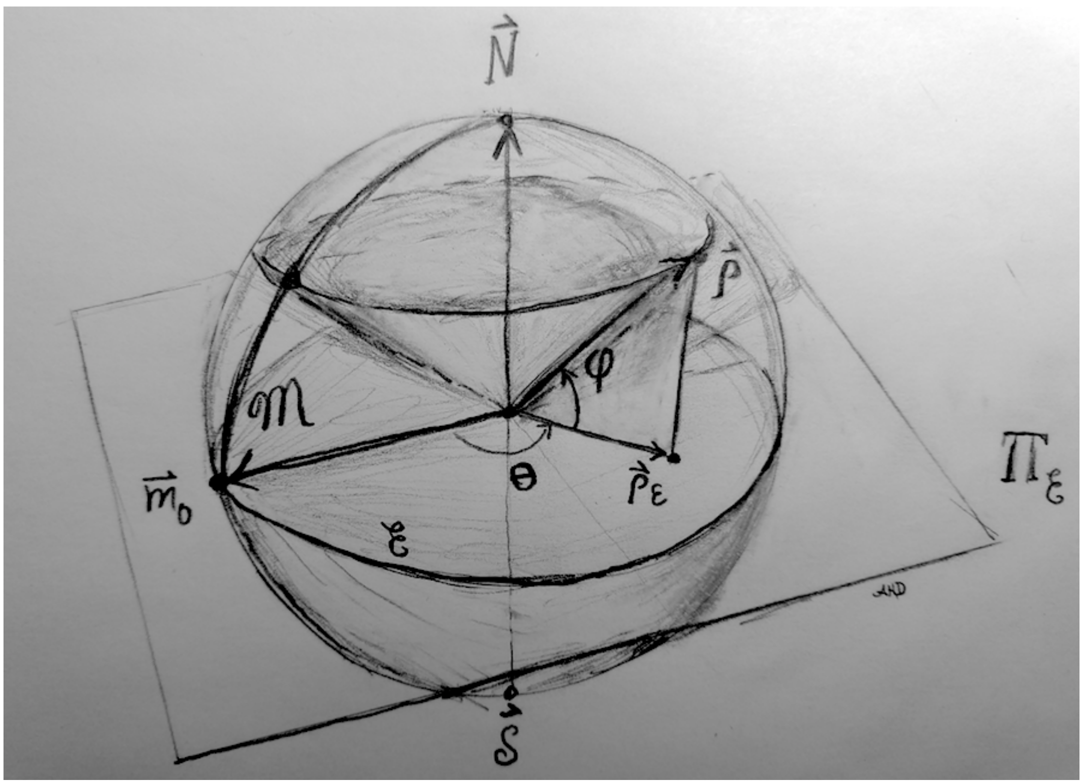


FIGURE 10. A form of spherical coordinates modeled loosely on geographic coordinates by longitude and latitude - note that these coordinates define φ as an *elevation* angle measured from the projection of $\hat{\mathbf{u}}_\varrho$ into the equatorial plane, rather than the common mathematical convention, in which that angle is defined instead as an *polar* or *inclination* angle measured between $\hat{\mathbf{k}}$ and $\hat{\mathbf{u}}_\varrho$. Since these competing angles are complementary, to recover the more common coordinate convention, merely swap $\sin\varphi$ and $\cos\varphi$ in the coordinate expressions.

Per step (i), we must first express $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$ as linear combinations of $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$ with coefficients dependent on the spherical coordinate variables (ϱ, θ, ϕ). A little linear algebra gives

$$\begin{aligned}
\hat{\mathbf{\imath}} &= \cos(\theta)\cos(\varphi)\,\hat{\mathbf{u}}_\varrho - \sin(\theta)\,\hat{\mathbf{u}}_\theta - \cos(\theta)\sin(\varphi)\,\hat{\mathbf{u}}_\varphi\,,\\
\hat{\mathbf{\jmath}} &= \sin(\theta)\cos(\varphi)\,\hat{\mathbf{u}}_\varrho + \cos(\theta)\,\hat{\mathbf{u}}_\theta - \sin(\theta)\sin(\varphi)\,\hat{\mathbf{u}}_\varphi\,,\\
\hat{\mathbf{k}} &= \sin(\varphi)\,\hat{\mathbf{u}}_\varrho + 0\,\hat{\mathbf{u}}_\theta + \cos(\varphi)\,\hat{\mathbf{u}}_\varphi\,.
\end{aligned}$$

Following step (ii), we employ the chain rule and write

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \varrho}{\partial x}\frac{\partial}{\partial \varrho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\,,\\
\frac{\partial}{\partial y} &= \frac{\partial \varrho}{\partial y}\frac{\partial}{\partial \varrho} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}\,,\\
\frac{\partial}{\partial z} &= \frac{\partial \varrho}{\partial z}\frac{\partial}{\partial \varrho} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi}\,.
\end{aligned}$$

Using the relations $\varrho^2 = x^2 + y^2 + z^2$, $x \tan\theta = y$, and $z = \varrho \sin\varphi$, one deduces

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos(\theta)\cos(\varphi)\frac{\partial}{\partial \varrho} - \frac{\sin(\theta)}{\varrho\cos(\varphi)}\frac{\partial}{\partial \theta} - \frac{\cos(\theta)}{\varrho}\frac{\partial}{\partial \varphi}\,,\\
\frac{\partial}{\partial y} &= \sin(\theta)\cos(\varphi)\frac{\partial}{\partial \varrho} + \frac{\cos(\theta)}{\varrho\cos(\varphi)}\frac{\partial}{\partial \theta} - \frac{\sin(\theta)}{\varrho}\frac{\partial}{\partial \varphi}\,,\\
\frac{\partial}{\partial z} &= \sin(\varphi)\frac{\partial}{\partial \varrho} + \frac{\cos(\varphi)}{\varrho}\frac{\partial}{\partial \varphi}\,.
\end{aligned}$$

And finally, we put it all together according to step (iii):

$$\begin{aligned}
\nabla &= \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\\
&= \hat{\mathbf{u}}_\varrho\frac{\partial}{\partial \varrho} + \frac{1}{\varrho\cos(\varphi)}\hat{\mathbf{u}}_\theta\frac{\partial}{\partial \theta} + \frac{1}{\varrho}\hat{\mathbf{u}}_\varphi\frac{\partial}{\partial \varphi}\,.
\end{aligned}$$

We've left this final calculation to problems (16) and (17); note that it can be accomplished via matrix multiplication, by recognizing the coefficients a_{ij} and b_{ik} as matrix elements for the corresponding linear transformations. In this instance, there are numerous cancellations and we are left with just three terms, each consisting of a scale factor, and a partial derivative which matches the basis vector. This is a consequence of the *orthogonality* of the coordinates, but need not happen for skew coordinates.

Once one is acquainted with integrals in vector calculus, it becomes possible to give a new definition of the gradient which is coordinate free, as described in §4.3. This integral approach allows one to more easily obtain expressions for the gradient and related differential operators by considering certain well adapted curves, surfaces and solids as domains of integration, and taking limits as these domains are shrunk to a point. Until then, we have the arduous process above, which is great practice with the chain rule and linear algebra!

§ 1.5. Problems

(1) (a) Recall the multivariate chain rule for a composition of a differentiable n-variable function $f : \mathcal{D} \to \mathbb{R}$ with a curve $\mathbf{r}(t) \subset \mathcal{D}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\mathbf{r}(t)\big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \dot{x}_i(t) = \nabla f(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\,.$$

Prove this chain rule directly using the limit definition of $\frac{\mathrm{d}f}{\mathrm{d}t}$ and the limit definitions of partials $\partial_{x_i} f$. Be sure to note how one needs the assumption of differentiability of f.

(b) Prove the coordinate formula

$$D_{\hat{\mathbf{u}}} f(\mathbf{r}_0) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(\mathbf{r}_0)$$

using the limit definition of the directional derivative and the multivariate chain rule.

(2) In the map $D_{\bullet} f(P) : \mathbb{S}^{n-1} \to \mathbb{R}$ sending $\hat{\mathbf{u}}$ to $D_{\hat{\mathbf{u}}} f(P) = \hat{\mathbf{u}} \cdot \nabla f(P) = \|\nabla f(P)\| \cos\varphi$, what is the interpretation of φ on $\mathbb{S}^{n-1}$? That is, interpret the function sending $\hat{\mathbf{u}}$ to φ as a map on the sphere $\mathbb{S}^{n-1}$. For a three variable function f, what are the level curves of φ as subsets of $\mathbb{S}^2$?

(3) Prove that the gradient is normal to level sets by showing that $\nabla f(P) \cdot \mathbf{v} = 0$ for all tangent vectors $\mathbf{v} \in T_P\mathcal{S}$ for any point P of a level set $\mathcal{S}$.

(4) For a position $\mathbf{r} = x\,\hat{\mathbf{\imath}} + y\,\hat{\mathbf{\jmath}} + z\,\hat{\mathbf{k}} \in \mathbb{R}^3$, let α be the angle between $\mathbf{r}$ and the x-axis, β the angle between $\mathbf{r}$ and the y-axis, and γ the angle between $\mathbf{r}$ and the z-axis. Find a function $f : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}$ such that $D_{\hat{\mathbf{u}}} f(\mathbf{r}) = (\hat{\mathbf{u}} \cdot \hat{\mathbf{\imath}}) \cos\alpha + (\hat{\mathbf{u}} \cdot \hat{\mathbf{\jmath}}) \cos\beta + (\hat{\mathbf{u}} \cdot \hat{\mathbf{k}}) \cos\gamma$.

(5) For a unit vector $\hat{\mathbf{u}} = u_1\,\hat{\mathbf{\imath}} + u_2\,\hat{\mathbf{\jmath}} + u_3\,\hat{\mathbf{k}} \in \mathbb{S}^2$, find functions $f : \mathbb{R}^3 \to \mathbb{R}$ such that $D_{\hat{\mathbf{u}}} f(\mathbf{r}) = u_i$, for each of $i = 1$, 2, and 3.

(6) Consider the surface $\mathcal{S}$ given as the locus of points in $\mathbb{R}^3$ satisfying the equation

$$xy - xz + yz = 2\,.$$

(a) Find an equation of the tangent plane $\mathcal{AT}_P\mathcal{S}$ at the point $P(\sqrt{2}, \sqrt{2}, 1/\sqrt{2})$.

(b) Exhibit a line through $P(\sqrt{2}, \sqrt{2}, 1/\sqrt{2})$ contained in $\mathcal{AT}_P\mathcal{S}$ as found above which is also contained in the surface $\mathcal{S}$, and then find another one.

(c) Show that at any point $P(x, y, z) \in \mathcal{S}$, $\mathcal{AT}_P\mathcal{S}$ contains a pair of lines through P which are both contained in $\mathcal{S}$. The surface $\mathcal{S}$ can be "built" out of lines, called rulings, in two different ways, and so is called a *doubly ruled surface.*

(d) The surface $\mathcal{S}$ is a quadric. Find a coordinate transformation of $\mathbb{R}^3$ putting it into a standard form, and identify the quadric.

(7) Consider a function $f(x, y, z)$ defined over a domain $\mathcal{D} \subseteq \mathbb{R}^3$, and let $\mathcal{S}$ be the level surface defined by $f(x, y, z) = 0$. Assume $0 \in F(\mathcal{D})$ so $\mathcal{S}$ is nonempty, and further assume that $\mathcal{S}$ is $\mathcal{C}^1$-smooth and contains a point $P(x_0, y_0, z_0)$ with a neighborhood satisfying conditions of the implicit function theorem:

Theorem (Implicit function theorem for a 3-variable scalar function)**.**

Let $f(\mathbf{r})$ be a function which is defined on the ball $B_R(\mathbf{r}_0) = \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r} - \mathbf{r}_0\| \leq R\}$ centered at $\mathbf{r}_0 = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$, and such that $f(\mathbf{r}_0) = 0$. Suppose that f is $\mathcal{C}^1$ on $B_R(\mathbf{r}_0)$, i.e., f is continuous throughout $B_R(\mathbf{r}_0)$ and each of $\partial_x f(\mathbf{r})$, $\partial_y f(\mathbf{r})$, and $\partial_z f(\mathbf{r})$ are continuous throughout $B_R(\mathbf{r}_0)$. If $\partial_z f(\mathbf{r}_0) \neq 0$, then $f(\mathbf{r}) = 0$ implicitly defines z as a differentiable function of x and y near $\mathbf{r}_0$, and the level set of f determined by $f(\mathbf{r}) = 0$ on $B_R(\mathbf{r}_0)$ is the surface of the graph of $z(x, y)$ near (x_0, y_0).

Moreover, the partial derivatives of z near (x_0, y_0) are given by

$$\partial_x z = -\frac{\partial_x f}{\partial_z f}, \quad \textit{and} \quad \partial_y z = -\frac{\partial_y f}{\partial_z f}.$$

(a) Write down the scalar equation of $\mathcal{AT}_P\mathcal{S}$ for a point $P(x_0, y_0, z_0) \in \mathcal{S}$ using that $\nabla f(P)$ is a normal vector to $\mathcal{S}$ at P.

(b) Assume that near P, z is locally a function of x and y. Use implicit differentiation to obtain an equation for $\mathcal{AT}_P\mathcal{S}$ and show that it is equivalent to the equation from part (a).

(8) Let Π be a plane with unit normal vector $\hat{\mathbf{n}} = \cos(\alpha)\,\hat{\mathbf{\imath}} + \cos(\beta)\,\hat{\mathbf{\jmath}} + \cos(\gamma)\,\hat{\mathbf{k}}$. Show that there exists a point $P(x_0, y_0, z_0)$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ such that the tangent plane to the ellipsoid at P is parallel to the plane Π. How many such points are there? Express the coordinates of such P in terms of the direction cosines $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ giving the components of $\hat{\mathbf{n}}$.

(9) Consider the functions below over the disk $x^2 + y^2 \leq 4$. For each of the functions, draw a picture illustrating together a family of level curves, the gradient vector field, and the field-lines for the gradient flow (you do not need to explicitly describe the gradient flow). Identify the critical points and their types.

(a) $f(x, y) = x^2 + y^2$,

(b) $f(x, y) = y^2 - x^2$,

(c) $f(x, y) = 4 - x^2 - y^2$,

(d) $f(x, y) = \dfrac{2x}{x^2 + y^2}$.

(10) For each of the functions above in problem (9), re-express the function in polar coordinates, and then recompute the gradient using the polar form of the del operator. (If you somehow thought to do this first, then go back and use rectangular coordinates instead.)

(11) This problem reconsiders the function $f(x, y) = \sqrt{36 - x^2 - y^2}$ and studies its gradient flow. However you should first consider an arbitrary (i.e., unspecified) function $f(x, y)$ for parts (a) and (b).

(a) To warm up: write down the differential equation for the gradient field-lines giving $\mathrm{d}y/\mathrm{d}x$ in terms of the partials of f. Assuming y is given implicitly as a function of x, show that y and x equivalently satisfy an equation in the form

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

for appropriate functions $M(x, y)$ and $N(x, y)$. By realizing the left hand side as the *total differential* $\mathrm{d}F(x, y) = \partial_x F\,\mathrm{d}x + \partial_y F\,\mathrm{d}y$, "integrate" to obtain $F(x, y)$, and recover the geometric description of the field-lines as line segments heading towards the origin. Note that the assumptions that $F(x, y) =$ constant implicitly determines y as a function of x, and that $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y$ is the total differential of F impose conditions on the partials of M and N: if $M(x,y) = \partial_x F(x,y)$ and $N(x,y) = \partial_y F(x,y)$ and both functions are continuously differentiable, then Clairaut's theorem requires that $\partial_y M = \partial_x N$. This will help you choose between ways in which to rewrite the differential equation for $\mathrm{d}y/\mathrm{d}x$ in terms of possible $M(x,y)$ and $N(x,y)$.

(b) The above procedure is equivalent in this case (but more powerful in general; see §2.2) to *separation of variables*: one can instead rearrange the equation for $\mathrm{d}y/\mathrm{d}x$ to put all terms involving y on the left side, and all terms involving x, including the differential $\mathrm{d}x$ on the right hand side. To solve the differential equation, one then integrates both sides separately, and by equating the indefinite integrals and combining constants of integration, one then has an implicit solution. If y can be solved for as a function of x, then one can obtain a general solution whose trajectories are curves of graphs.
Use the method of separation of variables to recover a general solution for this differential equation. Then, under the assumption that the trajectory is initialized at a regular point $(x(0), y(0)) = (x_0, y_0)$ in the domain of f, determine the corresponding values of any constants of integration. Separately treat the cases of x initially being 0 or y initially being 0.

(c) Recall that the distance $r = \|\mathbf{r}\| = \sqrt{x^2+y^2}$ of a point of the disk $0 < x^2 + y^2 < 36$ from the origin undergoing gradient flow for the function $f(x,y) = \sqrt{36 - r^2}$ satisfies the following differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{r}{\sqrt{36 - r^2}}\,.$$

Using separation of variables, find an expression for t as a function of r, with an appropriately determined constant of integration so that when $t = 0$, $r = r_0 \in (0,6)$.

(d) Argue that the limit of the trajectory of a regular point of f under the gradient flow as $t \to \infty$ is the origin. What happens to a regular point if we consider running the flow backwards (taking the limit as $t \to -\infty$)?

(e) Find how long it will take for a point to flow from $r_0 = 3$ to $r = 2$, and for a point to flow from $r_0 = 1$ to $r = 1/2$.

(12) This problem examines the geometry of Apollonian circles and the gradient flow of $f(x,y) = \frac{2x}{x^2+y^2+1}$, as discussed above in the example at the end of section 1.3.

(a) Using the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - y^2 - 1}\,,$$

show that circles of the form $x^2 + (y-h)^2 = 1 + h^2$ satisfy the differential equation. Thus, field-lines of $f(x,y)$ are arcs of such circles.

(b) Fix a number $\sigma \in [0, \pi]$. Argue that the set of all points (x,y) such that the angle between the vector $\mathbf{r} + \hat{\imath}$ and $\mathbf{r} - \hat{\imath}$ is σ determines a circle, except for the boundary cases $\sigma = 0$ or π, which you should argue give a line. Find the equation of the circle (or line) in terms of x, y and σ.

(c) Fix a number $\tau \in \mathbb{R}$. Argue that the set of all points (x,y) such that the ratio

$$\frac{\|\mathbf{r} + \hat{\imath}\|}{\|\mathbf{r} - \hat{\imath}\|}$$

equals e^τ determines a circle, except for the case when $\tau = 0$, which you should argue is a line. Find the equation of the circle (or line) in terms of x, y and τ.

(d) Show that any of the circles/lines in the family from part (b) are orthogonal to any of the circles/lines in the family of part (c).

(e) Show that a point (x, y) is uniquely determined as an intersection of a circle or line of constant τ and a circle or line of constant σ, and in particular show that

$$x = \frac{\sinh\tau}{\cosh\tau - \cos\sigma}\,, \quad y = \frac{\sin\sigma}{\cosh\tau - \cos\sigma}\,.$$

(f) Show that $f(x, y) = \tanh\tau$.

(g) By computing $\partial_\tau x$ as well as $\partial_\tau y$ and comparing with the equations for $\dot{x}$ and $\dot{y}$ in terms of $\partial_x f$ and $\partial_y f$ that determine the gradient flow, show that the flow is given by

$$\Phi_\tau\big(x(\tau_0,\sigma_0), y(\tau_0,\sigma_0)\big) = \frac{\sinh(\tau+\tau_0)}{\cosh(\tau+\tau_0)-\cos(\sigma_0)}\,\hat{\mathbf{\imath}} + \frac{\sin(\sigma_0)}{\cosh(\tau+\tau_0)-\cos(\sigma_0)}\,\hat{\mathbf{\jmath}}\,.$$

(h) Compute the limits $\lim_{\tau\to\infty} f\Big(\Phi_\tau\big(x(\tau_0,\sigma_0), y(\tau_0,\sigma_0)\big)\Big)$ and $\lim_{\tau\to-\infty} f\Big(\Phi_\tau\big(x(\tau_0,\sigma_0), y(\tau_0,\sigma_0)\big)\Big)$ to confirm the analysis given in the example above. Pay attention to exceptional cases of τ_0 and σ_0.

(13) Prove the validity of the procedure outlined in section 1.4 to convert ∇ between coordinates. That is, prove that given coordinates $(y_1,\ldots,y_n)$ with associated frame $(\hat{\mathbf{u}}_1,\ldots,\hat{\mathbf{u}}_n)$, the operator

$$\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ji}\partial_{x_i}(y_k)\mathbf{u}_j\partial_{y_k}$$

agrees with

$$\nabla = \sum_{i=1}^n \mathbf{e}_i\partial_{x_i}$$

when acting on functions $f\in\mathcal{C}^1(\mathcal{D},\mathbb{R})$, where $a_{ji}(y_1,\ldots y_n)$ are functions such that

$$\mathbf{e}_i = \sum_{j=1}^n a_{ji}(y_1,\ldots y_n)\hat{\mathbf{u}}_j(y_1,\ldots y_n)\,.$$

(14) Express $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ as linear combinations of $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ with coefficients that are functions of θ. Rewrite the linear equations to also express the coefficients as functions of x and y.

(15) Use the procedure outlined in section 1.4 to express ∇ in cylindrical coordinates, proving the proposition giving del in polar coordinates along the way.

(16) Verify the expressions of $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$ in the spherical frame given in section 1.4, and rewrite them to express the coefficients as functions of the rectangular variables x, y, and z. Apply the differential criterion from step (i) to check that these expressions are constant with respect to the spherical variables ϱ, θ and φ.

(17) Verify the remaining details used to express del in spherical coordinates. In particular, you should use the chain rule to verify the expressions given for ∂_x, ∂_y, ∂_z in terms of ∂_ϱ, ∂_θ, and ∂_φ, and do the work of step (iii) to arrive at the final expression

$$\nabla = \hat{\mathbf{u}}_\varrho\frac{\partial}{\partial\varrho} + \frac{1}{\varrho\cos(\varphi)}\hat{\mathbf{u}}_\theta\frac{\partial}{\partial\theta} + \frac{1}{\varrho}\hat{\mathbf{u}}_\varphi\frac{\partial}{\partial\varphi}\,.$$

(18) Compute the gradients of the coordinate functions for spherical coordinates, i.e. compute $\nabla\varrho$, $\nabla\theta$ and $\nabla\varphi$. Try the calculation both using spherical del, and using rectangular del together with the relations between ϱ, θ, and φ and rectangular coordinates.

(19) Fix a number $a \in \mathbb{R}_+$. Following the ideas of problem (12) above define a coordinate system with focal points at $(\pm a, 0)$ using an angle σ and a logarithm of a ratio τ to determine the location of any point in the plane. Then describe ∇ in this coordinate system. What are $\nabla\tau$ and $\nabla\sigma$?

(20) Find the gradients of the following functions, and express them in rectangular, cylindrical, and spherical coordinates.

(a) $f(x, y, z) = xyz$,

(b) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$,

(c) $f(x, y, z) = \dfrac{xy - xz + yz}{\sqrt{x^2 + y^2 + z^2}}$,

(d) $f(x, y, z) = z^2x - 2y + z - (x^2 + y^2)z$.

2. Vector Fields in Low Dimensions

§ 2.1. General Vector Fields in Domains of $\mathbb{R}^2$ and $\mathbb{R}^3$

Our introduction to vector fields was through the natural case of considering the gradient operator as attaching vectors to each point of the domain of some scalar field. Generalizing, we will now consider vector valued functions on domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. These general vector fields are just functions that assign vectors to every point of their domains. We work in low dimensions, where it is easy to visualize the fields and where important and concrete examples abound.

Definition. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be a domain. Then a 2-dimensional vector field on $\mathcal{D}$ is a vector-valued map

$$\mathbf{F} : \mathcal{D} \to \mathbb{R}^2 \,.$$

Given a coordinate system on the range of $\mathbf{F}$ in $\mathbb{R}^2$, the values $\mathbf{F}$ can resolved into components; we will generally also resolve the rule $\mathbf{F}$ into *component functions*, which are themselves scalar fields defined on $\mathcal{D}$ with values in $\mathbb{R}$. For example, using rectangular coordinates, we assume there are two bivariate functions $F_1(x, y)$ and $F_2(x, y)$ defined on $\mathcal{D}$, such that the vector field is described by the rule

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} \mapsto \mathbf{F}(\mathbf{r}) = F_1(x, y)\hat{\mathbf{\imath}} + F_2(x, y)\hat{\mathbf{\jmath}} \,.$$

We geometrically interpret this map as "attaching" a vector $\mathbf{F}(x, y)$ to the point $(x, y) \in \mathcal{D} \subseteq \mathbb{R}^2$. We will give a similar description for general $3D$ vector fields shortly. First, let us look at some examples of 2-dimensional vector fields.

Example 2.1. Let $\mathbf{F}(x, y) = y\hat{\mathbf{\imath}} - x\hat{\mathbf{\jmath}}$. We claim this is a "spin field", with vectors tangent to concentric origin-centered circles, rotating clockwise, and with magnitudes that grow with distance to the origin.

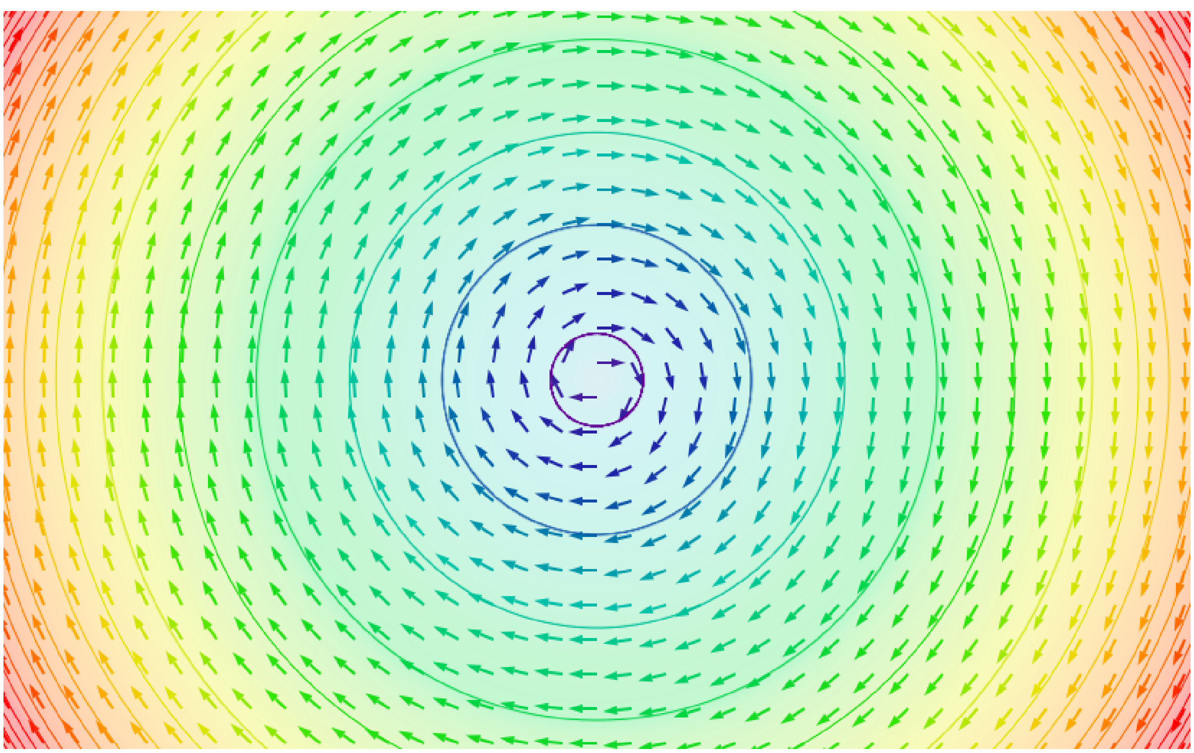


FIGURE 11. The clockwise spin field $\mathbf{F}(x, y) = y\hat{\mathbf{\imath}} - x\hat{\mathbf{\jmath}}$. The vectors are not drawn to scale, so as to avoid collisions; warmer colors indicate increased magnitude. field-lines shown are denser where the field is stronger.

This is easy to show: let $\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}$, which is the radial position vector. Then $\mathbf{r} \cdot \mathbf{F}(x,y) = xy - yx = 0$, which shows that $\mathbf{F}(x,y)$ is always perpendicular to the position vector of the point (x,y). Meanwhile, the magnitude of $\mathbf{F}(x,y)$ is $\|\mathbf{F}(x,y)\| = \sqrt{y^2+x^2} = \|\mathbf{r}\|$.

In fact, it is quite natural to express this field using the polar frame as

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(r,\theta) = -r\hat{\mathbf{u}}_\theta(\theta)\,,$$

which confirms that this is a clockwise spin field. Note that though $\hat{\mathbf{u}}_\theta$ is not defined at the origin, the field $\mathbf{F}$ is defined there using the rectangular coordinate expression, and $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ there, so we should perhaps think of our polar formula as needing to be given by a limit

$$\mathbf{F}(r_0,\theta_0) = \lim_{(r,\theta)\to(r_0,\theta_0)} -r\hat{\mathbf{u}}_\theta(\theta)$$

in order to use this to define $\mathbf{F}$ throughout the whole of $\mathbb{R}^2$.

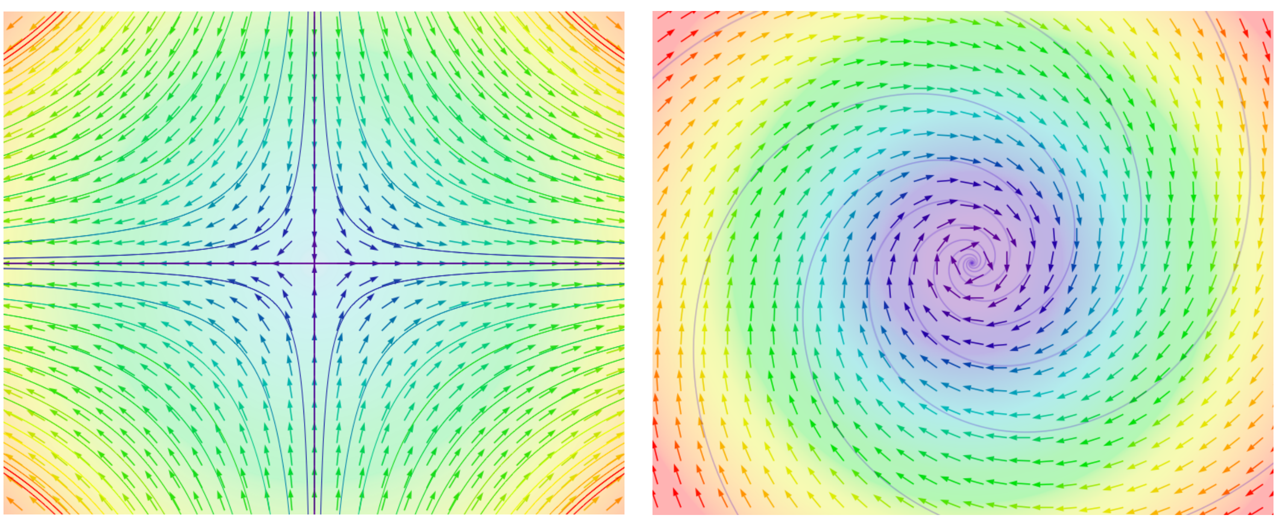
Example 2.2. A large class of simple to study but useful vector fields arise from the theory of linear systems. Consider a vector field of the form

$$\mathbf{F}(\mathbf{r}) = \mathrm{M}\mathbf{r}\,,$$

where $\mathrm{M} \in \mathbb{R}^{2\times 2}$ is a 2×2 real valued matrix. Recall, that a 2×2 matrix acts on a vector $\mathbf{r} = x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}}$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = (ax+by)\hat{\mathbf{\imath}} + (cx+dy)\hat{\mathbf{\jmath}}\,.$$

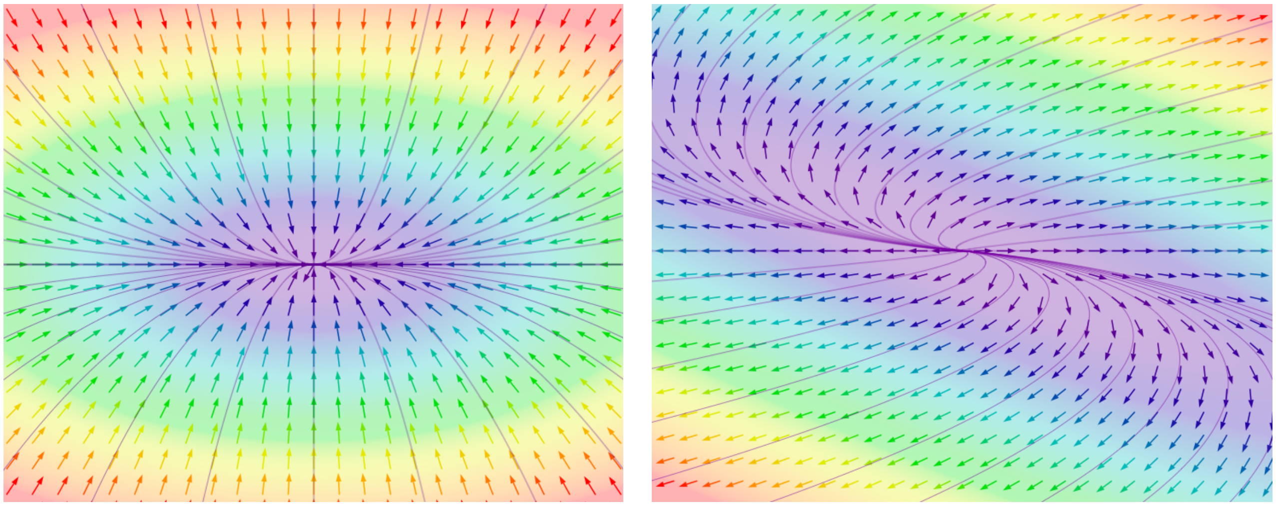
The spin vector field of the preceding example is thus of this form (can you write down the matrix for it?). Linear vector fields are classified by some simple properties of the matrices generating them (in particular, by their eigenvalues and eigenvectors). Figures 12 and 13 below show some of the possibilities; see (2) in the problems below to explore the classification of linear 2-dimensional vector fields in greater detail.



(A) (B)

FIGURE 12. (a) – A saddle vector field, corresponding to the linear transformation $x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}} \mapsto x\hat{\mathbf{\imath}}-y\hat{\mathbf{\jmath}}$. (b) – A spiral sink, arising from the linear transformation $x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}} \mapsto (3y-x)\hat{\mathbf{\imath}} + (3x+y)\hat{\mathbf{\jmath}}$.

In classifying linear vector fields, one is often interested in the *topology* of the field near the origin (and more generally, for nonlinear vector fields, the *local topology* of the field as well as its *global topology*[8]). By topology we are referring to characteristics of the vector field that are persistent



(A) (B)

FIGURE 13. (a) – A stable node vector field, determined by the linear transformation $x\hat{\imath} + y\hat{\jmath} \mapsto -x\hat{\imath} - 2y\hat{\jmath}$. (b) – An unstable degenerate node, given by the linear transformation $x\hat{\imath} + y\hat{\jmath} \mapsto (x + 2y)\hat{\imath} + y\hat{\jmath}$.

after continuous deformations of the domain. We briefly discuss some example topologies of linear fields.

In §2.2 below we will formally define flow for general vector fields, directly extending the idea of gradient flow to the larger class of vector fields. Intuitively, imagine that a vector field represents the velocity field of a fluid. To understand the topologies, we will discuss behavior in terms of the motion of a particle trapped in the flow, along *field-lines*, or *integral curves* which are tangent to the vector field.

We will use the terms *stable* and *unstable* in this context refer to the dynamics of a particle trapped in the flow near an *equilibrium point* of a vector field, which is a point $\mathbf{r}^*$ such that $\mathbf{F}(\mathbf{r}^*) = \mathbf{0}$. If the particle is at the origin subject to the flow of a linear field, it will not move, since linear maps send $\mathbf{0}$ to $\mathbf{0}$, and the origin thus corresponds to an equilibrium point. If under a small perturbation displacing the particle away from the origin, the particle begins to return to the origin under the flow, then the dynamics are *stable*, as in the spiral sink and the stable node depicted in (12.b) and (13.a) respectively. However, if the particle makes an escape away from the origin after such a perturbation, then the dynamics are said to be *unstable*. This is visually detectable from the directions of arrows along trajectories leading into/out of the origin. In figure (12.a) we see a *saddle node topology*,[9] characterized by a pair of stable paths leading towards the origin, and a pair of unstable paths leading out of $\mathbf{0}$, and all other paths running roughly along one of the stable paths before turning and running along an unstable path.

There are several other "topologies" for linear fields:

- *stars*, which occur e.g., for maps of the form $\mathbf{r} \mapsto c\mathbf{r}$ for a scalar c,
- centers, such as the spin field shown in figure 11, and

[8] Away from zeroes of a smooth two-dimensional vector field, one can always continuously deform a small rectangular neighborhood of a point to make all of the vectors parallel and all of the field-lines into lines parallel to the rectangular neighborhoods sides. Near zeroes however there may be more interesting features which distinguish the local structure–field lines converging along some directions, diverging along others, or spiraling either inwards or outwards–certain singular features cannot be smoothed away without fundamentally altering the field itself, beyond just deforming the domain in some small way. Local topology captures what flavor of neighborhood one has around a point. Global topology captures larger scale invariant information, such as whether there are field lines running from one zero to another, orbits and closed cycles, etc.

[9] The term node is sometimes dropped from the description "saddle node."

- fields arising from singular matrices, such as the *trivial map* $\mathbf{r} \mapsto \mathbf{0}$, projections $\mathbf{r} \mapsto \mathrm{proj}_{\hat{\mathbf{u}}}\, \mathbf{r}$, or *nilpotent* maps like $x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} \mapsto y\hat{\mathbf{\imath}}$.

See (2) below in the problems, where you will have a chance to study these maps along with saddles, nodes, centers, and spirals in context of the eigen-theory classification of linear fields in the plane.

Example 2.3. We now give an essential example of a non-linear vector field, a so-called *dipole field* $\mathbf{F}(x, y) = (x^2 - y^2)\,\hat{\mathbf{\imath}} + 2xy\,\hat{\mathbf{\jmath}}$, which is visualized in figure 14.

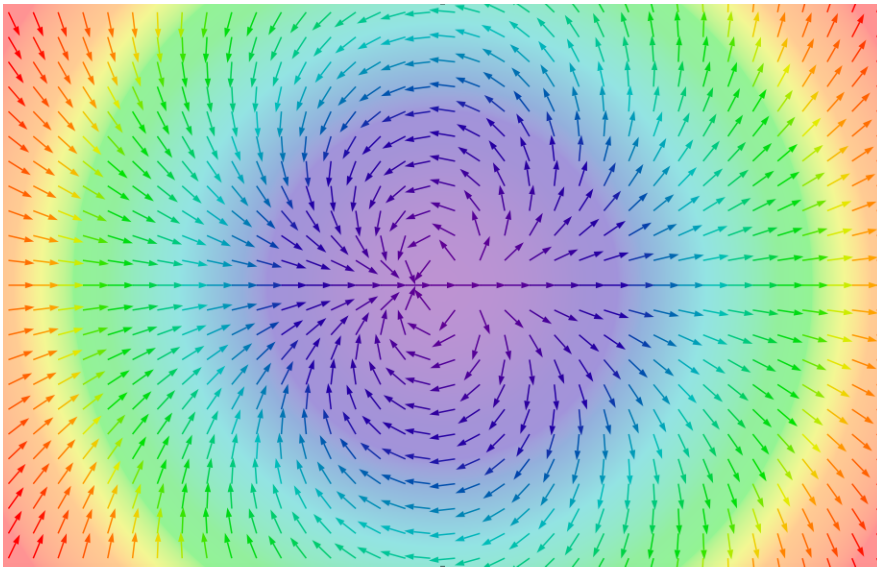


FIGURE 14. The topological dipole $\mathbf{F}(x, y) = (x^2 - y^2)\,\hat{\mathbf{\imath}} + 2xy\,\hat{\mathbf{\jmath}}$.

Observe however that the field strength increases away from the origin:

$$\|\mathbf{F}(\mathbf{r})\| = \sqrt{r^4 \cos^2(2\theta) + r^4 \sin^2(2\theta)} = r^2\,,$$

where $r = \|\mathbf{r}\|$. This isn't very physical; indeed, this field is really a *topological dipole*, while a true physical dipole arising e.g., in electromagnetic theory, has diminishing strength away from the dipole's center. See (10) below in the problems to study both topological and physical dipoles. Here, the term *topological* refers to the fact the the essential "dipole shape" and co-orientations[10] of the field lines are preserved under well behaved (continuous and continuously invertible) maps $\mathbb{R}^2 \to \mathbb{R}^2$.

We now consider 3-dimensional fields.

Definition. Let $\mathcal{D} \subseteq \mathbb{R}^3$ be a domain in 3-space. Then a 3-dimensional vector field on $\mathcal{D}$ is a vector-valued map

$$\mathbf{F} : \mathcal{D} \to \mathbb{R}^3\,.$$

[10]Here, the idea of co-orientations is simple to explain: the field lines are oriented by the vector field; and two lines in a neighborhood of a point are co-oriented around that point if they have "matching" directions near the point; more formally, two oriented curves are co-oriented near P if there is an open neighborhood U of P containing arcs of each curve, and a continuous bijective transformation from U to the open square $(-1, 1) \times (-1, 1)$ such that P is sent to the square's center, and the arcs of the curves, with orientations, are sent either to the parallel oriented open line segments $(-1, 1) \times \{\pm 1/2\}$, or to the pair of parabolae $\{(x, y) \in (-1, 1) \times (-1, 1) : y = \pm\frac{1}{2}x^2\}$, where the orientation is left to right in the square.

As in the two dimensional case, given a coordinate system on the range of $\mathbf{F}$ in $\mathbb{R}^3$, it can resolved into components. In rectangular coordinates, the components are three variable functions $F_1(x,y,z)$ $F_2(x,y,z)$ and $F_3(x,y,z)$ defined on $\mathcal{D}$, such that the vector field is described by the rule

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}} \mapsto \mathbf{F}(\mathbf{r}) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{\mathbf{k}}\,.$$

One can define linear fields in 3 dimensions:

$$\mathbf{F}(x,y,z) = (a_1x + a_2y + a_3z)\hat{\imath} + (b_1x + b_2y + b_3z)\hat{\jmath} + (c_1x + c_2y + c_3z)\hat{\mathbf{k}}\,,$$

which arise from the action of a 3×3 matrix on 3-vectors:

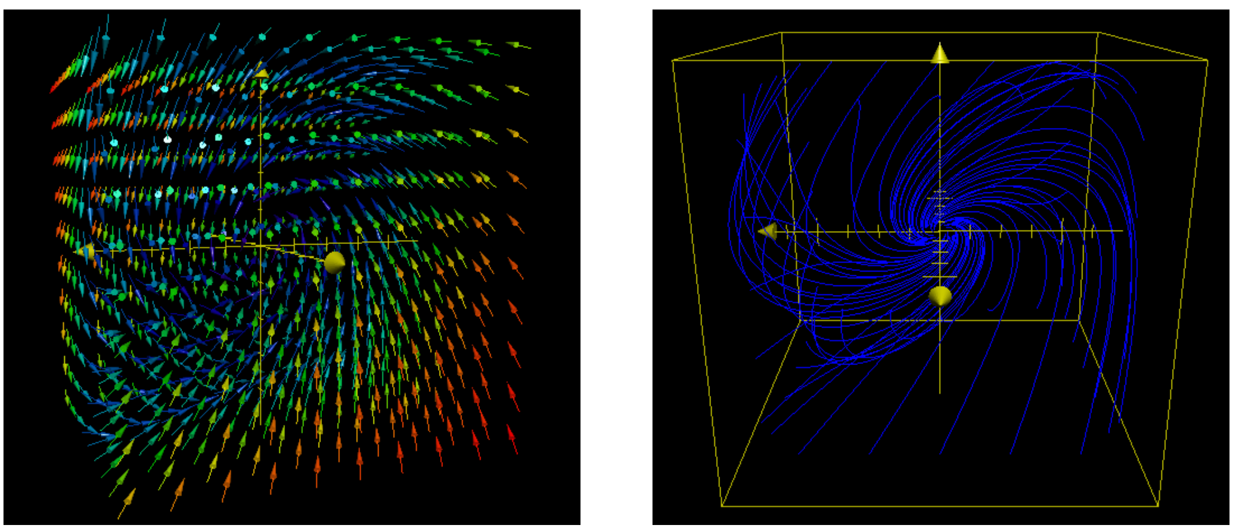
$$\mathbf{F}(\mathbf{r}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \mathbf{r} = (\mathbf{a}\cdot\mathbf{r})\hat{\imath} + (\mathbf{b}\cdot\mathbf{r})\hat{\jmath} + (\mathbf{c}\cdot\mathbf{r})\hat{\mathbf{k}}\,,$$

where $\mathbf{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{\mathbf{k}} + \mathbf{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\mathbf{k}}$ are the constant vectors corresponding to the rows of the matrix.

Example 2.4. Consider the linear vector field

$$\mathbf{F}(\mathbf{r}) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mathbf{r} = (z-y)\hat{\imath} + (x-3y+z)\hat{\jmath} - (x-y+z)\hat{\mathbf{k}}\,,$$

which is shown in figure 15, together with some field-lines, which are the trajectories particles would follow if $\mathbf{F}$ was their velocity field.



(A) (B)

FIGURE 15. (A) – The vector field $\mathbf{F}(x,y,z) = (z-y)\hat{\imath} + (x-3y+z)\hat{\jmath} - (x-y+z)\hat{\mathbf{k}}$. The vectors are not drawn to scale, so as to avoid collisions; warmer colors indicate increased magnitude. (B) – Some field-lines for this vector field.

In general, one needs the assistance of computers to efficiently visualize 3 dimensional vector fields, except when the component functions are exceptionally simple.

Example 2.5. Consider an object with mass M and fix the origin of our coordinate system at its center of mass, and let $m < M$ be the mass of a smaller object at position $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$. Newton's law of gravitation states that the magnitude of the force of gravitational attraction between these objects is proportional to the product of their masses, and inversely proportional to the square of the distance between them, and further, the force on each object is attractive, acting in the direction of displacement towards the other object. Thus, if $\mathbf{F}_g$ is the field describing the force exerted by the larger mass on the smaller mass, Newton's law of gravitation tells us that

$$\|\mathbf{F}_g(\mathbf{r})\| = \frac{GMm}{\|\mathbf{r}\|^2}\,,$$

where G is a constant, now known as the *universal gravitational constant*. Since this field is attractive, the force on the smaller object acts in negative radial direction, so the actual vector field describing the force is

$$\mathbf{F}_g(\mathbf{r}) = -\|\mathbf{F}_g(\mathbf{r})\|\frac{\mathbf{r}}{\|\mathbf{r}\|} = -GMm\frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\,.$$

Note that the field is undefined at the origin, and is strongest as one approaches the origin. In the notes on *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*, we combine this law with Newton's second law of motion and elementary calculus of curves to study the two body problem of celestial mechanics. In particular, we give a reproof of Kepler's Laws, from Hamilton's theorem on velocity circles. If you skipped over that, now is a great time to go learn about why planetary orbits are (approximately) ellipses!

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§ 2.2. Flows and Integral Curves

Recall, in our discussion of gradients in §1.3, we defined field-lines as the curves of the gradient flow which are everywhere tangent to the gradient vector field. This particular idea is not special to gradient fields; any vector field with sufficiently well behaved component functions will possess smooth field lines and an associated flow function. One recovers the field-line through a point $\mathbf{r}_0$ as the solution $\mathbf{r}(t)$ of the initial value problem

$$\dot{\mathbf{r}}(t) = \mathbf{F}\big(\mathbf{r}(t)\big)\,,\ \mathbf{r}(0) = \mathbf{r}_0\,.$$

This is a first order ordinary differential equation in n variables, where n is the dimension of the space over which our vector field is defined. This equation merely specifies that the curve parametrized by $\mathbf{r}$ has velocity vector given by the vector field $\mathbf{F}$ at the position $\mathbf{r}(t)$. The parameterized curve is called an *integral curve* of the vector field. One can also consider the field-line as an abstract curve, which is often easier; one merely has to find curves everywhere tangent to the vector field, and can ignore any worries about ensuring the particular parameterization yields the correct velocity vectors on the nose. For planar vector fields, this reduces to solving differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\hat{\mathbf{\jmath}}\cdot\mathbf{F}(\mathbf{r})}{\hat{\mathbf{\imath}}\cdot\mathbf{F}(\mathbf{r})} = \frac{F_2(x,y)}{F_1(x,y)}\,.$$

Example 2.6. Let $\mathbf{F}(x,y) = y\hat{\mathbf{\imath}} - x\hat{\mathbf{\jmath}}$. Recall this is the clockwise spin-field discussed at the beginning of section 2.1 (see figure 11). We can find an implicit description of the field-lines from the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}\,.$$

Indeed, this corresponds to the differential form

$$x\,\mathrm{d}x + y\,\mathrm{d}y = 0\,,$$

which is *exact*, meaning it arises as the total differential $\mathrm{d}f$ of a function f. Indeed, you may recognize this derivative as coming from implicitly differentiating the equation of a circle. We can write this as two separated differential one forms, equated:

$$y\,\mathrm{d}y = -x\,\mathrm{d}x\,.$$

Integrating both sides, and consolidating all constants of integration on the right:

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \implies x^2 + y^2 = 2C\,.$$

This is indeed the equation of a circle, of radius $\sqrt{2C}$. The method of solution we just used is called *separation of variables*, and generally works whenever we have a differential equation that can be written in differential form as $M(x)\,\mathrm{d}x + N(y)\,\mathrm{d}y = 0$. More generally, we can readily solve a differential equation of the form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ by partial integration whenever the differential is exact, see the discussion of conservative fields and exact differentials at the end of section 2.3.

If we wanted the explicit integral curves, we'd solve the system

$$\dot{\mathbf{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = y\hat{\imath} - x\hat{\jmath} \implies \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Using *eigentheory* of matrices and linear differential equations one can derive a solution which agrees with one's intuition. It's easy to check that a solution is of the form $\mathbf{r}(t) = r_0(\cos(\theta_0 - t)\hat{\imath} + \sin(\theta_0 - t)\hat{\jmath})$, where $r_0 = x_0^2 + y_0^2$ and $x_0 = r_0\cos\theta_0$, so θ_0 is the angle made by (x_0, y_0) with $\hat{\imath}$. See (2) in the problems below to learn about eigentheory and linear vector fields in greater detail.

As with gradient flow, we can define a flow function for any vector field:

Definition. The *flow* of n n-dimensional vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$ on a domain $\mathcal{D} \subseteq \mathbb{R}^n$ is the map given by

$$\Phi(t, \mathbf{r}) = \boldsymbol{\gamma}_{\mathbf{r}}(t)\,,$$

where $\boldsymbol{\gamma}_{\mathbf{r}}(t)$ is the *field-line* through $\mathbf{r}$ such that $\boldsymbol{\gamma}_{\mathbf{r}}(0) = \mathbf{r}$ and $\dot{\boldsymbol{\gamma}}_{\mathbf{r}}(t) = \mathbf{F}(\boldsymbol{\gamma}_{\mathbf{r}}(t))$ for all t for which $\boldsymbol{\gamma}_{\mathbf{r}}(t)$ is defined.

Example 2.7. One can use our description of the integral curves of $\mathbf{F}(x,y) = y\hat{\imath} - x\hat{\jmath}$ as trajectories to give an explicit description of the associated flow. Let $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} = r\cos(\theta)\,\hat{\imath} + r\sin(\theta)\,\hat{\jmath}$. Then the flow is

$$\begin{aligned}\Phi(t, \mathbf{r}) &= r(\cos(\theta - t)\hat{\imath} + \sin(\theta - t)\hat{\jmath}) \\ &= \big(r\cos(\theta)\cos(t) + r\sin(\theta)\sin(t)\big)\hat{\imath} + \big(r\sin(\theta)\cos(t) - r\cos(\theta)\sin(t)\big)\hat{\jmath}) \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.\end{aligned}$$

It is not hard to show that the matrix above is a rotation matrix performing an origin-centered rotation clockwise by an angle of t (hint: draw the vectors corresponding to the columns, and consider what angles they make with each coordinate axis). Thus the flow in this case is as one expects: clockwise rotation at unit angular speed.

Generally, there are analytical difficulties in explicitly describing the flow function for all but simple classes of vector fields. But as a conceptual tool, flow functions have useful applications in dynamics and geometry, and even aid in proving existence results in differential topology.

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§ 2.3. Conservative Vector Fields and Potentials

Gradient vector fields hold a prominent role among the vector fields one studies, in particular because of the geometry they admit due to their trajectories being orthogonal to a set of subspaces (curves for $2D$ fields, surfaces for $3D$ fields, and $(n-1)$-dimensional "hypersurfaces" for dimensions $n \geq 4$) given as the level sets of a differentiable scalar function. In fact, due to their connection with the physics of conservation of energy (as will be explained at the end of section 3.1, via the fundamental theorem of line integrals), they have earned the special name "conservative vector fields":

Definition. Suppose $\mathbf{F}$ is a differentiable vector field defined throughout a domain $\mathcal{D} \subseteq \mathbb{R}^n$. Then $\mathbf{F}$ is called *conservative in the domain* $\mathcal{D}$ if there exists a scalar field $f : \mathcal{D} \to \mathbb{R}$ such that for every point $\mathbf{r} \in \mathcal{D}$

$$\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r}) .$$

A scalar field f whose gradient throughout $\mathcal{D}$ equals $\mathbf{F}$ is called a *scalar potential for* $\mathbf{F}$. Note that potentials are not unique: if $f(\mathbf{r})$ is a potential for $\mathbf{F}$, then so is $f(\mathbf{r}) + C$ for any constant $C \in \mathbb{R}$.

We'll later connect the scalar potential to the notion of potential energy, and justify the name "conservative".

The question remains, how do we identify a conservative vector field, or rule out the existence of a potential? If $\mathbf{F}$ is *continuously differentiable*, then there is an easy criterion that must be satisfied for it to be conservative, though it is not a sufficient criterion:

Proposition 2.1. *Let* $\mathbf{F}(\mathbf{r})$ *be a continuously differentiable vector field defined on a domain* $\mathcal{D} \subseteq \mathbb{R}^n$. *Let*

$$\mathbf{F}(\mathbf{r}) = \sum_{i=1}^{n} F_i(\mathbf{r})\hat{\mathbf{e}}_i$$

be the decomposition of $\mathbf{F}(\mathbf{r})$ *into component functions in rectangular coordinates. Then if* $\mathbf{F}$ *is conservative, the component partial derivatives satisfy*

$$\frac{\partial F_i}{\partial x_j}(\mathbf{r}) = \frac{\partial F_j}{\partial x_i}(\mathbf{r}) ,$$

for all $i, j = 1, \ldots, n$ *and for all* $\mathbf{r} \in \mathcal{D}$.

Proof. If $\mathbf{F}$ is conservative, then for some potential $f : \mathcal{D} \to \mathbb{R}$, each component function $F_i(\mathbf{r})$ satisfies

$$F_i(\mathbf{r}) = \frac{\partial f}{\partial x_i}(\mathbf{r})$$

throughout $\mathcal{D}$. Since $\mathbf{F}$ is continuously differentiable throughout $\mathcal{D}$, by Clairaut's theorem for any j and any $\mathbf{r} \in \mathcal{D}$,

$$\frac{\partial F_i}{\partial x_j}(\mathbf{r}) = \frac{\partial^2 f}{\partial x_j \, \partial x_i}(\mathbf{r}) = \frac{\partial^2 f}{\partial x_i \, \partial x_j}(\mathbf{r}) = \frac{\partial F_j}{\partial x_i}(\mathbf{r}) .$$

□

Remark. For $i = j$ above the conditions are trivial, so one only has to concern themselves with the cases when the indices don't match. For 2-dimensional vector fields, this condition gives rise to a single equation that one can check: if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\hat{\mathbf{\imath}} + Q(\mathbf{r})\hat{\mathbf{\jmath}}$, one checks if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} .$$

Thus, if these partials do not match, one can be sure that the field is *not* conservative. If the equality holds, it *does not* guarantee that $\mathbf{F}$ is conservative; this equality is a necessary but not sufficient condition. A partial converse to this proposition is discussed in section 3.1.

For a 3-dimensional vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\hat{\mathbf{\imath}} + Q(\mathbf{r})\hat{\mathbf{\jmath}} + R(\mathbf{r})\hat{\mathbf{k}}$, we have 3 equalities to check:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} , \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} , \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} .$$

In the next section, we will look at a a differential operator on 3-dimensional vector fields $\mathbf{F}$ which vanishes precisely when these equations are satisfied. But again, it is important to remember that these conditions are necessary but not sufficient to determine if a 3-dimensional vector field is conservative.

For higher dimensions, we have many more equations to check, as the number of possible combinations of partials increases. For example, in four dimensions there are four equations relating eight different partial derivatives, while in five dimensions there are 10 equations relating 20 different partial derivatives.

To find a potential for a conservative vector field, we use a process called *indefinite partial integration*. The idea is that if the component functions of $\mathbf{F}$ in rectangular coordinates are just the partial derivatives with respect to the corresponding coordinate variables, then by integrating any component with respect to a corresponding coordinate variable we should recover the potential, up to adding undetermined functions that are constant with respect to that particular coordinate variable. Doing this for each component, and then "matching" to determine the unknown functions, we can fully recover the potential, up to a scalar constant.

Definition. Given a scalar function $g(\mathbf{r})$ integrable over its domain $\mathcal{D}$, the *partial integral* of g with respect to x_i is

$$\int g(\mathbf{r})\,\mathrm{d}x_i = G(x_1,\dots,x_n) + C(x_1,\dots,\hat{x}_i,\dots,x_n)\,,$$

where G is any x_i antiderivative of g, i.e., $\partial_{x_i}G(\mathbf{r}) = g(\mathbf{r})$ throughout $\mathcal{D}$, and $C(x_1,\dots,\hat{x}_i,\dots,x_n)$ is a function that depends only on the variables x_j, $j \neq i$ (the notation $(x_1,\dots,\hat{x}_i,\dots,x_n)$ means "omit x_i"), i.e., $\partial_{x_i}C(\mathbf{r}) = 0$ throughout $\mathcal{D}$. The function C is undetermined, playing a role analogous to the constants of integration appearing in indefinite integrals of single variable functions.

One computes a partial integral with respect to x_i by integrating, assuming x_j, $j \neq i$ are all constant. Thus, the usual rules of indefinite integration apply, and techniques such as substitution, integration by parts, and partial fractions can be used as needed.

Example 2.8. Let us find the partial integrals for the bivariate function $f(x,y) = y^2e^{xy}$ with respect to both x and y.

$$\int f(x,y)\,\mathrm{d}x = \int y^2e^{xy}\,\mathrm{d}x = ye^{xy} + C(y)\,,$$

$$\int f(x,y)\,\mathrm{d}y = \int y^2e^{xy}\,\mathrm{d}x = \frac{y^2}{x}e^{xy} - \frac{1}{x}\int 2ye^{xy}\,\mathrm{d}y = \left(\frac{y^2}{x} - \frac{2}{x^2} + \frac{2}{x^3}\right)e^{xy} + D(x)\,.$$

The first integral may be accomplished by the simple substitution $u(x) = xy$, $\mathrm{d}u = y\,\mathrm{d}x$. The second integral is done via integration by parts as well as repeated use of the substitution $w(y) = xy$, $\mathrm{d}w = x\,\mathrm{d}y$. Note that the undetermined function in the first integral depends only upon y, while in the second depends only upon x.

Now suppose we have a vector field $\mathbf{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, which is conservative and for which we are trying to determine a potential function $f(x,y)$. Then

$$f(x,y) = \int P(x,y)\,\mathrm{d}x = \int Q(x,y)\,\mathrm{d}y\,,$$

and we can use the equality of these two partial integrals to determine the unknown functions $C(x)$ and $D(y)$ that arise in the respective partial integrals.

Example 2.9. Let $\mathbf{F}(x,y) = (\cos y - y\cos x)\,\hat{\imath} - (\sin x + x\sin y)\,\hat{\jmath}$. Note that

$$\frac{\partial}{\partial y}(\cos y - y\cos x) = -\sin y - \cos x = \frac{\partial}{\partial x}(-\sin x - x\sin y)\,,$$

and so it is possible that $\mathbf{F}$ is conservative. Computing partial integrals:

$$\int \cos y - y\cos x\,\mathrm{d}x = x\cos y - y\sin x + C(y)\,,$$

$$\int -\sin x - x\sin y\,\mathrm{d}y = -y\sin x + x\cos y + D(x)\,.$$

Comparing these two, we see that $C(y) = D(x)$ which implies that they must be a common constant, say k. Thus

$$f(x,y) = x\cos y - y\sin x + k\,,$$

for any constant k is a potential for $\mathbf{F}(x, y)$. It is easy to check that $\nabla f(x, y) = \mathbf{F}(x, y)$ by differentiation.

Example 2.10. Let $\mathbf{G}(x, y) = 3(x^2 + y^2)\,\hat{\imath} + 2y(3x - e^{-y^2})\,\hat{\jmath}$. We'll show that $\mathbf{G}$ is conservative by finding a potential $g(x, y)$ for $\mathbf{G}$.

$$3\int x^2 + y^2 \,\mathrm{d}x = x^3 + 3xy^2 + C(y)\,,$$

$$\int 6xy - 2ye^{-y^2}\,\mathrm{d}y = 3xy^2 + e^{-y^2} + D(x)\,.$$

Comparing, we see that setting $C(y) = e^{-y^2}$ and $D(x) = x^3$ we can make these equations match. Thus $g(x, y) = x^3 + 3xy^2 + e^{-y^2}$ is a potential for $\mathbf{G}(x, y)$, as is any function that differs from this $g(x, y)$ by adding a constant.

We now connect potential theory to the theory of differentials. Recall the definition of a total differential:

Definition. The *total differential* of a scalar function $f(x_1, \dots x_n)$ is the *differential one form*

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i\,.$$

For a two variable function $f(x, y)$, the total differential is then a differential one-form that encodes the same information as the gradient ∇f in xy-coordinates, and is in a particular sense dual to the gradient.

A differential one-form $\alpha = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ is said to be exact if it is the total differential of some function, i.e., $\alpha = \mathrm{d}f$ for some f. In this case, if the functions P and Q are themselves continuously differentiable, then by Clairaut's theorem we recover the equality $P_y = Q_x$. Conversely, if $P_y = Q_x$ on some open disk, then α is *locally exact*, meaning it is exact in the open disk.

Thus, there is a correspondence between exact differentials and conservative fields on sufficiently simple domains. This relationship is deepened when we study line integrals in conservative vector fields, and the idea of *independence of path*. Fields have the path independence property when the the energy exerted by the field in transporting a particle along a path depends not on the particular shape of the path, but only the endpoints and direction in which the particle traverses the path.

Observe also that one encounters differentials when solving differential equations, e.g., to find the field lines of a two dimensional vector field. A field may be nonconservative, but its field lines may still be associated to an exact differential that can be extracted from the components of the vector field. Recall that the field lines of a two dimensional vector field satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\hat{\jmath}\cdot\mathbf{F}(\mathbf{r})}{\hat{\imath}\cdot\mathbf{F}(\mathbf{r})} = \frac{F_2(x, y)}{F_1(x, y)}\,,$$

whence, the field lines are solution curves corresponding to solutions of the differential equation

$$F_2(x, y)\,\mathrm{d}x - F_1(x, y)\,\mathrm{d}y = 0\,.$$

Now, if $\mathbf{F}$ is conservative, then the level curves of the potential are perpendicular to the integral curves satisfying this differential equation, and the differential forms corresponding to the total differential of the potential and the total differential of the function defining the field lines implicitly are seen to correspond to orthogonal vector fields. If both differentials are exact, then

$$(*)\qquad \begin{cases} \dfrac{\partial F_1}{\partial x} &= -\dfrac{\partial F_2}{\partial y} \\[2ex] \dfrac{\partial F_1}{\partial y} &= \dfrac{\partial F_2}{\partial x}\,. \end{cases}$$

This is reminiscent of the famous *Cauchy-Riemann equations* for a complex analytic function: if $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at $z_0 = x_0 + iy_0$ (meaning $f'(z_0) = \lim_{z\to z_0} \big(f(z) - f(z_0)\big)/(z - z_0)$ exists) then

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0,y_0) &= \frac{\partial v}{\partial y}(x_0,y_0)\,, \\ \frac{\partial u}{\partial y}(x_0,y_0) &= -\frac{\partial v}{\partial x}(x_0,y_0)\,, \end{aligned}$$

except *the signs of* $(*)$ *are backwards!* One can show that a function f is complex analytic at z_0 if and only if the vector field corresponding to its *complex conjugate*, $\overline{\mathbf{F}} = u(x,y)\hat{\mathbf{\imath}} - v(x,y)\hat{\mathbf{\jmath}}$ is conservative on a neighborhood of z_0, and in this case the flow differential is also exact.

For a thorough treatment of the connections between complex functions and vector fields, see the wonderfully illustrated *Visual Complex Analysis* by Tristan Needham, or check out the 1974 book of Pólya and Latta *Complex Variables*, where this connection was initially explored.

§ 2.4. Vector Fields from Frames*

(Note: this is an optional section which covers coordinate systems that are of great utility, but are not part of the curriculum for our current course–however these frames are used occasionally in optional examples and in the optional advanced sections.)

From any coordinate system, we can obtain a family of vector fields, representing the coordinate directions at points of the domain where the coordinates are defined. Such a collection is called a *frame* for the coordinate system. For example, in rectangular coordinates $(x, y, z)_{\mathscr{R}}$ on $\mathbb{R}^3$, we have the constant frame $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$ which give the directions of positive x, y and z respectively. We've also encountered the polar frame $(\hat{\mathbf{u}}_r, \hat{\mathbf{u}}_\theta)$ for polar coordinates $(r, \theta)_{\mathscr{P}}$ on $\mathbb{R}^2$, and the spherical frame $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$ for spherical coordinates $(\varrho, \theta, \varphi)_{\mathscr{S}}$ on $\mathbb{R}^3$. We'll now describe the general idea of coordinate frames.

Let $(x_1, \ldots, x_n)_{\mathscr{R}}$ denote rectangular coordinates in $\mathbb{R}^n$, and let $\hat{\mathbf{e}}_i$ be the usual unit vector pointing along the positive x_i axis:

$$\hat{\mathbf{e}}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \; i\text{th coordinate} \;.$$

The ordered tuple $(\hat{\mathbf{e}}_1, \ldots \hat{\mathbf{e}}_n)$ is called the *standard frame for rectangular coordinates on* $\mathbb{R}^n$, or the *standard basis for* $\mathbb{R}^n$. The latter term is used when it is thought of from the linear-algebra perspective as a set which can be used to make any vector in $\mathbb{R}^n$ as a linear combination of the vectors of $(\hat{\mathbf{e}}_1, \ldots \hat{\mathbf{e}}_n)$. The perspective of it as a frame is subtly different: we want to regard the tuple as a collection of n vector fields, each assigning a vector to each point of $\mathbb{R}^n$. In this case, the vector fields are all constant.

Now let $(y_1, \ldots, y_n)_{\mathscr{Y}}$ be a new collection of coordinates, possibly only defined locally on some subset $\mathcal{D} \subseteq \mathbb{R}^n$. On $\mathcal{D}$ each y_j is a function of $x_1, \ldots x_n$, and presumably some y_j's are non-constant as functions of the x_i's. If we hold all the y_j's constant except y_1, we obtain a curve parameterized by y_1. Under the assumption of regularity and smoothness of our new coordinates, we may define $\hat{\mathbf{u}}_1(y_1, \ldots y_n)$ to be the unit tangent vector to such a curve through the point with coordinates $(y_1, \ldots, y_n)_{\mathscr{Y}}$. Similarly we can define $\hat{\mathbf{u}}_j$ for any $j = 2, \ldots n$. Each $\hat{\mathbf{u}}_j$ thus determines a vector field, and the *induced frame for the coordinates* $(y_1, \ldots, y_n)_{\mathscr{Y}}$ is the ordered tuple $(\hat{\mathbf{u}}_1, \ldots \hat{\mathbf{u}}_n)$ of these vector fields. It may be the case that the fields are not universally defined even within $\mathcal{D}$, but are defined over some region(s) within $\mathcal{D}$ where the coordinates are smooth and unambiguous.

Example 2.11. In the notes *Curvature, Natural Frames, and Acceleration for Plane and Space Curves* a frame for working with polar coordinates on $\mathbb{R}^2$ was introduced:

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r} = \cos(\theta)\,\hat{\mathbf{\imath}} + \sin(\theta)\,\hat{\mathbf{\jmath}} \qquad \hat{\mathbf{u}}_\theta = \partial_\theta \hat{\mathbf{u}}_r = -\sin(\theta)\,\hat{\mathbf{\imath}} + \cos(\theta)\,\hat{\mathbf{\jmath}} = \frac{-y\hat{\mathbf{\imath}} + x\hat{\mathbf{\jmath}}}{r}\,.$$

It was then discussed again in section 1.4 above in the context of rewriting the del operator ∇ in polar coordinates. Observe the following facts about the polar frame:

- the frame is undefined at the origin, since θ is undefined at the origin and the frame vectors are dependent on θ,
- at a point P with polar coordinates $(r_0, \theta_0)_{\mathscr{P}}$, the vector $\hat{\mathbf{u}}_r(\theta_0)$ is the unit tangent vector along the ray $\theta = \theta_0$ from the origin through P, which is the curve of motion for varying r while keeping $\theta = \theta_0$ constant,
- similarly, at the point P with polar coordinates $(r_0, \theta_0)_{\mathscr{P}}$, the vector $\hat{\mathbf{u}}_\theta(\theta_0)$ is the unit tangent vector to origin centered circle $r = r_0$ through P, which is the curve of motion for varying θ while keeping $r = r_0$ constant,

- $\nabla r = \hat{\mathbf{u}}_r$, and the directional derivative of a function $f(r,\theta)$ along $\hat{\mathbf{u}}_r$ is
$$D_{\hat{\mathbf{u}}_r} f(r,\theta) = \hat{\mathbf{u}}_r \cdot \nabla f(r,\theta) = \hat{\mathbf{u}}_r \cdot \Big(\hat{\mathbf{u}}_r \partial_r f(r,\theta) + \hat{\mathbf{u}}_\theta \frac{1}{r}\partial_\theta f(r,\theta)\Big) = \frac{\partial f}{\partial r}(r,\theta)\,,$$
- $\nabla \theta = \frac{1}{r}\hat{\mathbf{u}}_\theta$ and the directional derivative of $f(r,\theta)$ along $\hat{\mathbf{u}}_\theta$ is
$$D_{\hat{\mathbf{u}}_\theta} f(r,\theta) = \frac{1}{r}\frac{\partial f}{\partial \theta}(r,\theta)\,.$$

Thus, we can see that the polar frame is indeed the pair of unit tangent vectors to the coordinate curves of the polar coordinate system. Moreover, the directional derivatives along the frame directions are proportional to the corresponding partial derivatives with respect to the polar variables. This will be true for any frame arising from an orthogonal coordinate system (meaning all of the frame vectors are mutually orthogonal.)

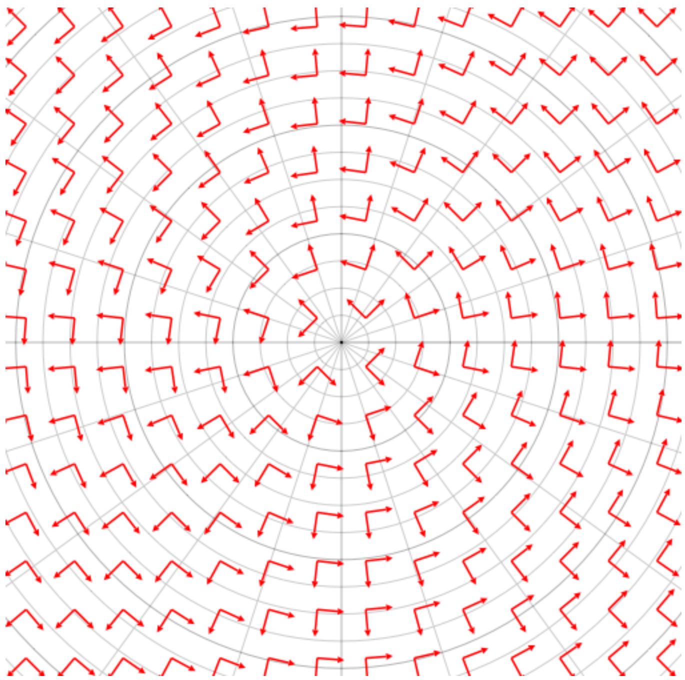

FIGURE 16. The polar frame, visualized as a a pair of orthogonal vector fields. Note that the frame is undefined at the origin, as neither $\hat{\mathbf{u}}_r$ nor $\hat{\mathbf{u}}_\theta$ can be defined there. The *field-lines* for the vector field $\hat{\mathbf{u}}_r$ are rays from the origin, while the field-lines for the vector field $\hat{\mathbf{u}}_\theta$ are concentric origin centered circles. Together they form a web of orthogonal curves which define the constant sets for the polar coordinate system; the rays and circles play the same roles as the gridlines of the rectangular Cartesian coordinate system on $\mathbb{R}^2$.

Example 2.12. Let us look at the spherical frame $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$ once again, this time regarding each frame element as a vector field. The first vector field, expressed in the rectangular frame, is
$$\hat{\mathbf{u}}_\varrho = \cos(\theta)\cos(\varphi)\,\hat{\mathbf{\imath}} + \sin(\theta)\cos(\varphi)\,\hat{\mathbf{\jmath}} + \sin(\varphi)\,\hat{\mathbf{k}} = \frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{\varrho}\,,$$
where $\varrho = \sqrt{x^2+y^2+z^2}$. Thus $\hat{\mathbf{u}}_\varrho$ is a radial field, defined over $\mathbb{R}^3 - \{\mathbf{0}\}$. It is pictured in figure (17).

The vector field $\hat{\mathbf{u}}_\theta$, which also appears in polar/cylindrical coordinates, is well defined whenever x and y are not both zero, i.e. it is well defined on $\mathbb{R}^3 - \{x = 0 = y\}$. This field is recognizable as a unit "spin field" with vectors tangent to the level circles of cylinders centered along the z-axis. It is pictured in figure (18).

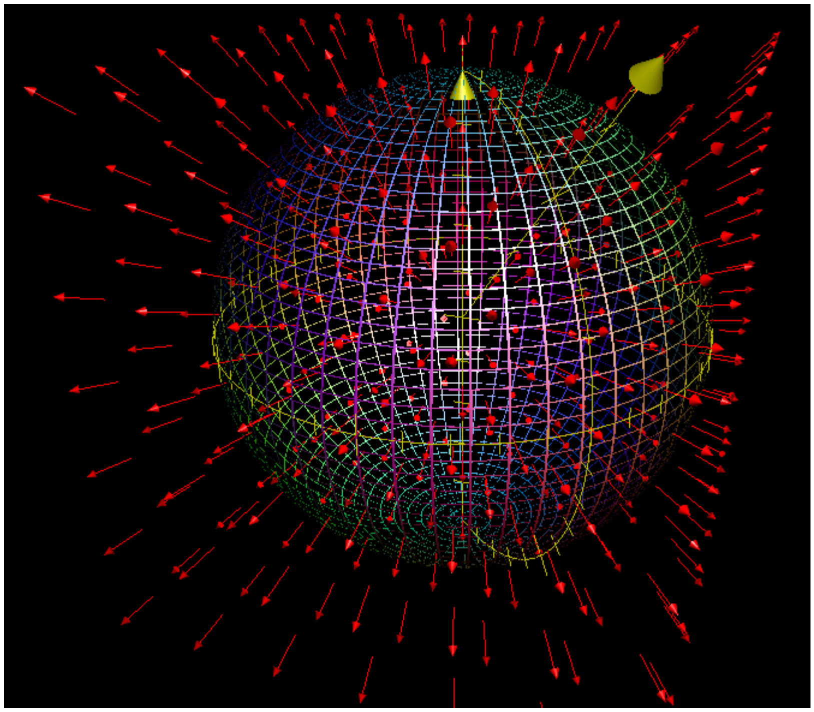

FIGURE 17. The spherical frame element $\hat{\mathbf{u}}_\varrho$ as a vector field on $\mathbb{R}^3 - \{\mathbf{0}\}$.

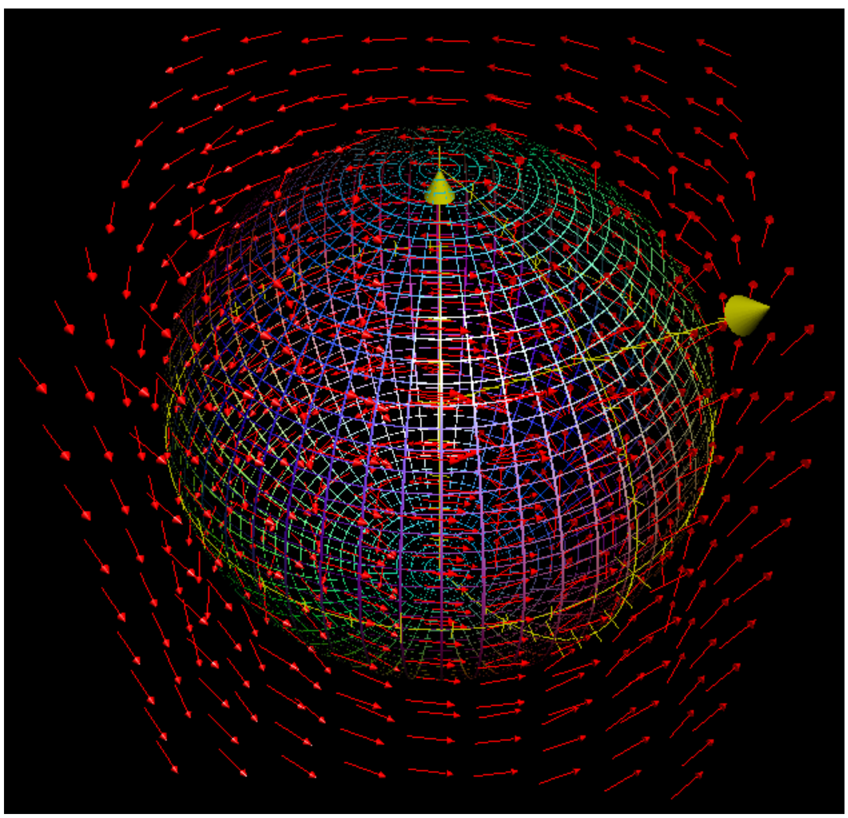

FIGURE 18. The frame element $\hat{\mathbf{u}}_\theta$ of polar/cylindrical and spherical coordinates, as a vector field on $\mathbb{R}^3 - \{x = 0 = y\}$.

Similarly, the vector field

$$\hat{\mathbf{u}}_\varphi = -\cos(\theta)\sin(\varphi)\,\hat{\mathbf{\imath}} - \sin(\theta)\sin(\varphi)\,\hat{\mathbf{\jmath}} + \cos(\varphi)\,\hat{\mathbf{k}}$$

is defined only on $\mathbb{R}^3$ minus the z-axis. These vectors are tangent to meridians of origin centered spheres, pointing "north," heading towards poles located on the z-axis. It is pictured in figure (19).

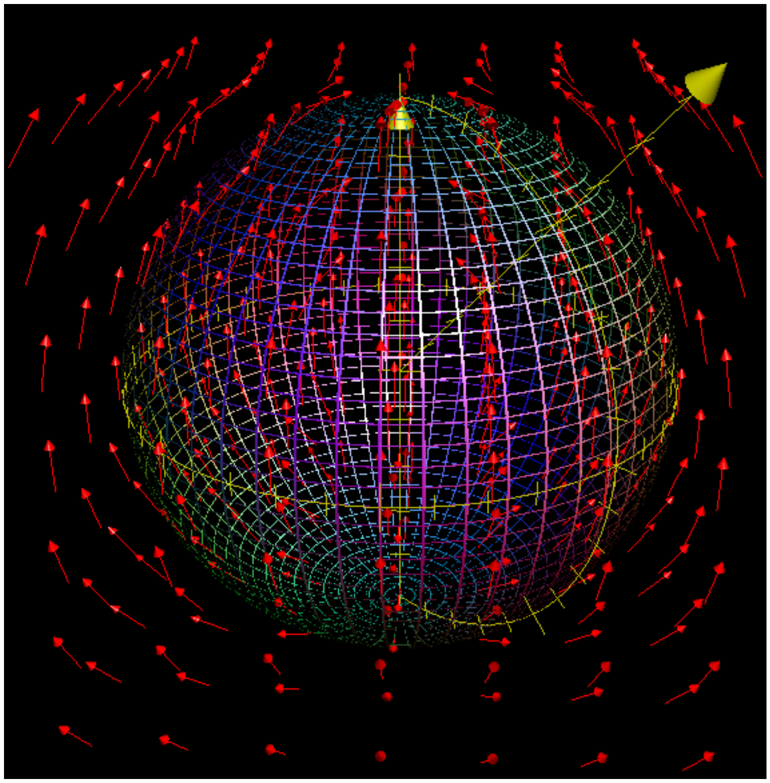

FIGURE 19. The spherical frame element $\hat{\mathbf{u}}_\varphi$ as a vector field on $\mathbb{R}^3 - \{x = 0 = y\}$.

The vector fields of a coordinate frame are often called *coordinate vector fields*. There is a close connection between coordinate systems, coordinate vector fields, and partial derivatives which leads to the modern perspective on tangent vectors. In the next section, we will consider differential operators on vector fields. In order to understand how to compute such operators in general coordinates, we have to consider how non-constant/non-global frames change along their own integral curves. Then in section 2.6 we will explore coordinates induced on a surface from a parametrization, and describe the resulting coordinate vector fields when viewed extrinsically as vectors in $\mathbb{R}^3$ attached to the surface, and after that we will explore the notions of tangent and normal vectors to curves and surfaces. Central to these discussions is the idea of building a frame adapted to a coordinate or parametric description of an object such as a curve or surface.

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§ 2.5. Divergence, Curl, Jacobians, and the Laplacian

For a scalar field depending on multiple variables, we defined various partial derivatives, as well as a notion of directional derivative, to quantify how fast the function's values changed locally when the input is perturbed in a particular direction within the domain. We would now like to define notions of derivatives for vector fields as well. For a vector field, however, there are more types of derivative operators, corresponding to different ways to capture the types of change occurring within a multi-dimensional image. For example, one can try measure how much a vector field is a *source or sink*, how much *vorticity* is present at a point, how a vector field is best approximated by a linear map/linear vector field, or how a vector field changes along another vector field. We will explore a few differential operations on vector fields which correspond to measuring some of the preceding forms of infinitesimal change.

The Divergence Operator. Recall, the del operator, introduced in section 1.4, is the partial differential operator that sends a function $f \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$ to its gradient $\nabla f \in \mathfrak{V}(\mathcal{D}, \mathbb{R}^n)$. In rectangular coordinates it can be expressed as

$$\nabla = \sum_{i=1}^{n} \mathbf{e}_i \partial_{x_i} = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle .$$

We also explored a procedure that allows the del operator to be expressed in other coordinate systems. Treating such an expression as a vector operator, we can define new operations on vector fields. The first operation we will define is called the *divergence*:

Definition 2.1. The divergence of a differentiable vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$ is the scalar function

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}(\mathbf{r}) = \frac{\partial F_1}{\partial x_1}(\mathbf{r}) + \ldots + \frac{\partial F_n}{\partial x_n}(\mathbf{r}) .$$

One geometric interpretation of the divergence at a point $\mathbf{r} \in \mathcal{D}$ is as a measure of how much the point $(\mathbf{r}) \in \mathcal{D}$ is a *source* or *sink* for the vector field. By source, we mean that the vector field has a positive net flow away from the point locally, whereas by sink, we mean it has a negative net flow. To better quantify this and properly define these ideas, we need the notion of flux, which is defined in terms of integration. Intuitively, flux is the limiting (infinitesimal) measure of the amount of flow out of a small volume, minus the amount of flow into that volume. Though we will not yet discuss the formal definition with integrals, we nevertheless can consider a few examples to illustrate this intuition.

Example 2.13. Consider the vector fields

$$\mathbf{F}(x, y) = y\hat{\imath} + x\hat{\jmath} .$$
$$\mathbf{G}(x, y) = (y - x)\hat{\imath} - (x + y)\hat{\jmath} .$$
$$\mathbf{H}(x, y) = \hat{\imath} + xy\hat{\jmath} .$$
$$\mathbf{K}(x, y) = \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{1 + x^2 + y^2}} .$$

We'll examine the divergence for each.

For $\mathbf{F}(x, y)$, the divergence is zero, since the $\hat{\imath}$-component is independent of x, and the $\hat{\jmath}$-component is independent of y. If this were a steady state fluid flow, it would be a fluid of constant density, as any parcel of fluid that enters a region is balanced by an equal parcel leaving that region.

For $\mathbf{G}(x, y)$, the divergence is -2. This vector field is a spiral sink, but note that the divergence is constantly equal to negative 2, so even away from the origin, the field behaves like a sink. In physical terms, if we place any small permeable spherical membrane into the flow, more fluid flows in than out of the spherical membrane. In this sense, every point is a sink in this field. The fluid density then must change, as more fluid is compressed into tighter spaces. If it were a charge field, we'd deduce that there was a uniform charge density of negative charges, such as electrons, distributed throughout the plane (though the rotation of the field would imply the charges are not static, perhaps due to the influence of a magnetic field.)

$\mathbf{H}(x,y)$ has divergence equal to x. Thus, the strength of the infinitesimal field flux is given by the x coordinate itself. See figure 20 for a visualization of this field.

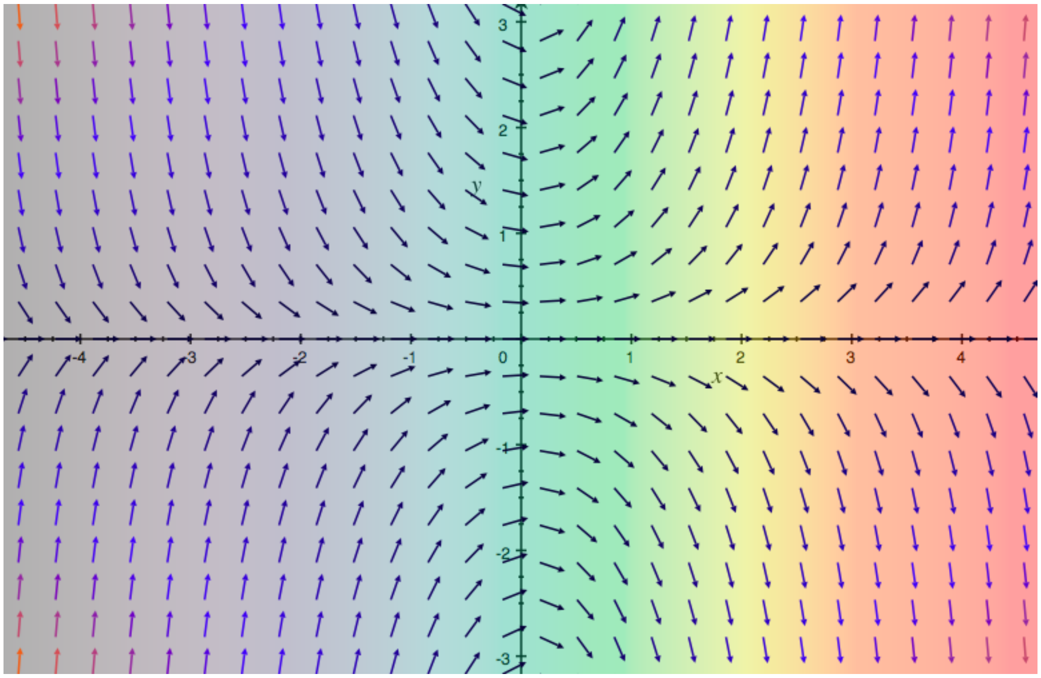


FIGURE 20. The vector field $\mathbf{H}(x,y) = \hat{\imath} + xy\hat{\jmath}$ has divergence $\nabla \cdot \mathbf{H}(x,y) = x$. The background color indicates the magnitude of the scalar field $\nabla \cdot \mathbf{H}(x,y) = x$, with warmer colors corresponding to larger values. Note that for $x < 0$, the field tends to have more net flow "inwards" in any given neighborhood, while for $x > 0$ the field tends to have more net flow "outwards" from any given neighborhood.

Finally, for $\mathbf{K}(x,y)$, we have

$$\begin{aligned}\nabla \cdot \mathbf{K}(x,y) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y}\right) \cdot \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{1+x^2+y^2}} \\ &= \frac{\partial}{\partial x}\left(\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{1+x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{1+x^2+y^2}}\right) \\ &= \frac{2+x^2+y^2}{(1+x^2+y^2)^{3/2}} = \frac{2+\mathbf{r}\cdot\mathbf{r}}{(1+\mathbf{r}\cdot\mathbf{r})^{3/2}}\,.\end{aligned}$$

Thus, observe that at $\mathbf{r} = \mathbf{0}$, the field has a divergence of 2, which indicates that there is net outward flow from the origin. The divergence decreases as $\|\mathbf{r}\|$ increases, since the denominator grows more rapidly in $\mathbf{r}\cdot\mathbf{r}$ than the numerator. Thus, this vector field has divergence that decreases with distance from the origin, even though the magnitude of the vectors increases as the radius increases.

Optional discussion of Divergence in other coordinates*. We would like to be able to compute divergence in other coordinate systems. We must caution that to use the other versions of the del operator, care must be taken when using the "dot product" mnemonic for divergence: to apply, e.g., the spherical form of ∇ to compute divergence $\nabla\cdot\mathbf{F}$, one must compute derivatives *first*, and then take the appropriate dot products between basis vectors. This is because the spherical frame itself is non-constant, and so change *along the frame itself* must be taken into account. In the case of spherical coordinates, the frame vectors $\hat{\mathbf{u}}_\varrho$, $\hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{u}}_\varphi$ are independent of ϱ but depend on the angular coordinates θ and φ. Then, consider for example how the spherical frame behaves under the action of the derivative $\frac{\partial}{\partial\theta}$:

$$\frac{\partial}{\partial\theta}\begin{bmatrix}\hat{\mathbf{u}}_\varrho\\ \hat{\mathbf{u}}_\theta\\ \hat{\mathbf{u}}_\varphi\end{bmatrix}=\begin{bmatrix}\cos(\varphi)\hat{\mathbf{u}}_\theta\\ -\cos(\varphi)\hat{\mathbf{u}}_\varrho+\sin(\varphi)\hat{\mathbf{u}}_\varphi\\ -\sin(\varphi)\hat{\mathbf{u}}_\theta\end{bmatrix}=\begin{bmatrix}0 & \cos(\varphi) & 0\\ -\cos(\varphi) & 0 & \sin(\varphi)\\ 0 & -\sin(\varphi) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_\varrho\\ \hat{\mathbf{u}}_\theta\\ \hat{\mathbf{u}}_\varphi\end{bmatrix},$$

as you will show in problem (13) in the problems in section 2.8 below. Similarly, one can show that

$$\frac{\partial\hat{\mathbf{u}}_\varrho}{\partial\varphi}=\hat{\mathbf{u}}_\varphi\,,\quad \frac{\partial\hat{\mathbf{u}}_\theta}{\partial\varphi}=0\,,\quad \frac{\partial\hat{\mathbf{u}}_\varphi}{\partial\varphi}=-\hat{\mathbf{u}}_\varrho\,.$$

Now let $\mathbf{F}=F^1\hat{\mathbf{u}}_\varrho+F^2\hat{\mathbf{u}}_\theta+F^3\hat{\mathbf{u}}_\varphi$ (where F^i are components, not powers). For convenience, write

$$\mathbf{F}=\begin{bmatrix}F^1\\ F^2\\ F^3\end{bmatrix}_{\mathscr{S}},$$

and observe that by the calculations above

$$\begin{aligned}\frac{\partial\mathbf{F}}{\partial\theta}&=\begin{bmatrix}F^1_\theta\\ F^2_\theta\\ F^3_\theta\end{bmatrix}_{\mathscr{S}}+\begin{bmatrix}0 & -\cos(\varphi) & 0\\ \cos(\varphi) & 0 & -\sin(\varphi)\\ 0 & \sin(\varphi) & 0\end{bmatrix}\begin{bmatrix}F^1\\ F^2\\ F^3\end{bmatrix}_{\mathscr{S}}\\ &=\begin{bmatrix}F^1_\theta-F^2\cos(\varphi)\\ F^2_\theta+F^1\cos(\varphi)-F^3\sin(\varphi)\\ F^3_\theta+F^2\sin(\varphi)\end{bmatrix}_{\mathscr{S}}\\ &=\big(F^1_\theta-F^2\cos(\varphi)\big)\hat{\mathbf{u}}_\varrho+\big(F^2_\theta+F^1\cos(\varphi)-F^3\sin(\varphi)\big)\hat{\mathbf{u}}_\theta+\big(F^3_\theta+F^2\sin(\varphi)\big)\hat{\mathbf{u}}_\varphi\,,\end{aligned}$$

since, e.g., $\partial_\theta(F^1\hat{\mathbf{u}}_\varrho)=F^1_\theta\hat{\mathbf{u}}_\varrho+F^1\partial_\theta\hat{\mathbf{u}}_\varrho=F^1_\theta+F^1\cos(\varphi)\hat{\mathbf{u}}_\theta$. Similarly one can compute an expression for $\frac{\partial\mathbf{F}}{\partial\varphi}$ in the spherical frame.

Putting the above ideas together, one can show that applying

$$\nabla\cdot\bullet=\left(\hat{\mathbf{u}}_\varrho\frac{\partial}{\partial\varrho}+\frac{1}{\varrho\cos\varphi}\hat{\mathbf{u}}_\theta\frac{\partial}{\partial\theta}+\frac{1}{\varrho}\hat{\mathbf{u}}_\varphi\frac{\partial}{\partial\varphi}\right)\cdot\bullet$$

to a vector field $\mathbf{F}$ by first computing $\frac{\partial\mathbf{F}}{\partial\varrho}$, $\frac{\partial\mathbf{F}}{\partial\theta}$ and $\frac{\partial\mathbf{F}}{\partial\varphi}$, followed by computing the necessary scalar products, one arrives at the following formula for the divergence in our version of spherical coordinates:

Proposition 2.2. *For* $\mathbf{F}(\varrho,\theta,\varphi)=F^1\hat{\mathbf{u}}_\varrho+F^2\hat{\mathbf{u}}_\theta+F^3\hat{\mathbf{u}}_\varphi$,

$$\begin{aligned}\operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F}(\varrho,\theta,\varphi)&=\frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\big(\varrho^2F^1\big)+\frac{1}{\varrho\cos(\varphi)}\frac{\partial F^2}{\partial\theta}+\frac{1}{\varrho\cos(\varphi)}\frac{\partial}{\partial\varphi}\big(\cos(\varphi)F^3\big)\\ &=\frac{2}{\varrho}F^1+\frac{\partial F^1}{\partial\varrho}+\frac{1}{\varrho\cos\varphi}\frac{\partial F^2}{\partial\theta}-\frac{\tan(\varphi)}{\varrho}F^3+\frac{1}{\varrho}\frac{\partial F^3}{\partial\varphi}\,.\end{aligned}$$

Example 2.14. We can easily compute the divergence of a radial vector field $\mathbf{F}=x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}}+z\hat{\mathbf{k}}=\varrho\hat{\mathbf{u}}_\varrho$ using the spherical divergence expression:

$$\nabla\cdot\mathbf{F}=\frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\big(\varrho^3\big)=\frac{3\varrho^2}{\varrho^2}=3\,,\varrho\neq0\,.$$

Of course, at the origin, our spherical coordinates are ill defined, but in the limit, this expression holds and agrees with the rectangular calculation.

One moral from the above work is that along general curvilinear coordinates, one must take care to account for the local change of the frame along various integral curves of the frame. The key that allowed us to move forward was to express the derivatives of the frame element as linear combinations of the frame elements, with weights given by functions of the variables. Reconsider the equation

$$\frac{\partial \mathbf{F}}{\partial \theta} = \big(F^1_\theta - F^2\cos(\varphi)\big)\hat{\mathbf{u}}_\varrho + \big(F^2_\theta + F^1\cos(\varphi) - F^3\sin(\varphi)\big)\hat{\mathbf{u}}_\theta + \big(F^3_\theta + F^2\sin(\varphi)\big)\hat{\mathbf{u}}_\varphi\,,$$

which we also could express in matrix form. The extra terms appearing which aren't of the form F^i_θ appear precisely as a consequence of the change of the frame along itself (in this case, along the directions of increasing θ).

In a general frame $(\hat{\mathbf{u}}_1, \dots \hat{\mathbf{u}}_n)$ adapted to orthogonal coordinates $(y_1, \dots, y_n)_{\mathscr{Y}}$ on $\mathbb{R}^n$, there exists some collection of functions Γ_{ijk}, $i, j, k \in \{1, \dots, n\}$ such that

$$\frac{\partial \hat{\mathbf{u}}_i}{\partial y_j} = \sum_{k=1}^n \Gamma_{ijk}\hat{\mathbf{u}}_k\,.$$

These functions are called *Cristoffel symbols* (of the first kind). Using Cristoffel symbols, one can express the y_j partial derivative of a vector field $\mathbf{F} = \sum_{i=1}^n F^i\hat{\mathbf{u}}_i$ as

$$\frac{\partial \mathbf{F}}{\partial y_j} = \sum_{i=1}^n \left(\frac{\partial F^i}{\partial y_j}\hat{\mathbf{u}}_i + \sum_{k=1}^n F^i\Gamma_{ijk}\hat{\mathbf{u}}_k\right) = \sum_{k=1}^n \left(\frac{\partial F^k}{\partial y_j} + \sum_{i=1}^n F^i\Gamma_{ijk}\right)\hat{\mathbf{u}}_k\ .$$

The Curl Operator. In three dimensions, we can also define a somewhat unique differential operator on vector fields, which returns a vector field that quantifies the infinitesimal vorticity of the original vector field. This operator is called the *curl*:

Definition. The curl of a vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^3$ is the vector field whose Cartesian coordinate expression is

$$\mathrm{curl}(\mathbf{F}) := \nabla \times \mathbf{F} = (\partial_y F_3 - \partial_z F_2)\hat{\mathbf{\imath}} + (\partial_z F_1 - \partial_x F_3)\hat{\mathbf{\jmath}} + (\partial_x F_2 - \partial_y F_1)\hat{\mathbf{k}}\,.$$

Perhaps the simplest fields to illustrate the meaning of the curl are spin fields like $\mathbf{F}_\pm = \pm(y\hat{\mathbf{\imath}} + x\hat{\mathbf{\jmath}}) + 0\hat{\mathbf{k}}$. It is easy to calculate that for these fields, $\nabla \times \mathbf{F}_\pm = \mp\hat{\mathbf{k}}$. Thus, the spin field that has right-handed (i.e., counter-clockwise) rotation has curl $\hat{\mathbf{k}}$, which points in the direction of the angular velocity vector for a particle rotating in the flow, while the clockwise spin field has curl $-\hat{\mathbf{k}}$, as one should expect.

Observe that curl can be nonzero even if the field doesn't have an obvious vortex:

Example 2.15. For the field $\mathbf{F} = x\hat{\mathbf{\jmath}}$, the curl is $\nabla \times \mathbf{F} = \hat{\mathbf{k}}$.

If you imagine this field as wind, then a flag placed initially parallel to $\hat{\mathbf{\imath}}$ will rotate counterclockwise on its way to being parallel to the stream-lines of the field. This counterclockwise rotation has angular momentum in the $\hat{\mathbf{k}}$ direction, so perhaps we shouldn't be too surprised. On the other hand, that the magnitude of the curl is constantly one is more subtle. Can you explain why the curl would have the same magnitude in an area near the plane $x = 0$, where the field is weak, as it does in an area far out along the x-axis, where the field is stronger? The important realization is that the curl is a limit, which can be calculated by considering independent circulations along small paths. To define the curl this way, we need *line integrals.*

Example 2.16. The curl of a vector field in written in the rectangular frame may be computed using the determinant trick for cross products, treating the components of ∇ as operators in the usual way:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}\,.$$

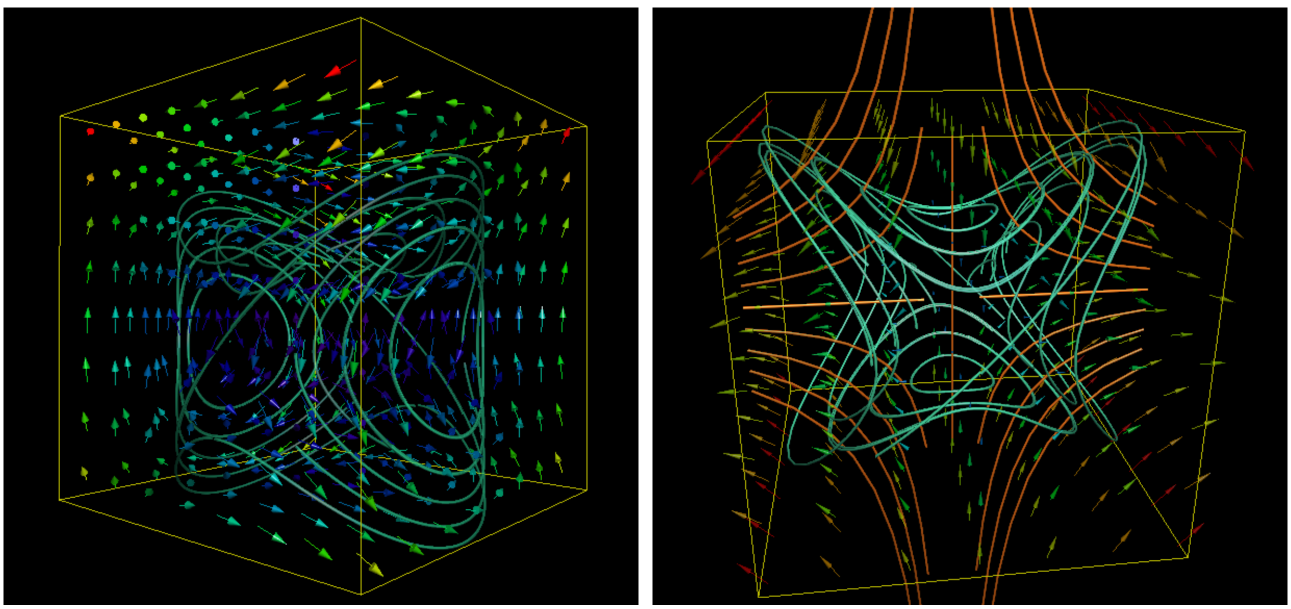
E.g., for the vector field $\mathbf{F}(x, y, z) = (y - z)\hat{\imath} + (z - x)\hat{\jmath} + xy\hat{\mathbf{k}}$, we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y - z & z - x & xy \end{vmatrix} \\ &= \big(\partial_y(xy) - \partial_z(z - x)\big)\hat{\imath} - \big(\partial_x(xy) - \partial_z(y - z)\big)\hat{\jmath} + \big(\partial_x(z - x) - \partial_y(y - z)\big)\hat{\mathbf{k}} \\ &= (x - 1)\hat{\imath} - (y + 1)\hat{\jmath} - 2\hat{\mathbf{k}}\,.\end{aligned}$$

Example 2.17. We'll compute the curl of $\mathbf{F} = yz\hat{\imath} - xz\hat{\jmath} + xy\hat{\mathbf{k}}$ and visualize the original field, some of its field lines, and the curl field as it relates to the original field.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & xy \end{vmatrix} = 2x\hat{\imath} - 2z\hat{\mathbf{k}}\,.$$

Note that the field lines of the curl field are hyperbolae in planes of constant y. As seen below in figure **??** they meet the trajectories of $\mathbf{F}$ at right angles, and are oriented so as to indicate the right-handed angular velocity of the rotational trajectories of $\mathbf{F}$.



(A) (B)

Figure 21. (A) – A view of the vector field $\mathbf{F} = yz\hat{\imath} - xz\hat{\jmath} + xy\hat{\mathbf{k}}$ and some of its trajectories. The vectors are not drawn to scale, so as to avoid collisions; warmer colors indicate increased magnitude. (B) – A view of the curl of $\mathbf{F}$, $\nabla \times \mathbf{F} = \nabla \times (yz\hat{\imath} - xz\hat{\jmath} + xy\hat{\mathbf{k}}) = 2x\hat{\imath} - 2z\hat{\mathbf{k}}$, and some of its trajectories (in orange), alongside the trajectories of $\mathbf{F}$.

One can show using Clairaut's theorem that the divergence of the curl is always zero for sufficiently smooth vector fields, and similarly, gradient vector fields are *irrotational*:

Proposition 2.3. *For any* 3*-dimensional vector field* $\mathbf{F}$ *whose components are continuously differentiable to second order on a domain* $\mathcal{D}$*,*

$$\operatorname{div}\operatorname{curl}(\mathbf{F}(\mathbf{r})) = \nabla \cdot \big(\nabla \times \mathbf{F}(\mathbf{r})\big) = 0\,,$$

and for any scalar function $f(x, y, z)$ *continuously differentiable to second order on* $\mathcal{D}$*,*

$$\operatorname{curl}\operatorname{grad}(f(\mathbf{r})) = \nabla \times \nabla \mathbf{f}(\mathbf{r}) = \mathbf{0}\,.$$

See (12) in the problems below.
It follows that a vector field with nonzero curl cannot be conservative.

The Jacobian. One of the most essential gifts of calculus is the ability to study non-linear phenomena via approximations provided by linearization. In vector calculus, such a linear approximation comes in the form of a matrix map, given by a matrix called the *Jacobian*:

Definition. The *Jacobian matrix* at $\mathbf{r} \in \mathcal{D} \subseteq \mathbb{R}^n$ of a differentiable n-dimensional vector field $\mathbf{F}(\mathbf{r}) = \sum_{i=1}^n F_i(\mathbf{r})\hat{\mathbf{e}}_i$ on $\mathcal{D}$ is the matrix

$$\mathrm{J}_{\mathbf{F}}(\mathbf{r}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{r}) & \frac{\partial F_1}{\partial x_2}(\mathbf{r}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{r}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{r}) & \frac{\partial F_2}{\partial x_2}(\mathbf{r}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{r}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{r}) & \frac{\partial F_n}{\partial x_2}(\mathbf{r}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{r}) \end{bmatrix}.$$

That is, the Jacobian matrix of $\mathbf{F}$ is the matrix whose expression with respect to the rectangular coordinate basis has kth row equal to the gradient of the kth scalar field component, $\hat{\mathbf{e}}_k^T \mathrm{J}_{\mathbf{F}}(\mathbf{r}) = \nabla\big(\hat{\mathbf{e}}_k \cdot \mathbf{F}\big)(\mathbf{r}) = \nabla F_k(\mathbf{r})$.

The Jacobian is also frequently notated $\mathrm{D}\mathbf{F}(\mathbf{r})$, and called the *total derivative* of $\mathbf{F}$. Observe that the matrix expression is coordinate dependent. We encountered Jacobians when expressing the general chain rule for multivariate maps. Recall, if $f(\mathbf{r})$ is a multivariate function and $\mathbf{G} : \mathbb{R}^k \to \mathbb{R}^n$ is a coordinate transformation for the map $\mathbf{v} \mapsto \mathbf{r}$, then the chain rule could be expressed as a composition of linear maps via matrix products:

$$\mathrm{D}_{\mathbf{v}}(f \circ \mathbf{G}\big)(\mathbf{v}) = \mathrm{D}_{\mathbf{x}} f\big(\mathbf{G}(\mathbf{v})\big) \circ \mathrm{D}_{\mathbf{v}}\mathbf{G}(\mathbf{v})\,,$$

where $\mathrm{D}_{\mathbf{x}} f$ is the Jacobian as above using variables x_i, and $\mathrm{D}_{\mathbf{v}} f$, $\mathrm{D}_{\mathbf{v}}\mathbf{G}$ are appropriate Jacobian matrices with respect to variables v_i. Though the form of the Jacobian matrix is dependent on the choice of variables, the linear map it determines which approximates f is unique and independent of coordinates:

Proposition 2.4. *For a differentiable vector field* $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$*, there is a unique "best" linear approximation to* $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$ *centered at a point* $\mathbf{r} \in \mathcal{D}$ *which is given by the map*

$$\mathbf{x} \mapsto \big[\mathrm{J}_{\mathbf{F}}(\mathbf{r})\big]\mathbf{x} = \sum_{i=1}^n \big([\mathrm{D}_{\mathbf{r}} F_i(\mathbf{r})]\mathbf{x}\big)\,\mathbf{e}_i = \sum_{i=1}^n \mathbf{x} \cdot \nabla F_i(\mathbf{r})\,\mathbf{e}_i\,.$$

If $\mathbf{r} = \mathbf{G}(\mathbf{v})$ *is a bijective coordinate transformation, then this linear map in the new coordinates is given as well by the Jacobian with respect to the new variables* $v_1, \ldots v_n$*, with the frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ *replaced by the new coordinate frame* $(\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_n)$ *determined via the new coordinates, and with* $\mathbf{x}$ *expressed in this frame:*

$$\mathbf{x} \mapsto \big[\mathrm{J}_{\mathbf{F}}(\mathbf{v})\big]\mathbf{x} = \sum_{i=1}^n \big([\mathrm{D}_{\mathbf{v}} F_i(\mathbf{v})]\mathbf{x}\big)\,\mathbf{u}_i = \sum_{i=1}^n \Big([\mathrm{D}_{\mathbf{r}} F_i\big(\mathbf{G}(\mathbf{v})\big) \circ \mathrm{D}_{\mathbf{v}}\mathbf{G}(\mathbf{v})]\mathbf{x}\Big)\,\mathbf{u}_i\,.$$

By "best" we mean that it minimizes local error among all possible linear vector fields approximating $\mathbf{F}$ near the point $\mathbf{r}$ around which the approximation is centered. We will not prove this theorem here. The claim about the Jacobian approximation under coordinate transformations is a consequence of the linear-algebraic rules for transforming matrices to re-express linear maps under a change of coordinates, which reduces in this context to the above chain rule. We will focus on the use of this theorem in capturing the local behavior of two-dimensional vector fields near zeroes: for two-dimensional vector fields, linearization allows us to determine the *local topology* of a zero $\mathbf{r}_0$ of a nonlinear vector field whenever the Jacobian is nonzero around $\mathbf{r}_0$.

Example 2.18. We will use the Jacobian to determine the topology of the zeros of the vector field $\mathbf{F}(x,y) = \sin y\,\hat{\imath} + \cos x\,\hat{\jmath}$ inside the disk $x^2 + y^2 \leq 4$.

The Jacobian is

$$\mathrm{J}_{\mathbf{F}}(x,y) = \begin{bmatrix} \frac{\partial F_1}{\partial x}(x,y) & \frac{\partial F_1}{\partial y}(x,y) \\ \frac{\partial F_2}{\partial x}(x,y) & \frac{\partial F_2}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 0 & \cos y \\ -\sin x & 0 \end{bmatrix}.$$

The zeros of $\mathbf{F}$ occur whenever $\sin y = 0 = \cos x$, which requires $y = k\pi$ and $x = (2l+1)\pi/2$ for integers k and l. Note that the only zeros occurring in the disk $x^2 + y^2 \leq 4$ are at $(\pm\pi/2, 0)$. For these points we have

$$\mathrm{J}_{\mathbf{F}}(\pi/2, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\mathrm{J}_{\mathbf{F}}(-\pi/2, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, the linearized fields are

$$\mathbf{L}_{\mathbf{F},(\pi/2,0)}(\mathbf{r}) = [\mathrm{J}_{\mathbf{F}}(\pi/2,0)]\mathbf{r} = y\hat{\imath} - x\hat{\jmath}\,,$$

$$\mathbf{L}_{\mathbf{F},(-\pi/2,0)}(\mathbf{r}) = [\mathrm{J}_{\mathbf{F}}(-\pi/2,0)]\mathbf{r} = y\hat{\imath} + x\hat{\jmath}\,.$$

The first is the clockwise spin field encountered at the beginning of section 2.1, and the second is a saddle field, as you will hopefully show when completing problem 1 in section 2.8.

The Jacobian matrix also makes an appearance in the study of integrals under coordinate changes. We briefly discuss change of variables for double integrals. For a bijective, continuously differentiable coordinate transformation $\mathbf{T} : \mathcal{V} \to \mathcal{D}$ of domains $\mathcal{D}, \mathcal{V} \subseteq \mathbb{R}^2$, one can assign a Jacobian, whose determinant measures the areal distortion imposed by the transformation.

The transformation can be viewed as a vector field $\langle u, v\rangle \mapsto \mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle \in \mathcal{V}$, and the Jacobian determinant of $\mathbf{T}$ is then

$$\frac{\partial(x,y)}{\partial(u,v)} := \det \mathrm{J}_{\mathbf{T}}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\,.$$

Then the differential 2-forms giving the area transform as

$$\mathrm{d}\mathcal{A}(x,y) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}\mathcal{A}(u,v)\,.$$

Thus the absolute value of the Jacobian determinant of the transformation gives the appropriate scale factor. For example, the transformation from Cartesian to polar coordinates has Jacobian equal to r, so we have the relation of area elements $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, provided r is non-negative.

The Laplace Operator. We can also define second order differential operators for vector fields. The most essential one is the Laplacian operator.

Definition. The Laplacian of a twice differentiable real scalar function f of n variables is

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}\,.$$

Definition. Laplace's equation for a scalar function is the partial differential equation $\nabla^2 u = 0$. A function u which satisfies Laplace's equation is called a *harmonic function.*

Example 2.19. We'll verify that $\Psi(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ is a 3-dimensional harmonic function.

$$\begin{aligned}\nabla^2\Psi(x,y,z) &= \nabla\cdot\nabla\Psi(x,y,z) = \nabla\cdot\left(\Psi_x\hat{\mathbf{\imath}} + \Psi_y\hat{\mathbf{\jmath}} + \Psi_z\hat{\mathbf{k}}\right)\\ &= \nabla\cdot\left[\left(\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{\imath}} - \left(\frac{3xy}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{\jmath}} - \left(\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{k}}\right]\\ &= \left(\frac{\partial}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial}{\partial y}\hat{\mathbf{\jmath}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot\frac{(y^2+z^2-2x^2)\hat{\mathbf{\imath}} - 3x(y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}})}{(x^2+y^2+z^2)^{5/2}}\\ &= \frac{(6x^2-9y^2-9z^2)x}{(x^2+y^2+z^2)^{7/2}} + \frac{(12y^2-3x^2-3z^2)x}{(x^2+y^2+z^2)^{7/2}} + \frac{(12z^2-3x^2-3y^2)x}{(x^2+y^2+z^2)^{7/2}}\\ &= 0\,.\end{aligned}$$

Another solution is to use the spherical form of the divergence given in proposition 2.2, together with the fact that in our version of spherical coordinates, Ψ may be re-expressed as

$$\Psi(\varrho,\theta,\varphi) = \frac{\varrho\cos\theta\cos\varphi}{\varrho^3} = \frac{\cos\theta\cos\varphi}{\varrho^2}\,.$$

Proposition 2.2 implies that the spherical form of the Laplacian is

$$\begin{aligned}\nabla^2\Psi(\varrho,\theta,\varphi) &= \nabla\cdot\left(\Psi_\varrho\hat{\mathbf{u}}_\varrho + \frac{1}{\varrho\cos\varphi}\Psi_\theta\hat{\mathbf{u}}_\theta + \frac{1}{\varrho}\Psi_\varphi\hat{\mathbf{u}}_\varphi\right)\\ &= \frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\left(\varrho^2\frac{\partial\Psi}{\partial\varrho}\right) + \frac{1}{\varrho^2\cos^2(\varphi)}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{\varrho^2\cos(\varphi)}\frac{\partial}{\partial\varphi}\left(\cos(\varphi)\frac{\partial\Psi}{\partial\varphi}\right)\end{aligned}$$

Thus, since $\Psi_\varrho = -2\Psi/\varrho$, $\Psi_\theta = -\tan(\theta)\Psi$, and $\Psi_\varphi = -\tan(\varphi)\Psi$, we have

$$\begin{aligned}\nabla^2\Psi(\varrho,\theta,\varphi) &= \frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\left(\varrho^2\Psi_\varrho\right) + \frac{1}{\varrho^2\cos^2(\varphi)}\left(\Psi_\theta\right)_\theta + \frac{1}{\varrho^2\cos(\varphi)}\frac{\partial}{\partial\varphi}\left(\cos(\varphi)\Psi_\varphi\right)\\ &= \frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\left(-2\varrho\Psi\right) + \frac{1}{\varrho^2\cos^2(\varphi)}\Big(-\tan(\theta)\Psi\Big)_\theta + \frac{1}{\varrho^2\cos(\varphi)}\frac{\partial}{\partial\varphi}\left(-\sin(\varphi)\Psi\right)\\ &= \frac{-2\Psi+4\Psi}{\varrho^2} - \frac{\sec^2(\theta)-\tan^2(\theta)}{\varrho^2\cos^2(\varphi)}\Psi + \frac{-\cos(\varphi)+\sin(\varphi)\tan(\varphi)}{\varrho^2\cos(\varphi)}\Psi\\ &= \frac{2\Psi - \sec^2(\varphi)\Psi - \Psi + \tan^2(\varphi)\Psi}{\varrho^2}\\ &= 0\end{aligned}$$

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§ 2.6. Parametrized Surfaces and Coordinate Vector Fields*

(Note: This is an optional section, which is incomplete; there are a number of figures and examples yet to add.)

In our study of vector fields, we've been able to view them as vector-valued maps of vectors or points, and in the common examples arising from gradients or frames, we have emphasized two main cases:

- 2-dimensional fields $\mathbf{F}:\mathcal{D}\to\mathbb{R}^2$, $\mathcal{D}\subset\mathbb{R}^2$,
- 3-dimensional fields $\mathbf{F}:\mathcal{D}\to\mathbb{R}^3$, $\mathcal{D}\subset\mathbb{R}^3$.

Special cases like normal vector fields arise by considering a restriction of one of the above types of maps (often coming from a gradient of a scalar field) to either a level curve or a level surface. Now we will consider a particularly useful interpretation of vector-valued maps from domains in $\mathbb{R}^2$ to $\mathbb{R}^3$. Fix a connected domain $\mathcal{V}\in\mathbb{R}^2$, and let $\boldsymbol{\sigma}:\mathcal{V}\to\mathbb{R}^3$ be a continuous vector-valued function on $\mathcal{V}$, the outputs of which are 3-dimensional vectors. By interpreting the image vectors as position vectors, we can ponder what geometric object the image $\boldsymbol{\sigma}(\mathcal{V})$ traces out in $\mathbb{R}^3$.

Conventionally, we will denote the input for such a vector-valued function by the vector $\hat{\mathbf{v}}$, or by writing its components as u and v:

$$\mathbf{v} = u\hat{\mathbf{\imath}} + v\hat{\mathbf{\jmath}} \in \mathcal{V}\,,$$

$$\boldsymbol{\sigma}(\mathbf{v}) = \boldsymbol{\sigma}(u,v) = \sigma_1(u,v)\hat{\mathbf{\imath}} + \sigma_2(u,v)\hat{\mathbf{\imath}} + \sigma_3(u,v)\hat{\mathbf{k}}\,.$$

Observe that $\boldsymbol{\sigma}(\mathbf{v})$ is composed of three components, each of which is a bivariate function.

We know that vector-valued functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^3$ trace out space curves. A natural guess would be that $\boldsymbol{\sigma}$ traces out a surface[11], and under our presumption that $\boldsymbol{\sigma}$ was continuous, this is usually true. But it could also be a constant map giving an image $\boldsymbol{\sigma}(\mathcal{V}) =$ a point, or, if it takes constant values along a family of curves building up $\mathcal{V}$, it could also trace out a space curve; in these cases you can imagine $\boldsymbol{\sigma}$ as collapsing $\mathcal{V}$ into a curve or point, which is then embedded into three dimensional space.

To ensure that our image is a surface, we need a few more conditions, and hence, definitions. The first is the idea of an open set in $\mathbb{R}^3$, which generalizes the idea of an open set in $\mathbb{R}^2$. Recall, a set $U \subset \mathbb{R}^2$ is open if it does not contain any of its boundary points, or if equivalently, for any point $P \in U$ there is an open disk around P that is contained in U. Here, an open disk is one which excludes its boundary points, i.e., a disk of the form $\{\mathbf{r} \in \mathbb{R}^2 : 0 \leq \|\mathbf{r} - \mathbf{r}_0\| < R\}$ for some radius R and center $\mathbf{r}_0$.

Definition. An open ball in $\mathbb{R}^3$ is a set of the form $\mathring{B}_R(\mathbf{r}_0) = \{\mathbf{r} \in \mathbb{R}^3 : 0 \leq \|\mathbf{r} - \mathbf{r}_0\| < R\}$. Any such ball may be thought of as the set of points interior to the sphere of radius R centered at a point $\mathbf{r}_0 \in \mathbb{R}^3$. A set $U \subseteq \mathbb{R}^3$ is called open if around every point $\mathbf{r} \in U$ there is some open ball $\mathring{B}_R(\mathbf{r})$ such that $\mathring{B}_R(\mathbf{r}) \subseteq U$.

To ensure that a vector-valued function gives us a genuine surface and not a curve or a point, and to prevent other sorts of singular behavior, we can ask that it behave nicely with respect to open sets of its domain. The intuition is that if the image of the map $\boldsymbol{\sigma}$ is a surface, then $\boldsymbol{\sigma}$ should act on the domain locally by lifting small open subsets of the domain into three dimensions in a *one-to-one* or *injective* manner, like carrying small open disks of $\mathbb{R}^2$ to analogous sets of $\mathbb{R}^3$. Such a set should be the intersection of an open set of $\mathbb{R}^3$ with the image of the disk by $\boldsymbol{\sigma}$. What this means is that the pre-image of an open set of $\mathbb{R}^3$ by $\boldsymbol{\sigma}$ should be an open set (possibly empty) of $\mathcal{V}$, and when the image $\boldsymbol{\sigma}(\mathcal{V})$ is a nice, continuous surface, there is an honest inverse function, at least locally away from any self-intersections, from the image $\boldsymbol{\sigma}(\mathcal{V})$ back to the domain $\mathcal{V}$. The condition on open sets is actually *equivalent to continuity*:

Proposition 2.5. *Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a domain for the vector-valued map $\boldsymbol{\sigma} : \mathcal{V} \to \mathbb{R}^3$. Then $\boldsymbol{\sigma}$ is continuous throughout $\mathcal{V}$ if and only if for any open set $\mathcal{U} \subseteq \mathbb{R}^3$, the pre-image $\boldsymbol{\sigma}^{-1}(\mathcal{U}) := \{\mathbf{v} \in \mathcal{V} : \boldsymbol{\sigma}(\mathbf{v}) \in \mathcal{U}\}$ is itself an open subset of $\mathcal{V}$, i.e., it is the intersection of an open subset of $\mathbb{R}^2$ with $\mathcal{V}$.*

Proof. See problem (19) below. □

A continuous map which is also continuously invertible has a special name: a *homeomorphism*. The most basic goal of the study of *topology* is understanding spaces with enough structure to define continuous functions, up to equivalence of spaces by homeomorphisms. For our purposes, we only need to understand homeomorphisms as they relate to defining surfaces in $\mathbb{R}^3$:

Definition. A *parametric surface patch* is a subset of $\mathbb{R}^3$ which is realized as the image of a continuous and continuously invertible vector-valued map $\boldsymbol{\sigma} : \mathcal{V} \to \mathbb{R}^3$ for a domain $\mathcal{V} \subseteq \mathbb{R}^2$, i.e., it is the homeomorphic image of a vector-valued map from a domain of $\mathbb{R}^2$ into $\mathbb{R}^3$. The map $\boldsymbol{\sigma}$ is called a *parameterization* of the patch. If resolved into components, then the resulting equations are called parametric equations for the surface patch.

[11] Note that I have not defined surface yet. Differential geometers often use a fairly restrictive definition of a *smooth* surface, such as "*a surface in $\mathbb{R}^3$ is a subspace $\mathcal{S}$ such that around each point $p \in \mathcal{S}$ there is a smooth chart giving a diffeomorphism from an open set U of the surface to an open set V of $\mathbb{R}^2$, and such that the transition functions on overlapping charts are diffeomorphisms of the corresponding open sets of $\mathbb{R}^2$.*" This definition is too restrictive and elaborate for us; it excludes surfaces with boundary, and surfaces with interesting singularities, like cone points, triple points, and self-intersections. See the next footnote for a rough but workable definition for our purposes.

The idea is that a general surface in $\mathbb{R}^3$ can be built up from a number of patches, or parameterizations, which may overlap, and together fill out the whole surface. One should visualize the homeomorphism condition on a surface patch $\boldsymbol{\sigma}$ as follows: if the domain $\mathcal{V} \subseteq \mathbb{R}^2$ is placed in the xy-plane within $\mathbb{R}^3$, then $\boldsymbol{\sigma}$ acts on $\mathcal{V}$ by stretching, bending, rotating, and translating the set $\mathcal{V}$ until it is in the position to make up a patch of the (possibly) larger surface, but it is not permitted to collapse multiple points, tear the image set, or cause it to pass through itself.

A general surface[12] in $\mathbb{R}^3$ of course might admit self intersections, but it can always be arranged that this occurs when patches intersect each other along some curve or in a point. One can also construct parameterizations of these sorts of self-intersecting and pinching behaviors, but to avoid difficulties in doing calculus on surfaces, it is often convenient to work with patches where the image is homeomorphic to the plane domain of the parameterization. It is also sometimes convenient to work with a single parameterization, even if it fails to be a homeomorphism (one just has to be careful around subsets of the image which are not homeomorphic images of a plane domain, like triple points, branch points, and curves of self-intersection).

We now turn to some examples of surfaces and parameterizations.

Example 2.20. For a domain $\mathcal{V} \subseteq \mathbb{R}^2$, let $f : \mathcal{V} \to \mathbb{R}$ be any continuously differentiable function. Let

$$\boldsymbol{\sigma}(u,v) = u\hat{\mathbf{\imath}} + v\hat{\mathbf{\jmath}} + f(u,v)\hat{\mathbf{k}}\,,\ \hat{\mathbf{v}} = u\hat{\mathbf{\imath}} + v\hat{\mathbf{\jmath}} \in \mathcal{V}\,.$$

Then $\boldsymbol{\sigma}$ parameterizes the graph of the function f. Since f is a function, it produces a single output for any $\hat{\mathbf{v}}$ which ensures that the map $\boldsymbol{\sigma} : \mathcal{V} \to \boldsymbol{\sigma}(\mathcal{V}) \subset \mathbb{R}^3$ is invertible. Thus the graph of a continuous function is homeomorphic as a surface to the domain of the function being graphed.

Example 2.21. Fix a positive real number R. Then we can use the spherical coordinate functions for when $\varrho = R$ to give a parameterization in rectangular coordinates of surface of the radius R sphere, minus the meridian with $y = 0$ and negative x coordinates connecting the two poles corresponding to $\pm R\,\hat{\mathbf{k}}$:

$$\boldsymbol{\sigma}(u,v) = R\cos u\cos v\,\hat{\mathbf{\imath}} + R\sin u\cos v\,\hat{\mathbf{\jmath}} + R\sin v\,,\quad -\pi < u < \pi,\ -\tfrac{\pi}{2} < v < \tfrac{\pi}{2}\,.$$

If we remove the restriction that the patch be a homeomorphism from an open set, we can cover the sphere except at the poles (where θ is not defined) by letting θ take the value π. However, to cover the whole sphere with patches giving honest homeomorphisms from open sets of $\mathbb{R}^2$ we would need one more patch (e.g., we could use $\tilde{\boldsymbol{\sigma}}(u,v) = R\sin u\cos v\,\hat{\mathbf{\imath}} + R\sin v\,\hat{\mathbf{\jmath}} + R\cos u\cos v\,\hat{\mathbf{k}}$, with u and v restricted as in the domain of $\boldsymbol{\sigma}$.)

Another parametric way to try to build a sphere is with hemispherical patches, which correspond to graphs of functions over open disks, switching the roles of the dependent variable. Each such patch also misses a circle (an equator dividing the sphere into the two hemispheres), and you should convince yourself you need six hemispherical patches to completely cover the sphere; see problem (19) below.

There exist other parameterizations which only miss one point of the sphere; see problem (20) below. It turns out this is the best we can do, and so to cover the whole sphere, we still need at least two patches.

Example 2.22. Surfaces of revolution are among the first surfaces encountered by students of calculus. Starting from a plane curve placed on a plane in $\mathbb{R}^3$, one can sweep out a surface by revolving that curve about an axis in the plane containing the curve. One of the most important

[12] We can define general surfaces from our ideas of chart and patches; around every point which is not a boundary, singularity, or a point along a curve of self intersection, we can construct a local chart to an open set of $\mathbb{R}^2$, and there are clear local models for self-intersections, boundaries, and various types of singularities. Alternatively and in more expert language, our notion of general surfaces in $\mathbb{R}^3$ is defined as any locus of points in $\mathbb{R}^3$ consisting of a smooth portion, which is an immersion of a 2-manifold with or without boundary, and a singular sub-locus, which is a discrete, measure zero (and possibly empty) collection of singular points, where there is no consistent definition of a tangent space. Note that we don't consider self intersections singular: for self intersections, we can define a tangent space at a point to each "branch" of the surface. Indeed, if we choose a pre-image of the multiple point in the original non-singular 2-manifold, we can select a tangent space and represent it by an affine plane in $\mathbb{R}^3$.

examples of a surface of revolution is a *torus*, which famously is the mathematical version of a doughnut.

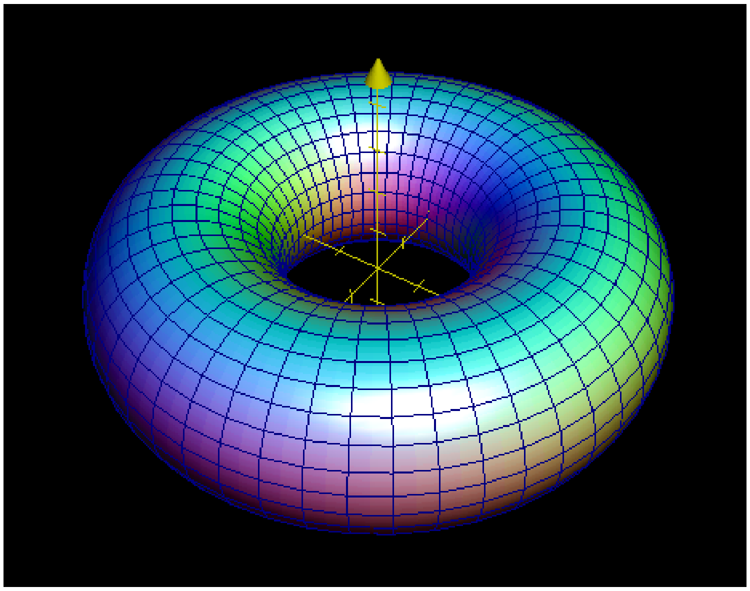

FIGURE 22. It's Torus!

To create a torus, one starts with a circle of radius a, centered at a point $b > a$ units from the axis of rotation. Then one sweeps this circle around a perpendicular circle centered on the chosen axis. Thus, the core circle has radius b, while a *meridional slice* is a circle of radius a. This procedure yields a simple parametrization:

$$\boldsymbol{\sigma}(u,v) = \big(a\cos(u)+b\big)\cos(v)\hat{\boldsymbol{\imath}} + \big(a\cos(u)+b\big)\sin(v)\hat{\boldsymbol{\imath}} + a\sin(u)\hat{\mathbf{k}}\,, (u,v)\in[0,2\pi]\times[0,2\pi]\,.$$

Here, u is the angular coordinate in the meridional direction, and v is a *longitudinal angle*. To make this into a proper patch giving a homeomorphism from an open set, one must delete one meridional circle and one longitudinal circle

Example 2.23. A ruled surface is a surface swept out by lines. A helicoid is a ruled surface obtained by rotating a line about an axis while simultaneously translating the line along the axis. Note that the surface $z = \arctan(y/x)$ is a portion of a helicoid. To parameterize the full helicoid of which this graph is a portion, one can use

$$\boldsymbol{\sigma}(\rho,\vartheta) = \rho\cos\vartheta\hat{\boldsymbol{\imath}} + \rho\sin\vartheta\hat{\boldsymbol{\jmath}} + \vartheta\hat{\mathbf{k}}\,, (\rho,\vartheta)\in\mathbb{R}^2\,.$$

Observe that the rulings are given by constant ϑ, while the space curves obtained by setting ρ equal to a constant are helices.

Let $\mathcal{S}$ be a surface given by a parameterization $\boldsymbol{\sigma}:\mathcal{V}\to\mathbb{R}^3$, and assume that the components of $\boldsymbol{\sigma}$ are all continuously differentiable. Since $\boldsymbol{\sigma}$ is vector-valued, the partial derivatives

$$\partial_u\boldsymbol{\sigma}(u_0,v_0) = \lim_{h\to u_0}\frac{\boldsymbol{\sigma}(u_0+h,v_0)-\boldsymbol{\sigma}(u_0,v_0)}{h} \quad\text{and}\quad \partial_v\boldsymbol{\sigma}(u_0,v_0) = \lim_{h\to 0}\frac{\boldsymbol{\sigma}(u_0,v_0+h)-\boldsymbol{\sigma}(u_0,v_0)}{h}$$

are vectors. What do they represent?

From their definition, these partial derivatives are the result of varying one variable, while holding the other constant, and then taking the limit of the corresponding difference quotient. If we vary just one variable and hold the other constant, then the corresponding image under $\boldsymbol{\sigma}$ is a curve on the surface. E.g., if we fix $v = v_0$ and let u vary, then we have that $\boldsymbol{\sigma}(u,v_0)$ is the curve given by mapping a line of constant v and varying u in the uv-plane into $\mathbb{R}^3$. Taking the derivative of this with respect to the mobile parameter u and evaluating at $u = u_0$ is merely extracting a tangent vector to this curve at the point $\boldsymbol{\sigma}(u_0,v_0)$. Since $\partial_u\boldsymbol{\sigma}(u_0,v_0)$ is tangent to this curve of constant v,

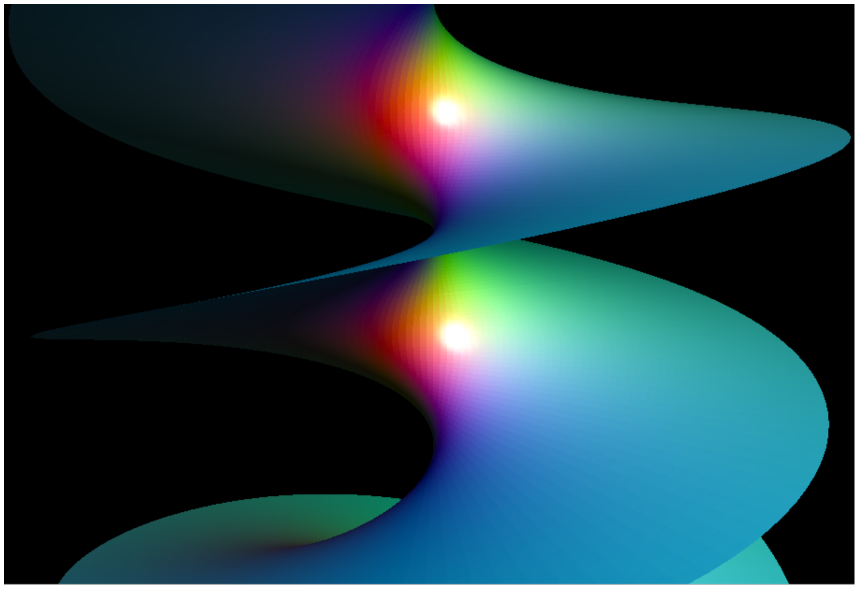

Figure 23. It's Helicoid!

it is necessarily tangent to the surface $\mathcal{S}$ itself. Thus the partials $\partial_u\boldsymbol{\sigma}(u_0, v_0)$ and $\partial_v\boldsymbol{\sigma}(u_0, v_0)$ are tangent vectors. They give rise to vector fields $\partial_u\boldsymbol{\sigma}$ and $\partial_v\boldsymbol{\sigma}$ defined on the surface $\mathcal{S}$, which are called the *coordinate vector fields* for the parameterized surface $\mathcal{S}$. Note that they depend on the coordinates used in the domain $\mathcal{V}$ and the parameterization given for $\mathcal{S}$. The field-lines of these fields are precisely the images of the grid-lines of constant coordinates u and v in the domain $\mathcal{V}$. See for example the toroidal grid in figure 22. One can view the coordinate vector fields as a lift of the (u, v) coordinate frame to the surface $\mathcal{S}$.

§ 2.7. Tangent Vectors, Normal Vectors, and Orientations*

In our study of curves (e.g., in the notes *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*) we've already encountered tangent vectors and natural frames for plane and space curves. We now build on that discussion for space curves in light of the notion of vector fields.

Tangent Vectors and Orientations for Space Curves For a space curve described by a parameterization $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, we know that the derivative $\dot{\boldsymbol{\gamma}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\gamma}(t)$ gives a tangent vector to the curve at the point $\boldsymbol{\gamma}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}}$. Thus, we can view the velocity $\dot{\boldsymbol{\gamma}}$ in two ways: as describing a new curve (since it is a vector valued function), or as specifying tangent vectors which we can attach to the position $\boldsymbol{\gamma}(t)$ to produce a *vector field along the curve* $\boldsymbol{\gamma}$. This is the *velocity field along the curve parameterized by* $\boldsymbol{\gamma}$. If we instead regard a space curve as strictly being the set of points in $\mathbb{R}^3$ given as the image of a vector valued function $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, then we see that there might be many velocity vector fields along the curve, corresponding to different parameterizations. Note that the tangent lines to the curve are independent of the particular tangent vectors we obtain as velocity vectors from possible parameterizations, though the possible directions are set in stone: the velocity vectors can point one of two ways, depending on whether the parameterization traces the curve out in one of two directions. Indeed, changing parameter by setting $\tau = -t$ and replacing I by $-I$, one can *reverse direction* along the curve. This leads to the notion of an *oriented curve*:

Definition. Let $\mathcal{C} \subset \mathbb{R}^n$ be a regular curve (thus, it admits at least one parameterization $\boldsymbol{\gamma} : I \to \mathbb{R}^n$ which is differentiable with no-where $\mathbf{0}$ velocity). By regularity, at each point P of $\mathcal{C}$, there is a well defined tangent line $\mathcal{AT}_P\mathcal{C}$. An *orientation* on $\mathcal{C}$ is a consistent (continuous[13]) choice for each P along C of identification of $\mathcal{AT}_P\mathcal{C}$ with $\mathbb{R}$ sending P to 0 and specifying which half line departing from P corresponds to the half line $[0, \infty) \subset \mathbb{R}$. Equivalently, it is a consistent choice in a small region of C around each point P of which portion of the curve in the region is "ahead" of P and which portion is "behind" P.

Any regular parameterization $\boldsymbol{\gamma}$ of C induces an orientation by choosing the half line in $\mathcal{AT}_P\mathcal{C}$ corresponding to the same direction as $\dot{\boldsymbol{\gamma}}$, and there are exactly two possible orientations. You can imagine an orientation as an arrow, telling a particle which way to travel along the curve.

Example 2.24. The orientations on a line segment correspond to choosing which endpoint is initial and which is terminal. Recall, if $\mathbf{p}$ and $\mathbf{q}$ are two position vectors for points P and Q, the line segment *from P to Q* can be parameterized as

$$\ell_{\overrightarrow{PQ}}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = (1-t)\mathbf{p} + t\mathbf{q}\,, \quad 0 \le t \le 1\,,$$

while the line segment *from Q to P* may be parameterized as

$$\ell_{\overrightarrow{QP}}(t) = \mathbf{q} + t(\mathbf{p} - \mathbf{q}) = (1-t)\mathbf{q} + t\mathbf{p}\,, \quad 0 \le t \le 1\,.$$

Note that changing t to $1-t$ and keeping I, we can pass from one parameterization to the other.

Example 2.25. Let $\boldsymbol{\gamma}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$. This induces a counterclockwise orientation on the unit circle. In terms of the tangent lines, we can express this explicitly. The tangent line at the point $\boldsymbol{\gamma}(t)$ has equation $(\cos t)(x - \cos t) - (\sin t)(y - \sin t) = 0$, which may be parameterized as

$$\ell(s,t) = \boldsymbol{\gamma}(t) + s\dot{\boldsymbol{\gamma}}(t) = \Big(\cos(t) - s\sin(t)\Big)\hat{\imath} + \Big(\sin t + s\cos(t)\Big)\hat{\jmath}\,, \quad 0 \le t \le 2\pi, s \in \mathbb{R}\,.$$

This is a continuous vector-valued function of both s and t, and the identification of $\mathbb{R}$ with the affine tangent lines lines is already given to us: s can be viewed as a coordinate map from the line to $\mathbb{R}$, such that $s = 0$ corresponds to points of the curve $\ell(0,t) = \boldsymbol{\gamma}(t)$ and such that positive s chooses the half line ahead of the motion (in the direction of the velocity vector $\dot{\boldsymbol{\gamma}}$).

The clockwise orientation can be obtained by changing t to $-t$, or considering other parameterizations, such as $\boldsymbol{\eta}(t) = \sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$.

For curves, we can define tangent vector fields along them from a given parameterization. We can also define normal fields, and more generally, we can consider vectors attached along each point of the curve, as suited to particular applications involving such a curve in the plane or in space.

Example 2.26. Recall, in the notes on *Curvature, Natural Frames, and Acceleration for Plane and Space Curves*, we became acquainted with a natural frame for motion along a curve, called the Frenet-Serret frame. In particular, given an oriented curve in space parameterized by a vector valued function $\mathbf{r} : I \to \mathbb{R}^3$, we have a frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ consisting of a *unit tangent vector* (or *normalized velocity vector*), the *unit normal vector* (or *normalized curvature vector*), and the *binormal vector* (which is $\mathbf{T} \times \mathbf{N}$). These satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix},$$

where κ is the curvature and τ is the torsion. This frame furnishes a triple of vector fields defined along the curve. If the curve lies in a surface, then the tangent and normal vectors will lie in affine tangent planes to the surface, while the binormal will be normal to the surface.

More generally, for a regular curve parameterization $\mathbf{r}(t)$, there is a unique normal plane to $\mathbf{r}(t)$ for any given t. We can then choose vectors parallel to each plane by a smooth function $\mathbf{n}(t)$ such that $\mathbf{n}(t) \cdot \dot{\mathbf{r}}(t) = 0$ for every t. This determines a *normal vector field* to the curve. Note that it need not be *the unit normal field* $\mathbf{N}$ determined by the Frenet-Serret frame–we allow $\mathbf{n}$ to twist around the curve, even if there is no curvature and $\mathbf{N}$ is undefined. If we keep the lengths of $\mathbf{n}(t)$ small, then we get a copy of the original curve, called a *push-off*, which need not be strictly parallel, as

[13] The choice amounts to defining a map from the set of all affine tangent lines of $\mathcal{C}$ to $I \times \mathbb{R}$ which is continuous. The notion of continuity here should be as that of the previous section: the pre-image of any open set of $I \times \mathbb{R}$ should be an open set of the space of all affine tangent lines to $\mathscr{C}$. If you are interested in the subject of topology, you should try to convince yourself that the conditions given leave you with only two possibilities for the orientation.

it may wind around the original curve. Such push-offs are quite useful in studying *knotted curves*, and in the study of low-dimensional topology (such as studying 3 and 4 dimensional spaces).

In the plane, a push-off of a simple (i.e., non-self-intersecting) closed curve along a normal direction cannot cross the curve–Jordan's curve theorem says any simple closed curve divides the plane into two regions, and so the normal direction we choose confines us to a push-off that runs either "outside" or "inside" the original curve. In the plane an outward pointing unit normal field to a closed curve allows us to study a notion of *flux*: we can try to measure how much a given vector field $\mathbf{F}$ flows out of a region by measuring the net change in incoming and outgoing flow. To carefully define this, we would need the notion of *line-integrals in vector fields*, which is introduced in §3.2.

Vector Fields and Orientations on Surfaces. We now move on to consider tangent and normal vectors to surfaces. In the previous section, we encountered coordinate vector fields for a parameterized surface, which are tangent to the surface. We can use these to construct more general tangent vector fields by taking linear combinations of coordinate vector fields:

Definition. A tangent vector field $\mathbf{X}$ to a surface $\mathcal{S} \subset \mathbb{R}^3$ is an assignment of a vector in $\mathbb{R}^3$ to each point of $\mathcal{S}$, such that each vector can be realized as a tangent vector to some curve lying on $\mathcal{S}$. If $\mathcal{S}$ is parameterized by $\boldsymbol{\sigma}(\mathbf{v}) = \sigma_1(\mathbf{v})\hat{\imath} + \sigma_2(\mathbf{v})\hat{\imath} + \sigma_3(\mathbf{v})\hat{\mathbf{k}}$, then at any non-singular point $\boldsymbol{\sigma}(\mathbf{v})$,

$$\mathbf{X}(\mathbf{v}) = X_1(\mathbf{v})\boldsymbol{\sigma}_u(\mathbf{v}) + X_2(\mathbf{v})\boldsymbol{\sigma}_v(\mathbf{v})$$

for some functions X_1 and X_2 defined in domain of $\boldsymbol{\sigma}$ within the $\mathbf{v}$-plane.

Example 2.27. On the unit sphere minus its poles, $\mathbb{S}^2 - \{\pm\hat{\mathbf{k}}\}$ parameterized by

$$\boldsymbol{\sigma}(\theta, \phi) = \cos\theta\cos\varphi\,\hat{\imath} + \sin\theta\cos\varphi\,\hat{\jmath} + \sin\varphi\,\hat{\mathbf{k}} = \hat{\mathbf{u}}_\varrho(\theta, \varphi)$$

consider the vector field $\mathbf{X}(\theta, \phi) = \frac{1}{\sqrt{2}}\big(\hat{\mathbf{u}}_\theta(\theta) + \hat{\mathbf{u}}_\varphi(\theta, \varphi)\big)$. Here, the coefficient functions are constant. The integral curves of this field are depicted in figure 24. Note that the vector field cannot be defined at the poles, but one can define a vector field which vanishes at each pole and everywhere else has the same field-lines (but with different velocities along the trajectories) by multiplying $\mathbf{X}$ by $\cos^2\varphi$, and defining a limiting vector field: $\tilde{\mathbf{X}}(\theta, \varphi) = \lim_{(u,v)\to(\theta,\varphi)} \cos^2(v)\mathbf{X}(u, v)$.

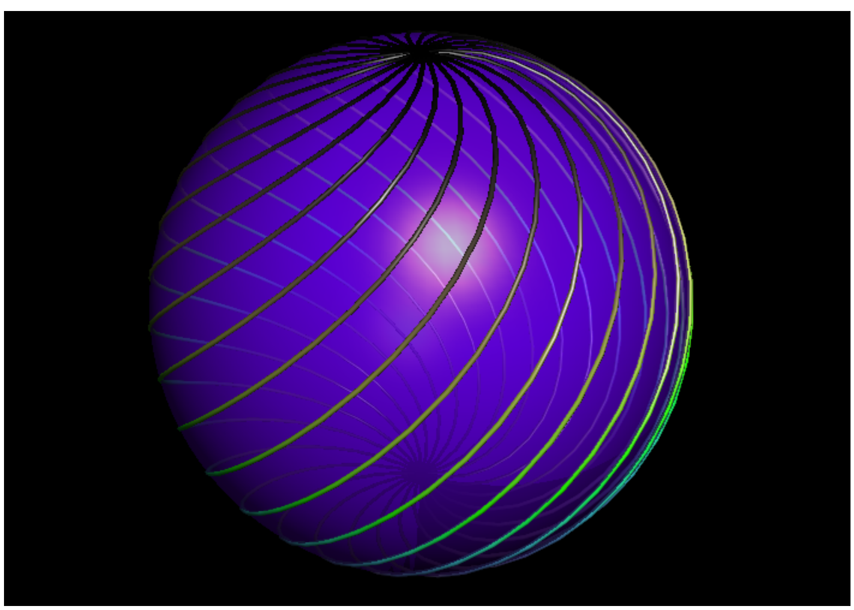

FIGURE 24. The integral curves of the vector field $\mathbf{X}(\theta, \phi) = \frac{1}{\sqrt{2}}\big(\hat{\mathbf{u}}_\theta(\theta) + \hat{\mathbf{u}}_\varphi(\theta, \varphi)\big)$ on $\mathbb{S}^2 - \{\pm\hat{\mathbf{k}}\}$.

We can also use the cross product to construct normal vectors where ever $\boldsymbol{\sigma}_u(\mathbf{v})$ and $\boldsymbol{\sigma}_v(\mathbf{v})$ are linearly independent. We can normalize the cross product to obtain a unit normal vector:

$$\mathbf{N}(\mathbf{v}) = \frac{\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})}{\|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\|}\,.$$

The choice of a differentiable parameterization induces a map from the image $\boldsymbol{\sigma}(U)$ to the unit sphere via the assignment $\boldsymbol{\sigma}(\mathbf{v}) \mapsto \mathbf{N}(\mathbf{v})$. Suppose the surface can be covered by patches such that on the overlaps, the choices of unit normals agree (meaning, they choose normals of the same sign, since there are only two possible choices of unit normal at a regular point.) Then there is a well defined map from the surface itself to the unit sphere:

$$\begin{aligned}\mathbf{N} : \mathcal{S} &\to \mathbb{S}^2 \\ p \mapsto \mathbf{N}(p) &= \frac{\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})}{\|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\|}\,,\end{aligned}$$

where $\boldsymbol{\sigma} : U \to \mathcal{S}$ is any parametric surface patch for $\mathcal{S}$ with p a point in its image. When such a map is well defined, we say the surface is *orientable*. Such a surface is always *two sided*; we can choose a consistent continuous normal vector field which points away from one side of the surface, while the negative of this normal vector field points to the other side.

Definition. A surface is *orientable* if and only if there exists a continuous map $\mathbf{N} : \mathcal{S} \to \mathbb{S}^2$ giving a consistent choice of unit normal vector, i.e., $\mathbf{N}(p)$ is orthogonal to the tangent plane to $\mathcal{S}$ at p, and if one follows any closed path $\boldsymbol{\gamma} : [0, 1] \to \mathcal{S}$ on the surface, $\mathbf{N}\big(\boldsymbol{\gamma}(0)\big) = \mathbf{N}\big(\boldsymbol{\gamma}(1)\big)$. The map $\mathbf{N}$ is then called the *Gauss map* of the surface[14].

Choosing an orientation for a surface amounts to deciding which normal vector field is the "positive one". For a closed surface such as a sphere or a torus which divides $\mathbb{R}^3$ into an interior region and an exterior region, it is conventional to choose orientations consistently. For this class, we use the convention that the outer pointing normal is the positive one in such cases. Note that one can always alter a parameterization (swapping the roles of u and v) to reverse the orientation associated to the parameterization.

Example 2.28. Perhaps the simplest example of tangent vectors we can write down arises in the context of graph surfaces. Let $f(x, y)$ be a bivariate function. Recall that the graph $\mathcal{G}_f$ of $f(x, y)$ is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$. Thus, the graph may be parameterized using x and y as parameters: $\boldsymbol{\sigma}(x, y) = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + f(x, y)\hat{\mathbf{k}}$. Let (x_0, y_0) be some point in the domain of f and let $z_0 = f(x_0, y_0)$ be the height of the graph over the point $(x_0, y_0, 0)$. Then the curves $\boldsymbol{\sigma}(x, y_0)$ and $\boldsymbol{\sigma}(x_0, y)$ are curves through (x_0, y_0, z_0) along constant x and y directions respectively, and the corresponding derivatives give us a basis of tangent vectors for the tangent space to the graph surface at the point (x_0, y_0, z_0):

$$\boldsymbol{\sigma}_x(x_0, y_0) = \hat{\mathbf{\imath}} + f_x(x_0, y_0)\hat{\mathbf{k}}\,, \quad \boldsymbol{\sigma}_x(x_0, y_0) = \hat{\mathbf{\jmath}} + f_y(x_0, y_0)\hat{\mathbf{k}}\,.$$

We thus get coordinate vector fields on the surface (which are fields of tangent vectors) of the forms

$$\boldsymbol{\sigma}_x(x, y) = \hat{\mathbf{\imath}} + f_x(x, y)\hat{\mathbf{k}}\,, \quad \boldsymbol{\sigma}_y(x, y) = \hat{\mathbf{\jmath}} + f_y(x, y)\hat{\mathbf{k}}\,.$$

Let t_1 and t_2 be arbitrary scalars, and fix a point (x_0, y_0) on the surface of the graph. Then observe that $\{t_1\boldsymbol{\sigma}_x(x_0, y_0) + t_2\boldsymbol{\sigma}_y(x_0, y_0) : t_1, t_2 \in \mathbb{R}\}$ determines the set of all tangent vectors to the point (x_0, y_0) on the graph. Replacing t_1 and t_2 by scalar functions of (x, y) defined on the same domain as the function f, and varying (x, y) over the domain of f allows one to construct arbitrary vector fields on the surface $z = f(x, y)$. Note that the cross product of the graph's coordinate vector fields yields the normal vector field $\boldsymbol{\sigma}_x \times \boldsymbol{\sigma}_y = -f_x\hat{\mathbf{\imath}} - f_y\hat{\mathbf{\jmath}} + \hat{\mathbf{k}}$, which should look familiar if you recall

[14] The Gauss map is important in the study of the differential geometry of surfaces, and is named in honor of (who else) Johann Karl Friedrich Gauß, who initiated the study of such maps in 1825, later publishing a discussion of this map and its applications, e.g. in computing curvature, in his 1827 paper *Disquisitiones generales circa superficies curvas*.

our earlier discussions of the tangent plane to a graph. This vector is an "upward normal" to the graph (note that the $\hat{\mathbf{k}}$-component is $+1$). The Gauss map is then

$$\boldsymbol{\sigma}(x,y) \mapsto \mathbf{N}(x,y) = \frac{-f_x(x,y)\hat{\mathbf{\imath}} - f_y(x,y)\hat{\mathbf{\jmath}} + \hat{\mathbf{k}}}{\sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}}\,.$$

Not so coincidentally, the denominator of this expression appears when one wants to compute *surface areas*; see (24) in the problems below.

Fascinatingly, there also exist non-orientable surfaces, which when placed in $\mathbb{R}^3$ are *one-sided*, as demonstrated by the following well known example:

Example 2.29. Consider the surface $\mathcal{M}$ given by the parameterization

$$\boldsymbol{\sigma}(u,v) = \big(2 + v\cos(u/2)\big)\hat{\mathbf{u}}_r(u) + v\sin(u/2)\,\hat{\mathbf{k}}\,,\ \ (u,v) \in [0,2\pi] \times [-1,1]\,.$$

This surface is shown in figure 25, and is none other than the famous *Möbius band*. It is a "one-sided" band. One can make a model of a Möbius band by adding one half-twist to a rectangular strip of paper, and gluing the two ends[15].

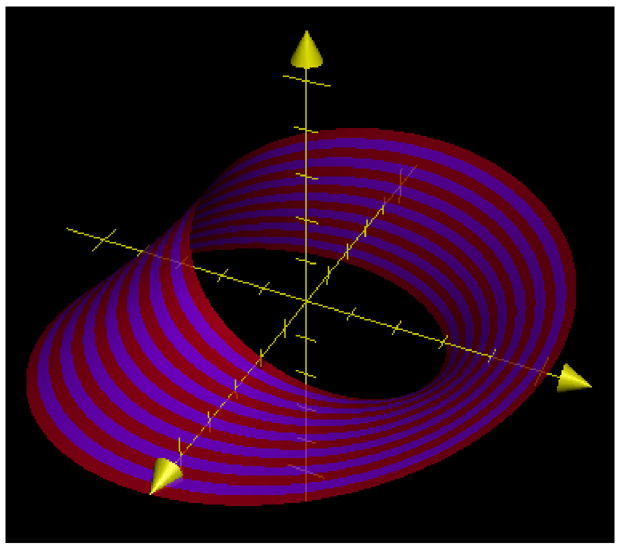

FIGURE 25. A Möbius band. Note that following the boundary stripe takes you along the whole boundary: it may seem as though you go from the "inner edge" to the "outer edge" and back, but there is actually just one edge! Imagine now what happens as you carry a normal vector around the core circle–is there any non-contractible loop around which you can give a consistent Gauss map?

Example 2.30. Another famous example of a non-orientable surface is the *projective plane*. The projective plane is what one gets if one takes a Möbius band and tries to "cap it off" with a disk (observe that the boundary of the band is a single loop; one can stitch a disk to this loop along the disk's boundary). In three dimensions, one can only do this by contorting the disk or the band, and allowing the surface to self-intersect. Provided the self intersections happen in a regular fashion, with a clear choice of independent tangent planes on each branch of surface along a curve of intersection, the surface is said to be *immersed* in $\mathbb{R}^3$. A famous immersion of the projective plane, which is called the Boy's surface, is depicted in figure 26.

[15]The paper model of the Möbius band is geometrically different from the parameterization given; the paper model though curved in space has "intrinsically flat geometry" meaning that the product of the minimum and maximum curvatures of slices of the band by normal planes is zero–this is a consequence of the fact that the original paper strip is flat, and it doesn't need to be stretched considerably to form the Möbius strip. On the other hand, the parameterization given above has *nonzero Gaussian curvature*: one can show that the product of the maximum and minimum curvatures of curves sliced by normal planes through any point on is nonzero.

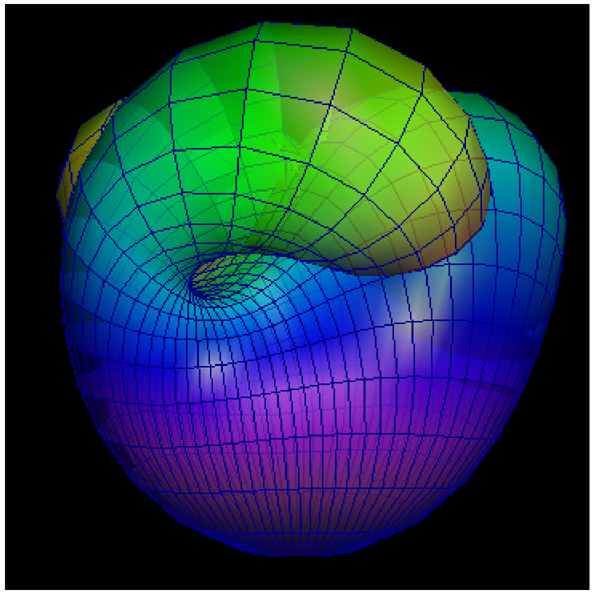

Figure 26. One view of a Boy's surface.

In figure 27 several "slices" of the surface are shown side by side, indicating the structure of the surface, and in particular showing how it can be viewed as a Möbius band with three half-twists capped by a warped disk which triply intersects itself. From the left, it starts with a small disk, bent into a cup shape. Subsequent slices show stages of growth off of the original disk, as bands embedded or immersed in $\mathbb{R}^3$. In the next slice, such a band may be viewed as a *collar* of the disk (meaning a neighborhood of the boundary), which begins to bend in a 3-fold symmetric way. This bending is so as to introduce full twists in the band. Growing the disk further, the next slice shows the band self-intersect to form a triple point. After this, a collar of the growing disk is a knotted trefoil ribbon, with 3 full twists, which were introduced when the ribbon passed through itself. Finally, the band is pulled into a form where one boundary meets itself, leaving a Möbius band with 3 half-twists. This closes the surface up, since the other boundary of this Möbius band meets the preceding slice, which is just a collar of the (now self-intersecting) disk's boundary.

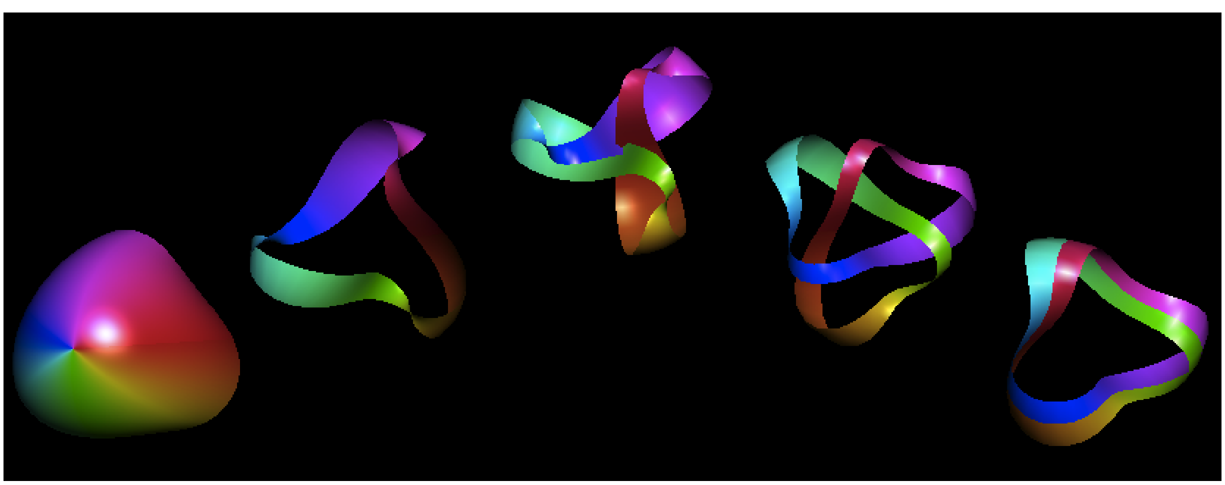

Figure 27. A dissection of the Boy's surface, with slices being like the frames of a movie.

§ 2.8. Problems

(1) For each of the following vector fields, provide a sketch. Then determine equations for the field-lines, and sketch a sufficient family of oriented field-lines to capture the behavior of the corresponding flow.

(a) $\mathbf{F}(x, y) = y\hat{\imath} + x\hat{\jmath}\,,$

(b) $\mathbf{F}(x, y) = x\hat{\imath} + 2y\hat{\jmath}\,,$

(c) $\mathbf{F}(x, y) = \hat{\imath} - x\hat{\jmath}\,,$

(d) $\mathbf{F}(x, y) = x\hat{\imath} - \hat{\jmath}\,,$

(e) $\mathbf{F}(x, y) = (x - y)(\hat{\imath} + \hat{\jmath})\,.$

(2) This is a small project which explores vector fields that arise from linear maps $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by matrix multiplication

$$\mathbf{F}(\mathbf{r}) = \mathrm{M}\mathbf{r} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = (ax + by)\hat{\imath} + (cx + dy)\hat{\jmath}\,,$$

which were introduced in example 2.2.

This collection of problems is highly recommended for those studying or planning to study linear algebra or differential equations.

(a) (You may skip this part if you are already well acquainted with eigentheory.)
The preliminary idea we need is that of *eigenvalues and eigenvectors*, which play a role for linear transformations analogous to fixed points in the study of functions, except they are perhaps as important (if not more so) to the study of linear algebra as zeros are to the theory of polynomial functions. A nonzero vector $\mathbf{v}$ is called an *eigenvector for* M if $\mathrm{M}\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ; the scalar λ is called *the eigenvalue of* M *associated to* $\mathbf{v}$. Assume M is a real 2×2 matrix.
Show the following:

(i) If $\mathbf{v}$ is an eigenvector with eigenvalue λ, then so is $c\mathbf{v}$ for any scalar c. Thus, the eigenvalue λ is associated with a whole *subspace*, which is at least a line (and possibly the whole plane–if it is a *repeated eigenvalue*, it is possible that λ is associated to two linearly independent vectors). Such a subspace is called an *eigenspace*.

(ii) A scalar λ is an eigenvalue of M if and only if $\det(\mathrm{M} - \lambda\mathrm{I}) = 0$, where I is the 2×2 identity matrix.

(iii) The equation $\det(\mathrm{M} - \lambda\mathrm{I}) = 0$ is called the *characteristic equation* of the matrix M. Check that $\det(\mathrm{M} - \lambda\mathrm{I}) = \lambda^2 - \tau\lambda + \Delta$ where $\tau = \mathrm{trace}(\mathrm{M}) = a + d$ and $\Delta = \det \mathrm{M} = ad - bc$.

(iv) If $\tau^2 > 4\Delta$ then there are two distinct eigenvalues, and thus two distinct eigen-lines. If $\tau^2 = 4\Delta$, there's only one eigenvalue. If $\tau^2 < 4\Delta$ then the matrix has two distinct complex eigenvalues, which are conjugates of each other.

From the above, describe an algorithm to compute the eigenvalues and determine eigenvectors of a real 2×2 matrix, assuming the eigenvalues are real.

(b) A 2D linear vector field has *saddle topology* at the origin if there exists a stable eigenline and an unstable eigenline through the origin, i.e., there is one positive real eigenvalue and one negative real eigenvalue. Determine conditions on Δ and τ such that $\mathbf{r} \mapsto \mathrm{M}\mathbf{r}$ has saddle topology at the origin, and then check that $\mathbf{F}(x, y) = (2x + 4y)\hat{\imath} + (3x - y)\hat{\jmath}$ has saddle topology at the origin. Provide a sketch of the vector field $\mathbf{F}$, indicating the eigenlines with orientations, and showing additional oriented field-lines.

(c) There are several types of *nodal topologies* for a 2D linear vector field $\mathbf{F}$:
- $\mathbf{F}$ has *stable node topology* at the origin if there exist two stable eigenlines with distinct eigenvalues associated to each of them.
- $\mathbf{F}$ has *unstable node topology* at the origin if there exist two unstable eigenlines with distinct eigenvalues associated to each of them.
- $\mathbf{F}$ has *stable degenerate node topology* if there is just one eigenline, which is stable.
- $\mathbf{F}$ has *unstable degenerate node topology* if there is just one eigenline, which is unstable.

For each nodal topology, determine conditions on M such that $\mathbf{r} \mapsto \mathrm{M}\mathbf{r}$ possesses that topology at the origin. Phrase conditions in terms of Δ and τ when possible, and in terms of eigenvalues and eigenvectors otherwise. Construct example vector fields for each, and provide sketches including field-lines.

(d) Stars occur as topology of 2D linear vector fields when every line through the origin is an eigenline. A stable star is also called a *star sink* and an unstable star is also called a *star source*. What types of matrices give stable/unstable stars?

(e) Suppose M has *complex eigenvalues*. Then the 2D linear vector field given by the linear map $\mathbf{r} \mapsto \mathrm{M}\mathbf{r}$ has one of three topologies at the origin: a *center*, a *stable spiral* (also called a *spiral sink*), or an *unstable spiral* (also called a *spiral source*). These are distinguished as follows: centers possess only closed field-lines ("trajectories are orbits"), while stable spirals possess field-lines that spiral towards the origin, and unstable spirals possess field-lines that spiral away from the origin. Determine the conditions on Δ and τ which distinguish these topologies. Then construct example fields for each topology, and provide sketches including field-lines.

(f) Determine any edge cases by considering the possibilities for Δ and τ not covered by the preceding parts. In particular, you should assess what possible vector field topologies occur for linear vector fields determined by *singular matrices* M (which have nontrivial null spaces). Construct example vector fields for each, and provide sketches including field-lines.

(3) By computing the linearizations around the zeros for the vector field $\mathbf{F}(x, y) = \sin(y)\,\hat{\mathbf{\imath}} + \cos(x)\,\hat{\mathbf{\jmath}}$, determine the general pattern in the topologies of the zeros, and use this information to sketch the vector field and its field lines without the aid of a computer.

(4) Consider the vector field $\mathbf{F}(x, y) = (y^2 - x)\hat{\mathbf{\imath}} + (y - yx^2)\hat{\mathbf{\jmath}}$.

(a) Find all the zeros of $\mathbf{F}$.

(b) For the zeros found in (a), determine the local topology (saddles, spiral sources, spiral sinks, centers, nodes, dipoles, etc).

(c) Sketch the vector field $\mathbf{F}(x, y)$ in the square $R = [-2, 2] \times [-2, 2]$, and separately sketch a sufficient family of oriented field-lines to capture the behavior of a corresponding flow.

(5) Consider the vector field $\mathbf{F}(x, y) = \sin\left(\frac{\pi}{4}(y - x)\right)\hat{\mathbf{\imath}} + \cos\left(\frac{\pi}{4}(x + y)\right)\hat{\mathbf{\jmath}}$.

(a) Find all zeros of $\mathbf{F}$ in the rectangle $R = [0, 4] \times [0, 4]$.

(b) For the zeros found in (a), determine the local topology (saddles, spiral sources, spiral sinks, centers, dipoles, etc). Determine if the zeros are sources, sinks, or neither.

(c) Sketch the vector field $\mathbf{F}(x, y)$ in the rectangle R, and separately sketch a sufficient family of oriented field-lines to capture the behavior of a corresponding flow.

(6) For each of the vector fields in problem (1) above, determine if the vector field is conservative. For each conservative vector field, find a potential function.

(7) For an n-dimensional vector field $\mathbf{F} : \mathcal{V} \to \mathbb{R}^n$, how many equations of the type $\partial_{x_i} F_j = \partial_{x_j} F_i$ arising from Clairaut's theorem does one have to check to test if $\mathbf{F}$ is not conservative? First examine the pattern for dimensions 2, 3, 4, and 5, and then conjecture a pattern and a formula in terms of n. Use this to guess how many partial derivatives one might need to compute and how many equations might need to checked to apply the criterion to show that a 17-dimensional vector field is not conservative.

(8) For the gravitational force field $\mathbf{F}_g$ induced by a central large object of mass M acting on a test mass object with mass m, show explicitly that the field is conservative by carefully computing all necessary partial integrals, and reconciling the undetermined functions.

(9) Coulomb's law states that the electrical force exerted by a charged particle of charge q_0 positioned at $\mathbf{r}_0$ on a charged particle of charge q_1 positioned at $\mathbf{r}_1$ is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_0 q_1}{\|\mathbf{r}_1 - \mathbf{r}_0\|^3}(\mathbf{r}_1 - \mathbf{r}_0)\,,$$

where $\epsilon_0 = 8.854 \times 10^{-12}\text{coulombs}^2$ per newton $-$ meters^2 is the *free permittivity of space*, charge is measured in coulombs, and distance is measured in meters. An electrostatic field $\mathbf{E}(\mathbf{r})$ associated to a collection of fixed charges $q_1, \ldots, q_n$ placed at positions $\mathbf{r}_1, \ldots, \mathbf{r}_n$ is the net force exerted on a unit test charge placed at position $\mathbf{r}$.

(a) Write down an expression for the electric field from Coulomb's law, assuming of course the superposition principle for summing forces.

(b) Argue that an electrostatic field is always conservative, and find an appropriate potential.

(c) Compute the divergence of the electrostatic field produced by fixing a unit positive charge at $\hat{\mathbf{\imath}}$ and a unit negative charge at $-\hat{\mathbf{\imath}}$. Explain why your result is physically sensible.

(10) In this problem we study both the topological dipole introduced above, and electric dipole fields.

(a) Compute the field-lines of the topological dipole field $\mathbf{F}(x, y) = (x^2 - y^2)\,\hat{\mathbf{\imath}} + 2xy\,\hat{\mathbf{\jmath}}$, and sketch them.

(b) Rewrite the topological dipole field $\mathbf{F}$ using the polar frame.

(c) Show that the topological dipole field $\mathbf{F}$ is not conservative.

(d) Compute the Jacobian of $\mathbf{F}$, and use it to compute the linearization of the topological dipole field at the origin. Why doesn't the linearization capture the topology adequately?

(e) Find a continuous family $\mathbf{F}_t$, $0 \le t \le 1$, of fields whose limit $\mathbf{F}_1$ is the topological dipole, such that the local topologies near the zeros of the initial field are captured by the linearizations. Sketch a "movie" of the transformation from $\mathbf{F}_0$ to $\mathbf{F}_1$. Consider how the limiting topology arises from the initial topologies, and explain the name "topological dipole".

(f) A real physical dipole field should diminish in strength at large distances, and should model a physical field. Using Coulomb's law and the results of problem 9, construct an electrostatic field equation for a pair of charged particles, one of positive charge q and the other of negative charge $-q$, with the positive charge displaced from the negative charge by a displacement vector $\mathbf{d}$. Let $\mathbf{p} = q\mathbf{d}$, which is called the *dipole moment* of the charge pair. Consider the limiting electric field as $\mathbf{d} \to \mathbf{0}$ while q increases so that the dipole moment $\mathbf{p}$ remains constant. Show that this limiting dipole "electret" field is

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\hat{\mathbf{u}}_r)\hat{\mathbf{u}}_r - \mathbf{p}}{4\pi\epsilon_0 r^3}\,.$$

(g) Find a potential for $\mathbf{E}(\mathbf{r})$.

(11) Determine which of the following vector fields are conservative, and for each conservative vector field, find a general potential function. For additional credit, determine which vector fields are *solenoidal*, i.e., which vector fields have divergence equal to 0. Then find *vector potentials* for the solenoidal fields, i.e., find $\mathbf{A}$ such that the field is given as $\nabla \times \mathbf{A}$.

(a) $\mathbf{F}(x,y,z) = e^{-y^2-z^2}\,\hat{\mathbf{\jmath}} + e^{-x^2-y^2}\,\hat{\mathbf{k}}$,

(b) $\mathbf{G}(x,y,z) = (ye^{-z} - ze^{y} - yze^{-x})\,\hat{\mathbf{\imath}} + (xe^{-z} + ze^{-x} - xze^{y})\,\hat{\mathbf{\jmath}} + (ye^{-x} - xe^{y} - xye^{-z})\,\hat{\mathbf{k}}$,

(c) $\mathbf{H}(x,y,z) = (e^{-x} + xe^{-z})\,\hat{\mathbf{\imath}} + (e^{-y} + ye^{-x})\,\hat{\mathbf{\jmath}} + (e^{-z} + ze^{-y})\,\hat{\mathbf{k}}$,

(d) $\mathbf{K}(x,y,z) = (3x^2 - yz^2)\,\hat{\mathbf{\imath}} + (3y^2 - xz^2)\,\hat{\mathbf{\jmath}} - xy\,\hat{\mathbf{k}}$.

(12) Verify the proposition:

Proposition. *For any 3-dimensional vector field $\mathbf{F}$ whose components are continuously differentiable to second order on a domain $\mathcal{D}$,*

$$\operatorname{div}\operatorname{curl}(\mathbf{F}(\mathbf{r})) = \nabla\cdot\big(\nabla\times\mathbf{F}(\mathbf{r})\big) = 0\,,$$

and for any scalar function $f(x,y,z)$ continuously differentiable to second order on $\mathcal{D}$,

$$\operatorname{curl}\operatorname{grad}(f(\mathbf{r})) = \nabla\times\nabla\mathbf{f}(\mathbf{r}) = \mathbf{0}\,.$$

(13) This problem works with the spherical frame.

(a) Work out the details of calculating $\frac{\partial\hat{\mathbf{u}}_\varrho}{\partial\theta}$, $\frac{\partial\hat{\mathbf{u}}_\varrho}{\partial\varphi}$, $\frac{\partial\hat{\mathbf{u}}_\theta}{\partial\theta}$, $\frac{\partial\hat{\mathbf{u}}_\theta}{\partial\varphi}$, $\frac{\partial\hat{\mathbf{u}}_\varphi}{\partial\theta}$ and $\frac{\partial\hat{\mathbf{u}}_\varphi}{\partial\varphi}$ and expressing the results in the spherical frame.

(b) Use the preceding calculations to compute directional derivatives $D_{\hat{\mathbf{u}}_\varrho}\hat{\mathbf{u}}_\varrho$, $D_{\hat{\mathbf{u}}_\varrho}\hat{\mathbf{u}}_\theta$, $D_{\hat{\mathbf{u}}_\varrho}\hat{\mathbf{u}}_\varphi$, $D_{\hat{\mathbf{u}}_\theta}\hat{\mathbf{u}}_\varrho$, $D_{\hat{\mathbf{u}}_\theta}\hat{\mathbf{u}}_\theta$, $D_{\hat{\mathbf{u}}_\varrho}\hat{\mathbf{u}}_\varphi$, $D_{\hat{\mathbf{u}}_\varphi}\hat{\mathbf{u}}_\varrho$, $D_{\hat{\mathbf{u}}_\varphi}\hat{\mathbf{u}}_\theta$, and $D_{\hat{\mathbf{u}}_\varphi}\hat{\mathbf{u}}_\varphi$, given that

$$D_{\hat{\mathbf{u}}}\mathbf{F}(\mathbf{r}) := \lim_{h\to 0}\frac{\mathbf{F}(\mathbf{r}+h\hat{\mathbf{u}}) - \mathbf{F}(\mathbf{r})}{h} = (\hat{\mathbf{u}}\cdot\nabla)\mathbf{F}(\mathbf{r})\,,$$

where one applies the del operator to both components and basis vectors prior to computing dot products with $\hat{\mathbf{u}}$.

(c) Recall, for $(\hat{\mathbf{\imath}},\hat{\mathbf{\jmath}},\hat{\mathbf{k}})$ the standard global frame for rectangular coordinates (x,y,z), we have that $D_{\hat{\mathbf{\imath}}}\mathbf{F} = \frac{\partial\mathbf{F}}{\partial x}$, $D_{\hat{\mathbf{\jmath}}}\mathbf{F} = \frac{\partial\mathbf{F}}{\partial y}$, and $D_{\hat{\mathbf{k}}}\mathbf{F} = \frac{\partial\mathbf{F}}{\partial z}$. By analogy, compare the resulting directional derivatives of the spherical frame with respect to spherical frame elements to the spherical frame partial derivatives with respect to spherical coordinate variables. Is it true, e.g., that $\frac{\partial\mathbf{F}}{\partial\theta} = D_{\hat{\mathbf{u}}_\theta}\mathbf{F}$ for a differentiable vector field $\mathbf{F}$?

(14) (a) Show that the divergence of a vector field $\mathbf{F}(r,\theta,z) = F^1\hat{\mathbf{u}}_r + F^2\hat{\mathbf{u}}_\theta + F^3\hat{\mathbf{k}}$ given in cylindrical coordinates may be expressed as

$$\begin{aligned}\nabla\cdot\mathbf{F}(r,\theta,z) &= \frac{1}{r}\frac{\partial}{\partial r}\big(rF^1\big) + \frac{1}{r}\frac{\partial F^2}{\partial\theta} + \frac{\partial F^3}{\partial z}\\ &= \frac{1}{r}F^1 + \frac{\partial F^1}{\partial r} + \frac{1}{r}\frac{\partial F^2}{\partial\theta} + \frac{\partial F^3}{\partial z}\,.\end{aligned}$$

(b) Verify the expression given in proposition 2.2 for the divergence of a vector field $\mathbf{F}(\varrho,\theta,\varphi)$ given in spherical coordinates:

$$\begin{aligned}\nabla\cdot\mathbf{F}(\varrho,\theta,\varphi) &= \frac{1}{\varrho^2}\frac{\partial}{\partial\varrho}\big(\varrho^2F^1\big) + \frac{1}{\varrho\cos(\varphi)}\frac{\partial F^2}{\partial\theta} + \frac{1}{\varrho\cos(\varphi)}\frac{\partial}{\partial\varphi}\big(\cos(\varphi)F^3\big)\\ &= \frac{2}{\varrho}F^1 + \frac{\partial F^1}{\partial\varrho} + \frac{1}{\varrho\cos\varphi}\frac{\partial F^2}{\partial\theta} - \frac{\tan(\varphi)}{\varrho}F^3 + \frac{1}{\varrho}\frac{\partial F^3}{\partial\varphi}\,,\end{aligned}$$

by working out the details of the calculation of $\nabla\cdot\mathbf{F}(\varrho,\theta,\varphi)$.

(c) Use either expression above to compute the divergence of the following vector fields:

(i) $\mathbf{F}(r,\theta,z) = \dfrac{r\hat{\mathbf{u}}_r + z\hat{\mathbf{k}}}{\sqrt{r^2+z^2}}\,,$

(ii) $\mathbf{G}(\varrho,\theta,\varphi) = \dfrac{\sin\theta\hat{\mathbf{u}}_\varrho - \cos\varphi\hat{\mathbf{u}}_\theta + \hat{\mathbf{u}}_\varphi}{\varrho}\,,$

(iii) $\mathbf{K}(x,y,z) = \dfrac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\,.$

(15) Derive expressions for the curl in cylindrical and spherical coordinates, and use them to compute the curl of the following vector fields:

(a) $\mathbf{F}(r,\theta,z) = z\hat{\mathbf{u}}_r + r\hat{\mathbf{u}}_\theta - \theta\hat{\mathbf{k}}$

(b) $\mathbf{G}(\varrho,\theta,\varphi) = \frac{\varrho}{\sqrt{2}}(\hat{\mathbf{u}}_\theta + \hat{\mathbf{u}}_\varphi)$.

(16) Compute the Laplacians of the following functions:

(a) $u(x,y,z) = x\cos(yz) - y\sin(xz) + z\tan(xy)\,,$

(b) $v(\varrho,\theta,\varphi) = \varrho^3\sin^3\varphi - 2\varrho^2\cos\theta\sin\theta\cos\varphi\,,$

(c) $f(r,\theta,z) = \dfrac{r^2\cos 2\theta}{r^2+z^2}\,.$

(17) Let $\mathbf{F}$ and $\mathbf{G}$ be sufficiently differentiable 3-dimensional vector fields.

(a) Verify the following identities:

(i) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})\,,$

(ii) $\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot G) - (\nabla\cdot\mathbf{F})\mathbf{G}\,,$

(iii) $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$,

where $\nabla^2\mathbf{F}$ is the *vector Laplacian*, whose rectangular components are the Laplacians of the rectangular components of $\mathbf{F}$.

(b) Let $\mathbf{F}$, and $\mathbf{G}$ be sufficiently differentiable 3-dimensional vector fields, and let $f(x,y,z)$ be a scalar field. Find coordinate free expressions using divergence, curl, and gradient to compute product rules for each of the following derivatives:

(i) $\nabla \cdot (f\mathbf{F})$, (iii) $\nabla(\mathbf{F}\cdot\mathbf{G})$,
(ii) $\nabla \times (f\mathbf{F})$, (iv) $\nabla^2(\mathbf{F}\times\mathbf{G})$.

(18) This exercise will help you construct a proof of the proposition in § 2.6.

(a) Write down a limit definition of continuity at a point for a vector-valued function $\boldsymbol{\sigma} : \mathcal{V} \to \mathbb{R}^3$. Note that you need to express the limit condition in terms of an ε-δ formalism and the respective notions of distance for $\mathbb{R}^2$ and $\mathbb{R}^3$. (Compare with the limit definitions of continuity for a vector-valued function from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^3$, and for a two variable function $f(x, y)$ from a domain $\mathcal{V} \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$).

(b) Show that if a function $\boldsymbol{\sigma} : \mathcal{V} \to \mathbb{R}^3$ meets the condition that the pre-image of any open set $\mathcal{W} \subseteq \mathbb{R}^3$ is an open set of $\mathcal{V}$, then the function is continuous at every point.

Hints: First, argue that it suffices to think about open balls and open disks. Consider what happens to pre-images as you take a ball of smaller and smaller radius ε around a point P of the image.

(c) Show that a function which is continuous in the limit sense at every point is continuous in the sense of the proposition, namely that it meets the condition on pre-images of open sets. Again, you should try to make use of open balls and open disks.

(19) Write down the six hemispherical patches necessary to cover a sphere of radius R, such that pairs of opposite hemispheres correspond to pairs of graphs over disks in the xy, xz or yz planes.

(20) This problem concerns *stereographic projection*, which gives a map of the sphere minus a pole to the plane $\mathbb{R}^2$ in such a way as to faithfully preserve the angles between tangent vectors to curves. Thus, stereographic projection is an example of what is called a *conformal map*. The intuitive idea of stereographic projection is to trace a ray of light emanating from the north pole. Such a ray passes through a unique point on the sphere away from the north pole. Then one can associate to a point on the sphere the point where the associated light ray strikes the xy plane. The details are developed below.

For simplicity, we will work with the unit sphere centered at the origin in $\mathbb{R}^3$. To avoid some conflicts of notation, use capital letters (X, Y, Z) for the coordinates in $\mathbb{R}^3$, and lowercase letters (x, y) for coordinates on $\mathbb{R}^2$. Thus, the sphere for which we are building a coordinate chart is given algebraically by the equation $X^2 + Y^2 + Z^2 = 1$.

(a) Let $N(0, 0, 1)$ denote the "north pole" of the sphere on the Z-axis. Identify $\mathbb{R}^2$ with the plane $Z = 0$, so that a point $(X, Y, 0) = (x, y, 0)$ corresponds uniquely to the point (x, y). Parameterize the line between $N(0, 0, 1)$ and $(x, y, 0)$, and determine the point (X, Y, Z) where it strikes the sphere, in terms of x and y. Thus, give a parameterization of $\mathbb{S}^2 - N$ with domain (x, y). Argue that this is a homeomorphism.

(b) Find a formula for a chart which is the inverse of the homeomorphism constructed in the previous part, i.e., determine a formula for an ordered pair (x, y) given in terms of the coordinates (X, Y, Z) of a point on the sphere. Verify that this is a homeomorphism. This chart is the stereographic projection map.

(c) Thus, deduce that there is a one-to-one correspondence between points of the plane and points of the sphere with a point removed, given by stereographic projection and its inverse. What point on the sphere corresponds to $(0, 0)$? What is the image on the sphere of a line in $\mathbb{R}^2$?

(d) Re-express the stereographic projection map using polar coordinates on $\mathbb{R}^2$ and spherical coordinates defined by
$$\varrho^2 = X^2+Y^2+Z^2,\ (X,Y,Z) = (\varrho\sin\varphi\cos\theta, \varrho\sin\varphi\sin\theta, \varrho\cos\varphi)\,.$$

(e) Reprove the formulae from parts (a), (b), and (c) using diagrams of the sphere and a ray in profile, and appealing to similar triangles and Euclidean geometry, rather than linear and vector algebra. You may also find the following results of Euclidean geometry useful:

Theorem 2.1 (Thale's theorem)**.** *A triangle inscribed in a circle is a right triangle if and only if its hypotenuse is a diameter.*

Theorem 2.2 (Inscribed Angle Theorem)**.** *Let $\triangle ABC$ be a triangle with vertices A, B, and C, which is inscribed in a circle with center O. If $\alpha = \angle BAC$ is the inscribed angle of the triangle at vertex A, then $2\alpha = \angle BOC$ is the central angle subtending the corresponding arc of the circle.*

Note that Thale's Theorem is a special case of the inscribed angle theorem.

(f) Compute coordinate vector fields for the sphere coming from the stereographic coordinates in parts (b) and (c), and draw pictures of these coordinate vector fields on the sphere.

(21) Consider the parameterization of a torus given by
$$\boldsymbol{\sigma}(u,v) = \left(2+\cos\left(\tfrac{3}{2}u\right)\right)\hat{\mathbf{u}}_r(u+v) + \sin\left(\tfrac{3}{2}u\right)\,\hat{\mathbf{k}}\,,\ (u,v)\in[0,4\pi]\times[0,2\pi/3]\,.$$
Note that the argument of $\hat{\mathbf{u}}_r$ is $u+v$, hence $\theta = u+v$.

(a) Express the parameterization in rectangular coordinates.

(b) Compute the coordinate vector fields $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, and express them in both cylindrical and rectangular frames. Sketch the torus and the coordinate vector fields.

(c) Describe the curves of constant u and the curves of constant v. How do they relate to the usual meridional and longitudinal curves determined by the parameterization given in section 2.6 (for appropriate a and b)? Sketch the curve $\boldsymbol{\sigma}(u,0)$.

(d) Generalizing the above, for fixed coprime integers p, and q, study the (u,v) coordinate system for the parameterization
$$\boldsymbol{\sigma}(u,v) = \left(2+\cos\left(\tfrac{q}{p}u\right)\right)\hat{\mathbf{u}}_r(u+v) + \sin\left(\tfrac{q}{p}u\right)\,\hat{\mathbf{k}}\,,\ (u,v)\in[0,2\pi p]\times[0,2\pi/q]\,.$$
In particular, explain the dynamics of the coordinate vector fields, and describe the coordinate curves for constant u and for constant v. What are the images of these coordinate curves under a chart from the torus to $[0,2\pi]^2$ which inverts the parameterization given in section 2.6?

(22) Compute the Gauss map of the torus corresponding to the outward pointing normal for the parameterization given in section 2.6. Does the map cover $\mathbb{S}^2$?

(23) Use the parameterization $\boldsymbol{\sigma}(u,v)$ of the Möbius band given in section 2.7 to compute a normal vector $\boldsymbol{\sigma}_u(u,v)\times\boldsymbol{\sigma}_v(u,v)$. Show explicitly that this parameterization cannot be used to define a Gauss map for the band by considering the limiting values of your normal vector field around the core loop. What happens if you traverse such a loop twice, allowing u to range from 0 to 4π? Argue that there is no way to create a set of alternate patches that have well defined Gauss map, and thus deduce the non-orientability of the Möbius band.

(24) Give a geometric argument that the area of a smooth surface patch $\boldsymbol{\sigma} : \mathcal{V} \to \mathbb{R}^3$ is given by

$$\mathcal{A}\big(\boldsymbol{\sigma}(\mathcal{V})\big) = \iint_{\mathcal{V}} \|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\| \, \mathrm{d}\mathcal{A}(\mathbf{v}) \,,$$

where $\mathrm{d}\mathcal{A}(\mathbf{v})$ becomes either $\mathrm{d}u\,\mathrm{d}v$ or $\mathrm{d}v\,\mathrm{d}u$ when passing to an iterated integral. Use this formula to compute surface areas for the sphere, and the torus using the parameterizations given above. Set up the surface area calculation for the Möbius band, then use a computer and a preferred choice of numerical method to obtain an approximate result.

3. Line Integrals

§ 3.1. Defining Scalar Line Integrals

We'd like to define a way of accumulating change along paths, e.g., computing the work a force field does on a particle moving along a curve, or the mass of a wire given a density function defined along its length. To do this, we need to define a new integral object. First we define such an integral object for scalar fields in a way which generalizes the comfortable notion of an integral of a function along an interval. Recall, the definite integral in single variable calculus is an object $\int_a^b f(x)\,\mathrm{d}x$ that associates a number to a (well-behaved) single variable function $f : \mathcal{D} \to \mathbb{R}$ given an interval $[a, b]$ in (the closure[16] of) its domain, and this number is interpreted as the "net" area bounded by the graph of $y = f(x)$ over $[a, b]$ and the x-axis. Similarly, a *line integral* (unfortunately named) will associate to a multivariable function $f : \mathcal{D} \to \mathbb{R}^n$ and a *curve* $\boldsymbol{\gamma} : I \to \mathcal{D}$ a number, which in the case of a two variable function $f(x, y)$ can be visually interpreted as the net area between the curve $\boldsymbol{\gamma}$ in $\mathcal{D} \subseteq \{(x, y, 0) \in \mathbb{R}^3\}$ and the graph $\mathcal{G}_f \subset \mathbb{R}^3$ of $z = f(x, y)$. See figure (28).

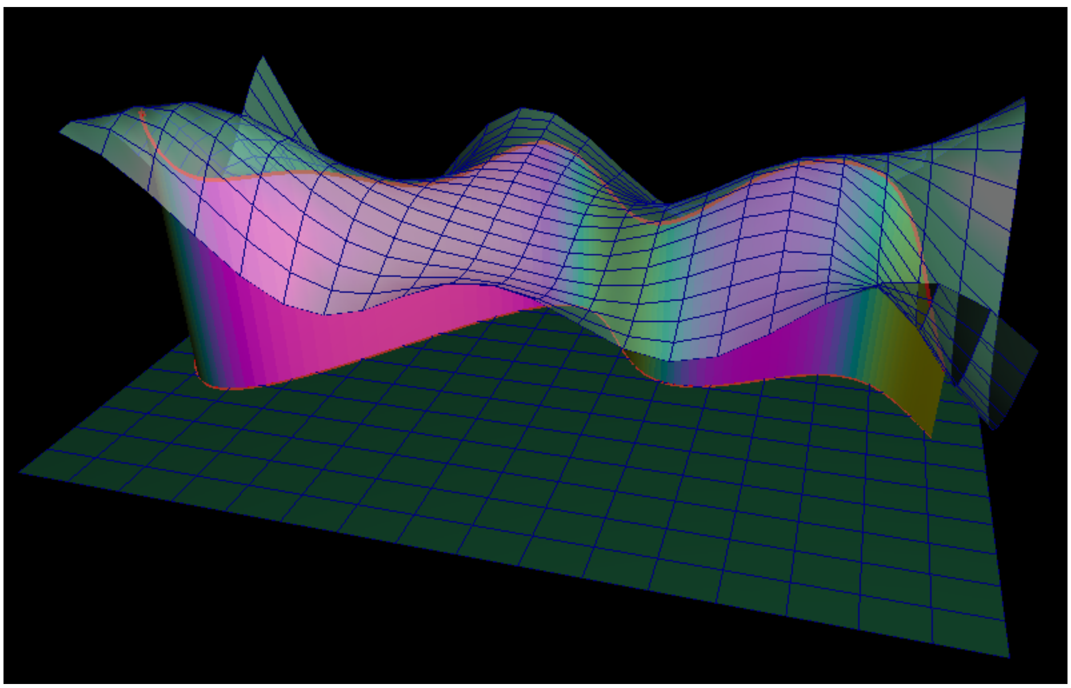

FIGURE 28. A vertical ribbon between a curve $\boldsymbol{\gamma}(t)$ sitting in the xy-plane and the surface of a graph $z = f(x, y)$; the scalar line integral $\int_{\boldsymbol{\gamma}} f(x, y)\,\mathrm{d}s$ is geometrically interpreted as the net area of such a ribbon.

For the general set up, fix a function $f : \mathcal{D} \to \mathbb{R}$ of n variables and a continuous vector-valued function $\boldsymbol{\gamma} : I \to \mathcal{D}$ describing a connected curve $\mathcal{C}$ in the domain of the function. We want to capture the net area along a "vertical ribbon" between the curve $\mathcal{C} \subset \mathcal{D} \subseteq \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and the graph $\mathcal{G}_f \subset \mathbb{R}^{n+1}$. The image $f(\boldsymbol{\gamma}(t))$ gives the x_{n+1} coordinate of the top of the slice sitting on the graph, and so $\boldsymbol{\gamma}(t) + f(\boldsymbol{\gamma}(t))\hat{\mathbf{e}}_{n+1}$ for $t \in I$ traces out a curve on the graph at a "height" of $f(\boldsymbol{\gamma}(t))$ above (or below) the curve $\mathcal{C}$ in the the domain.

[16]The function f should be "continuous almost everywhere" along $[a, b]$; what this means in practice is that we can allow a bounded function to be undefined on a discrete subset of $[a, b]$, and to have a discrete set of jump discontinuities. By closure, we mean that the actual domain of f $\mathcal{D}$ should meet $[a, b]$ in a possibly disjoint collection of intervals, such that if we add in all of the boundary points of $\mathcal{D}$ that are also in $[a, b]$, we get back all of $[a, b]$. To formalize what is really meant by "well behaved" and "discrete", we need the notion of Lebesgue measure and Lebesgue integrals; the discussion of these topics belongs to a good course on modern real analysis.

We will build an integral to calculate the net area by considering a limit of Riemann sums, as one often does to construct new integrals. Let $I = [a, b]$ and choose a partition

$$P_m : a = t_0 < t_1 < \ldots < t_m = b$$

of I such that we may subdivide the interval I into m subintervals $[t_0, t_1], \ldots, [t_{m-1}, t_m]$. Choosing a sample point $t_j^* \in [t_{j-1}, t_j]$ for each $j = 1, \ldots, m$, we make the partition P_m into a *marked partition.* This partitions the curve $\mathcal{C}$ into m arcs, which have lengths $\Delta s_j = \int_{t_{j-1}}^{t_j} \|\dot{\mathbf{r}}(\tau)\| \,\mathrm{d}\tau$. Now we want to consider the sum

$$\sum_{j=1}^{m} f\big(\boldsymbol{\gamma}(t_j^*)\big)\Delta s_j \,.$$

Each term of this sum can be interpreted geometrically as taking the product of a a little bit of arc-length along $\boldsymbol{\gamma}$ and with the value of f at a point within the little arc being approximated, which gives the signed area of a "bent rectangle" or vertical ribbon approximating a piece of the surface between $\boldsymbol{\gamma} \subset \mathcal{D}$ and the graph of f along $\boldsymbol{\gamma}$. Thus, the sum is itself an approximation of the net area bounded between $\boldsymbol{\gamma}$ and the curve $\{\boldsymbol{\gamma}(t) + f(\boldsymbol{\gamma}(t))\hat{\mathbf{e}}_{n+1} : t \in I\} \subset \mathcal{G}_f$ in $\mathbb{R}^{n+1}$. Increasing the number of subdivisions, we can play the usual limit game to define an integral that gives us the net area we desire:

Definition. Let $f : \mathcal{D} \to \mathbb{R}$ be a function of n variables on a domain $\mathcal{D} \subseteq \mathbb{R}^n$ and $\boldsymbol{\gamma} : I \to \mathcal{D}$ a continuous vector-valued function parameterizing a curve in $\mathcal{D}$. Given a marked partition P_m of I into m subintervals such that as $m \to \infty$ the maximum length of a subinterval tends to 0, yielding a sequence of Riemann sums

$$R_m(f, \boldsymbol{\gamma}) = \sum_{j=1}^{m} f\big(\boldsymbol{\gamma}(t_j^*)\big)\Delta s_j \,,$$

the line integral of f along $\boldsymbol{\gamma}$ is the limit of the Riemann sums

$$\int_{\boldsymbol{\gamma}} f(\mathbf{r}) \,\mathrm{d}s = \lim_{m\to\infty} R_m(f, \boldsymbol{\gamma}) = \lim_{m\to\infty} \sum_{j=1}^{m} f\big(\boldsymbol{\gamma}(t_j^*)\big)\Delta s_j \,,$$

if this limit exists.

By standard arguments, if f is continuous on $\boldsymbol{\gamma}$, this limit exists and is well defined, i.e., it does not depend on the choice of partition of I or the sequence of sample points marking the partitions. Note also that we defined this limit from a given parameterization, but in the end the integral itself, which represents the geometric quantity of net area between the curve $\boldsymbol{\gamma}$ and the graph of f, is an object which morally should depend only upon the curve itself, and not the choice of map $\boldsymbol{\gamma} : I \to \mathcal{D}$ realizing the curve as its image. We are fortunate that this is true:

Proposition 3.1. *Let $f : \mathcal{D} \to \mathbb{R}$ be a function of n variables on a domain $\mathcal{D} \subseteq \mathbb{R}^n$ and suppose $\boldsymbol{\gamma} : I \to \mathcal{D}$ and $\boldsymbol{\eta} : J \to \mathcal{D}$ are two continuous vector-valued functions with a common image curve $\mathcal{C} \subset \mathcal{D}$ traversed exactly once by each of the parameterizations. If the line integrals*

$$\int_{\boldsymbol{\gamma}} f(\mathbf{r}) \,\mathrm{d}s \quad \text{and} \quad \int_{\boldsymbol{\eta}} f(\mathbf{r}) \,\mathrm{d}s$$

both exist, then they are equal. We can thus write

$$\int_{\mathcal{C}} f(\mathbf{r}) \,\mathrm{d}s$$

to mean the line integral of f over the curve $\mathcal{C}$, regardless of the parameterization chosen.

Proof. See exercise (4) below. □

The matter of evaluation however still often invokes a particular parameterization. Moreover, observe that the differential in the integral is $\mathrm{d}s$ rather than $\mathrm{d}t$; even though we subdivided the domain of $\boldsymbol{\gamma}$ using whatever parameter t was given, the Riemann sums used the lengths of the displacement vectors in the image, and so approximated length along the curve (which is intrinsic to the curve $\mathcal{C}$, and independent of the choice of parameterization). But for a given parameterization, the arc-length element is $\mathrm{d}s = \|\dot{\boldsymbol{\gamma}}(t)\|\,\mathrm{d}t$, and so to evaluate such an integral when given a parameterization $\boldsymbol{\gamma}(t)$, we will work to re-express everything in terms of the given parameterization.

For our first examples, we will consider two-variable functions $f : \mathcal{D} \to \mathbb{R}$ and plane curves $\mathcal{C} \subset \mathcal{D} \subset \mathbb{R}^2$. Consider a curve $\mathcal{C}$ parameterized by a vector valued function $\boldsymbol{\gamma}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}}$. Then since $\mathrm{d}s = \|\dot{\boldsymbol{\gamma}}(t)\|\,\mathrm{d}t = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}\,\mathrm{d}t$, the line integral in terms of the parameterization is given by

$$\int_{\mathcal{C}} f(x,y)\,\mathrm{d}s = \int_a^b f\big(x(t),y(t)\big)\sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}\,\mathrm{d}t\,,$$

where a and b are the endpoints for the interval of parameterization for $\boldsymbol{\gamma} : [a,b] \to \mathcal{D}$. We'll use this formula to compute some manageable line integrals.

Example 3.1. Let $\mathcal{C}$ be the line segment from $(-2,6)$ to $(4,-2)$, and let $f(x,y) = xy$. We wish to compute the line integral of $f(x,y)$ along $\mathcal{C}$.

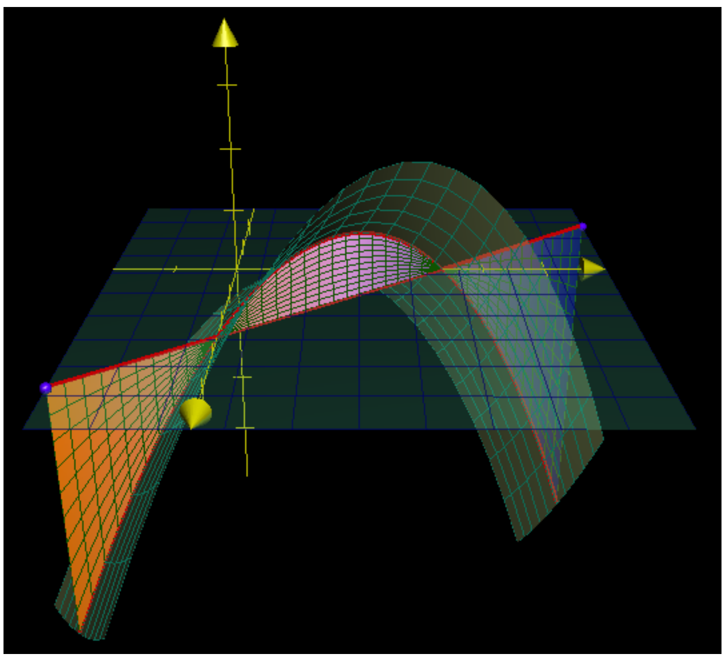

FIGURE 29. The net area computed by the line integral $\int_{\mathcal{C}} xy\,\mathrm{d}s$ over the line from $(-2,6)$ and $(4,-2)$. Note that since most of it is below the plane $z = 0$, the value of the integral is negative.

Let $\mathbf{p} = -2\hat{\mathbf{\imath}} + 6\hat{\mathbf{\jmath}}$ and $\mathbf{q} = 4\hat{\mathbf{\imath}} - 2\hat{\mathbf{\jmath}}$ be the respective position vectors of $(-2,6)$ and $(4,-2)$. Then

$$\boldsymbol{\gamma}(t) = (1-t)\mathbf{p} + t\mathbf{q} = (6t-2)\hat{\mathbf{\imath}} + (6-8t)\,, \quad 0 \le t \le 1$$

parameterizes $\mathcal{C}$ as a vector valued function. Thus the parametric equations are

$$x(t) = 6t-2\,, \quad y(t) = 6-8t\,, \quad 0 \le t \le 1\,,$$

giving

$$\mathrm{d}s = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}\,\mathrm{d}t = \sqrt{6^2 + (-8)^2}\,\mathrm{d}t = \sqrt{36+64}\,\mathrm{d}t = 10\,\mathrm{d}t\,.$$

Thus,

$$\begin{aligned}\int_{\mathcal{C}} f(x,y)\,\mathrm{d}s &= \int_0^1 x(t)\,y(t)\,(10)\,\mathrm{d}t = 10\int_0^1 (6t-2)(6-8t)\,\mathrm{d}t\\ &= 10\int_0^1 -48t^2+52t-12\,\mathrm{d}t = 10\left[-16t^3+26t^2-12t\right]_0^1\\ &= 10(-16+26-12+0-0+0) = -20\,.\end{aligned}$$

Example 3.2. Let $\mathcal{C}$ be the portion of the parabola $y = 4 - x^2$ in the first quadrant, and let $f(x,y) = \sqrt{4x - y + 8}$. We will exploit that the curve is a graph of a function of x to compute the line integral of $f(x,y)$ along $\mathcal{C}$.

Since $y = 4 - x^2$, $\mathcal{C}$ is described parametrically by $\mathbf{r}(x) = x\hat{\imath} + (4-x^2)\hat{\jmath}$, with $0 \le x \le 2$, since $0 \le x$ and $0 \le 4 - x^2$. From this, we can rewrite the arc-length differential as $\mathrm{d}s = \sqrt{1+4x^2}\,\mathrm{d}x$.

Next, we rewrite the function in terms of x:

$$f(x,y) = f(x, 4-x^2) = \sqrt{4x-(4-x^2)+8} = \sqrt{x^2+4x+4} = |x+2|\,.$$

Note that since $0 \le x \le 2$ along $\mathcal{C}$, we can drop the absolute value symbols. From here, we can assemble a single variable integral in terms of x:

$$\int_{\mathcal{C}} f(x,y)\,\mathrm{d}s = \int_{\mathcal{C}} \sqrt{4x-y+8}\,\mathrm{d}s = \int_0^2 (x+2)\sqrt{1+4x^2}\,\mathrm{d}x\,.$$

We can rewrite the integrand as $x\sqrt{1+4x^2} + 2\sqrt{1+4x^2}$. The antiderivative of the first term can be found using the simple substitution $u = 1+4x^2$, giving $\mathrm{d}u = 8x\,\mathrm{d}x$ so

$$\int x\sqrt{1+4x^2}\,\mathrm{d}x = \int \frac{1}{8}\sqrt{u}\,\mathrm{d}u\,.$$

The antiderivative of $2\sqrt{1+4x^2}$ is found by using the trigonometric substitution $2x = \tan v$, yielding $2\,\mathrm{d}x = \sec^2 v\,\mathrm{d}v$ and $\sqrt{1+4x^2} = |\sec v|$. Since $0 \le x \le 2$ along $\mathcal{C}$, the range of values of v should be taken to be $[0, \arctan(4)]$, over which both $\tan v$ and $\sec v$ are positive. Thus the tangent substitution turns $\int 2\sqrt{1+4x^2}\,\mathrm{d}x$ into $\int \sec^3 v\,\mathrm{d}v$, which can be computed via integration by parts.

Using the antiderivatives one gets by back-substituting after following the above substitutions, we can compute the line integral as a definite integral:

$$\begin{aligned}\int_{\mathcal{C}} \sqrt{4x-y+8}\,\mathrm{d}s &= \int_0^2 (x+2)\sqrt{1+4x^2}\,\mathrm{d}x\\ &= \left[\tfrac{1}{12}\left(1+4x^2\right)^{3/2} + x\sqrt{1+4x^2} + \tfrac{1}{2}\ln\left|2x+\sqrt{1+4x^2}\right|\,\right]_0^2\\ &= \frac{1}{12}\sqrt{2} + 2\sqrt{17} + \frac{1}{2}\ln(4+\sqrt{17})\,.\end{aligned}$$

One of the chief difficulties of evaluating line integrals is finding an easy to work with parameterization of the curve $\mathcal{C}$, and as we'll see in the examples, having an easy to work with arc-length parameterization $\boldsymbol{\gamma}(s)$ can simplify things immensely.

A line integral on a closed curve $\mathcal{C}$ is often written in the notation

$$\oint_{\mathcal{C}} f(\mathbf{r})\,\mathrm{d}s\,.$$

Our next example is over a closed curve.

Example 3.3. Let $\mathcal{C}$ be a circle of radius R centered at $(0,0) \in \mathbb{R}^2$, and let $f(x,y) = ax^2 + by^2$, where a, b are constants. We wish to compute $\oint_{\mathcal{C}} f(x,y)\,\mathrm{d}s$ in terms of a, b and R. We will use the fact that it is easy to arc-length parameterize a circle in order to rewrite x and y as functions of the arc-length s along $\mathcal{C}$

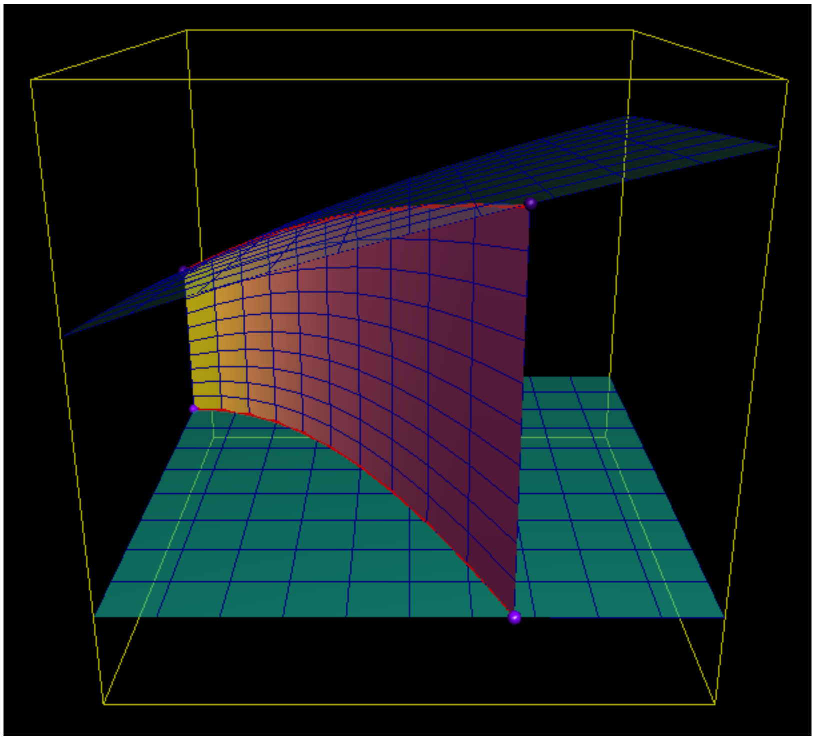

FIGURE 30. The area computed by the line integral $\int_{\mathcal{C}} \sqrt{4x-y+8}\,\mathrm{d}s$ over the parabola from $y = 4 - x^2$.

Let s be the arc-length along $\mathcal{C}$ at a point $(x, y) \in \mathcal{C}$ measured from the point $(R, 0)$. Then s is just R times the angle between the position vector $\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$ and $\hat{\imath}$, and the total arc-length of $\mathcal{C}$ is just $2\pi R$. Thus we have parametric equations

$$x(s) = R\cos(s/R)\ , \quad y(s) = R\sin(s/R)\ , \quad 0 \le s \le 2\pi R\,.$$

It is easy to check that $[x'(s)]^2 + [y'(s)]^2 = 1$, confirming that this is an arc-length parameterization.

The integral can now be computed using our arc-length parameterization:

$$\begin{aligned}
\oint_{\mathcal{C}} ax^2 + by^2\,\mathrm{d}s &= \int_0^{2\pi R} a\cos^2(s/R) + b\sin^2(s/R)\,\mathrm{d}s \\
&= \int_0^{2\pi R} a\left(\frac{1+\cos(2s/R)}{2}\right) + b\left(\frac{1-\cos(2s/R)}{2}\right)\mathrm{d}s \\
&= \int_0^{2\pi R} \frac{a+b}{2} + \frac{a-b}{2}\cos(2s/R)\,\mathrm{d}s \\
&= \pi R\,(a+b)\ .
\end{aligned}$$

For the sake of completeness, let us also look at one example of a line integral for a 3-variable function $f(x, y, z)$ along a space curve.

Example 3.4. We will calculate $\int_{\mathcal{C}} xyz\,\mathrm{d}s$ for the helix $\mathbf{r}(t) = \sin(2t)\,\hat{\imath} - \cos(2t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$ for $t \in [-\pi, \pi]$ by re-parameterizing with arc-length. First, we compute the arc-length function $s(t) =$

$\int_{-\pi}^{t} \|\dot{\mathbf{r}}(\tau)\|\,\mathrm{d}\tau$:

$$\begin{aligned} s(t) &= \int_{-\pi}^{t} \sqrt{[\dot{x}(\tau)]^2 + [\dot{y}(\tau)]^2 + [\dot{z}(\tau)]^2}\,\mathrm{d}\tau \\ &= \int_{-\pi}^{t} \sqrt{[2\cos(2\tau)]^2 + [-2\sin(2\tau)]^2 + [1]^2}\,\mathrm{d}\tau \\ &= \int_{-\pi}^{t} \sqrt{4\cos^2(2\tau) + 4\sin^2(2\tau) + 1}\,\mathrm{d}\tau \\ &= \int_{-\pi}^{t} \sqrt{8+1}\,\mathrm{d}\tau = 3\int_{-\pi}^{t} \mathrm{d}\tau = 3(t+\pi) \\ &\implies t = \frac{s}{3} - \pi\,. \end{aligned}$$

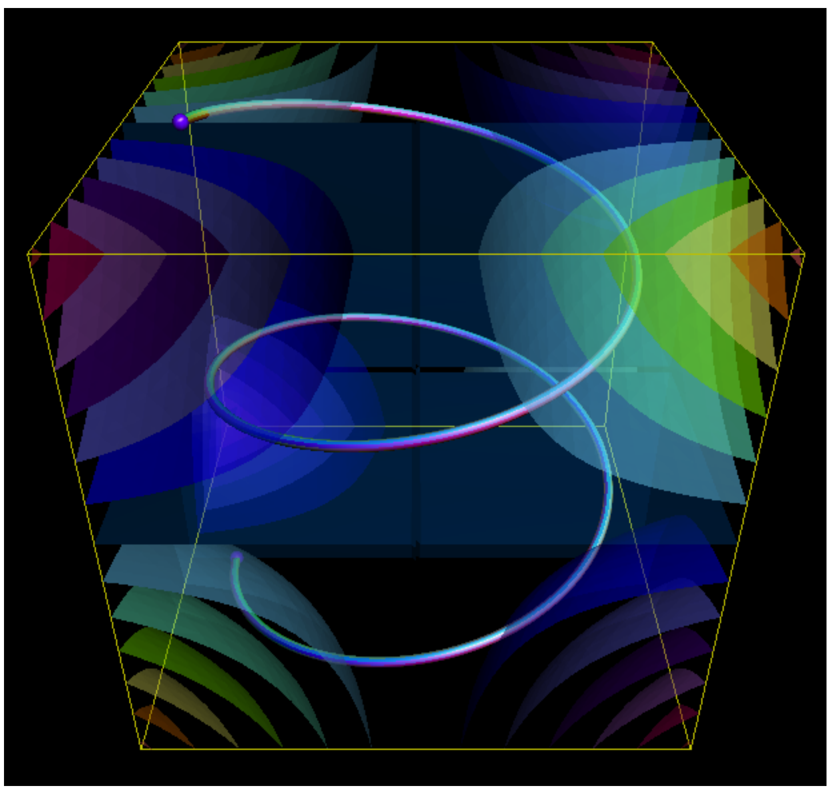

FIGURE 31. The helix $\mathbf{r}(t) = \sin(2t)\,\hat{\imath} - \cos(2t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$, for $t \in [-\pi, \pi]$, together with some level sets of $f(x, y, z) = xyz$.

Thus, let

$$\gamma(s) = \sin\left(\tfrac{2s}{3}\right)\,\hat{\imath} - \cos\left(\tfrac{2s}{3}\right)\,\hat{\jmath} + \left(\tfrac{s}{3} - \pi\right)\,\hat{\mathbf{k}}\,, \quad 0 \le s \le 6\pi\,.$$

$$\begin{aligned}
\int_{\mathcal{C}} xyz\,\mathrm{d}s &= \int_0^{6\pi} -\left(\tfrac{s}{3}-\pi\right)\sin\left(\tfrac{2s}{3}\right)\cos\left(\tfrac{2s}{3}\right)\,\mathrm{d}s \\
&= \int_0^{6\pi} \left(\tfrac{\pi}{2}-\tfrac{s}{6}\right)\sin\left(\tfrac{4s}{3}\right)\,\mathrm{d}s \\
&= \left(\tfrac{s}{8}-\tfrac{3\pi}{8}\right)\cos\left(\tfrac{4s}{3}\right)\Big|_0^{6\pi} - \frac{3}{8}\int_0^{6\pi}\cos\left(\tfrac{4s}{3}\right)\,\mathrm{d}s \\
&= \left(\tfrac{3\pi}{4}-\tfrac{3\pi}{8}\right)\cos(8\pi) + \frac{3\pi}{8}\cos(0) - \frac{9}{32}\sin(8\pi) + \frac{9}{32}\sin(0) \\
&= \frac{3\pi}{4}\,.
\end{aligned}$$

We can also define scalar line integrals where the differential is not an arc-length differential: given a collection of n multivariable functions $G_1, \dots G_n$ defined on $\mathcal{D} \subset \mathbb{R}^n$ and an *oriented curve* $\mathcal{C}$ in $\mathcal{D}$, we can define a scalar line integral along $\mathcal{C}$ of the differential one-form $G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$.

Definition. Let $G_1, \dots G_n$ be n functions of n variables defined on a domain $\mathcal{D} \subset \mathbb{R}^n$, and let $\mathcal{C}$ be an oriented curve in $\mathcal{D}$ parameterized by $\boldsymbol{\gamma} : I \to \mathcal{D}$. Let P_m be a sequence of partitions of I into m subintervals, such that as $m \to \infty$ the maximum length of a subinterval tends to 0. Choose sample points t_j^* in each subinterval $[t_{j-1}, t_j]$ and set $\mathbf{r}_j = \boldsymbol{\gamma}(t_j^*)$. Let $\Delta x_{ij} = (\mathbf{r}_j - \mathbf{r}_{j-1})\cdot\hat{\mathbf{e}}_i = x_i(t_j^*) - x_i(t_{j-1}^*)$. Then the line integral of the differential form $G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$ is

$$\int_{\mathcal{C}} G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n = \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} G_i(\mathbf{r}_j)\Delta x_{ij}$$

if the limit exists.

Again, when this exists it is well defined and independent of the parameterization of the curve $\mathcal{C}$. The geometric meaning of such a line integral will become more clear when we discuss line integrals in vector fields. Observe, however, that we defined it for an *oriented curve*, and in fact, you should convince yourself that if you reverse the orientation of the curve, the value of the integral is negated.

We'll consider again a few examples using functions of two variables. Let $P, Q : \mathcal{D} \to \mathbb{R}$ be scalar fields and $\mathcal{C}$ an oriented curve in $\mathcal{D}$. Given a parameterization $\boldsymbol{\gamma}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ of $\mathcal{C}$ defined on an interval $[a, b]$, we can rewrite the integral of $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ along $\mathcal{C}$ in terms of t:

$$\begin{aligned}
\int_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y &= \int_a^b P\big(x(t),y(t)\big)\,\mathrm{d}x(t) + Q\big(x(t),y(t)\big)\,\mathrm{d}y(t) \\
&= \int_a^b \Big(P\big(x(t),y(t)\big)\dot{x}(t) + Q\big(x(t),y(t)\big)\dot{y}(t)\Big)\,\mathrm{d}t
\end{aligned}$$

Example 3.5. Let $\mathcal{C}$ be the segment of the plane curve $y^2 = x^3$ joining the point $(1, 1)$ to $(2, 2\sqrt{2})$. We will compute

$$\int_{\mathcal{C}} y^{1/3}\,\mathrm{d}x + x^{1/2}\,\mathrm{d}y\,.$$

To parameterize $\mathcal{C}$ we could simply rewrite y as a function of x since $\mathcal{C}$ lies completely in the first quadrant. However, a more interesting parameterization to work with is the monomial parameterization

$$x(t) = t^2\,,\quad y(t) = t^3\,, 1 \le t \le \sqrt{2}\,.$$

One way to arrive at this is to start with $y = x^{3/2}$ and ask what power of t should be used in place of x to ensure that y is also an integral power of t. Of course, setting $x = t^2$ works since $(t^2)^{3/2} = t^3$.

Now, we need to express the differentials $\mathrm{d}x$ and $\mathrm{d}y$ in terms of t:

$$\mathrm{d}x = \mathrm{d}(t^2) = 2t\,\mathrm{d}t \quad \mathrm{d}y = \mathrm{d}(t^3) = 3t^2\,\mathrm{d}t\,.$$

Thus,

$$\begin{aligned}\int_{\mathcal{C}} y^{1/3}\,\mathrm{d}x + x^{1/2}\,\mathrm{d}y &= \int_1^{\sqrt{2}} (t^3)^{1/3}\,2t\,\mathrm{d}t + (t^2)^{1/2}\,3t^2\,\mathrm{d}t \\ &= \int_1^{\sqrt{2}} 2t^2 + 3t^3\,\mathrm{d}t = \left[\tfrac{2}{3}t^3 + \tfrac{3}{4}t^4\right]_1^{\sqrt{2}} = \tfrac{4}{3}\sqrt{2} + 3 - \left(\tfrac{2}{3} + \tfrac{3}{4}\right) \\ &= \frac{16\sqrt{2}+19}{12}\,.\end{aligned}$$

Our next example shows that endpoints and orientations are not enough to determine the value of a line integral with respect to the coordinate variables.

Example 3.6. Let $\mathcal{C}_1$ be the portion of the parabola $y = 2 - x^2$ where $y \geq x$. Let $\mathcal{C}_2$ be the line segment connecting the points of intersection of $\mathcal{C}_1$ and the line $y = x$. We consider three line integrals: the integral of $\frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)$ over $\mathcal{C}_1$ oriented "left to right", the integral of the same form over $-\mathcal{C}_1$, by which we mean oriented from "right to left", and the line integral of $\frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)$ over $\mathcal{C}_2$, oriented from "left to right".

First, we need to locate the intersection points of $y = 2 - x^2$ and $y = x$. A little algebra shows these are at $(1,1)$ and $(-2,-2)$. Since $\mathcal{C}_1$ is oriented right to left, the initial point is $(-2,-2)$ and the terminal point is $(1,1)$. We can evaluate by using that $\mathcal{C}_1$ is a portion of a graph:

$$\begin{aligned}\int_{\mathcal{C}_1} \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y) &= \frac{1}{2}\int_{-2}^{1} (2-x^2)\,\mathrm{d}x - x\,\mathrm{d}(2-x^2) \\ &= \frac{1}{2}\int_{-2}^{1} (2-x^2) + 2x^2\,\mathrm{d}x = \frac{1}{2}\int_{-2}^{1} 2 + x^2\,\mathrm{d}x \\ &= \frac{1}{2}\left[2x + \frac{x^3}{3}\right]_{-2}^{1} = \frac{1}{2}\left[2 + \frac{1}{3} - \left(-4 - \frac{8}{3}\right)\right] \\ &= \frac{9}{2}\end{aligned}$$

If we reverse orientations, we merely reverse the limits of the corresponding integral with respect to x. Thus,

$$\int_{-\mathcal{C}_1} \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y) = -\frac{9}{2} = -\int_{\mathcal{C}_1} \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)\,.$$

For the line integral over $\mathcal{C}_2$, we can use that $y = x$ along the line segment to deduce that the differential form vanishes:

$$\frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y) = \frac{1}{2}(x\,\mathrm{d}x - x\,\mathrm{d}x) = 0\,.$$

Thus, despite having the same endpoints and orientation as $\mathcal{C}_1$, the line integral over $\mathcal{C}_2$ is not equal to the line integral over $\mathcal{C}_1$:

$$\int_{\mathcal{C}_2} \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y) = 0\,.$$

The following proposition lists some useful properties of scalar line integrals.

Proposition 3.2. *Let ω and ψ represent differential one-forms on a domain $\mathcal{D} \subseteq \mathbb{R}^n$ either of the form $f(\mathbf{r})\,\mathrm{d}s$ or $G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$, and let $\mathcal{C}$, $\mathcal{C}'$ represent oriented curves in the domain $\mathcal{D}$, and $-\mathcal{C}$ denote the same curve as $\mathcal{C}$ but with opposite orientation. Let $\mathcal{C}+\mathcal{C}'$ be the curve which is the (possibly disjoint) union of the curves $\mathcal{C}$ and $\mathcal{C}'$. Let a and b be any real constants. Then the following identities hold for scalar line integrals:*

(*i*) $\displaystyle\int_{\mathcal{C}} a\omega + b\psi = a\int_{\mathcal{C}} \omega + b\int_{\mathcal{C}} \psi,$

(*ii*) *if* $\omega = f(\mathbf{r})\,\mathrm{d}s$ *then* $\displaystyle\int_{-\mathcal{C}} \omega = \int_{\mathcal{C}} \omega$,

(*iii*) *if* $\omega = G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$, *then* $\displaystyle\int_{-\mathcal{C}} \omega = -\int_{\mathcal{C}} \omega$,

(*iv*) $\displaystyle\int_{\mathcal{C}+\mathcal{C}'} \omega = \int_{\mathcal{C}} \omega + \int_{\mathcal{C}'} \omega.$

Proof. See exercise (5) below. □

Example 3.7. We will compute the line integral

$$\int_{\mathcal{S}} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y$$

where $\mathcal{S}$ is the boundary of the unit square $[0,1]\times[0,1]$ in the first quadrant of $\mathbb{R}^2$, oriented counter-clockwise.

Let $\mathcal{S}_1$ be the line segment parameterized by $\mathbf{r}_1(t) = t\hat{\imath}$, $\mathcal{S}_2$ be the line segment parameterized by $\mathbf{r}_2(t) = \hat{\imath} + t\hat{\jmath}$, $\mathcal{S}_3$ be the line segment parameterized by $\mathbf{r}_3(t) = (1-t)\hat{\imath} + \hat{\jmath}$ and $\mathcal{S}_4$ be the line segment parameterized by $\mathbf{r}_4(t) = (1-t)\hat{\jmath}$. Using property (*iv*) we can re-express the integral over $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4$ as

$$\begin{aligned}\oint_{\mathcal{S}} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y &= \sum_{k=1}^{4} \int_{\mathcal{S}_k} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y \\ &= \int_{\mathcal{S}_1} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y + \int_{\mathcal{S}_2} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y \\ &\quad + \int_{\mathcal{S}_3} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y + \int_{\mathcal{S}_4} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y\end{aligned}$$

For each line segment $\mathcal{S}_k$, $k = 1,\ldots,4$ we use the parameterization $\mathbf{r}_k(t)$ to re-express the differential one-form $\cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y$ in terms of the parameter t:

- along $\mathcal{S}_1$ $\cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y = \cos(0\pi)\,\mathrm{d}(t) - \sin(\pi t)\,\mathrm{d}(0) = \mathrm{d}t$,
- while along $\mathcal{S}_2$ $\cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y = \cos(\pi t)\,\mathrm{d}(1) - \sin(1\pi)\,\mathrm{d}(t) = 0$,
- and along $\mathcal{S}_3$: $\cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y = \cos(1\pi)\,\mathrm{d}(1-t) - \sin((1-t)\pi)\,\mathrm{d}(1) = -1\,\mathrm{d}(-t) = \mathrm{d}t$,
- and finally along $\mathcal{S}_4$: $\cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y = \cos((1-t)\pi)\,\mathrm{d}(0) - \sin(0\pi)\,\mathrm{d}(1) = 0$.

Thus

$$\begin{aligned}\oint_{\mathcal{S}} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y &= \int_{\mathcal{S}_1+\mathcal{S}_3} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y \\ &= \int_0^1 2\,\mathrm{d}t = 2\,.\end{aligned}$$

Another way to see this is to observe that along $\mathcal{S}_1$ and $\mathcal{S}_3$, y is constant and along $\mathcal{S}_2$ and $\mathcal{S}_4$, x is constant. On $\mathcal{S}_1$, since $y = 0$, the integral reduces to

$$\int_{\mathcal{S}_1} \cos(\pi y)\,\mathrm{d}x - \sin(\pi x)\,\mathrm{d}y = \int_0^1 \cos(0)\,\mathrm{d}x = \int_0^1 \mathrm{d}x = 1$$

and along $\mathcal{S}_3$ we similarly obtain a simplified integral with value 1, since $\pi y = \pi(1) = \pi$ and $\cos(\pi) = -1$, and the orientation is negative with respect to increasing x. The other two sides end up having zero integrand since only y is changing, and $\sin(\pi x)$ vanishes whenever x is an integer, as it is along these sides of the square.

§ 3.2. Line Integrals in Vector Fields

We are now interested in defining line integrals for curves in a vector field $\mathbf{F}: \mathcal{D} \to \mathbb{R}^n$. As before, let $\mathcal{C}$ be an oriented continuous curve parameterized by a vector-valued function $\boldsymbol{\gamma}: I \to \mathcal{D}$. We can repeat the process of subdivision of the domain of $\boldsymbol{\gamma}$, and analogously define a Riemann sum, this time dotting the vector field with displacement vectors $\Delta\mathbf{r}_j = \boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})$, $j = 1, \ldots, m$, giving a piecewise linear/polygonal approximation of the curve $\boldsymbol{\gamma}$. Observe that for a sufficiently fine partition of the curve, $\|\Delta\mathbf{r}_j\| \approx \Delta s_j$.

Definition. Let $\mathbf{F}: \mathcal{D} \to \mathbb{R}$ be a vector field on a domain $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathcal{C}$ be an oriented continuous curve parameterized by a vector-valued function $\boldsymbol{\gamma}: I \to \mathcal{D}$. Given a sequence of marked partitions P_m of I into m subintervals with sample points t_j^* in each subinterval $[t_{j-1}, t_j]$ of a partition P_m, yielding Riemann sums

$$\sum_{j=1}^{m} \mathbf{F}\big(\boldsymbol{\gamma}(t_j^*)\big) \cdot \Delta\mathbf{r}_j \,,$$

the line integral of $\mathbf{F}$ along $\mathcal{C}$ is the limit of the Riemann sums

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \lim_{m\to\infty} \sum_{j=1}^{m} \mathbf{F}\big(\boldsymbol{\gamma}(t_j^*)\big) \cdot \Delta\mathbf{r}_j \,,$$

if the limit exists.

As above, you should deduce that this is well defined when the limit exists, that it is independent of the choice of parameterization of $\mathcal{C}$, and that the expected properties hold. In particular, it is useful to note that

$$\int_{\mathcal{C}+\mathcal{C}'} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}'} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

for any curves $\mathcal{C}$ and $\mathcal{C}'$ in the domain $\mathcal{D}$ of $\mathbf{F}$. This allows us to describe methods of computation of line integrals in vector fields when the curve of integration can be decomposed as a collection of regular curves (that is, curves whose tangent vectors are defined and non-zero).

Presume that $\mathcal{C}$ is a curve admitting a regular parameterization $\boldsymbol{\gamma}: I \to \mathcal{D}$. Regularity implies that the unit tangent vector

$$\hat{\mathbf{T}}(t) = \frac{\dot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\|} = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s}$$

is well defined along the length of $\mathcal{C}$. It is easy to argue that the Riemann sum

$$\sum_{j=1}^{m} \mathbf{F}\big(\boldsymbol{\gamma}(t_j^*)\big) \cdot \Delta\mathbf{r}_j$$

is approximately equal to

$$\sum_{j=1}^{m} \mathbf{F}\big(\boldsymbol{\gamma}(t_j^*)\big) \cdot \hat{\mathbf{T}}(t_j^*)\Delta s_j \,,$$

using that $\Delta s_j = \int_{t_{j-1}}^{t_j} \|\mathbf{r}'(\tau)\|\,\mathrm{d}\tau \approx \|\Delta\mathbf{r}_j\|$. If the limits of these sums exists, one can show that the limits are actually equal. But the latter sequence of Riemann sums converges to a scalar line integral of the form

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}(\mathbf{r})\,\mathrm{d}s \,,$$

where $\hat{\mathbf{T}}(\mathbf{r})$ is evaluated at points $\mathbf{r}$ along $\mathcal{C}$, and $\mathrm{d}s$ is the usual arc-length element as before. This is often abbreviated as

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}\,\mathrm{d}s \,.$$

Now, since

$$\hat{\mathbf{T}} = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s} = \sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}s}\hat{\mathbf{e}}_i$$

we have that

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}\, \mathrm{d}s &= \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \left(\sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}s}\hat{\mathbf{e}}_i\right) \mathrm{d}s \\ &= \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \left(\sum_{i=1}^{n} \mathrm{d}x_i \hat{\mathbf{e}}_i\right) \\ &= \int_{\mathcal{C}} \sum_{i=1}^{n} (\mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{e}}_i)\, \mathrm{d}x_i = \int_{\mathcal{C}} \sum_{i=1}^{n} F_i(\mathbf{r})\, \mathrm{d}x_i \\ &= \int_{\mathcal{C}} F_1(\mathbf{r})\, \mathrm{d}x_1 + \ldots + F_n(\mathbf{r})\, \mathrm{d}x_n\,,\end{aligned}$$

which is the second kind of scalar line integral. Writing

$$\mathrm{d}\mathbf{r} = \hat{\mathbf{T}}\, \mathrm{d}s = \sum_{i=1}^{n} \hat{\mathbf{e}}_i \frac{\mathrm{d}x_i}{\mathrm{d}s}\, \mathrm{d}s = \sum_{i=1}^{n} \hat{\mathbf{e}}_i\, \mathrm{d}x_i$$

the notation $\mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} := F_1(\mathbf{r})\, \mathrm{d}x_1 + \ldots + F_n(\mathbf{r})\, \mathrm{d}x_n$ now presents us with a unified geometric meaning for line integrals in vector fields and the second kind of scalar line integral we defined.

The above arguments applied when the curve $\mathcal{C}$ was regular, but if we had a piecewise curve $\mathcal{C} = \sum_{k=1}^{l} \mathcal{C}_k$ whose sub-pieces $\mathcal{C}_k$ were regular curves, then each sub-piece can be evaluated as a scalar line integral by any of the previous techniques (involving parameterizations or realizing a differential form as the total derivative of some function). Thus, applying the property

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \sum_{k=1}^{l} \int_{\mathcal{C}_k} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}\,,$$

we can evaluate general line integrals in vector fields by utilizing a parameterization or, when we are fortunate, by identifying a function whose total derivative is the differential form $\mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_1(\mathbf{r})\, \mathrm{d}x_1 + \ldots + F_n(\mathbf{r})\, \mathrm{d}x_n$.

Example 3.8. Let $\mathcal{T}$ be the triangle in $\mathbb{R}^2$ with vertices $A(0,0)$, $B(\sqrt{3},-1)$ and $C(\sqrt{3},1)$ oriented counterclockwise, and let $\mathbf{F}(x,y) = -2xy\,\hat{\mathbf{\imath}} + (x^2 - y^2)\,\hat{\mathbf{\jmath}}$. Let us compute the line integral

$$\oint_{\mathcal{T}} \mathbf{F}(x,y) \cdot \hat{\mathbf{T}}\, \mathrm{d}s\,.$$

Note that $\mathcal{T}$ is an equilateral triangle with sides of length 2. Write $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ where $\mathcal{T}_1$ is the line segment from $A(0,0)$ to $B(\sqrt{3},-1)$, $\mathcal{T}_2$ is the line segment from $B(\sqrt{3},-1)$ to $C(\sqrt{3},1)$, and $\mathcal{T}_3$ is the line segment from $C(\sqrt{3},-1)$ to $A(0,0)$. Let $\hat{\mathbf{T}}_i$, $i = 1,2,3$ denote the unit tangent vectors to these segments. We have arc-length parameterizations

$$\mathbf{r}_1(s) = \mathbf{a} + \tfrac{s}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{s}{2}(\sqrt{3}\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}}) = s\left(\tfrac{\sqrt{3}}{2}\hat{\mathbf{\imath}} - \tfrac{1}{2}\hat{\mathbf{\jmath}}\right)\,,\ 0 \le s \le 2\,,$$

$$\mathbf{r}_2(s) = \mathbf{b} + \tfrac{s}{2}(\mathbf{c} - \mathbf{b}) = (\sqrt{3}\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}}) + s\,\hat{\mathbf{\jmath}} = \sqrt{3}\hat{\mathbf{\imath}} + (s-1)\hat{\mathbf{\jmath}}\,,\ 0 \le s \le 2\,,$$

$$\mathbf{r}_3(s) = \mathbf{c} + \tfrac{s}{2}(\mathbf{a} - \mathbf{c}) = (\sqrt{3}\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}) - \frac{s}{2}(\sqrt{3}\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}) = (2-s)\left(\tfrac{\sqrt{3}}{2}\hat{\mathbf{\imath}} + \tfrac{1}{2}\hat{\mathbf{\jmath}}\right)\,,\ 0 \le s \le 2\,,$$

for $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$, respectively. Each of the above parameterizations can be obtained by adding an initial position vector to $s/2$ times a displacement vector between endpoints, and so the unit tangent vectors are just normalizations of these displacements. Thus, the unit tangents are

$$\hat{\mathbf{T}}_1 = \tfrac{\sqrt{3}}{2}\hat{\mathbf{\imath}} - \tfrac{1}{2}\hat{\mathbf{\jmath}}\ ,\quad \hat{\mathbf{T}}_2 = \hat{\mathbf{\jmath}}\,,\quad \hat{\mathbf{T}}_3 = -\tfrac{\sqrt{3}}{2}\hat{\mathbf{\imath}} - \tfrac{1}{2}\hat{\mathbf{\jmath}}\,.$$

We can compute the line integral we want as a sum of line integrals over each of the segments:

$$\oint_{\mathcal{T}} \mathbf{F}(x,y) \cdot \hat{\mathbf{T}}\, \mathrm{d}s = \int_{\mathcal{T}_1} \mathbf{F}(x,y) \cdot \hat{\mathbf{T}}\, \mathrm{d}s + \int_{\mathcal{T}_2} \mathbf{F}(x,y) \cdot \hat{\mathbf{T}}\, \mathrm{d}s + \int_{\mathcal{T}_3} \mathbf{F}(x,y) \cdot \hat{\mathbf{T}}\, \mathrm{d}s\,.$$

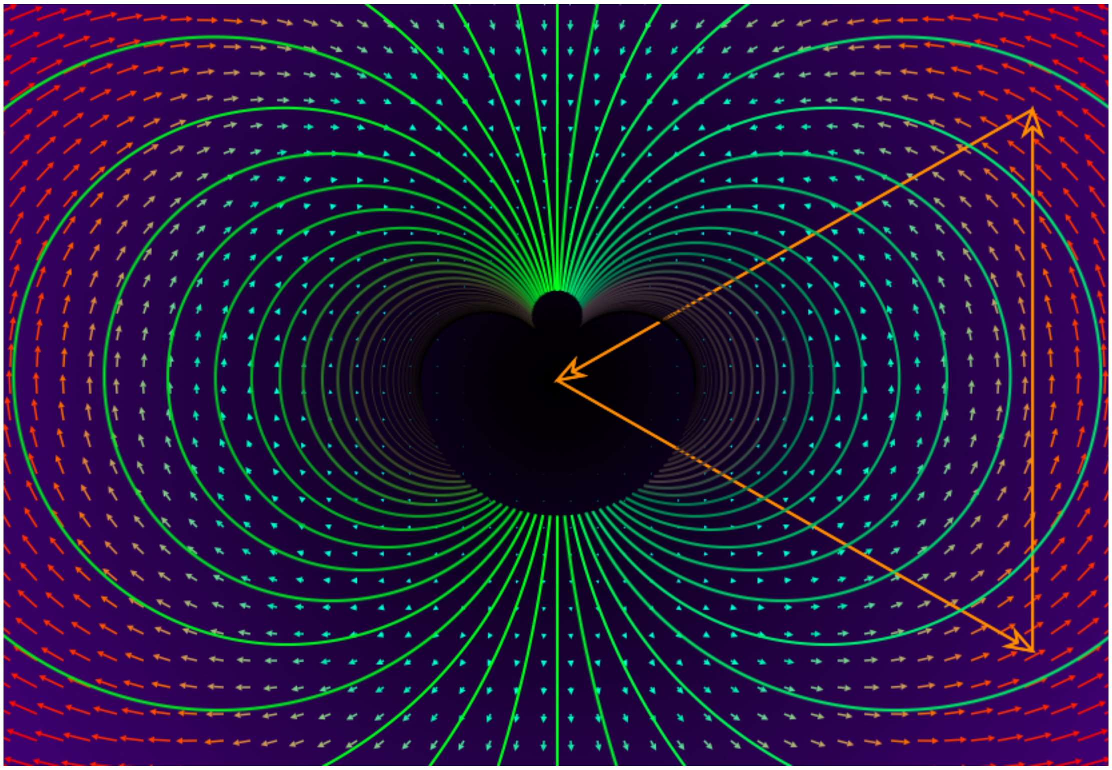

Figure 32. The dipole field $\mathbf{F}(x,y) = -2xy\,\hat{\imath} + (x^2 - y^2)\,\hat{\jmath}$ as well as some of its field-lines, together with the triangle $\mathcal{T}$.

Along $\mathcal{T}_1$ we have parametric equations $x(s) = s\sqrt{3}/2$, $y(s) = -s/2$, with $s \in [0,2]$, so

$$\mathbf{F}\big(x(s),y(s)\big) = -2x(s)y(s)\,\hat{\imath} + \big([x(s)]^2 - [y(s)]^2\big)\,\hat{\jmath} = \frac{s^2\sqrt{3}}{2}\hat{\imath} + \frac{s^2}{2}\hat{\jmath} = -s^2\,\hat{\mathbf{T}}_3$$

$$\begin{aligned}\int_{\mathcal{T}_1} \mathbf{F}(x(s),y(s))\cdot\hat{\mathbf{T}}\,\mathrm{d}s &= \int_0^2 -s^2\,\hat{\mathbf{T}}_3\cdot\hat{\mathbf{T}}_1\,\mathrm{d}s \\ &= \int_0^2 -s^2\cos\left(\frac{2\pi}{3}\right)\,\mathrm{d}s = \frac{1}{2}\left[\frac{s^3}{3}\right]_0^2 \\ &= \frac{4}{3}\,,\end{aligned}$$

where we've used that the dot product $\hat{\mathbf{T}}_1\cdot\hat{\mathbf{T}}_3$ is the cosine of the angle between them measured when they are both placed with tails at a common point (such as the origin).

Along $\mathcal{T}_2$, we have parametric equations $x(s) = \sqrt{3}$, $y(s) = s-1$, with $s \in [0,2]$, which gives

$$\mathbf{F}\big(x(s),y(s)\big) = -2\sqrt{3}(s-1)\hat{\imath} + (2+2s-s^2)\hat{\jmath}\,.$$

Since the unit tangent vector along $\mathcal{T}_2$ is $\hat{\jmath}$, $\mathbf{F}(x(s),y(s))\cdot\hat{\mathbf{T}} = 2+2s-s^2$, whence

$$\begin{aligned}\int_{\mathcal{T}_2} \mathbf{F}(x(s),y(s))\cdot\hat{\mathbf{T}}\,\mathrm{d}s &= \int_0^2 2+2s-s^2\,\mathrm{d}s \\ &= 2(2) + 2^2 - \frac{2^3}{3} = \frac{16}{3}\,.\end{aligned}$$

Finally, along $\mathcal{T}_3$ we have parametric equations $x(s) = \sqrt{3} - s\sqrt{3}/2$, $y(s) = 1 - s/2$, with $s \in [0,2]$, and

$$\mathbf{F}\big(x(s),y(s)\big) = -2\sqrt{3}\left(1-\frac{s}{2}\right)^2\hat{\imath} + 2\left(1-\frac{s}{2}\right)^2\hat{\jmath} = -4\left(1-\frac{s}{2}\right)^2\hat{\mathbf{T}}_1\,.$$

Thus,

$$\int_{\mathcal{T}_3} \mathbf{F}(x(s), y(s)) \cdot \hat{\mathbf{T}}\,\mathrm{d}s = \int_0^2 -4\left(1-\frac{s}{2}\right)^2 \hat{\mathbf{T}}_1 \cdot \hat{\mathbf{T}}_3\,\mathrm{d}s$$
$$= -4\int_0^2 \left(1-\frac{s}{2}\right)^2 \cos\left(\frac{2\pi}{3}\right)\,\mathrm{d}s = \left[-\frac{4}{3}\left(1-\frac{s}{2}\right)^3\right]_0^2$$
$$= \frac{4}{3}\,,$$

where again we've used that $\hat{\mathbf{T}}_1 \cdot \hat{\mathbf{T}}_3 = \cos(2\pi/3) = -1/2$.

Putting it all together:

$$\oint_{\mathcal{T}} \mathbf{F}(x,y)\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_{\mathcal{T}_1} \mathbf{F}(x,y)\cdot\hat{\mathbf{T}}\,\mathrm{d}s + \int_{\mathcal{T}_2} \mathbf{F}(x,y)\cdot\hat{\mathbf{T}}\,\mathrm{d}s + \int_{\mathcal{T}_3} \mathbf{F}(x,y)\cdot\hat{\mathbf{T}}\,\mathrm{d}s$$
$$= \frac{4}{3} + \frac{16}{3} + \frac{4}{3}$$
$$= 8\,.$$

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§ 3.3. Work in a Force Field

As far as giving a physical intuition to line integrals in vector fields, and thus to the line integrals of differential forms involving the coordinate differentials, we recall the notion of *work* in physics: the work done on a particle by a force is equal to the magnitude of the *effective force* times the displacement produced. For a constant linear force $\mathbf{F}$ producing a displacement $\Delta\mathbf{r}$ in the position of a particle, this is calculated as the dot product $W = \mathbf{F}\cdot\Delta\mathbf{r}$. However, if the direction of motion and force both vary, then we can imagine our particle as having trajectory given by some oriented curve, and the force being given at different points of the particle's path by a vector field $\mathbf{F}$ defined along the trajectory. The *infinitesimal work* would then be $\mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r} := F_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + F_n(\mathbf{r})\,\mathrm{d}x_n$ where $\mathrm{d}\mathbf{r} = \hat{\mathbf{T}}\,\mathrm{d}s$ is a directed differential along the trajectory. The total work contributed by the field to the particle's motion along its trajectory $\mathcal{C}$ is then the line integral

$$\mathcal{W}[\mathbf{F},\mathcal{C}] := \int_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}\,.$$

Example 3.9. Let $\mathbf{F}(x,y) = (x-y)\hat{\mathbf{\imath}} + (y-x)\hat{\mathbf{\jmath}}$, and let $\mathcal{C}$ be the unit circle $\mathbb{S}^1$. Then the work done by $\mathbf{F}$ on a particle completing one counterclockwise circuit around $\mathbb{S}^1$ is

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \oint_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\,\mathrm{d}\mathbf{r} = \int_{\mathbb{S}^1} (x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y$$
$$= \int_0^{2\pi} (\cos\theta - \sin\theta)\,\mathrm{d}(\cos\theta) + (\cos\theta + \sin\theta)\,\mathrm{d}(\sin\theta)$$
$$= \int_0^{2\pi} (\cos\theta - \sin\theta)(-\sin\theta) + (\cos\theta+\sin\theta)(\cos\theta)\,\mathrm{d}\theta$$
$$= \int_0^{2\pi} \sin^2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta + \cos^2\theta\,\mathrm{d}\theta$$
$$= \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

Another way we could have found this is to use that $\mathbf{r}(\theta) = \cos(\theta)\,\hat{\mathbf{\imath}} + \sin(\theta)\,\hat{\mathbf{\jmath}}$ is an arc-length parameterization, whence $\hat{\mathbf{T}}(\theta) = \mathrm{d}\mathbf{r}/\mathrm{d}\theta = -\sin(\theta)\,\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}$ and $\mathrm{d}s = \mathrm{d}\theta$, so

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \oint_{\mathbb{S}^1} \mathbf{F}(\mathbf{r})\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_0^{2\pi} 1\,\mathrm{d}s\,,$$

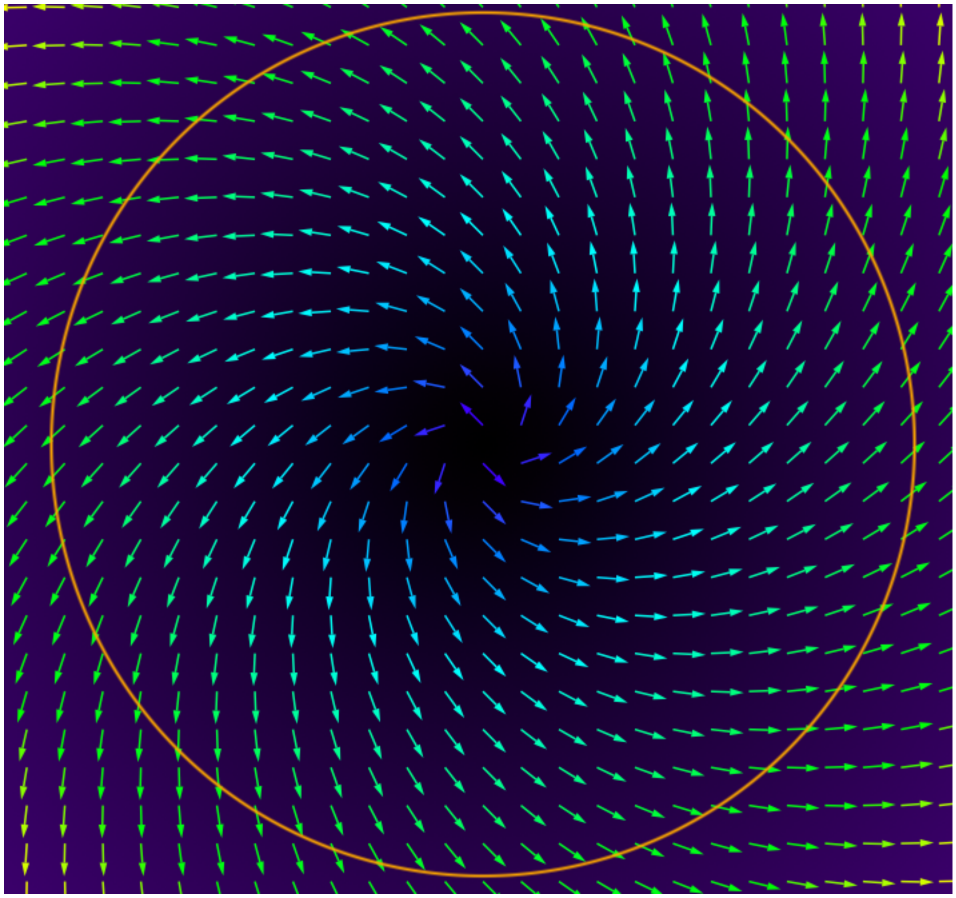

FIGURE 33. The spiral source vector field $\mathbf{F} = r\hat{\mathbf{u}}_r(\theta) + r\hat{\mathbf{u}}_\theta(\theta) = (x-y)\hat{\mathbf{\imath}} + (x+y)\hat{\mathbf{\jmath}}$ and the unit circle $\mathbb{S}^1$. Vectors are not drawn to scale to avoid cluttering the image; colors indicate magnitude, with warmer hues indicating larger magnitude.

since $\mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}} = (\cos\theta - \sin\theta)(-\sin\theta) + (\cos\theta + \sin\theta)(\cos\theta) = 1$, as seen above. What this says is that the components of the field $\mathbf{F}$ along the tangential direction to $\mathbb{S}^1$ are all of length $1 = \|\hat{\mathbf{T}}(\theta)\|$ and oriented compatibly (circulating counterclockwise). Indeed, the field can be rewritten in polar coordinates:

$$\mathbf{F} = (x-y)\hat{\mathbf{\imath}} + (y-x)\hat{\mathbf{\jmath}} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + (-y\hat{\mathbf{\imath}} + x\hat{\mathbf{\jmath}}) = r\hat{\mathbf{u}}_r(\theta) + r\hat{\mathbf{u}}_\theta(\theta)\,,$$

where $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ are the vectors of the polar frame, $\hat{\mathbf{u}}_r$ being a unit vector parallel to the position at (x, y), and $\hat{\mathbf{u}}_\theta$ being a unit vector tangential to the circle $x^2 + y^2 = r^2$. We see then that for $\mathbb{S}^1 = \{(r,\theta)_{\mathscr{P}} : r = 1)\}$, $\hat{\mathbf{T}}(\theta) = \hat{\mathbf{u}}_\theta(\theta)$, and it is immediate that $\mathbf{F}$ can be written as $\hat{\mathbf{u}}_r(\theta) + \hat{\mathbf{T}}(\theta)$ along $\mathbb{S}^1$. It is thus unsurprising that the work done in this case is the arc-length along the trajectory.

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§ 3.4. The Fundamental Theorem of Line Integrals

Working with parameterizations can be difficult and tedious, and so you may wonder if there is a swifter way to compute line integrals, more in line with the fundamental theorem of calculus. Why can't we just take something like an antiderivative right from the start, and evaluate at the endpoints of our path, and subtract?

Several of the previous examples demonstrate why this isn't always a possible approach. If it were always the case that only endpoints of paths and orientations mattered, then closed paths, which begin and end at the same point, would necessarily lead to vanishing line integrals. But we've seen examples of closed curves $\mathcal{C}$ such that line integrals around them have non-zero values.

But there is a large class of vector fields whose line integrals can be computed without resorting to parameterizations. Suppose a vector field $\mathbf{F}$ over $\mathcal{D}$ is conservative, i.e., $\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r})$ for some scalar function $f : \mathcal{D} \to \mathbb{R}$ called a *potential*. Then in fact, line integrals of $\mathbf{F}$ over curves $\mathcal{C} \subset \mathcal{D}$ are *independent of path*, meaning they only depend on the choice initial and terminal point.

Theorem 3.1 (Fundamental Theorem of Line Integrals)**.** *If a vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$ is conservative with potential $f : \mathcal{D} \to \mathbb{R}$, then for any oriented curve $\mathcal{C} \subset \mathcal{D}$ with the initial point $\mathbf{r}_1$ and the terminal point $\mathbf{r}_2$,*

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1)\,,$$

whenever the line integral is defined for such a $\mathcal{C}$.

For closed curves $\mathcal{C}$, i.e., curves such that $\mathbf{r}_1 = \mathbf{r}_2$,

$$\oint_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0\,.$$

Proof. Let $\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r})$. Then if $\mathbf{r}(t)$ is any parameterization of $\mathcal{C}$ defined on an interval $I = [a, b]$ with $\mathbf{r}(a) = \mathbf{r}_1$ and $\mathbf{r}(b) = \mathbf{r}_2$, by the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\mathbf{r}(t)\big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(\mathbf{r}(t)\big)\dot{x}_i(t) = \nabla f\big(\mathbf{r}(t)\big) \cdot \dot{\mathbf{r}}(t)\,.$$

On the other hand:

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \dot{\mathbf{r}}(t)\,\mathrm{d}t = \int_{\mathcal{C}} \nabla f\big(\mathbf{r}(t)\big) \cdot \dot{\mathbf{r}}(t)\,\mathrm{d}t\,.$$

Thus, by the second fundamental theorem of calculus,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f\big(\mathbf{r}(t)\big)\,\mathrm{d}t \\ &= f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big) \\ &= f(\mathbf{r}_2) - f(\mathbf{r}_1)\,.\end{aligned}$$

As the path was arbitrary and the final difference depends only upon the potential and the endpoints together with the orientation determining which is initial and final, we conclude that for any conservative vector field $\mathbf{F}$, the line integral

$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

is independent of path.

Finally, if $\mathcal{C}$ is closed, then this final expression is a difference of identical terms, and must therefore be zero. □

We can formalize the idea of path independence for vector fields as follows:

Definition. Given a fixed vector field $\mathbf{F}$ defined on a domain $\mathcal{D}$, a line integral $\int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$ is said to be path independent if its value depends only upon the endpoints and orientation of $\mathcal{C} \subset \mathcal{D}$, and not on the particular path $\mathcal{C}$.

A vector field $\mathbf{F}$ defined on a domain $\mathcal{D}$ is said to have the property of *independence of path* in $\mathcal{D}$ if for any curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{D}$, whose initial endpoints coincide and whose terminal endpoints coincide, the equality

$$\mathcal{W}[\mathbf{F}, \mathcal{C}_1] = \mathcal{W}[\mathbf{F}, \mathcal{C}_2]$$

holds; equivalently, the work of $\mathcal{F}$ on any particle undergoing motion on a trajectory in $\mathcal{D}$ depends only upon the starting and ending points of the particle's motion.

Example 3.10. Let $\mathbf{F}(x,y) = (y^2 - x^2)\hat{\imath} + 2xy\hat{\jmath}$, and let $\mathcal{C}$ be the curve given by the portion of the graph of $y = 2\ln\cos x + \ln 4 - \ln 3$ with nonnegative y coordinates, starting on the positive x axis and terminating on the negative x axis. What is the work done by $\mathbf{F}$ on a particle moving from a starting position on the positive x axis and ending on the y axis, following $\mathcal{C}$?

In this case, $\mathbf{F}$ happens to be conservative. Indeed, if we partially integrate we find:

$$\int y^2 - x^2 \,\mathrm{d}x = xy^2 - \frac{x^3}{3} + C(y)\,,$$

for some $C(y)$. Taking the derivative of this partial integral with respect to y, we have

$$\frac{\partial}{\partial y}\left(xy^2 - \frac{x^3}{3} + C(y)\right) = 2xy + C'(y)\,.$$

Comparing with $\mathbf{F}\cdot\hat{\jmath}$, we see that we should take $C'(y) = 0$, and so $C(y)$ can really be chosen as any constant. Thus, $f(x,y) = xy^2 - \frac{x^3}{3}$ is a potential for $\mathbf{F}(x,y)$.

Since $\mathbf{F}$ is conservative, we only need to find the endpoints of our curve. Setting $y = 0$ gives that $\ln\cos x = \ln\frac{\sqrt{3}}{2}$. There are many such points, but as we want $\mathcal{C}$ to be the portion of the graph which travels from the $+x$-axis to the $-x$-axis, we have to pick the least positive value of x satisfying $\cos x = \sqrt{3}/2$ to get the initial point; using that the graph has y-axis symmetry, we can then find the terminal point by negating this x. Of course $x = \pi/6$ is the least such value. Thus our endpoints are $(\pi/6, 0)$ and $(-\pi/6, 0)$.

Thus, using the fundamental theorem of line integrals

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}(x,y)\cdot\,\mathrm{d}\mathbf{r} &= f(-\pi/6,0) - f(\pi/6,0)\\ &= 0 - \frac{1}{3}\left(-\frac{\pi}{6}\right)^3 - 0 + \frac{1}{3}\left(\frac{\pi}{6}\right)^3\\ &= \frac{\pi^3}{324}\,.\end{aligned}$$

Example 3.11. Let C be one arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}$, starting at the origin and ending at the point $(2\pi, 0)$. Consider the line integral

$$\int_{\mathcal{C}} e^{-y^2}\,\mathrm{d}x - 2xye^{-y^2}\,\mathrm{d}y\,.$$

The differential $e^{-y^2}\,\mathrm{d}x - 2xye^{-y^2}\,\mathrm{d}y$ is actually the total differential $\mathrm{d}f(x,y)$ of the function $f(x,y) = xe^{-y^2}$. Thus

$$\begin{aligned}\int_{\mathcal{C}} e^{-y^2}\,\mathrm{d}x - 2xye^{-y^2}\,\mathrm{d}y &= \int_{\mathcal{C}} \left(e^{-y^2}\,\hat{\imath} - 2xye^{-y^2}\,\hat{\jmath}\right)\cdot\,\mathrm{d}\mathbf{r}\\ &= \int_{\mathcal{C}} \nabla\left(xe^{-y^2}\right)\cdot\,\mathrm{d}\mathbf{r}\\ &= f(2\pi,0) - f(0,0)\\ &= 2\pi\,.\end{aligned}$$

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§ 3.5. Motion in Conservative Force Fields Conserves Energy

Equipped with the fundamental theorem of line integrals, we are finally in a position to explain the terminology "conservative vector field" and "potential" in terms of a connection to the physics of conservation of energy. Let $\mathbf{F}(\mathbf{r})$ be a force field acting on a particle with a trajectory $\mathcal{C}$ described by the time-dependent vector-valued function $\mathbf{r}(t)$, beginning at $\mathbf{r}(t_0)$ and ending at $\mathbf{r}(t_1)$, $t_0 \le t \le t_1$. If $\mathbf{F}$ represents the *net force* acting on the particle, then by Newton's second law of motion, the trajectory of the particle is determined by an initial value problem:

$$\mathbf{F}(\mathbf{r}(t)) = m\ddot{\mathbf{r}}(t)\,,\quad \dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0\,,\quad \mathbf{r}(t_0) = \mathbf{r}_0\,,$$

where m is the particle's mass, $\dot{\mathbf{r}}_0$ is its initial velocity, and $\mathbf{r}_0$ is the starting point of the trajectory measured at time $t = t_0$. Now, the net work of $\mathbf{F}$ on the particle is

$$\begin{aligned}\mathcal{W}[\mathbf{F},\mathcal{C}] = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot \mathrm{d}\mathbf{r} &= \int_{t_0}^{t_1} m\ddot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\big)\,\mathrm{d}t \\ &= \frac{m}{2}\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\Big|_{t_0}^{t_1}.\end{aligned}$$

We recognize the result as a difference in kinetic energy values for the particle at the end and beginning of the its trajectory. Writing $v_i = \|\dot{\mathbf{r}}(t_i)\|$, $i = 0, 1$ for the initial and final speeds of the particle, we can express the net work as

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \Delta K\,,$$

where ΔK is the net change in kinetic energy.

Since $\mathbf{F}$ is conservative, we can repeat this calculation using that $\mathbf{F}(\mathbf{r}) = -\nabla P(\mathbf{r})$ for a scalar field $P(\mathbf{r})$ called the *potential energy*. By the fundamental theorem of line integrals

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \int_{\mathcal{C}} -\nabla P(\mathbf{r})\cdot\mathrm{d}\mathbf{r} = -\big(P(\mathbf{r}_1) - P(\mathbf{r}_0)\big) = P(\mathbf{r}_0) - P(\mathbf{r}_1)\,.$$

Writing $\Delta P = P(\mathbf{r}_1) - P(\mathbf{r}_0)$, we have that

$$\Delta K = \mathcal{W}[\mathbf{F},\mathcal{C}] = -\Delta P\,.$$

The *net change in total energy* is $\Delta E := \Delta K + \Delta P$, and by the above equality $\Delta K = -\Delta P$ we see that for $\mathbf{F}$ conservative, $\Delta E = 0$, which is the statement of the principle of *conservation of energy.*

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§ 3.6. Path Independence and Corollaries of the Fundamental Theorem

In the final portion of this section, we explore the connections between path independence of vector fields and the existence of potentials. The following result is a corollary of the fundamental theorem of line integrals:

Proposition 3.3. *Let $\mathbf{F} : \mathcal{D}$ be a continuous vector field on an open path-connected domain of $\mathbb{R}^2$. Suppose $\mathbf{F}$ is independent of path in $\mathcal{D}$. Then $\mathbf{F}$ is conservative.*

Proof. Fix a point $\mathbf{r}_0 \in \mathcal{D}$, and for any $\mathbf{r} \in \mathcal{D}$, select a path $\mathcal{C}_{\mathbf{r}}$ starting at $\mathbf{r}_0$ and ending at $\mathbf{r}$. Let

$$f(\mathbf{r}) = \int_{\mathcal{C}_{\mathbf{r}}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}\,.$$

Since $\mathbf{F}$ is path independent throughout $\mathcal{D}$, this is well defined independent of the choice of a path $\mathcal{C}_{\mathbf{r}}$. We now must show that $\nabla f(\mathbf{r}) = \mathbf{F}(\mathbf{r})$. This detail is left to (7) in the problems below. □

Recall that for a continuously differentiable 2-dimensional conservative vector field $\mathbf{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$, the component functions P and Q satisfy the partial differential equation

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}\,.$$

We now explore the conditions under which a converse holds. Namely, we can give a condition on a domain $\mathcal{D}$ of $\mathbf{F}$ such that given a continuously differentiable vector field $\mathbf{F}(x, y)$ whose components $P(x, y)$ and $Q(x, y)$ satisfy $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $\mathcal{D}$, $\mathbf{F}$ will be conservative over $\mathcal{D}$.

First, we need a pair of definitions. The first pertains to curves, and the second to domains. Both are. *topological* in nature; they concern properties of curves and domains that are invariant under continuous deformations, but do not depend on the exact geometric shapes involved.

Definition. A curve $\mathcal{C} \subset \mathbb{R}^2$ is said to be a *simple closed curve* if it is a closed curve that admits no self-intersections. Recall, a closed curve in $\mathbb{R}^2$ is one for which there exists a continuous parameterization $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ such that $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. Equivalently, a continuous closed curve can be viewed as a map from the circle $\mathbb{S}^1 = \{\mathbf{r} \in \mathbb{R}^2 : \|\mathbf{r}\| = 1\}$ to $\mathbb{R}^2$, and so a curve is simple if it can be realized as a continuous *embedded* image of the circle, meaning that the map from $\mathbb{S}^1$ admits a continuous inverse from its image.

A famous theorem regarding the topology of simple closed curves in the plane bears mentioning:

Theorem 3.2 (The Jordan Curve Theorem)**.** *Any simple closed curve $\mathcal{C}$ in $\mathbb{R}^2$ divides the plane into two disjoint regions, called* the interior of $\mathcal{C}$ *and the* exterior of $\mathcal{C}$. *The interior* $\operatorname{int}(\mathcal{C})$ *is a bounded region of $\mathbb{R}^2$ (there exists a disk of sufficiently large radius which covers* $\operatorname{int}(\mathcal{C})$*) with boundary $\partial\operatorname{int}(\mathcal{C}) = \mathcal{C}$, while the exterior* $\operatorname{ext}(\mathcal{C})$ *is an unbounded region, with boundary $\partial\operatorname{ext}(\mathcal{C}) = \mathcal{C}$.*

Despite the intuitive nature of this theorem, it is quite difficult to prove, and belongs to the study of topology. However, we'd like to be able to assume its result in the remainder of our discussion of domains of $\mathbb{R}^2$.

Definition. A connected plane region $\mathcal{D} \subseteq \mathbb{R}^2$ is said to be *simply connected* if the interior of every simple closed curve in $\mathcal{D}$ is contained entirely in $\mathcal{D}$. That is, $\mathcal{D}$ is simply connected if and only if it is connected and given any continuous embedding $\boldsymbol{\gamma} : \mathbb{S}^1 \to \mathcal{D}$, with image $\mathcal{C} = \boldsymbol{\gamma}(\mathbb{S}^1)$, $\operatorname{int}(\mathcal{C}) \subset \mathcal{D}$.

Example 3.12. By definition, the interior of a simple closed curve in $\mathbb{R}^2$ is itself a simply connected region, and so in particular, any disk is simply connected. On the other hand, a *punctured disk*, like $\{\mathbf{r} \in \mathbb{R}^2 : 0 < \|\mathbf{r}\| < 1\}$ is not simply connected, nor is any annular region $\{\mathbf{r} \in \mathbb{R}^2 : a \le \|\mathbf{r} - \mathbf{r}_0\| \le b\}$. Simple-connectivity can be colloquially stated as the property that the a connected region is "free of holes." Another way to describe it is that any closed curve within a simply connected region $\mathcal{D}$ can be contracted to a single point without tracing through any points not lying within $\mathcal{D}$.

Proposition 3.4. *Suppose a continuously differentiable vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^2$ is defined over a simply connected region $\mathcal{D}$. Then $\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{\imath}} + Q(x, y)\hat{\mathbf{\jmath}}$ is conservative in $\mathcal{D}$ if and only if*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

holds throughout $\mathcal{D}$.

We will defer the proof of this result to our discussion of Green's Theorem.

Another interesting application of line integrals of differential forms is that we can use them to express the area of a closed region as a line integral. Let $\mathcal{C}$ be a continuous simple closed curve, oriented counter-clockwise. $\mathcal{C}$ bounds a region $\mathcal{D}$ whose area $\mathcal{A}$ we wish to express using a line integral (rather than a double integral).

Select $m + 1$ points $(x_0, y_0), \ldots (x_m, y_m)$ spaced around $\mathcal{C}$. Let $\Delta x_j = x_j - x_{j-1}$ and $\Delta y_j = y_j - y_{j-1}$. As one might recall from single variable calculus, $\sum_{j=1}^m y_j \Delta x_j$ and $\sum_{j=1}^m x_j \Delta y_j$ are sums of signed areas of rectangles bounded by the x and y axes respectively, which can be used to approximate the area inside $\mathcal{C}$. In particular, you can cut the region $\mathcal{D}$ into pieces which are either Type I or Type II regions, meaning that they are described as areas between curves that are either locally graphs $y = f(x)$ or $x = g(y)$. As $m \to \infty$, we can ensure that all regions are both type I and type II, except for a *vanishingly small proportion*. It follows that each of the integrals

$$\oint_{\mathcal{C}} x \, \mathrm{d}y \quad \text{and} \quad \oint_{\mathcal{C}} -y \, \mathrm{d}x$$

give the area of the region $\mathcal{D}$. Note the minus sign needed for the second integral: since $\mathcal{C}$ has a counterclockwise orientation, x is decreasing for the "upper" portions of $\mathcal{C}$. By averaging these line integrals and employing property (i), we arrive at the interesting formula for the area $\mathcal{A}(\mathcal{D})$ of the interior $\mathcal{D}$ of the curve $\mathcal{C}$:

$$\mathcal{A}(\mathcal{D}) = \frac{1}{2} \oint_{\mathcal{C}} x \, \mathrm{d}y - y \, \mathrm{d}x \,.$$

One can easily prove this formula using Green's theorem, to be discussed in section 3.7 below.

§ 3.7. Green's Theorem

We can now connect line integrals along closed curves to double integrals over regions bounded by the curves. First, we give the theorem of Green that connects line integrals along simple closed curves in domains of $\mathbb{R}^2$ to area integrals over the interiors of simple closed curves.

Theorem 3.3 (Green's Theorem for simply connected regions)**.** *Let $\mathcal{C} \subset \mathbb{R}^2$ be a piecewise smooth, simple closed plane curve oriented counterclockwise. Suppose $P(x, y)$ and $Q(x, y)$ are continuously differentiable functions on an open set $\mathcal{R}$ containing $\mathcal{C}$ and such that* $\mathrm{int}(\mathcal{C}) \subset \mathcal{R}$. *Then*

$$\oint_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_{\mathrm{int}(\mathcal{C})} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A}\,.$$

Before partially proving Green's theorem, we show two example applications of the theorem.

Example 3.13. Let $\mathbf{F}(x, y) = \langle \sin(\pi y) - e^{-x^2}, e^{y^2} + \cos(\pi y) + \cos(\pi x/4)\rangle$, and let $\mathcal{S}$ be the unit square $[0, 1] \times [0, 1] = \{(x, y) : 0 \le x \le 1\,,\ 0 \le y \le 1\}$. We can use Green's Theorem to calculate

$$\oint_{\partial\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_{\partial\mathcal{S}} \left(\sin(\pi y) - e^{-x^2}\right)\mathrm{d}x + \left(e^{y^2} + \cos(\pi y) + \cos(\pi x/4)\right)\mathrm{d}y\,,$$

without resorting to computing a difficult collection of line integrals along each of the sides of the square. Note that the function is continuously differentiable on $\mathbb{R}^2$, and so also on $\mathcal{S} \subset \mathbb{R}^2$. Observe also that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right) - \pi\cos(\pi y)\,.$$

Thus, applying Green's theorem:

$$\begin{aligned}\int_{\partial\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_{\mathcal{S}} -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right) - \pi\cos(\pi y)\,\mathrm{d}\mathcal{A}\\ &= \int_0^1\int_0^1 -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right) - \pi\cos(\pi y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^1 -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)\,\mathrm{d}x\\ &= \frac{\pi}{4}\cos\frac{\pi}{4} - \frac{\pi}{4}\cos 0 = \frac{\pi(\sqrt{2}-2)}{8}\,.\end{aligned}$$

Example 3.14. We will compute the line integral

$$\oint_{\mathcal{C}} \left(\cos(x^2) - 4y^3\right)\mathrm{d}x + \left(\sqrt{1+y^3} + 4x^3\right)\mathrm{d}y\,,$$

where $\mathcal{C}$ is the curve bounding the semi-annular region $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \,|\, 1 \le x^2 + y^2 \le 4,\ y \ge 0\}$ illustrated in figure 34.

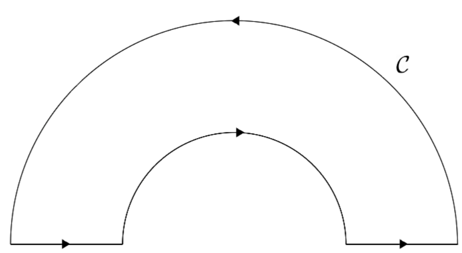

FIGURE 34. The semiannular region of integration and oriented boundary curve $\mathcal{C}$.

Observe that $\mathcal{C}$ has four pieces, and at least two of them yield particularly difficult integrals if we use standard parameterizations. However, $\mathcal{C}$ bounds a polar rectangle, and

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 12x^2 + 12y^2 = 12r^2\,,$$

so that by Green's theorem

$$\oint_{\mathcal{C}} (e^{-x^2} - 4y^3)\,\mathrm{d}x + (\sqrt{1+y^3} + 4x^3)\,\mathrm{d}y = 12 \iint_{\mathcal{D}} x^2 + y^2\,\mathrm{d}\mathcal{A}\,.$$

This double integral is simple to evaluate:

$$\begin{aligned} 12 \iint_{\mathcal{D}} x^2 + y^2\,\mathrm{d}\mathcal{A} &= 12 \int_0^{\pi} \int_1^2 r^3\,\mathrm{d}\mathcal{A} \\ &= 12\pi \frac{r^4}{4}\Big|_1^2 = 12\pi(4 - \tfrac{1}{4}) = 45\pi\,. \end{aligned}$$

To prove Green's theorem we'll need the following lemma:

Lemma. *Suppose a simply connected region $\mathcal{D}$ is bounded by a piecewise smooth curve $\mathcal{C}$. Then there exists a decomposition of $\mathcal{D}$ into finitely many subregions $\mathcal{D}_i$ with boundaries $\partial\mathcal{D}_i = \mathcal{C}_i$ such that each $\mathcal{D}_i$ is expressible as both a type I region and a type II region, and such that*

$$\oint_{\mathcal{C}} \omega = \sum_i \oint_{\mathcal{C}_i} \omega\,,$$

for any continuous differential form ω defined throughout an open set containing $\mathcal{D}$.

Proof sketch. Orient $\mathcal{C}$ counterclockwise. Divide the region $\mathcal{D}$ by cutting along any simple curve $\mathcal{K} \in \mathcal{D}$ connecting a pair of points A and B on $\mathcal{C}$. We can realize $\mathcal{C}$ as a piecewise curve: $\mathcal{C} = \mathcal{C}_{\overrightarrow{AB}} + \mathcal{C}_{\overrightarrow{BA}}$ where $\mathcal{C}_{\overrightarrow{AB}}$ is the piece of $\mathcal{C}$ starting at A and terminating at B, while $\mathcal{C}_{\overrightarrow{BA}}$ starts at B and terminates at A. Orient $\mathcal{K}$ so that $\mathcal{C}_1 := \mathcal{C}_{\overrightarrow{AB}} + \mathcal{K}$ is a counterclockwise simple closed curve. Let $\mathcal{C}_2 = \mathcal{C}_{\overrightarrow{AB}} - \mathcal{K}$, and observe this is also counterclockwise oriented. Now, for any continuous differential form defined on $\mathcal{D}$:

$$\begin{aligned} \oint_{\mathcal{C}} \omega = \int_{\mathcal{C}_{\overrightarrow{AB}} + \mathcal{C}_{\overrightarrow{BA}}} \omega &= \int_{\mathcal{C}_{\overrightarrow{AB}}} \omega + \int_{\mathcal{C}_{\overrightarrow{BA}}} \omega + \int_{\mathcal{K}} \omega - \int_{\mathcal{K}} \omega \\ &= \int_{\mathcal{C}_{\overrightarrow{AB}}} \omega + \int_{\mathcal{K}} \omega + \int_{\mathcal{C}_{\overrightarrow{BA}}} \omega + \int_{-\mathcal{K}} \omega \\ &= \oint_{\mathcal{C}_{\overrightarrow{AB}} + \mathcal{K}} \omega + \oint_{\mathcal{C}_{\overrightarrow{BA}} - \mathcal{K}} \omega \\ &= \oint_{\mathcal{C}_1} \omega + \oint_{\mathcal{C}_2} \omega \end{aligned}$$

This shows that any simply connected region $\mathcal{D}$ decomposes into simply connected subregions such that the line integrals of a form ω over the positively oriented boundaries of the subregions sum to the line integral over the positively oriented boundary $\partial\mathcal{D}$. It follows that if we cut $\mathcal{D}$ into finitely many regions $\mathcal{D}_i$, then

$$\oint_{\mathcal{C}} \omega = \sum_i \oint_{\mathcal{C}_i} \omega\,,$$

for any continuous differential form ω defined throughout $\mathcal{D}$. It remains to show we can perform the cuts so that all the $\mathcal{D}_i$ are regions which can be expressed both as type I and type II regions. Call such a region *elementary*. Note that rectangles with edges parallel to the x and y coordinate axes are elementary.

We can show, in a method reminiscent of using Riemann sums to define integrals, that $\mathcal{D}$ is covered and approximated by a region composed of rectangles, and the boundary of this collection of rectangles is itself an approximation of $\mathcal{C} = \partial\mathcal{D}$. At the boundary, replace rectangles $\mathcal{R}_i$ which cover portions of $\mathcal{C}$ by $\mathcal{R}_i \cap \mathcal{D}$; this at worst alters one or more boundaries of $\mathcal{R}_i$ by incorporating points on the boundary curve and portions of the boundary curve into the boundary of the new, smaller region. We can choose a sufficiently fine partition of $\mathcal{D}$ into subregions such that all such boundary-adjacent regions are elementary. This last step relies on the assumption that $\mathcal{C}$ is piecewise smooth, and so admits a decomposition into finitely many smooth pieces. That the resulting edge regions can be chosen to be elementary requires some deep results from topology involving compactness and smoothness: each smooth piece of the curve $\mathcal{C}$ is compact, and smoothness + compactness[17] guarantee that the number of places where tangent lines are either vertical or horizontal is finite, and these points are therefore isolated. It follows that we can choose a set of cuts to ensure that each subregion adjacent to the boundary can be chosen to be simple.

□

We will now begin to outline the proof of Green's Theorem.

Proof. Let $\mathcal{D}$ be the closure of the interior of $\mathcal{C}$, i.e. $\mathcal{D} = \mathcal{C} \cup \mathrm{int}(\mathcal{C})$. With the above lemma, it now suffices to prove that Green's theorem holds for elementary regions. Let $\mathcal{E}$ be a closed, simply connected, elementary region with $\partial\mathcal{E}$ a piecewise smooth, simple closed curve, and $P(x,y)$, $Q(x,y)$ continuously differentiable functions over an open set $\mathcal{U}$ containing $\mathcal{E}$. Since $\mathcal{E}$ is elementary, there are constants $a, b, c, d \in \mathbb{R}$ and functions $f_i : [a,b] \to \mathbb{R}$, $g_i : [c,d] \to \mathbb{R}^2$, $i = 1, 2$, such that

$$\mathcal{E} = \{(x,y) \in \mathbb{R}^2 : a \le x \le b\,,\ f_1(x) \le y \le f_2(x)\} = \{(x,y) \in \mathbb{R}^2 : c \le y \le d\,,\ g_1(y) \le x \le g_2(y)\}\,.$$

Then

$$\begin{aligned}
\iint_{\mathcal{E}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A} &= \iint_{\mathcal{E}} \frac{\partial Q}{\partial x}\,\mathrm{d}\mathcal{A} - \iint_{\mathcal{E}} \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A}\\
&= \int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y - \int_a^b \int_{f_1(y)}^{f_2(x)} \frac{\partial P}{\partial y}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_c^d Q\big(g_2(y), y\big) - Q\big(g_1(y), y\big)\,\mathrm{d}y - \int_a^b P\big(x, f_2(x)\big) - P\big(x, f_1(x)\big)\,\mathrm{d}x\\
&= \oint_{\partial\mathcal{E}} Q(x,y)\,\mathrm{d}y - \oint_{\partial\mathcal{E}} -P(x,y)\,\mathrm{d}x\\
&= \oint_{\partial\mathcal{E}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y\,,
\end{aligned}$$

where the equalities

$$\begin{aligned}
\int_c^d Q\big(g_2(y), y\big) - Q\big(g_1(y), y\big)\,\mathrm{d}y &= \oint_{\partial\mathcal{E}} Q(x,y)\,\mathrm{d}y\,, \text{and}\\
\int_a^b P\big(x, f_2(x)\big) - P\big(x, f_1(x)\big)\,\mathrm{d}x &= \oint_{\partial\mathcal{E}} -P(x,y)\,\mathrm{d}x
\end{aligned}$$

follow from the elementary cases handled in (11) in the problems below. □

Note that the integrand $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ of the double integral in Green's theorem is identically zero if $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ is the total differential of some scalar function $f(x,y)$. Put another way, we can rewrite the equation of Green's theorem for a line integral over a piecewise smooth simple closed curve $\mathcal{C}$ in a continuously differentiable vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\hat{\imath} + Q(\mathbf{r})\hat{\jmath}$ defined on an open set containing $\mathcal{D} := \mathrm{int}(\mathcal{C})$:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A}\,.$$

If $\mathbf{F}$ is conservative, then we know that the left hand side is zero, while the right hand side is also clearly zero as the integrand vanishes. Since $\mathrm{int}(\mathcal{C})$ is simply connected, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ being zero throughout would also imply that $\mathbf{F}$ was conservative and path independent in $\mathcal{D}$. This gives rise to the following interpretation when $\frac{\partial Q}{\partial x} \ne \frac{\partial P}{\partial y}$: the integrand $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ of the double integral is a differential that gives a measure of the failure of $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\hat{\imath} + Q(\mathbf{r})\hat{\jmath}$ to be conservative, since it is zero throughout a simply connected region if and only if $\mathbf{F}$ is conservative. The area integral

[17]This is essentially a result from *Morse Theory* (named for Marston Morse), though it is closely related to the Morse-Sard theorem (named for the unrelated Anthony Morse who proved the 1-dimensional version, and Arthur Sard who generalized it). The formal statements concern critical points of smooth functions, but the take away is that for a compact smooth curve $\mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ neither $\dot{y}/\dot{x}$ nor $\dot{x}/\dot{y}$ can admit infinitely many zeros or singularities.

then accumulates this differential over the interior of the curve $\mathcal{C}$, and the result is precisely the work the field does moving a particle counterclockwise around this curve $\mathcal{C}$! This is reminiscent of the usual fundamental theorem of calculus: we pass from an integral on the *interior* of a set to an integral that produces a value associated to the *boundary* of the set. The thing we evaluate on the boundary has to be, in some sense, an anti-derivative of the thing evaluated on the interior. Perhaps a better perspective is to think that accumulated change of a function along a boundary of a simply-connected region is related to the accumulated change of a *differential* of that function on the interior of the region.

In light of this interpretation as a connection between integrals of vector fields on boundaries and integrals of a derivative object associated to them on interiors, Green's theorem is commonly rewritten as

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A} = \oint_{\partial\mathcal{D}} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}\,,$$

where $\mathcal{D}$ is taken as any simply connected region such that $\mathbf{F}$ is continuously differentiable on an open set containing $\mathcal{D}$, and $\partial\mathcal{D}$ is the boundary curve of $\mathcal{D}$.

We can extend Green's theorem to regions which aren't simply connected, as long as we appropriately orient boundaries. For example, if a region $\mathcal{D}$ has an "inner" boundary and an "outer boundary", we should orient the outer boundary counterclockwise, and the inner boundary clockwise. If a region is bounded by a closed, possibly disconnected curve which is not simple, then one can break it up into pieces which are bounded by a collection of simple closed curves. This gives rise to the general form of Green's theorem for a bounded plane region $\mathcal{D}$ with oriented boundary $\partial\mathcal{D} = \sum_i \mathcal{C}_i$, where each $\mathcal{C}_i$ is orientated such that points of $\mathcal{D}$ lie to the left of a particle following $\mathcal{C}_i$ with its orientation:

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A} = \sum_i \oint_{\mathcal{C}_i} P\,\mathrm{d}x + Q\,\mathrm{d}y\,.$$

We can now appeal to Green's Theorem to sketch a proof of the proposition in the previous section:

Proposition 3.5. *Suppose a continuously differentiable vector field $\mathbf{F}:\mathcal{D}\to\mathbb{R}^2$ is defined over a simply connected region $\mathcal{D}$. Then $\mathbf{F}(x,y) = P(x,y)\hat{\mathbf{\imath}} + Q(x,y)\hat{\mathbf{\jmath}}$ is conservative in $\mathcal{D}$ if and only if*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

holds throughout $\mathcal{D}$.

Proof. Let $\mathcal{C}$ be any closed piecewise smooth curve within the region $\mathcal{D}$. Since $\mathcal{D}$ is simply connected, $\mathcal{C}$ decomposes into a finite collection of simple closed curves $\mathcal{C}_i$ bounding simply connected subregions $\mathcal{D}_i$ inside $\mathcal{D}$. By Green's theorem, along any such $\mathcal{C}_i$ taken with counter-clockwise orientation:

$$\oint_{\mathcal{C}_i} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\mathcal{D}_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A} = 0\,,$$

since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout $\mathcal{D} \supseteq \mathcal{D}_i$. It follows that, given appropriate orientations realizing $\mathcal{C} = \sum_i \mathcal{C}_i$:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \sum_i \oint_{\mathcal{C}_i} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0\,,$$

whence $\oint_{\mathcal{C}} \mathbf{F}\,\mathrm{d}\mathbf{r} = 0$ for any closed curve $\mathcal{C}\subset\mathcal{D}$. But then, considering a pair of arbitrary paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{D}$, both emanating from $\mathbf{r}_0\in\mathcal{D}$ and terminating in $\mathbf{r}_1\in\mathcal{D}$, we have that

$$\int_{\mathcal{P}_1} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} - \int_{\mathcal{P}_2} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \oint_{\mathcal{P}_1-\mathcal{P}_2} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = 0\,,$$

Since the union of the paths with one taking the opposite orientation is itself a closed curve. This implies that

$$\int_{\mathcal{P}_1} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_{\mathcal{P}_2} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$$

But then, since $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathcal{P}_1$ and $\mathcal{P}_2$ were all arbitrary, it follows that $\mathbf{F}$ is path independent, and thus conservative, throughout $\mathcal{D}$. □

We now discuss a corollary of Green's theorem that allows us to compute planar areas via line integrals.

Corollary. *Suppose $\mathcal{C}$ is a piecewise smooth closed curve bounding a collection of simply connected regions $\mathcal{D}_i$, $i = 1, \dots, k$, disjoint except at "corners" where $\mathcal{C}$ self-intersects. Let $\mathcal{A}_i$ be the area of the region $\mathcal{D}_i$, and $\mathcal{C}_i = \partial\mathcal{D}_i$, oriented counterclockwise. Writing $\mathcal{C} = \sum_{i=1} \varepsilon_i \mathcal{C}_i$, where $\varepsilon_i = +1$ if the orientation of $\mathcal{C}_i$ agrees with that of $\mathcal{C}$, and -1 otherwise, one can express the* signed area $\mathcal{A} = \sum_{i=1}^k \varepsilon_i \mathcal{A}_i$ *of the regions as*

$$\mathcal{A} = \oint_C x\,\mathrm{d}y = \oint_C -y\,\mathrm{d}x = \frac{1}{2}\oint_{\mathcal{C}} x\,\mathrm{d}y - y\,\mathrm{d}x\,.$$

The simplest case of this proposition is when $\mathcal{C}$ is a simple closed curve bounding a simply connected region $\mathcal{D}$, of area $\mathcal{A}$, in which case the geometric (positive) area is recovered by any of the line integrals above. We prove only this simpler case; the general case follows by subdivision and repeated applications of Green's theorem, minding orientations for each region.

Proof. If $\mathcal{C} = \partial\mathcal{D}$ is a piecewise smooth simple closed curve bounding a simply connected region $\mathcal{D}$. Then Observe that by Green's theorem:

$$\begin{aligned}\iint_{\mathcal{D}} \mathrm{d}\mathcal{A} &= \iint_{\mathcal{D}} 1\,\mathrm{d}\mathcal{A}\\ &= \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}(x) - 0\right]\,\mathrm{d}\mathcal{A} = \oint_{\mathcal{C}} x\,\mathrm{d}y\\ &= \iint_{\mathcal{D}} \left[0 - \frac{\partial}{\partial y}(-y)\right]\,\mathrm{d}\mathcal{A} = \oint_{\mathcal{C}} -y\,\mathrm{d}x\\ &= \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{2}\right)\right]\,\mathrm{d}\mathcal{A} = \frac{1}{2}\oint_{\mathcal{C}} x\,\mathrm{d}y - y\,\mathrm{d}x\,.\end{aligned}$$

□

Example 3.15. We'll compute the area bounded by the ellipse $\mathcal{E}$ with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using a line integral. Observe that one can parameterize this ellipse by $\mathbf{r}(t) = a\cos(t)\hat{\mathbf{\imath}} + b\sin(t)\hat{\mathbf{\jmath}}$, $0 \le t \le 2\pi$. Then the area $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\oint_{\mathcal{E}} x\,\mathrm{d}y - y\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{2\pi} (a\cos t)\,\mathrm{d}(b\sin t) - (b\sin t)\,\mathrm{d}(a\cos t)\\ &= \frac{1}{2}\int_0^{2\pi} \left[ab\cos^2(t) + ab\sin^2(t)\right]\mathrm{d}t\\ &= \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}t\\ &= \pi ab\,.\end{aligned}$$

§ 3.8. Problems

(1) Compute the following line integrals by parametrizing the given curves.

(a) $\mathcal{C}$ is the portion of the parabolic graph $y = 3x - x^2$ in the first quadrant,

$$\int_{\mathcal{C}} \frac{(3-2x)^2 + y + 1}{\sqrt{1+(3-2x)^2}}\,\mathrm{d}s\,.$$

(b) $\mathcal{C}$ is the pair of line segments from $(1/4, -1/4)$ to $(1,1)$, and from $(1,1)$ to $(-1/2, 1/2)$,

$$\int_{\mathcal{C}} \cos(\pi x)\,\mathrm{d}x + \sin(\pi y)\,\mathrm{d}y\,.$$

(c) The line integral computing the area of the vertical ribbon above the circle of radius R centered on the $+x$-axis that passes through the origin, and below the upper surface of the cylinder $(x-R)^2 + z^2 = R^2$.

(d) $\mathcal{T}$ is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise,

$$\int_{\mathcal{T}} y e^{\sin x}\cos x\,\mathrm{d}x + (e^{\sin x} - \sin y)\,\mathrm{d}y\,.$$

(2) For each of the above line integrals in (1), exhibit a vector field so that the integral is realized as the work by the field on a particle with a trajectory along the given curve.

(3) Determine whether the vector fields discovered in (2) are conservative. If a field is conservative, specify a maximal domain over which that field is conservative and find a potential function for it, and use this potential to verify the results from problem (1). If a field is not conservative, show explicitly that no potential can exist.

(4) Prove the proposition on the independence of parameterization of a line integral. Namely, show that for any differential one-form ω on $\mathbb{R}^n$ either of the form $F(\mathbf{r})\,\mathrm{d}s$ or $G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$, if $\boldsymbol{\gamma} : I \to \mathcal{D}$ and $\boldsymbol{\eta} : J \to \mathcal{D}$ are two continuous vector-valued functions with a common image curve $\mathcal{C} \subset \mathcal{D}$, traversed exactly once by each parameterization (with the same orientation, if $\omega = G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$) and if the line integrals of ω over $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ both exist, then

$$\int_{\boldsymbol{\gamma}} \omega = \int_{\boldsymbol{\eta}} \omega\,.$$

Hint: Appeal to the change of variables theorem for Riemann integrals of a single variable (i.e., find suitable substitutions) to show how to transform an integral under change of parameterization.

(5) Prove that scalar line integrals satisfy the properties

(*i*) $\displaystyle\int_{\mathcal{C}} a\omega + b\psi = a\int_{\mathcal{C}} \omega + b\int_{\mathcal{C}} \psi$,

(*ii*) if $\omega = f(\mathbf{r})\,\mathrm{d}s$ then $\displaystyle\int_{-\mathcal{C}} \omega = \int_{\mathcal{C}} \omega$,

(*iii*) if $\omega = G_1(\mathbf{r})\,\mathrm{d}x_1 + \ldots + G_n(\mathbf{r})\,\mathrm{d}x_n$, then $\displaystyle\int_{-\mathcal{C}} \omega = -\int_{\mathcal{C}} \omega$,

(*iv*) $\displaystyle\int_{\mathcal{C}+\mathcal{C}'} \omega = \int_{\mathcal{C}} \omega + \int_{\mathcal{C}'} \omega$, for any differential one-forms ω and ψ on a domain $\mathcal{D} \subseteq \mathbb{R}^n$, curves $\mathcal{C}$ and $\mathcal{C}'$ in the domain $\mathcal{D}$, and real constants a and b. As above, $-\mathcal{C}$ denotes the same curve as $\mathcal{C}$ but with opposite orientation, and $\mathcal{C} + \mathcal{C}'$ denotes the oriented curve which is the union of the oriented curves $\mathcal{C}$ and $\mathcal{C}'$.

(6) Let $\mathbf{F}(x, y) = -2xy\,\hat{\mathbf{\imath}} + (x^2 - y^2)\,\hat{\mathbf{\jmath}}$, the topological dipole field seen above in example 3.8.
  (a) Compute the work done by $\mathbf{F}$ on a particle that traverses a full circuit on any of the circles that are field-lines, by parameterizing to evaluate a line integral.
  (b) Recompute the line integral from the previous part using Green's theorem.
  (c) Use Green's Theorem to recompute and verify the line integral of the example above:
$$\oint_{\mathcal{T}} \mathbf{F}(x, y) \cdot \hat{\mathbf{T}}\,\mathrm{d}s = 8\,.$$
  where $\mathcal{T}$ is the triangle in $\mathbb{R}^2$ with vertices $A(0, 0)$, $B(\sqrt{3}, -1)$ and $C(\sqrt{3}, 1)$ oriented counterclockwise.

(7) Finish the proof of the proposition:

Proposition. *Let $\mathbf{F} : \mathcal{D}$ be a continuous vector field on an open path-connected domain of $\mathbb{R}^2$. Suppose $\mathbf{F}$ is independent of path in $\mathcal{D}$. Then $\mathbf{F}$ is conservative.*

In particular, for some fixed $\mathbf{r}_0 \in \mathcal{D}$, and for any $\mathbf{r} \in \mathcal{D}$, select a path $\mathcal{C}_{\mathbf{r}}$ starting at $\mathbf{r}_0$ and ending at $\mathbf{r}$ let
$$f(\mathbf{r}) = \int_{\mathcal{C}_{\mathbf{r}}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}\,.$$
Use that $\mathcal{D}$ is an open region to describe a path from $\mathbf{r}_0$ to $\mathbf{r}$ that allows you to determine
$$\frac{\partial f}{\partial x}(\mathbf{r})\,,$$
and then relate this to $\hat{\mathbf{\imath}} \cdot \mathbf{F}(\mathbf{r})$ using a parametrization and the fundamental theorem of calculus. Similarly calculate $\partial f/\partial y$ and relate it to $\hat{\mathbf{\jmath}} \cdot \mathbf{F}(\mathbf{r})$ to conclude that $\nabla f(\mathbf{r}) = \mathbf{F}$.

(8) State whether the following are true or false, and provide justification. In particular, either prove the statements or provide counterexamples.
  (a) For a vector field $\mathbf{F}$ continuous in a region $\mathcal{D} \subseteq \mathbb{R}^2$, if $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every piecewise smooth simple closed curve $\mathcal{C}$ in $\mathcal{D}$, then $\mathbf{F}$ is conservative in $\mathcal{D}$.
  (b) A domain $\mathcal{D}$ with piecewise smooth boundary is simply connected if and only if its boundary is a union of simple closed curves whose interiors are non-intersecting.
  (c) If $\mathbf{F}$ is conservative in regions $\mathcal{D}_1$ and $\mathcal{D}_2$, then $\mathbf{F}$ is conservative in $\mathcal{D}_1 \cup \mathcal{D}_2$.
  (d) Suppose $\mathbf{F}$ is a force field that has the path-independence property in $\mathcal{D}$. If there is a curve $\mathcal{C}$ from $\mathbf{r}_0$ to $\mathbf{r}_1$ such that $\mathbf{F}$ is perpendicular to $\mathcal{C}$ at every point along $\mathcal{C}$, then $\mathbf{F}$ does no work in moving a particle from $\mathbf{r}_0$ to $\mathbf{r}_1$.

(9) Let $\mathbf{F}(x,y) = \dfrac{-y}{x^2+y^2}\hat{\mathbf{\imath}} + \dfrac{x}{x^2+y^2}\hat{\mathbf{\jmath}}$.

(a) Let $\mathcal{C}$ be the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, oriented counter-clockwise. Compute

$$\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

by suitably parametrizing $\mathcal{C}$.

(b) Let D be the unit disk bounded by $\mathcal{C}$. Compute the double integral

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}\mathcal{A}\,,$$

and compare to the result of part (b). Why does this not contradict Green's Theorem?

(c) Is $\mathbf{F}$ conservative? If so, specify a maximal domain in $\mathbb{R}^2$ over which it is conservative and find a potential function for $\mathbf{F}$ if one exists. If $\mathbf{F}$ is not conservative, show explicitly that no potential can exist.

(d) Explain the results of (a) and (b) geometrically by arguing that $\mathbf{F}\cdot \mathrm{d}\mathbf{r} = \mathrm{d}\theta$.

(e) Give two proofs, one by Green's theorem, and one by the fundamental theorem of line integrals, that for any simple, closed, rectifiable curve $\mathcal{C} \subset \mathbb{R}^2 - \{(0,0)\}$ not enclosing the origin,

$$\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0\,.$$

(10) Let $\mathbf{F}(x,y) = \dfrac{x-y}{\sqrt{x^2+y^2}}\hat{\mathbf{\imath}} + \dfrac{x+y}{\sqrt{x^2+y^2}}\hat{\mathbf{\jmath}}$.

(a) Sketch the vector field $\mathbf{F}$ over the region $0 < x^2+y^2 \le 16$.

(b) Show $\mathbf{F}$ is not conservative.

(c) Let D be the region in the first quadrant bounded by the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ together with the x and y axes. Let $\mathcal{C} = \partial D$ be the boundary of D. Without integrating, use the geometry of $\mathbf{F}$ over the region D to compute the work done by $\mathbf{F}$ on a particle completing a counter-clockwise circuit around $\mathcal{C}$.

(d) By parametrizing $\mathcal{C}$, explicitly compute the work

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

done by $\mathbf{F}$ on a particle completing a counter-clockwise circuit around $\mathcal{C}$.

(e) Apply Green's Theorem to recompute the work $\mathcal{W}[\mathbf{F},\mathcal{C}]$ as a double integral over D. (Hint: choose appropriate coordinates, and the integral will become quite simple).

(f) Show that

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \oint_{\mathcal{C}} \mathrm{d}r + r\,\mathrm{d}\theta\,,$$

and recompute the work using this line integral, further justifying the calculation of part (c).

(11) Prove that for a simply connected type I region

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : a \le x \le b\,,\ f_1(x) \le y \le f_2(x)\}$$

with boundary a simple closed curve $\partial\mathcal{R}$ that

$$\int_a^b P\big(x, f_2(x)\big) - P\big(x, f_1(x)\big)\,\mathrm{d}x = \oint_{\partial\mathcal{R}} -P(x,y)\,\mathrm{d}x\,.$$

Is it necessary that $\partial\mathcal{R}$ be simple?

Argue similarly that if $\mathcal{R}$ is a simply connected type II region

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : c \le y \le d\,,\ g_1(y) \le x \le g_2(y)\}$$

then

$$\int_c^d Q\big(g_2(y), y\big) - Q\big(g_1(y), y\big)\,\mathrm{d}y = \oint_{\partial\mathcal{R}} Q(x,y)\,\mathrm{d}y\,.$$

This completes the proof of Green's Theorem for elementary regions.

(12) Let $\mathbf{u} = \langle u_1, u_2, 0\rangle$ and $\mathbf{v} = \langle v_1, v_2, 0\rangle$ be two nonzero vectors in $\mathbb{R}^3$ lying in the xy-plane with their tails placed at the origin. These *span* a parallelogram with vertices positioned at $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$. The pair $(\mathbf{u},\mathbf{v})$ is right-handed if, when reading the vertices of the parallelogram off going counterclockwise from $\mathbf{0}$, the order is $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{v}$.

Recall that $\mathbf{u}\times\mathbf{v} = (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$ has length equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, and points in the $+\hat{\mathbf{k}}$ direction if $(\mathbf{u},\mathbf{v},\hat{\mathbf{k}})$ is *right-handed* and in the $-\mathbf{k}$ direction $(\mathbf{u},\mathbf{v},\hat{\mathbf{k}})$ is *left handed* (in which case, $(\mathbf{u},\mathbf{v},-\hat{\mathbf{k}})$ is *right-handed*).

By regarding $\mathbf{u}$ and $\mathbf{v}$ as being in $\mathbb{R}^2$, use Green's Theorem to prove the above statement about $\mathbf{u}\times\mathbf{v}$ by showing that the signed area of the parallelogram is $u_1v_2 - u_2v_1$, with sign positive if and only if $(\mathbf{u},\mathbf{v})$ is right handed. You may use symmetry to simplify your integral calculations.

4. Surface Integrals, Flux, and Fundamental Theorems

§ 4.1. Surface Integrals of Scalar Fields

For a surface $\mathcal{S}$ arising as a graph of a differentiable bivariate function $z = f(x, y)$, the area of the portion of the surface over a region $\mathcal{D}$ in the xy plane is given by the double integral

$$\mathcal{A}(\mathcal{S}) = \iint_{\mathcal{D}} \mathrm{d}\mathcal{A}_{\mathcal{S}} = \iint_{\mathcal{D}} \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,.$$

To arrive at such an expression, one can start by approximating a small patch of $\mathcal{S}$ above a rectangle by a small piece of tangent plane. Forming an appropriate Riemann sum and taking the limit, one arrives at the above expression. One can view this integral expression as a special case of a more general type of multiple integral, called a *surface integral*.

Consider a scalar field $F(x, y, z)$ which is defined at every point of a surface $\mathcal{S}$. In physical applications such a function may represent density, temperature, or a distribution of charges on the surface S. In any case, it is fruitful to be able to integrate a scalar field over a (usually compact) surface. To define such an object, one partitions the surface $\mathcal{S}$ into small pieces S_{ij}, and within each piece, chooses a sample point $\mathbf{r}^*_{ij} = \langle x^*_{ij}, y^*_{ij}, z^*_{ij}\rangle$. Then evaluating a sum of products of the scalar field $F(\mathbf{r})$ evaluated at sample points $\mathbf{r}^*_{ij}$ multiplied by the corresponding areas of the S_{ij}'s, one obtains a Riemann sum which in the usual limit as the partition becomes infinitely fine gives a surface integral

$$\iint_S F(x, y, z)\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} F(x^*_{ij}, y^*_{ij}, z^*_{ij})\mathcal{A}(S_{ij})\,,$$

provided the limit exists. Here $\mathcal{A}(S_{ij})$ represents surface area of the piece S_{ij}, and $\mathrm{d}\mathcal{A}_S$ represents the *differential element of surface area.* For a graph surface, as above, we have

$$\mathrm{d}\mathcal{A}_S = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,,$$

where $\mathcal{A}_D$ is the usual area element $\mathrm{d}x\,\mathrm{d}y$ (or $\mathrm{d}y\,\mathrm{d}x$) for the region $\mathcal{D}$ of the plane which is the domain of the function whose graph is S. It is not uncommon for the surface area element to be denoted $\mathrm{d}S$, though we will use $\mathrm{d}\mathcal{A}_S$ to emphasize that it is an area element on the surface S, analogous to an area element $\mathcal{A}_{\mathcal{D}}$ on a piece $\mathcal{D}$ of a coordinate plane.

Example 4.1. Compute the surface integral $\displaystyle\iint_{\mathcal{S}} x^2 y\,\mathrm{d}\mathcal{A}_{\mathcal{S}}$ where $\mathcal{S}$ is the portion of the plane $2x - 2y + z = 4$ above the region of the xy plane where $0 \le y \le 4 - x^2$.

Solution: Let $\mathcal{D} = \{(x, y) : -2 \le x \le 2, 0 \le y \le 4 - x^2\}$ be the plane region in the plane $z = 0$ onto which $\mathcal{S}$ projects, and observe that this region is described in as a type I plane region, for which the order of iterated integration is first with respect to y and then with respect to x. Note that for the plane $2x - 2y + z = 4$

$$\frac{\partial z}{\partial x} = -2\,, \quad \frac{\partial z}{\partial y} = 2 \implies \mathrm{d}\mathcal{A}_{\mathcal{S}} = \sqrt{1 + 4 + 4}\,\mathrm{d}y\,\mathrm{d}x = 3\,\mathrm{d}y\,\mathrm{d}x\,.$$

Thus

$$\begin{aligned}
\iint_{\mathcal{S}} x^2 y\,\mathrm{d}\mathcal{A}_{\mathcal{S}} &= \iint_{\mathcal{D}} 3x^2 y\,\mathrm{d}\mathcal{A}_{\mathcal{D}} = 3\int_{-2}^{2}\int_{0}^{4-x^2} x^2 y\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{3}{2}\int_{-2}^{2} x^2(4 - x^2)^2\,\mathrm{d}x = 3\int_0^2 16x^2 - 8x^4 + x^6\,\mathrm{d}x \\
&= 3\Big[\frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7\Big]_0^2 = 3\Big(\frac{2^7}{3} - \frac{2^8}{5} + \frac{2^7}{7}\Big) \\
&= \frac{2^{10}}{35} = \frac{1024}{35}\,.
\end{aligned}$$

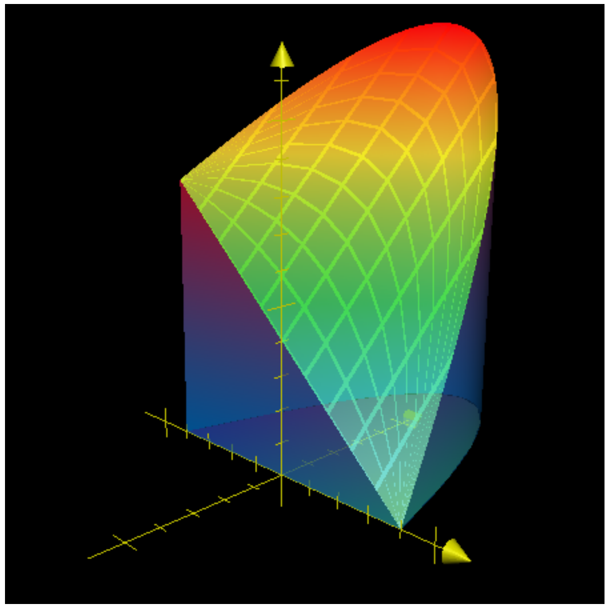

FIGURE 35. The portion of the plane $2x - 2y + z = 4$ above the parabolic region $\mathcal{D} = \{(x, y) : -2 \le x \le 2, 0 \le y \le 4 - x^2\}$ giving the surface $\mathcal{S}$ of example 4.1.

Example 4.2. Let $\mathcal{S}$ be the portion of the paraboloid $z = 2 - x^2 - y^2$ above the xy plane. Compute $\iint_{\mathcal{S}} x^2 + y^2 \,\mathrm{d}\mathcal{A}_{\mathcal{S}}$.

Solution: Let $\mathcal{D}$ be the disk contained in the xy plane and within the paraboloid $z = 2 - x^2 - y^2$. Since

$$\frac{\partial z}{\partial x} = -2x\,, \quad \frac{\partial z}{\partial y} = -2y\,,$$

the surface area differential is

$$\mathrm{d}\mathcal{A}_{\mathcal{S}} = \sqrt{1 + 4x^2 + 4y^2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,.$$

Using polar coordinates, one has $\mathrm{d}\mathcal{A}_{\mathcal{D}} = r\,\mathrm{d}r\,\mathrm{d}\theta$, and

$$\begin{aligned}
\iint_{\mathcal{S}} x^2 + y^2 \,\mathrm{d}\mathcal{A}_{\mathcal{S}} &= \iint_{\mathcal{D}} r^2\sqrt{1+4r^2}r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{2\pi}{8}\int_0^{\sqrt{2}} r^2(8r)\sqrt{1+4r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{\pi}{4}\int_0^{\sqrt{2}} \frac{(1+4r^2)-1}{4}\,\sqrt{1+4r^2}\,(8r)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{\pi}{16}\left[\frac{2}{5}(1+4r^2)^{5/2} - \frac{2}{3}(1+4r^2)^{3/2}\right]_0^{\sqrt{2}} \\
&= \frac{\pi}{8}\left(\frac{3^5}{5} - \frac{3^3}{3} - \frac{1}{5} + \frac{1}{3}\right) \\
&= \frac{149\pi}{30}\,.
\end{aligned}$$

One can of course consider surface integrals over surfaces which are not graphs. If a surface is given implicitly or in terms of several pieces, one faces a choice in setting up a surface integral:

(1) break the surface down into disjoint pieces which are locally graphs given by a choice of dependent variable as a function with respect to the remaining variables in your preferred 3-dimensional coordinate system, or
(2) parameterize the surface (possibly into multiple, disjoint patches) and for each parameterization $\boldsymbol{\sigma}(u,v)$, use the expression

$$\mathrm{d}\mathcal{A}_{\mathcal{S}} = \|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\|\,\mathrm{d}\mathcal{A}_{\mathcal{V}}\,,$$

derived in problem (24) of the problems in subsection 2.8. Here $\boldsymbol{\sigma}(\mathbf{v}) = \boldsymbol{\sigma}(u,v)$ is the parameterization, and V will be the region of the planar $\mathbf{v} = \langle u, v\rangle$ parameter space over which one integrates for this parametric patch, so $\mathrm{d}\mathcal{A}_{\mathcal{V}}$ is either $\mathrm{d}u\,\mathrm{d}v$ or $\mathrm{d}v\,\mathrm{d}u$ depending on whether $\mathcal{V}$ is a type I or two region. Note that the preceding case of splitting into graphs is a special case of this general procedure.

Then one adds together any surface integrals over disjoint patches of the surface to compute the original surface integral over the whole surface.

Example 4.3. To calculate the surface area of a sphere, one could split the sphere up into two hemispherical patches, and evaluate the corresponding surface integrals for each patch. By symmetry, one should get the same result on each patch, so one may instead double the surface integral over one hemisphere. Note this does not work, e.g., if one intends to double the integral $\iint_{\mathcal{U}} z\,\mathrm{d}\mathcal{A}_{\mathcal{U}}$ over the upper hemisphere $\mathcal{U}$ to compute the surface integral $\iint_{\mathcal{S}} z\,\mathrm{d}\mathcal{A}_{\mathcal{S}}$, because the coordinate function z is not completely symmetric over the sphere (it is "antisymmetric" with respect to reflection in the equatorial circle where $z = 0$!) Alternatively, one could use a parameterization arising from a choice of spherical coordinates. For example, one could use the mathematician's standard spherical coordinates, and write

$$\boldsymbol{\sigma}(u,v) = \langle a\cos(u)\cos(v), a\sin(u)\cos(v), a\sin(v)\rangle$$

to parameterize the sphere $\varrho = \sqrt{x^2+y^2+z^2} = a$ for a choice of a positive constant a. Then a quick calculation gives

$$\mathrm{d}\mathcal{A}_S = \|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\|\,\mathrm{d}\mathcal{A}_V = a^2 \sin v\,\mathrm{d}u\,\mathrm{d}v\,.$$

We thus find that the surface area of a radius R sphere is

$$\iint_{\mathcal{S}} \mathrm{d}\mathcal{A}_S = \int_0^{2\pi}\int_0^{\pi} a^2 \sin v\,\mathrm{d}u\,\mathrm{d}v = 4\pi a^2\,.$$

For the surface integral $\iint_{\mathcal{U}} z\,\mathrm{d}\mathcal{A}_{\mathcal{U}}$ over the upper hemisphere, one obtains

$$\iint_{\mathcal{S}} z\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \int_0^{2\pi}\int_0^{\pi/2} a^3 \sin v\cos v\,\mathrm{d}u\,\mathrm{d}v = \frac{\pi}{2}a^3\,,$$

while over the whole sphere the result is 0 by the aforementioned anti-symmetry.

Example 4.4. Let $\mathcal{S}$ be the radius $1\sqrt{2}$ cylinder centered on the z axis, of height $\sqrt{2}$, capped off with pieces of the unit sphere at each end. Consider the problem of computing

$$\iint_{\mathcal{S}} \frac{\mathrm{d}\mathcal{A}_{\mathcal{S}}}{\sqrt{r^2+z^2}}\,.$$

Along the spherical caps, $r^2+z^2 = 1$, so the integral would simply become the surface area $\iint_{\mathcal{S}} \mathrm{d}\mathcal{A}_{\mathcal{S}}$. Since the cylinder meets the sphere at heights $z = \pm 1\sqrt{2}$, the surface area integral of a single cap, let's call it $\mathcal{S}_2$, may be calculated as

$$\iint_{\mathcal{S}_2} \mathrm{d}\mathcal{A}_{\mathcal{S}_2} = \int_0^{2\pi}\int_0^{\pi/4} \sin v\,\mathrm{d}v\,\mathrm{d}u = (2-\sqrt{2})\pi\,,$$

whence the two spherical caps contribute $(4-2\sqrt{2})\pi$ to the surface integral. Now, for the cylindrical piece $\mathcal{S}_1$, one can use the parameterization

$$\boldsymbol{\sigma}(u,v) = \langle (1/\sqrt{2})\cos u, (1/\sqrt{2})\sin u, v\rangle\,,\quad 0 \le u \le 2\pi, -1\sqrt{2} \le v \le 1\sqrt{2}\,.$$

This leads to

$$\iint_{\mathcal{S}_2} \mathrm{d}\mathcal{A}_{\mathcal{S}_2} = \int_{-1/\sqrt{2}}^{1\sqrt{2}} \int_0^{2\pi} \frac{(1/\sqrt{2})\,\mathrm{d}u\,\mathrm{d}v}{\sqrt{1/2+v^2}}\,,$$

where we've used that the surface area element is

$$\|\boldsymbol{\sigma}_u(\mathbf{v}) \times \boldsymbol{\sigma}_v(\mathbf{v})\|\,\mathrm{d}\mathcal{A}_{\mathcal{V}} = \|\langle -(1/\sqrt{2})\sin u, (1/\sqrt{2})\cos u, 0\rangle \times \langle 0,0,1\rangle\|\,\mathrm{d}u\,\mathrm{d}v = (1/\sqrt{2})\,\mathrm{d}u\,\mathrm{d}v\,.$$

The final surface integral may be completed using the trigonometric substitution $\sqrt{2}v = \tan t$:

$$\iint_{\mathcal{S}_2} \mathrm{d}\mathcal{A}_{\mathcal{S}_2} = \int_{-1/\sqrt{2}}^{1\sqrt{2}} \int_0^{2\pi} \frac{\mathrm{d}u\,\mathrm{d}v}{\sqrt{1+2v^2}}$$

$$= 2\pi \int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{2}} \sec t\,\mathrm{d}t = 4\pi\ln(1+\sqrt{2})\,.$$

Thus

$$\iint_{\mathcal{S}} \frac{\mathrm{d}\mathcal{A}_{\mathcal{S}}}{\sqrt{r^2+z^2}} = \big(4 - 2\sqrt{2} + 4\ln(1+\sqrt{2})\big)\pi\,.$$

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§ 4.2. Flux

The word flux derives from *fluxus*, which is a Latin noun meaning "flow." For now we will describe only the scalar flux of a field $\mathbf{F}$ through either a curve or surface, which is defined by integrating the scalar component of $\mathbf{F}$ perpendicular to the curve or surface, relative to a chosen normal field.

Planar Flux. We consider first flux in two dimensions, as it pertains to the net flow of a planar vector field $\mathbf{F}$ through a curve $\mathcal{C}$.

Definition. Let $\mathbf{F}$ be a vector field defined on a planar domain $\mathcal{D} \subseteq \mathbb{R}^2$. Let $\mathcal{C}$ be an oriented curve in $\mathcal{D}$ which has a well defined co-oriented tangent vector field along it, except possibly at finitely many disjoint points (consider for example piecewise smooth curve, which is regular at all points except the joints between pieces). Let $\hat{\mathbf{N}}_s$ be the *signed unit normal field to* $\mathcal{C}$ obtained by rotating co-oriented unit tangent vectors $\hat{\mathbf{T}}$ by $\pi/2$ radians counter-clockwise. Then the *flux* of the vector field $\mathbf{F}$ through the oriented curve $\mathcal{C}$ is

$$\mathcal{F}[\mathbf{F},\mathcal{C}] := \int_{\mathcal{C}} \mathbf{F}\cdot\hat{\mathbf{N}}_s\,\mathrm{d}s\,.$$

Observe that $\mathcal{F}[\mathbf{F},-\mathcal{C}] = -\mathcal{F}[\mathbf{F},\mathcal{C}]$, since reversing the tangent vector reverses $\hat{\mathbf{N}}_s$. This line integral thus measures the net flow across the curve $\mathcal{C}$, with flow counted as positive if it is along the chosen normal field relative to the orientation. For closed curves, note that one normal field will point "inward", and the other will point "outward". Our convention for orientations of a closed curve $\mathcal{C}$ is that we take counterclockwise orientation to be positive, and the resulting signed normal field is an inward normal field. Thus, for simple closed curves $\mathcal{C}$ with positive orientation, flux measures how much $\mathbf{F}$ flows *into* the region bounded by $\mathcal{C}$. This is contrary to the usual flux desired in physics applications, but only in that it defers by a sign from the outward flux. However, if one wishes to measure the flux out of a region bounded by a non-simple closed curve, then choosing normal vectors in this fashion is disastrous, as illustrated by figure 36.

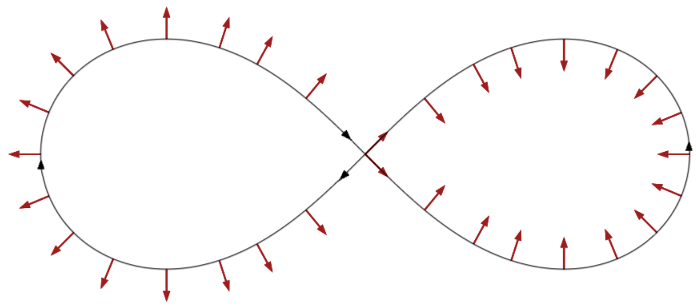

FIGURE 36. A lemniscate curve and the signed normal vector field determined by the orientation of the lemniscate.

Thus, we establish a separate convention for flux out of a region:

Definition. Let $\mathcal{D}$ be a bounded region (not necessarily simply connected), and $\mathcal{C} = \partial\mathcal{D}$ the boundary curve(s) of $\mathcal{D}$, and suppose that a well defined normal direction exists at all but finitely many disjoint points along $\mathcal{C}$. Let $\hat{\mathbf{n}}$ denote the *outward unit normal field* to $\mathcal{D}$, defined by ensuring that a path with velocity $\hat{\mathbf{n}}(\mathbf{r})$ at any point $\mathbf{r}$ along $\mathcal{C}$ is exiting $\mathcal{D}$. Then for a vector field $\mathbf{F}$ defined on an open set containing $\mathcal{D}$, we define the net flux of $\mathbf{F}$ out of $\mathcal{D}$ to be

$$\mathcal{F}[\mathbf{F}, \mathcal{D}] := \int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s \, .$$

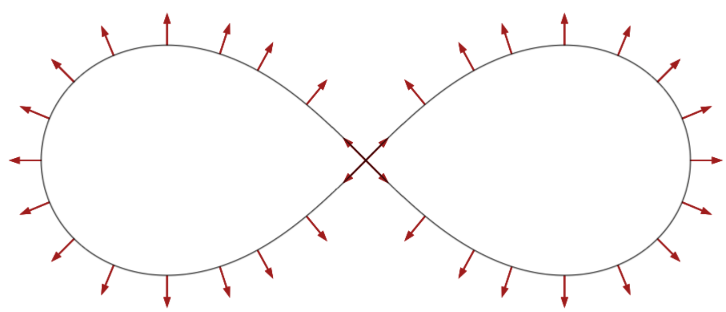

FIGURE 37. A lemniscate curve and its outward normal vector field. Observe the ambiguity in defining normals at the self-crossing. Since this ambiguity only occurs at an isolated point, it is still possible to use the illustrated normals to compute outward flux via line integration for this lemniscate.

Note that the formula appears nearly identical, but the convention about how $\hat{\mathbf{n}}$ is defined yields a definition of flux that behaves very differently from the initial definition when handling multiply connected regions or regions with non-simple boundaries. In particular, reversing the curve $\mathcal{C}$ has no effect on the value of flux out of a region. Since counterclockwise is the common positive orientation, it is sensible to choose the flux line differential $\hat{\mathbf{n}}\,\mathrm{d}s = \mathrm{d}y\,\hat{\mathbf{\imath}} - \mathrm{d}x\,\hat{\mathbf{\jmath}}$ in circumstances when one wants the flux out of a simply connected region. More generally, if $\hat{\mathbf{n}}$ is a choice of a preferred normal field along $\mathcal{C}$, we can define the flux relative to our chosen normal:

$$\mathcal{F}[\mathbf{F}, \mathcal{C}, \hat{\mathbf{n}}] = \int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s$$

One should compare the definition of flux to the definition of work of $\mathbf{F}$ along $\mathcal{C}$:

$$\mathcal{W}[\mathbf{F}, \mathcal{C}] := \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}} \, \mathrm{d}s = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \, .$$

If we write $\mathbf{F}(x, y) = P(x, y)\,\hat{\mathbf{\imath}} + Q(x, y)\,\hat{\mathbf{\jmath}}$ and take $\hat{\mathbf{n}}\,\mathrm{d}s = \mathrm{d}y\,\hat{\mathbf{\imath}} - \mathrm{d}x\,\hat{\mathbf{\jmath}}$ then

$$\mathcal{F}[\mathbf{F}, \mathcal{C}] = \int_{\mathcal{C}} P(x, y)\,\mathrm{d}y - Q(x, y)\,\mathrm{d}x \, ,$$

while for work one has

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \int_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y\,.$$

We will later compare these two scalar measures of a field's behavior along and across $\mathcal{C}$ by applying Green's Theorem to flux.

Example 4.5. Let $\mathcal{C}$ be the portion of the parabola $y = 1 - x^2$ above the x axis, traversed from $(-1, 0)$ to $(1, 0)$, and let $\mathbf{F}(x,y) = xy\,\hat{\mathbf{\imath}} + (1+y)\hat{\mathbf{\jmath}}$. Compute the flux

$$\mathcal{F}[\mathbf{F},\mathcal{C}] = \int_{\mathcal{C}} \mathbf{F}\cdot\hat{\mathbf{N}}_s\,\mathrm{d}s\,.$$

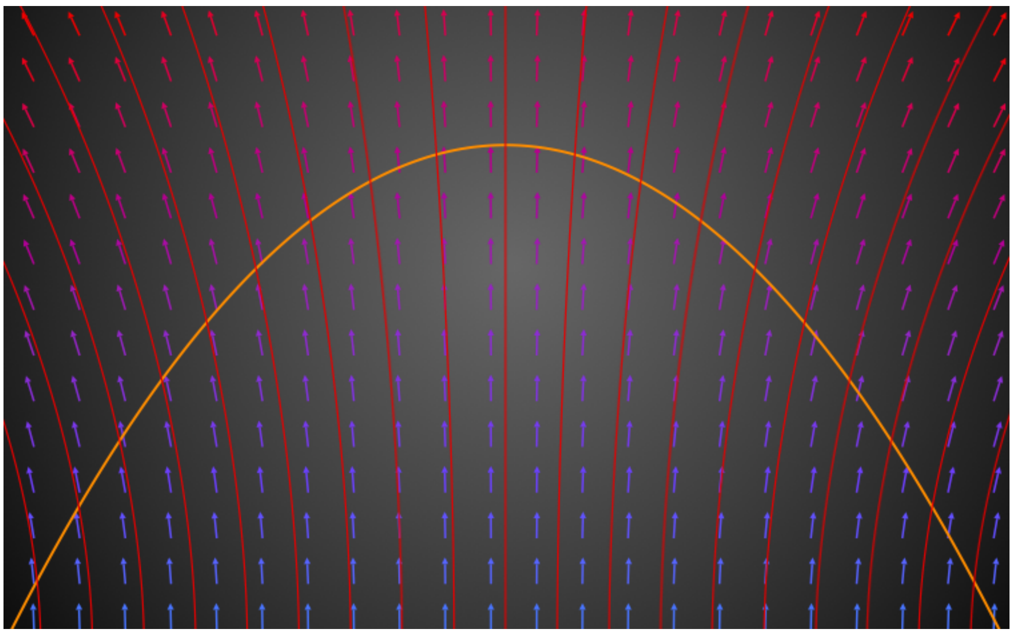

FIGURE 38. The curve $\mathcal{C}$ given by the parameterization $\mathbf{r}(x) = x\hat{\mathbf{\imath}} + (1-x^2)\,\hat{\mathbf{\jmath}}$ (orange), together with the vector field $\mathbf{F}(x,y) = xy\,\hat{\mathbf{\imath}} + (1+y)\hat{\mathbf{\jmath}}$ and some if its stream lines.

Solution: We first parameterize $\mathcal{C}$, in this case using x as the parameter:

$$\mathcal{C} : \mathbf{r}(x) = x\hat{\mathbf{\imath}} + (1-x^2)\,\hat{\mathbf{\jmath}}\,,\ -1 \le x \le 1\,,$$

$$\mathbf{r}'(x) = \hat{\mathbf{\imath}} - 2x\hat{\mathbf{\jmath}} \implies \hat{\mathbf{T}}(x) = \frac{\hat{\mathbf{\imath}} - 2x\,\hat{\mathbf{\jmath}}}{\sqrt{1+4x^2}}\,.$$

One can view the xy-plane as the plane $z = 0$ in $\mathbb{R}^3$ to calculate the signed normal as $\hat{\mathbf{N}}_s = \hat{\mathbf{k}} \times \hat{\mathbf{T}}$, but this just has the effect of swapping the $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ components of $\hat{\mathbf{T}}$, and negating the $\hat{\mathbf{\imath}}$ component (equivalently, one can multiply $\hat{\mathbf{T}}$ by the matrix whose first and second columns are the vectors obtained by rotating $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ by $\pi/2$ clockwise, respectively). Thus

$$\hat{\mathbf{N}}_s(x) = \frac{2x\,\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}}{\sqrt{1+4x^2}}\,.$$

Now, since $\mathrm{d}s = \|\mathbf{r}'(x)\|\,\mathrm{d}x = \sqrt{1+4x^2}\,\mathrm{d}x$, we get $\hat{\mathbf{N}}_s(x)\,\mathrm{d}s = (2x\,\hat{\imath} + \hat{\jmath})\,\mathrm{d}x$, whence

$$\begin{aligned}
\mathcal{F}[\mathbf{F},\mathcal{C}] &= \int_{\mathcal{C}} \mathbf{F}\cdot\hat{\mathbf{N}}_s\,\mathrm{d}s \\
&= \int_{-1}^{1} \Big(x(1-x^2)\,\hat{\imath} + \big(1+(1-x^2)\big)\,\hat{\jmath}\Big)\cdot(2x\,\hat{\imath}+\hat{\jmath})\,\mathrm{d}x \\
&= \int_{-1}^{1} 2x^2(1-x^2) - x^2 + 2\,\mathrm{d}x = \int_{-1}^{1} x^2 - 2x^4 + 2\,\mathrm{d}x \\
&= \frac{2}{3} - \frac{4}{5} + 4 = \frac{58}{15}\,.
\end{aligned}$$

Example 4.6. Recall the field in example 3.9, $\mathbf{F}(x,y) = (x-y)\hat{\imath} + (y-x)\hat{\jmath}$, for which the work around the unit circle was calculated. The flux of this field out of the unit disk $\mathbb{D}$, through its boundary is

$$\mathcal{F}[\mathbf{F},\mathbb{D}] = \int_{\mathbb{S}^1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = 2\pi\,,$$

as can be easily verified parametrically. In this case geometric insight allows quick computation of both work and flux. Indeed, since $\mathbf{F} = r\hat{\mathbf{u}}_r(\theta) + r\hat{\mathbf{u}}_\theta(\theta)$ when expressed in polar coordinates, it is seen to restrict to $\hat{\mathbf{n}} + \hat{\mathbf{T}}$ along $\mathbb{S}^1$ where $\hat{\mathbf{n}}$ is the unit outward normal and $\hat{\mathbf{T}}$ is the unit tangent vector for clockwise motion. Thus

$$\mathcal{W}[\mathbf{F},\mathbb{S}^1] = \int_{\mathbb{S}^1} \mathbf{F}\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_{\mathbb{S}^1} (\hat{\mathbf{n}}+\hat{\mathbf{T}})\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_{\mathbb{S}^1} \hat{\mathbf{T}}\cdot\hat{\mathbf{T}}\,\mathrm{d}s = \int_{\mathbb{S}^1} \mathrm{d}s = s(\mathbb{S}^1)\,,$$

$$\mathcal{F}[\mathbf{F},\mathbb{D}] = \int_{\mathbb{S}^1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_{\mathbb{S}^1} (\hat{\mathbf{n}}+\hat{\mathbf{T}})\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_{\mathbb{S}^1} \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \int_{\mathbb{S}^1} \mathrm{d}s = s(\mathbb{S}^1)\,.$$

Flux through surfaces. We now move on to describe flux of a 3 dimensional vector field across a surface. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$ such that there is a well defined tangent plane at all points of $\mathcal{S}$, except possibly a finite collection of disjoint points and piecewise smooth curves. At any regular point we can choose a unit normal vector perpendicular to the tangent plane, and locally we can choose normals in a fashion as to construct a normal vector field. Recall, as in section 2.6, we say a surface is oriented if there is a consistent *global* choice of such a normal vector field. We adopt the following conventions regarding orientations and normals:

- If $\mathcal{S}$ is a closed surface (meaning it bounds a compact region $\mathcal{E} \subset \mathbb{R}^3$) then a *positively oriented normal vector field* on $\mathcal{S}$ is the *outward normal vector field*, meaning that any curve crossing $\mathcal{S}$ whose velocity at a point of $\mathcal{S}$ is given by the normal vector there is exiting the region $\mathcal{E}$.
- If the surface is a graph $z = f(x,y)$ not giving a facet of a closed surface, we take the positive unit normal at a regular point $\big(x, y, f(x,y)\big)$ to be the unit normal vector $\hat{\mathbf{n}}$ such that $\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} > 0$. Similarly, if the surface is a graph $y = g(x,z)$ not giving a facet of a closed surface, we take the positive unit normal at a regular point $\big(x, g(x,z), z\big)$ to be the unit normal vector $\hat{\mathbf{n}}$ such that $\hat{\mathbf{n}}\cdot\hat{\jmath} > 0$, and if the surface is a graph $x = h(y,z)$ not giving a facet of a closed surface, we take the positive unit normal at a regular point $\big(h(y,z), y, z\big)$ to be the unit normal vector $\hat{\mathbf{n}}$ such that $\hat{\mathbf{n}}\cdot\hat{\imath} > 0$.
- If $\mathcal{S}$ is non-orientable, we can make no choice of positively oriented normal, but we can associate a *normal line* to every regular point.

Definition. Let $\mathcal{S}$ be an oriented surface in $\mathbb{R}^3$, and let $\mathbf{F}$ be a vector field defined on an open set of $\mathbb{R}^3$ containing $\mathcal{S}$. Let $\hat{\mathbf{n}}$ be a positively oriented unit normal vector field to $\mathcal{S}$. Then the flux of $\mathbf{F}$ through the surface $\mathcal{S}$ is the surface integral of the scalar component of $\mathbf{F}$ along the positive normal direction to $\mathcal{S}$:

$$\mathcal{F}[\mathbf{F},\mathcal{S}] := \iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,.$$

Observe that in the non-orientable case though we cannot choose a consistent positive normal allowing us to define the scalar flux as an integral $\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}$, there does exist a unique choice of *normal line* at each point of the surface, and one can project a vector field onto the normal directions using *any locally defined normal vector field* $\mathbf{N}$, and so one can still define and compute a *vector valued flux* through, e.g., a Möbius band by integrating such projections over the surface. One can also address only how much flows across the surface without regard to direction, and integrate the lengths of such projections onto normal lines, obtaining a strictly nonnegative scalar flux for a non-orientable surface, which we may call a *gross flux*. These notions are, strictly speaking, useless quantities, but many a useless mathematical calculation can be performed for sheer joy. However, our examples will henceforth focus on orientable surfaces, though you may try problem (??) below to explore flux for non-orientable surfaces.

Example 4.7. Let $\mathcal{S}$ be the portion of the plane $6x+2y+3z=6$ in the first octant, oriented via the "upward" normal (ie., the normals with $\hat{\mathbf{k}}$ component positive). Compute the flux of the vector field $\mathbf{F}(x,y,z) = z\hat{\mathbf{\imath}} + x\hat{\mathbf{\jmath}} - y\hat{\mathbf{k}}$ through $\mathcal{S}$.
Solution: Since $\mathcal{S}$ is a portion of a plane, it is simple to read off an upward normal vector from the coefficients in the given linear equation. Normalizing we get

$$\hat{\mathbf{n}} = \tfrac{6}{7}\,\hat{\mathbf{\imath}} + \tfrac{2}{7}\,\hat{\mathbf{\jmath}} + \tfrac{3}{7}\,\hat{\mathbf{k}}\,.$$

Note that we can solve for z in terms of x and y, and rewrite $\mathbf{F}$ restricted to $\mathcal{S}$ as a function of just x and y, and thus the integral we wish to compute is

$$\mathcal{F}[\mathbf{F},\mathcal{S}] = \iint_{\mathcal{S}} \left\langle 2-2x-\tfrac{2}{3}y, x, -y\right\rangle \cdot \left\langle \tfrac{6}{7},\tfrac{2}{7},\tfrac{3}{7}\right\rangle \mathrm{d}\mathcal{A}_{\mathcal{S}}\,.$$

Our domain of integration in the xy plane is bounded by the x and y axes and the line $y=3-3x$ where $\mathcal{S}$ meets the plane $z=0$. Treating $\mathcal{D}$ as a type I region, the surface area element is

$$\mathrm{d}\mathcal{A}_{\mathcal{S}} = \sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}\,\mathrm{d}y\,\mathrm{d}x = \tfrac{7}{3}\,\mathrm{d}y\,\mathrm{d}y\,.$$

Thus the flux is

$$\begin{aligned}\mathcal{F}[\mathbf{F},\mathcal{S}] &= \int_0^1\int_0^{3-3x} \left\langle 2-2x-\tfrac{2}{3}y, x, -y\right\rangle \cdot \left\langle \tfrac{6}{7},\tfrac{2}{7},\tfrac{3}{7}\right\rangle \tfrac{7}{3}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^1\int_0^{3-3x} 4-\tfrac{10}{3}x-\tfrac{5}{3}y\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^1 4(3-3x)-\tfrac{10}{3}x(3-3x)-\tfrac{5}{6}(3-3x)^2\,\mathrm{d}x\\ &= 12x-6x^2-5x^2+\tfrac{10}{3}x^3+\tfrac{5}{2}(1-x)^3\Big|_0^1\\ &= \frac{11}{6}\end{aligned}$$

Notice that it is no coincidence that in the preceding example oriented the surface area element could be written as

$$\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \left\langle \tfrac{6}{7},\tfrac{2}{7},\tfrac{3}{7}\right\rangle \tfrac{7}{3}\,\mathrm{d}\mathcal{A}_{\mathcal{D}} = \left\langle 2,\tfrac{2}{3},1\right\rangle \mathrm{d}y\,\mathrm{d}x\,.$$

Indeed, if $\mathcal{S}$ a graph of a function $z=f(x,y)$ over a region $\mathcal{D}$ in the xy plane, we have a preferred positive normal vector

$$\hat{\mathbf{n}} = \frac{-f_x(x,y)\,\hat{\mathbf{\imath}} - f_y(x,y)\,\hat{\mathbf{\jmath}} + \hat{\mathbf{k}}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}}\,,$$

whence the flux through such a graph surface becomes

$$\mathcal{F}[\mathbf{F},\mathcal{S}] = \iint_{\mathcal{D}} \mathbf{F}(x,y,z)\cdot\left(\frac{-f_x(x,y)\,\hat{\imath} - f_y(x,y)\,\hat{\jmath} + \hat{\mathbf{k}}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}}\right)\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}$$
$$= \iint_{\mathcal{D}} -P(x,y,f(x,y))f_x(x,y) - Q(x,y,f(x,y))f_y(x,y) + R(x,y,f(x,y))\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,,$$

where $\mathbf{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{\mathbf{k}}$. Similar formulae emerge for surfaces defined as graphs using x or y as dependent variables.

Example 4.8. Compute the flux of the spherically radial field $\mathbf{F}(x,y,z) = \varrho\hat{\mathbf{u}}_\varrho = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ through the portion of the paraboloid $4x + y^2 + z^2 = 4$ where $x \geq 0$.

Solution: Let $\mathcal{S}$ be the surface with the given equation, and note that we can rewrite the equation as

$$x = 1 - \frac{1}{4}(y^2+z^2)\,.$$

The normal vector can be calculated as a function of y and z via

$$\hat{\mathbf{n}}(y,z) = \frac{\hat{\imath} - \frac{\partial x}{\partial y}\,\hat{\jmath} - \frac{\partial x}{\partial z}\,\hat{\mathbf{k}}}{\sqrt{1+(\partial x/\partial y)^2+(\partial x/\partial z)^2}} = \frac{\hat{\imath} + (y/2)\,\hat{\jmath} + (z/2)\,\hat{\mathbf{k}}}{\sqrt{1+y^2/4+z^2/4}} = \frac{2\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}}{\sqrt{4+y^2+z^2}}\,.$$

The surface area element is $\mathrm{d}\mathcal{A}_{\mathcal{S}} = \sqrt{1+y^2/4+z^2/4}\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\sqrt{4+y^2+z^2}\,\mathrm{d}y\,\mathrm{d}z$. Let $\mathcal{D}$ be the disk of radius 2 in the yz plane centered at $(0,0,0)$. The surface integral we wish to compute is then given by the integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \iint_{\mathcal{D}} \left((1-(y^2+z^2)/4)\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}\right)\cdot\frac{2\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}}{\sqrt{4+y^2+z^2}}\;\frac{\sqrt{4+y^2+z^2}}{2}\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} 1 + \tfrac{1}{4}(y^2+z^2)\,\mathrm{d}y\,\mathrm{d}z\,.$$

Given the bounds, this integral is certainly better suited to being computed using polar coordinates. We can adapt polar coordinates to the yz plane by setting $y = u\cos v$ and $z = u\sin v$, for u a radial parameter and v an angular parameter. Then $u^2 = y^2 + z^2$, and $\mathrm{d}y\,\mathrm{d}z = u\,\mathrm{d}u\,\mathrm{d}v$, and

$$\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} 1 + \tfrac{1}{4}(y^2+z^2)\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\int_0^2 u + \tfrac{u^3}{4}\,\mathrm{d}u\,\mathrm{d}v\,.$$

This integral works out straightforwardly to 6π, whence

$$\iint_{\mathcal{S}} (x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = 6\pi\,.$$

Example 4.9. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}$$

where $\mathcal{S}$ is a closed circular cylinder of radius 2 and height 4 centered at $(0,0,0)$, and

$$\mathbf{F}(x,y,z) = \langle x^2 - y^2, 2xy, z^2 - x^2 - y^2\rangle\,.$$

Solution: Since $\mathcal{S}$ is a closed cylinder we can use cylindrical coordinates. We have to compute the flux through the top, bottom, and side. Let $\mathcal{S}_1$ denote the top of the cylinder, $\mathcal{S}_2$ denote the bottom, and $\mathcal{S}_\ni$ denote the side. Then

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \iint_{\mathcal{S}_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_1} + \iint_{\mathcal{S}_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_2} + \iint_{\mathcal{S}_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_3}\,.$$

For the top $\mathcal{S}_1$ note that the outward normal is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and

$$\mathbf{F}(x,y,z)\cdot\hat{\mathbf{n}} = z^2 - x^2 - y^2 = z^2 - r^2\,,$$

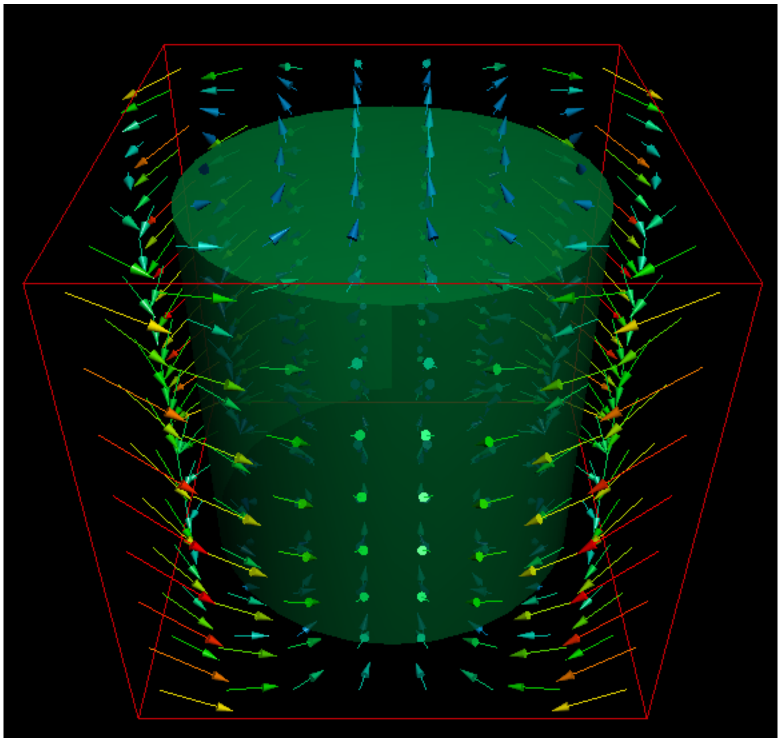

FIGURE 39. The field $\mathbf{F}(r,\theta,z) = r^2\hat{\mathbf{u}}_r(2\theta) + (z^2 - r^2)\hat{\mathbf{k}}$ and the origin centered cylinder $\mathcal{S}$.

and along $\mathcal{S}_1$ this restricts to $\mathbf{F}(x,y,2)\cdot\hat{\mathbf{n}} = (2)^2 - r^2 = 4 - r^2$. Since $\mathcal{S}_1$ is planar and parallel to the xy plane the area element expressed in cylindrical coordinates is just the usual polar area element $\mathrm{d}\mathcal{A}_{\mathcal{S}} = r\,\mathrm{d}r\,\mathrm{d}\theta$. Thus

$$\iint_{\mathcal{S}_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_1} = \int_0^{2\pi}\int_0^2 4r - r^3\,\mathrm{d}r\,\mathrm{d}\theta = 8\pi\,.$$

Similarly along the bottom $\mathcal{S}_2$, we have outward normal $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and $\mathbf{F}(x,y,-2)\cdot\hat{\mathbf{n}} = r^2 - 4$, and so

$$\iint_{\mathcal{S}_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_2} = -\iint_{\mathcal{S}_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_1} = -8\pi\,,$$

whence the net flux, if nonzero, is determined solely by the flux through the cylinder's side $\mathcal{S}_3$.

For $\mathcal{S}_3$, the outward unit normal is the polar radial unit vector $\hat{\mathbf{u}}_r(\theta) = \frac{x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}}}{r} = \cos(\theta)\,\hat{\mathbf{\imath}} + \sin(\theta)\,\hat{\mathbf{\jmath}}$. Since $\mathbf{F}$ was given in cartesian coordinates, it is straightforward to calculate $\mathbf{F}\cdot\hat{\mathbf{u}}_r$ using the Cartesian variables and then to convert:

$$\begin{aligned}\mathbf{F}(x,y,z)\cdot\hat{\mathbf{u}}_r &= (x^2-y^2)\frac{x}{r} + (2xy)\frac{y}{r}\\ &= \frac{x^3}{r} - \frac{xy^2}{r} + \frac{2xy^2}{r} = \frac{x^3+xy^2}{r}\\ &= \frac{xr^2}{r} = x\,r\\ &= r^2\cos\theta\,.\end{aligned}$$

Alternatively, one can express $\mathbf{F}$ in terms of cylindrical coordinates by observing that

$$x^2 - y^2 = r^2\cos^2(\theta) - r^2\sin^2(\theta) = r^2\cos(2\theta)\,,$$
$$2xy = r^2\sin(\theta)\cos(\theta) = r^2\sin(2\theta)\,,$$
$$z^2 - x^2 - y^2 = z^2 - r^2\,,$$

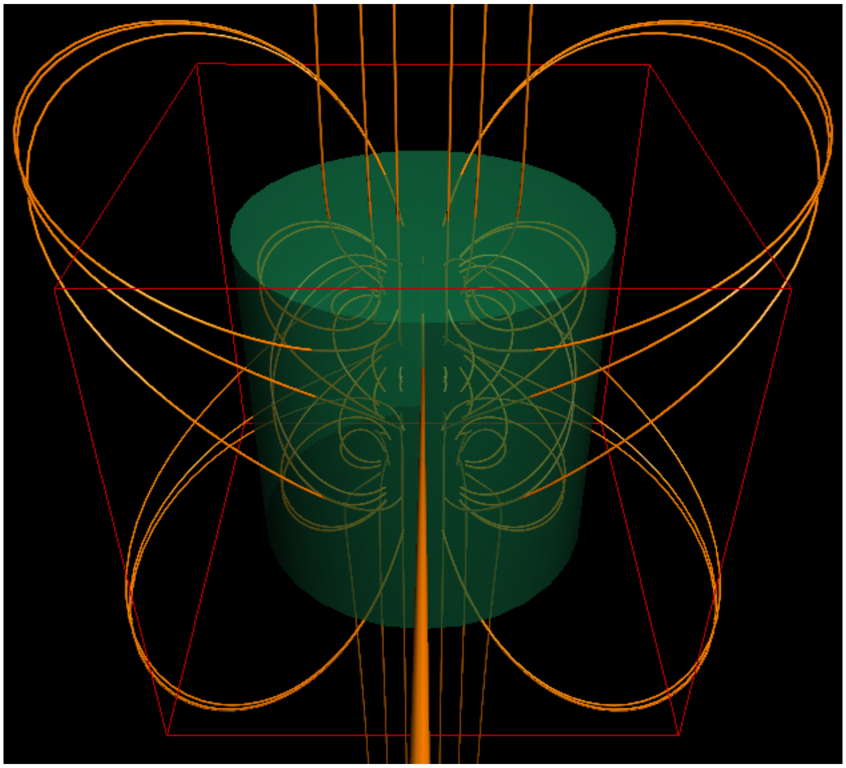

Figure 40. Some integral curves of $\mathbf{F}(r,\theta,z) = r^2\cos(2\theta)\hat{\mathbf{u}}_r + (z^2 - r^2)\hat{\mathbf{k}}$.

whence

$$\mathbf{F}(r,\theta,z) = r^2\cos(2\theta)\hat{\mathbf{u}}_r + (z^2 - r^2)\hat{\mathbf{k}}\,.$$

Then taking $\hat{\mathbf{n}} = \hat{\mathbf{u}}_r(\theta)$, note that

$$\begin{aligned}\mathbf{F}(r,\theta,z)\cdot\hat{\mathbf{n}}(\theta) &= \left(r^2\cos(2\theta)\hat{\mathbf{u}}_r + (z^2 - r^2)\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{u}}_r(\theta)\\ &= r^2\hat{\mathbf{u}}_r(2\theta)\cdot\hat{\mathbf{u}}_r(\theta)\\ &= r^2\cos(2\theta-\theta) = r^2\cos(\theta)\,,\end{aligned}$$

where we've used that the dot product of two unit vectors is merely the cosine of the angle between them. Now, the surface area element is $r\,\mathrm{d}\theta\,\mathrm{d}z = 2\,\mathrm{d}\theta\,\mathrm{d}z$ since the cylinder has radius $r = 2$. Thus the flux is

$$\iint_{\mathcal{S}}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \iint_{\mathcal{S}_3}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}_3} = \int_{-2}^{2}\int_0^{2\pi} 8\cos\theta\,\mathrm{d}\theta\,\mathrm{d}z = 0\,.$$

That the flux is zero means that as much of the field flows into the cylinder as flows out, as may be partially understood visually; see figures 39 and 40.

§ 4.3. The Gradient, Divergence, and Curl Operators Via Limits*

We will modify our understandings of flux and circulation in order to construct limit definitions of the three most important differential operators in vector calculus: the gradient, the curl, and the divergence.

This optional section could certainly benefit from some pictures. The good news is that if you've built up sufficient background by working through and understanding the preceding sections of these notes, taking the time to absorb the visual information that previous figures convey, then you are well equipped to draw your own pictures moving forward! As you read through about gradient, curl, and divergence, it is strongly encouraged that you sketch simple visualizations and work out the details of using these limit definitions in polar, cylindrical, or spherical coordinates.

The Gradient via vector flux.

We will motivate ourselves with an example application. Let $T(x,y)$ be a function defined on a region $\mathcal{D}$ of the plane, representing the temperature of a plate or lamina at a moment in time. Let k be a constant representing the *thermal conductivity* of the plate. Heat energy flows from regions of high temperature to lower temperature, and the more thermally conductive the medium, the more efficiently heat is transported or diffused to the lower temperature regions. This heat diffusion phenomenon is modeled by a partial differential equation called the heat equation. However at the moment we are only interested in the direction heat will flow, given the temperature distribution at this moment. We thus define the *heat flux density* to be

$$\mathbf{q}(x,y) := -k\nabla T(x,y)\,.$$

Assume now that $\mathcal{E}$ is a subregion of the plate. Integrating the components of heat flux density in the outward normal direction to the boundary of $\mathcal{E}$ gives us the (scalar) outward heat flux of the subregion $\mathcal{E}$:

$$\mathcal{F}[\mathbf{q},\mathcal{E}] = \oint_{\partial\mathcal{E}} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \oint_{\partial\mathcal{E}} -k\nabla T\,\cdot\hat{\mathbf{n}}\,\mathrm{d}s\,.$$

The heat flux density $\mathbf{q}$, being a vector valued quantity given by a gradient, can be thought of as a *vector form of flux*. It is alternately definable by considering the limiting value of a temperature-weighted "average" of normal vectors leaving a shrinking region:

$$\mathbf{q}(\mathbf{r}_0) := \lim_{\mathcal{A}(\mathcal{E})\to 0}\frac{1}{\mathcal{A}(\mathcal{E})}\oint_{\partial\mathcal{E}} -kT(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}s\,,$$

where the limit is taken over smooth simply connected subregions $\mathcal{E}$ bounded by simple closed curves $\partial\mathcal{E}$ and with $\mathbf{r}_0$ interior to every $\mathcal{E}$ in the sequence, $\mathcal{A}(\mathcal{E})$ is the area bounded inside $\partial\mathcal{E}$, and $\hat{\mathbf{n}}(\mathbf{r})$ is an outward unit normal to $\mathcal{E}$ at the position $\mathbf{r}$ on $\partial\mathcal{E}$.

An intuition for this definition is as follows: at any position $\mathbf{r}$ along a curve $\mathcal{C}$, we first associate a normal vector whose length is determined by the temperature $T(\mathbf{r})$, and then we rescale all such normals by $-k$. Note that near warmer portions of the curve $\mathcal{C}$ the vectors $T(\mathbf{r})\hat{\mathbf{n}}(\mathbf{r})$ are longer, and so upon rescaling and integrating along the curve $\mathcal{C}$, the warmer directions dominate, so the resulting "average flux vector" points from the warmer regions towards the colder regions. As we shrink the curves and take the limit, this limiting average captures the direction along which heat flows most rapidly, which is the heat flux density at the point $\mathbf{r}_0$.

We can similarly define heat flux density $\mathbf{q}$ in 3 dimensions in terms of the gradient of temperature, and in terms of a limit of vector valued flux:

$$\mathbf{q}(\mathbf{r}_0) = -k\nabla T(x_0,y_0,z_0) = \lim_{\mathcal{V}(\mathcal{S})\to 0}\frac{1}{\mathcal{V}(\mathcal{S})}\oiint_{\mathcal{S}} -kT(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,,$$

where the limit on the right is taken over a family of orientable closed smooth surfaces $\mathcal{S}$ shrinking to the point $\mathbf{r}_0$, and $\mathbf{nr}$ is an outward unit normal field to $\mathcal{S}$. For these definitions of heat flux density to agree, the gradient must be given by such a limit.

Theorem. *Given a differentiable bivariate function $f(\mathbf{r}) = f(x,y)$ defined on a region $\mathcal{D}\subseteq\mathbb{R}^2$, the gradient $\nabla f(\mathbf{r})$ is the unique vector field in $\mathcal{D}$ determined point-wise at any $\mathbf{r}_0\in\mathcal{D}$ by the limit*

$$\nabla f(\mathbf{r}_0) = \lim_{\mathcal{A}(\mathcal{C})\to 0}\frac{1}{\mathcal{A}(\mathcal{C})}\oint_{\mathcal{C}} f(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}s\,,$$

where the limit is taken over a continuum of smooth simple closed curves $\mathcal{C}$ in $\mathcal{D}$ each bounding a region around the position $\mathbf{r}_0$ and shrinking to the point at position $\mathbf{r}_0$, $\mathcal{A}(\mathcal{C})$ is the area of the region bounded by $\mathcal{C}$, and $\hat{\mathbf{n}}(\mathbf{r})$ is an outward unit normal to $\mathcal{C}$ at the position $\mathbf{r}$ on $\mathcal{C}$.

Given a differentiable trivariate function $f(\mathbf{r}) = f(x,y,z)$ on a domain $\mathcal{D}\subseteq\mathbb{R}^3$, the gradient $\nabla f(\mathbf{r})$ is the unique vector field in $\mathcal{D}$ determined point-wise at any $\mathbf{r}_0\in\mathcal{D}$ by the limit

$$\nabla f(\mathbf{r}_0) = \lim_{\mathcal{V}(\mathcal{S})\to 0}\frac{1}{\mathcal{V}(\mathcal{S})}\oiint_{\mathcal{S}} f(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,,$$

where the limit is taken over a continuum of orientable closed smoothly embedded surfaces $\mathcal{S}$ bounding solid regions of volume $\mathcal{V}(\mathcal{S})$ containing $\mathbf{r}_0$ and shrinking to the point $\mathbf{r}_0$, and $\mathbf{nr}$ is an outward unit normal field to $\mathcal{S}$.

Any battle-hardened mathematician would point out that we should address well-definedness, for it is not immediately clear that the above limits exist independent of any choices made in selecting families of curves or surfaces shrinking to a desired point. Correctly addressing such a question requires tools of analysis and topology, and some subtle argumentation. Nonetheless, there is utility in these coordinate free definitions, for they provide a means of *obtaining coordinate expressions in new coordinate systems*, without the fuss and tedium of the approach discussed in §1.4* which required copious applications of the chain rule and linear algebra.

Example 4.10. We can apply this idea to arrive at the two dimensional formula for the gradient in rectangular coordinates, appealing to our definition and the assumption that the resulting operator is independent of the choice of curves over which the limit is taken. Let $f(x, y)$ be a differentiable function on an open disk around a point (x_0, y_0), and consider the problem of determining $\nabla f(x_0, y_0)$ via the limit construction of the gradient. Since we are working in rectangular coordinates, we choose for $\mathcal{C}$ a small square of area h^2 centered at (x_0, y_0), the corners of which are

$$\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}\right), \quad \left(x_0 - \frac{h}{2}, y_0 + \frac{h}{2}\right), \quad \left(x_0 - \frac{h}{2}, y_0 - \frac{h}{2}\right), \text{ and } \left(x_0 + \frac{h}{2}, y_0 - \frac{h}{2}\right).$$

Along the bottom edge, the outward unit normal is constantly $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, while along the top edge the unit normal is constantly $\hat{\mathbf{n}} = \hat{\mathbf{j}}$. Meanwhile, for the left edge of the square the unit normal is the constant vector $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, whereas on the right edge the unit normal is $\hat{\mathbf{n}} = \hat{\mathbf{i}}$. Label the edges $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ going counterclockwise starting from the bottom edge. Each line integral may be approximated by evaluating $f(x, y)$ at the midpoint of the corresponding edge, and multiplying by the length of the edge. Grouping by horizontal and vertical normals:

$$\begin{aligned}
\frac{1}{h^2}\int_{\mathcal{C}} f(x,y)\,\hat{\mathbf{n}}\,\mathrm{d}s &= \frac{\hat{\mathbf{i}}}{h^2}\int_{\mathcal{C}_2+\mathcal{C}_4} f(x,y)\,\mathrm{d}s + \frac{\hat{\mathbf{j}}}{h^2}\int_{\mathcal{C}_1+\mathcal{C}_3} f(x,y)\,\mathrm{d}s \\
&\approx \frac{\hat{\mathbf{i}}}{h^2}\left(f\left(x_0+\tfrac{h}{2}, y_0\right)h - f\left(x_0-\tfrac{h}{2}, y_0\right)h\right) \\
&\qquad + \frac{\hat{\mathbf{j}}}{h^2}\left(f\left(x_0, y_0+\tfrac{h}{2}\right)h - f\left(x_0, y_0-\tfrac{h}{2}\right)h\right) \\
&= \frac{f\left(x_0+\tfrac{h}{2}, y_0\right) - f\left(x_0-\tfrac{h}{2}, y_0\right)}{h}\,\hat{\mathbf{i}} + \frac{f\left(x_0, y_0+\tfrac{h}{2}\right) - f\left(x_0, y_0-\tfrac{h}{2}\right)}{h}\,\hat{\mathbf{j}}\,.
\end{aligned}$$

In the limit as h shrinks to 0, the difference quotients in the approximation converge to the partial derivatives of $f(x, y)$ at (x_0, y_0). This may not be an immediately convincing argument, for one could reasonably protest "why should a single midpoint times arc length generate an approximation of the line integral that holds in the limit?" By the mean value theorem for integrals, along any given edge $\mathcal{E}$ of the square, $1/h^2 \int_{\mathcal{E}} f(x, y)\,\mathrm{d}s$ is equal to $f(x_1, y_1)$ for some $(x_1, y_1) \in \mathcal{E}$. In the limit as h approaches 0, the point (x_1, y_1) must converge to the midpoint, and so too must the value of $f(x_1, y_1)$ by continuity, and so if the overall limit exists, it must be true that the midpoint approximations become arbitrarily accurate in the limit.

We thus conclude that

$$\begin{aligned}\nabla f(x_0,y_0) &= \lim_{\mathcal{A}(\mathcal{C})\to 0}\frac{1}{\mathcal{A}(\mathcal{C})}\oint_{\mathcal{C}} f(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}s\\ &= \lim_{h\to 0}\frac{\hat{\mathbf{\imath}}}{h^2}\int_{\mathcal{C}_2+\mathcal{C}_4} f(x,y)\,\mathrm{d}s + \frac{\hat{\mathbf{\jmath}}}{h^2}\int_{\mathcal{C}_1+\mathcal{C}_3} f(x,y)\,\mathrm{d}s\\ &= \lim_{h\to 0}\frac{f\Big(x_0+\frac{h}{2},y_0\Big)-f\Big(x_0-\frac{h}{2},y_0\Big)}{h}\,\hat{\mathbf{\imath}} + \frac{f\Big(x_0,y_0+\frac{h}{2}\Big)-f\Big(x_0,y_0-\frac{h}{2}\Big)}{h}\,\hat{\mathbf{\jmath}}\\ &= \frac{\partial f}{\partial x}(x_0,y_0)\,\hat{\mathbf{\imath}} + \frac{\partial f}{\partial y}(x_0,y_0)\,\hat{\mathbf{\jmath}}\,.\end{aligned}$$

See the problems below to develop formulae for the gradient in other coordinate systems by a similar procedure.

Curl via limiting circulation.

Before we address three dimensional curl, we return to the plane and re-examine Green's theorem, which allows us to compute the work

$$\mathcal{W}[\mathbf{F},\mathcal{C}] = \oint_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\,\mathrm{d}\mathbf{r} = \oint_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

around a simple closed curve bounding a simply connected region $\mathcal{D}$ around and over which $\mathbf{F}$ is $\mathcal{C}^1$ by instead computing a double integral

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A}\,.$$

One interpretation of this is that the circulation of $\mathbf{F}$ around $\mathcal{C}=\partial\mathcal{D}$ is given as the area of $\mathcal{D}$ times the average value of $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ in $\mathcal{D}$. In particular, circulation vanishes when $Q_x - P_y = 0$, and for $\mathcal{D}$ simply connected, this is enough to conclude path independence in $\mathcal{D}$ and hence to show that $\mathbf{F}$ is conservative. We then begin to suspect that the quantity $Q_x - P_y$, which measures the failure of $\mathbf{F}$ to be conservative, may also measure a kind of local circulation. Indeed, we can argue using the mean value theorem for integrals that

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(\mathbf{r}_0) = \lim_{\mathcal{A}(\mathcal{D})\to 0}\frac{1}{\mathcal{A}(\mathcal{D})}\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A}\,,$$

where the regions $\mathcal{D}$ shrink around the point $\mathbf{r}_0$, like in the limiting construction of the gradient. Then by Green's theorem, we conclude that in fact $Q_x - P_y$ measures "infinitesimal circulation", since

$$\lim_{\mathcal{A}(\mathcal{D})\to 0}\frac{1}{\mathcal{A}(\mathcal{D})}\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}\mathcal{A} = \lim_{\mathcal{A}(\mathcal{D})\to 0}\frac{1}{\mathcal{A}(\mathcal{D})}\oint_{\partial\mathcal{D}} \mathbf{F}(\mathbf{r})\cdot\,\mathrm{d}\mathbf{r}\,.$$

This motivates us to call this quantity $Q_x(\mathbf{r}_0) - P_y(\mathbf{r}_0)$ the *planar curl* of $\mathbf{F}$:

Theorem. *The planar curl of a* $\mathcal{C}^1(\mathcal{D},\mathbb{R}^2)$ *vector field* $\mathbf{F}$ *is the scalar field* $\operatorname{curl}\mathbf{F}(\mathbf{r})$ *determined point-wise at any* $\mathbf{r}_0$ *in the domain* $\mathcal{D}$ *of* $\mathbf{F}$ *by*

$$\operatorname{curl}\mathbf{F}(\mathbf{r}_0) = \lim_{\mathcal{A}(\mathcal{C})\to 0}\frac{1}{\mathcal{A}(\mathcal{C})}\oint_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\,\mathrm{d}\mathbf{r}\,,$$

where the limit is taken over a continuum of smooth simple closed curves $\mathcal{C}$ *in* $\mathcal{D}$ *each bounding a simply connected subregion of* $\mathcal{D}$ *around the position* $\mathbf{r}_0$ *and shrinking to the point at position* $\mathbf{r}_0$*, and* $\mathcal{A}(\mathcal{C})$ *is the area of the region bounded by* $\mathcal{C}$*. If the expression of* $\mathbf{F}$ *in the rectangular frame is* $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\,\hat{\mathbf{\imath}} + Q(\mathbf{r})\hat{\mathbf{\jmath}}$ *then as a consequence of Green's theorem the planar curl is given by*

$$\operatorname{curl}\mathbf{F}(\mathbf{r}_0) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(\mathbf{r}_0)\,.$$

We now recall how the curl was defined for 3 dimensional vector fields:

Definition. The curl of a vector field $\mathbf{F} : \mathcal{D} \to \mathbb{R}^3$ expressed in the rectangular frame as $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\,\hat{\imath} + Q(\mathbf{r})\,\hat{\jmath} + R(\mathbf{r})\,\hat{\mathbf{k}}$ is the vector field whose rectangular coordinate expression is

$$\mathrm{curl}(\mathbf{F}) := \nabla \times \mathbf{F} = (\partial_y R - \partial_z Q)\hat{\imath} + (\partial_z P - \partial_x R)\hat{\jmath} + (\partial_x Q - \partial_y P)\hat{\mathbf{k}}$$

We will prefer the notation $\nabla \times \mathbf{F}$ for the 3 dimensional curl. *Note that each component of the rectangular expression for $\nabla \times \mathbf{F}$ appears as a planar curl relative to the coordinate plane normal to the respective frame vector.* This suggests the following:

Theorem. *Let $\hat{\mathbf{n}}$ be any unit vector, and let $\Pi_{\hat{\mathbf{n}},\mathbf{r}_0}$ be the plane through $\mathbf{r}_0$ normal to $\hat{\mathbf{n}}$. Then the curl of a $\mathcal{C}^1(\mathcal{D}, \mathbb{R}^3)$ vector field is the unique vector field $\nabla \times \mathbf{F}(\mathbf{r})$ such that for any $\hat{\mathbf{n}}$ and any point $\mathbf{r}_0 \in \mathcal{U}$ an open subset of $\mathcal{D}$,*

$$\hat{\mathbf{n}} \cdot \big(\nabla \times \mathbf{F}(\mathbf{r}_0)\big) = \lim_{\mathcal{A}(\mathcal{C}) \to 0} \frac{1}{\mathcal{A}(\mathcal{C})} \oint_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \,\mathrm{d}\mathbf{r}\,,$$

where the limit is taken over a continuum of smooth simple closed curves $\mathcal{C}$ in the plane $\Pi_{\hat{\mathbf{n}},\mathbf{r}_0}$ each bounding a simply connected subregion of $\Pi_{\hat{\mathbf{n}},\mathbf{r}_0}$ around the position $\mathbf{r}_0$ and shrinking to the point at position $\mathbf{r}_0$, and $\mathcal{A}(\mathcal{C})$ is the area of the region bounded by $\mathcal{C}$.

See the problems below to explore the use of this formalism to obtain expression for the three dimensional curl operator in cylindrical and spherical coordinates.

We can try to connect the idea of curl as a limit of circulation to the theory of integration in three dimensions. For a moment, let us regard Green's Theorem as applying to a vector field defined not just in the plane, but in an open set of $\mathbb{R}^3$ that contains $\mathcal{D}$. You may recognize that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the $\hat{\mathbf{k}}$-component of the curl of such an $\mathbf{F}$. If we rewrite Green's theorem with this in mind, we get the following equation:

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}}\,\mathrm{d}\mathcal{A} = \iint_{\mathcal{D}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}\mathcal{S}_{\mathcal{A}}\,,$$

or

$$\mathcal{W}[\mathbf{F}, \partial\mathcal{D}] = \mathcal{F}[\nabla \times \mathbf{F}, \mathcal{D}]\,.$$

That is, we get an equality relating work of a vector field $\mathbf{F}$ along a closed curve $\mathcal{C} = \partial\mathcal{D}$ to the flux of the curl of $\mathbf{F}$ on a surface bounded by $\mathcal{C}$, in the special case where the surface is just a simply-connected region in the xy-plane. We may wonder if this is more generally true, namely, will it still hold if we use a different "capping surface" for $\mathcal{C}$, or if $\mathcal{C}$ is a piecewise smooth simple closed *space* curve not confined to a plane.

It turns out that such a generalization does exist: the Stokes-Kelvin theorem explored in §4.4.

Divergence via limiting Flux per unit area.

We analogously will arrive at an interpretation of *planar* divergence as "infinitesimal flux" by considering the limiting value of flux out of a shrinking simply connected region around a point $\mathbf{r}_0$ in a planar $\mathcal{C}^1$ field. We will generalize this idea to three dimensions, which leads us naturally towards the divergence theorem discussed in §4.5.

Recall that the planar flux of a 2-dimensional vector field $\mathbf{F}$ out of $\mathcal{D}$ is given by finding a unit outward normal field $\hat{\mathbf{n}}$ to $\partial\mathcal{D}$ and then integrating the scalar component of $\mathbf{F}$ in this normal direction around the boundary:

$$\mathcal{F}[\mathbf{F}, \mathcal{D}] := \int_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}s\,.$$

Writing $\mathbf{F}(x, y) = P(x, y)\,\hat{\imath} + Q(x, y)\,\hat{\jmath}$ and using the expression $\hat{\mathbf{n}}\,\mathrm{d}s = \,\mathrm{d}y\,\hat{\imath} - \,\mathrm{d}x\,\hat{\jmath}$ the flux becomes

$$\mathcal{F}[\mathbf{F}, \mathcal{D}] = \int_{\partial\mathcal{D}} P(x, y)\,\mathrm{d}y - Q(x, y)\,\mathrm{d}x\,.$$

Applying Green's theorem to this expression we conclude

$$\mathcal{F}[\mathbf{F},\mathcal{D}] = \int_{\partial\mathcal{D}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \iint_{\mathcal{D}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,\mathrm{d}\mathcal{A}\,.$$

In particular, by arguing about averages as we did for the planar curl we deduce

Theorem. *The planar divergence of a* $\mathcal{C}^1(\mathcal{D},\mathbb{R}^2)$ *vector field* $\mathbf{F}$ *is the scalar field* $\operatorname{div}\mathbf{F}(\mathbf{r})$ *determined point-wise at any* $\mathbf{r}_0$ *in the domain* $\mathcal{D}$ *of* $\mathbf{F}$ *by*

$$\operatorname{div}\mathbf{F}(\mathbf{r}_0) = \lim_{\mathcal{A}(\mathcal{C})\to 0} \frac{1}{\mathcal{A}(\mathcal{C})}\oint_{\mathcal{C}} \mathbf{F}(\mathbf{r})\cdot\hat{\mathbf{n}}\,\mathrm{d}s\,,$$

where the limit is taken over a continuum of smooth simple closed curves $\mathcal{C}$ *in* $\mathcal{D}$ *each bounding a simply connected subregion of* $\mathcal{D}$ *around the position* $\mathbf{r}_0$ *and shrinking to the point at position* $\mathbf{r}_0$*, and* $\mathcal{A}(\mathcal{C})$ *is the area of the region bounded by* $\mathcal{C}$*. If the expression of* $\mathbf{F}$ *in the rectangular frame is* $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\,\hat{\imath} + Q(\mathbf{r})\hat{\jmath}$ *then as a consequence of Green's theorem the planar divergence is given by*

$$\operatorname{div}\mathbf{F}(\mathbf{r}_0) = \nabla\cdot\mathbf{F}(\mathbf{r}_0) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)(\mathbf{r}_0)\,.$$

Generalizing this idea to surfaces seems straightforward: we could choose to *define* the 3-dimensional divergence operator as the differential operator that produces the uniquely determined scalar field which is obtained point-wise as the limiting value of flux through a family of closed surfaces shrinking to a position $\mathbf{r}_0$:

Definition. The divergence $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$ of a $\mathcal{C}^1(\mathcal{D},\mathbb{R}^3)$ vector field is the unique scalar field determined point-wise by at any $\mathbf{r}_0 \in \mathcal{D}$ by the limit

$$\operatorname{div}\mathbf{F}(\mathbf{r}_0) := \lim_{\mathcal{V}(\mathcal{S})\to 0} \frac{1}{\mathcal{V}(\mathcal{S})} \oiint_{\mathcal{S}} \mathbf{F}(\mathbf{r})\cdot\hat{\mathbf{n}}(\mathbf{r})\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,,$$

where the limit is taken over a continuum of orientable closed smoothly embedded surfaces $\mathcal{S}$ bounding solid regions of volume $\mathcal{V}(\mathcal{S})$ containing $\mathbf{r}_0$ and shrinking to the point $\mathbf{r}_0$, and $\mathbf{nr}$ is an outward unit normal field to $\mathcal{S}$.

Note that the planar statement of Green's theorem applied to flux has the form

$$\iint_{\mathcal{D}} \nabla\cdot\mathbf{F}\,\mathrm{d}\mathcal{A} = \int_{\partial\mathcal{D}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s\,,$$

and we may expect a similar result of the form

$$\iiint_{\mathcal{E}} \nabla\cdot\mathbf{F}\,\mathrm{d}\mathcal{V} = \iint_{\partial\mathcal{E}} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\partial\mathcal{E}}\,.$$

Indeed this is the *divergence theorem* discussed in §4.5.

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§ 4.4. The Stokes-Kelvin Theorem

We now arrive at another generalization of the fundamental theorem of calculus, which relates line integrals to surface integrals, work to flux and curl, boundaries to interiors.

Theorem 4.1 (Stokes-Kelvin Theorem)**.** *Let* $\mathbf{F}$ *be a continuously differentiable vector field on an open domain* $\mathcal{D}$ *in* $\mathbb{R}^3$*. For a given a piecewise smooth simple closed curve* $\mathcal{C}$ *bounding at least one simply connected orientable surface* $\mathcal{S}$ *in* $\mathcal{D}$*, orient* $\mathcal{C}$ *and* $\mathcal{S}$ *so that* $\mathcal{S}$ *is "to the left" as one traverses* $\mathcal{C}$*. Then the circulation of* $\mathbf{F}$ *on* $\mathcal{C}$ *equals the flux of the curl through* $\mathcal{S}$*:*

$$\int_{\mathcal{C}} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,.$$

The surface $\mathcal{S}$ is often called a *capping surface* for $\mathcal{C}$. Thus, in words the theorem states "the work of a vector field on a particle moving along a simple loop is equal to the flux of the curl of that vector field through any co-oriented capping surface, provided the vector field is continuously differentiable in an open set containing the surface and its boundary loop." Here, the co-orientation of $\mathcal{S}$ and $\mathcal{C}$ amounts to requiring that the normal vector field $\hat{\mathbf{n}}$ for $\mathcal{S}$ is chosen so that, if viewing $\mathcal{C}$ from the tip of such a normal vector, it appears that $\mathcal{C}$ is traversed counter-clockwise. One often then calls the curve $\mathcal{C}$ *positively oriented with respect to the orientation of the surface* $\mathcal{S}$. A topologist would say that the curve $\mathcal{C}$ has the *induced orientation*; observe that reversing the orientation of $\mathcal{S}$ reverses the orientation of $\mathcal{C}$ which is considered positive.

Example 4.11. We will verify the Stokes-Kelvin theorem in the case of the line integral

$$\oint_{\mathcal{T}} (z-y)\,\mathrm{d}x + (x-z)\,\mathrm{d}y + (x-y)\,\mathrm{d}z$$

where $\mathcal{T}$ is the loop of the edges of the triangle in $\mathbb{R}^3$ with vertices $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$, oriented so that the vertices are encountered in the order listed (see figure 41).

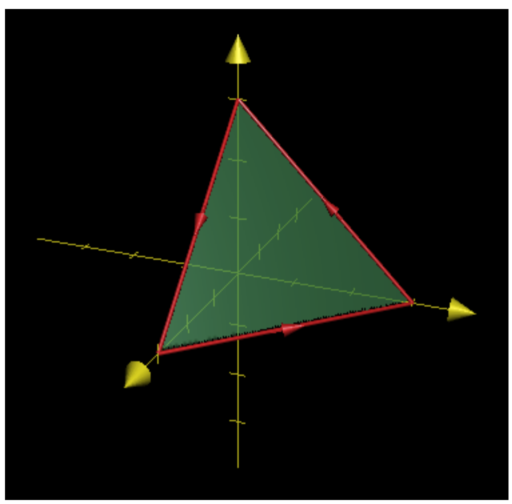

FIGURE 41. The curve $\mathcal{T}$ and the planar surface capping it, $\mathcal{S}$.

To compute the line integral directly by parameterization, one first must parameterize each edge of the triangle. Let $\mathcal{T}$ decompose as the edges $\mathcal{T}_1$ in the xy-plane, $\mathcal{T}_2$ in the yz-plane, and $\mathcal{T}_3$ in the xz-plane, and let $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}_3(t)$ be parameterizations of these edges given by

$$\mathbf{r}_1(t) = (3-3t)\,\hat{\imath} + 3t\,\hat{\jmath}\,, \quad \mathbf{r}_1(t) = (3-3t)\,\hat{\jmath} + 3t\,\hat{\mathbf{k}}\,, \text{ and } \mathbf{r}_3(t) = 3t\,\hat{\jmath} + (3-3t)\,\hat{\mathbf{k}}\,,$$

where $0 \le t \le 1$ for each parameterization.

Then

$$\begin{aligned}\oint_{\mathcal{T}}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z &= \oint_{\mathcal{T}_1}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z\\ &\quad+\oint_{\mathcal{T}_2}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z\\ &\quad+\oint_{\mathcal{T}_3}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z\\ &=\int_0^1 -3t\,\mathrm{d}(3-3t)+(3-3t)\,\mathrm{d}(3t)+(3-6t)\,\mathrm{d}(0)\\ &\quad+\int_0^1 3t\,\mathrm{d}(0)-3t\,\mathrm{d}(3-3t)-(3-3t)\,\mathrm{d}(3t)\\ &\quad+\int_0^1 (3-3t)\,\mathrm{d}(3t)+(6t-3)\,\mathrm{d}(0)+3t\,\mathrm{d}(3-3t)\\ &=\int_0^1 9\,\mathrm{d}t=9.\end{aligned}$$

On the other hand, we could have avoided the pain of writing down these parameterizations and setting up three separate integrals for the price of making the following observations before computing a simple double integral:

- $\mathcal{T}$ bounds a triangular surface $\mathcal{S}$ which is a portion of the plane $x+y+z=3$ in the first octant, with unit normal $\hat{\mathbf{n}}=\frac{1}{\sqrt{3}}(\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}})$, and surface area element $\mathrm{d}\mathcal{A}_{\mathcal{S}}=\sqrt{3}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}$ where $\mathcal{D}=\{(x,y):0\le x\le 3, 0\le y\le 3-x\}$,
- The desired line integral can be written as $\oint_{\mathcal{T}=\partial\mathcal{S}}\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F}=(z-y)\,\hat{\mathbf{\imath}}+(x-z)\,\hat{\mathbf{\jmath}}+(x-y)\,\hat{\mathbf{k}}$, which has curl

$$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ z-y & x-z & y-z\end{vmatrix}=2(\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}})\,.$$

The Stokes-Kelvin theorem then allows us to calculate the line integral as follows:

$$\begin{aligned}\oint_{\mathcal{T}}(z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z &= \oint\mathbf{F}\cdot\,\mathrm{d}\mathbf{r}=\iint_{\mathcal{S}}\nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\\ &=\iint_{\mathcal{D}}2(\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}})\cdot\tfrac{1}{\sqrt{3}}(\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}})\sqrt{3}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\\ &=\int_0^3\int_0^{3-x}2\,\mathrm{d}y\,\mathrm{d}x\\ &=2\mathcal{A}(\mathcal{D})=9\,,\end{aligned}$$

in agreement with our other calculation.

Of course, the choice of a valid capping surface meeting the conditions of the Stokes-Kelvin theorem does not affect the resulting value of the double integral. Our next example shows that one can turn a potentially difficult surface integral of curl into an easier one by switching capping surfaces:

Example 4.12. We compute the flux of the curl field $\nabla\times(yz\hat{\mathbf{\imath}}-xz\hat{\mathbf{\jmath}}+xy\hat{\mathbf{k}})=2x\hat{\mathbf{\imath}}-2z\hat{\mathbf{k}}$ through the portion $\mathcal{S}$ of the cone $z=1-\sqrt{x^2+y^2}$ above the plane $z=0$, oriented with the upward unit normal. This unit upward normal to the cone is given by

$$\hat{\mathbf{n}}=\frac{x\hat{\mathbf{\imath}}+y\hat{\mathbf{\jmath}}+\sqrt{x^2+y^2}\,\hat{\mathbf{k}}}{\sqrt{2(x^2+y^2)}}\,,$$

as one can readily check using the formula

$$\hat{\mathbf{n}} = \frac{-f_x(x,y)\,\hat{\mathbf{\i}} - f_y(x,y)\,\hat{\mathbf{\j}} + \hat{\mathbf{k}}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}}\,.$$

The surface area element works out to $\sqrt{2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}$, where $\mathcal{D}$ is the unit disk in the xy-plane centered at the origin. Thus

$$\begin{aligned}\iint_{\mathcal{S}} \left(2x\hat{\mathbf{\i}} - 2z\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} &= \iint_{\mathcal{D}} \left(2x\hat{\mathbf{\i}} - 2z\hat{\mathbf{k}}\right)\cdot\left(\frac{x\hat{\mathbf{\i}}+y\hat{\mathbf{\j}}+\sqrt{x^2+y^2}\,\hat{\mathbf{k}}}{\sqrt{2(x^2+y^2)}}\right)\sqrt{2}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\\ &= \iint_{\mathcal{D}} \frac{2x^2}{\sqrt{x^2+y^2}} - 2(1-\sqrt{x^2+y^2})\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,.\end{aligned}$$

On the other hand, we know from the Stokes-Kelvin theorem that

$$\iint_{\mathcal{S}} \nabla\times(yz\hat{\mathbf{\i}} - xz\hat{\mathbf{\j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \oint_{\mathcal{C}} (yz\hat{\mathbf{\i}} - xz\hat{\mathbf{\j}} + xy\hat{\mathbf{k}})\cdot\,\mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} (2x\hat{\mathbf{\i}} - 2z\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\,\mathrm{d}\mathcal{A}_{\mathcal{D}}\,,$$

since the unit disk is itself a capping circle of the unit circle $\mathcal{C}$, which gives the boundary of $\mathcal{S}$. But of course, $z=0$ on the disk, whence

$$\iint_{\mathcal{S}} \nabla\times(yz\hat{\mathbf{\i}} - xz\hat{\mathbf{\j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \iint_{\mathcal{D}} 0\,\mathrm{d}\mathcal{A}_{\mathcal{D}} = 0\,.$$

Computing the original surface integral over the cone is a bit more work. Indeed, to directly verify that the surface integral over the cone is zero, we switch to polar coordinates:

$$\begin{aligned}\iint_{\mathcal{S}} \left(2x\hat{\mathbf{\i}} - 2z\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} &= \int_0^{2\pi}\int_0^1 \left(\frac{2r^2\cos^2\theta}{r} - 2 + 2r\right) r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^1 2r^2\cos^2\theta - 2r + 2r^2\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \int_0^{2\pi} \frac{2}{3}r^3(1+\cos^2\theta) - r^2\Big|_0^1\,\mathrm{d}\theta\\ &= \int_0^{2\pi} \frac{2}{3}\cos^2\theta - \frac{1}{3}\,\mathrm{d}\theta\\ &= \int_0^{2\pi} \frac{2}{3}\left(\frac{1+\cos(2\theta)}{2}\right) - \frac{1}{3}\,\mathrm{d}\theta\\ &= \int_0^{2\pi} \frac{1}{3}\cos(2\theta)\,\mathrm{d}\theta = 0\,.\end{aligned}$$

To put this theorem in context with the fundamental theorems encountered so far, we briefly discuss the parallels between those fundamental theorems encountered in single variable calculus, the fundamental theorem of line integrals, and Green's theorem.

Consider the problem of computing the flux of some vector field $\mathbf{G}$ through an oriented surface $\mathcal{S}$ with co-oriented boundary $\partial\mathcal{S}$. If there exists a vector field $\mathbf{F}$ such that $\mathbf{G} = \nabla\times\mathbf{F}$, then we can calculate the surface integral by

$$\iint_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}} = \oint_{\partial\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{T}}\,\mathrm{d}s\,.$$

Many texts think of $\hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}$ as an *oriented surface area element*, and $\hat{\mathbf{T}}\,\mathrm{d}s$ as an *oriented line element*, and write both in vector notation:

$$\mathrm{d}\mathbf{r} = \hat{\mathbf{T}}\,\mathrm{d}s\,,\quad \mathrm{d}\mathbf{A} = \hat{\mathbf{n}}\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,,$$

and write the equality of Stokes-Kelvin theorem as

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \oint_{\partial \mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}\,.$$

This parallels the fundamental theorems encountered thus far as follows: on the left we have an integral over a space (in this case a surface), with an associated oriented differential element, and on the right we evaluate an integral on *the boundary of the original space*, again using an oriented differential element, and with integrand given in terms of an "antiderivative" of the left-hand integrand. In particular, the relation between the integrand on the left and the right is one of anti-differentiation in the sense of finding a *vector potential* $\mathbf{F}$ realizing $\mathbf{G}$ as the curl of $\mathbf{F}$, or viewed the other way, a relation of partial differentiation, turning $\mathbf{F}$ into its curl $\mathbf{G}$. But the essence is common to all such fundamental theorems of calculus: if one integrates a field over a space, then one equivalently can accumulate data about an antiderivative on the boundary of the space, and if one integrates a form on a closed boundary of some space, one obtains the same information as an integral of a differential of that form over the interior. The divergence theorem also follows this pattern, though the differential appears different from the curl. In truth, they can be unified as a common differential if we change how we think about integrals and vector fields. But that is the topic of the modern theory of differential forms and integrals on manifolds.

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§ 4.5. The Divergence Theorem

The divergence theorem relates the flux of a vector field $\mathbf{F}$ through a closed surface $\mathcal{S}$ to the divergence of the vector field $\mathbf{F}$ on region bounded inside the surface $\mathcal{S}$.

Theorem 4.2 (The Divergence Theorem)**.** *Let* $\mathbf{F}$ *be a continuously differentiable vector field on an open domain* $\mathcal{D}$ *which contains a simply connected solid region* $\mathcal{E}$ *which is bounded by a piecewise smooth, orientable closed surface* $\mathcal{S}$*. Let* $\hat{\mathbf{n}}$ *be the outward unit normal vector field to* $\mathcal{S}$*. Then the flux of* $\mathbf{F}$ *through* $\mathcal{S}$ *is equal to the integral of the divergence of* $\mathbf{F}$ *over the solid region* $\mathcal{E}$*:*

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}\mathcal{A}_{\mathcal{S}} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}\mathcal{V}\,.$$

As with the Stokes-Kelvin theorem, this theorem can be phrased in terms of boundaries and interiors, differentiation and anti-differentiation. The Divergence theorem in words states *the flux of a vector field through an oriented piecewise smooth closed surface is equal to the integral of divergence over the volume the surface encloses, provided the divergence is of that vector field is continuous over a neighborhood enveloping the surface and the solid region interior to it.* One may write the equality of theorem in the form

$$\iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}\mathcal{V} = \oiint_{\partial \mathcal{E}} \mathbf{F} \cdot \mathrm{d}\mathbf{A}\,,$$

where as in our discussion of Stokes-Kelvin, $\mathrm{d}\mathbf{A} = \hat{\mathbf{n}}\, \mathrm{d}\mathcal{A}_{\mathcal{S}}$ is the *oriented surface area element*, in this case positively oriented relative to the interior of the region $\mathcal{E}$ (hence $\hat{\mathbf{n}}$ is an outward normal).

Example 4.13. We compute the flux of the vector field $\mathbf{F}(x, y, z) = \langle x^2 - y, y^2 - x, z^2 - x^2 - y^2 \rangle$ through the tetrahedral surface $\mathcal{T}$ given as the boundary of the region $\mathcal{E}$ in the first octant below the plane $x + y + z = 1$. Computing the flux directly would be slightly horrendous, given that there are four triangular faces over which we need to compute surface integrals. Instead, we compute

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z\,,$$

and by the divergence theorem

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}\mathcal{A}_{\mathcal{S}} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}\mathcal{V} = \iiint_{\mathcal{E}} 2x + 2y + 2z\, \mathrm{d}\mathcal{V}\,.$$

This triple integral may be evaluated readily:

$$\begin{aligned}
\iiint_{\mathcal{E}} 2x+2y+2z\,\mathrm{d}\mathcal{V} &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} 2x+2y+2z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1\int_0^{1-x} \Big[2(x+y)z+z^2\Big]_0^{1-x-y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1\int_0^{1-x} 2(x+y)(1-x-y)+(1-x-y)^2\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1\int_0^{1-x} 2(x+y+1-x-y)(1-x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1\int_0^{1-x} 2-2x-2y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \Big[(2-2x)y-y^2\Big]_0^{1-x}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 (x-1)^2\,\mathrm{d}x \\
&= \int_0^1 x^2-2x+1\,\mathrm{d}x \\
&= \frac{1}{3}\,.
\end{aligned}$$

§ 4.6. Problems

(1) Compute the following scalar surface integrals:

(a) $\displaystyle\iint_{\mathcal{S}} \frac{\mathrm{d}\mathcal{A}_{\mathcal{S}}}{1+4x^2+4y^2}$, where the surface $\mathcal{S}$ is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$,

(b) $\displaystyle\iint_{\mathcal{S}} x^2 + y^2 \,\mathrm{d}\mathcal{A}_{\mathcal{S}}$ where $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = R^2$ for any constant $R > 0$,

(c) $\displaystyle\iint_{\mathcal{S}} xyz \,\mathrm{d}\mathcal{A}_{\mathcal{S}}$ where $\mathcal{S}$ is still the sphere $x^2 + y^2 + z^2 = R^2$ for any constant $R > 0$,

(d) $\displaystyle\iint_{\mathcal{Q}} x + y + z \,\mathrm{d}\mathcal{A}_{\mathcal{S}}$ where $\mathcal{Q} = [0,1]^3$ is the unit cube with vertices $\mathbf{0}$, $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, $\hat{\mathbf{k}}$, $\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}$, $\hat{\mathbf{\imath}}+\hat{\mathbf{k}}$, $\hat{\mathbf{\jmath}}+\hat{\mathbf{k}}$, and $\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}}$.

(2) Compute the following planar fluxes using parametric descriptions of the curves. Where possible, appeal to the Green's form of the divergence theorem and verify the equality of the outward flux through closed curves with the double integral of divergence over the region interior to the curve.

(a) The flux of the radial field $\mathbf{r}$ out of the unit square,

(b) The flux of the vector field $\mathbf{F} = x\hat{\mathbf{\imath}} + 2xy\hat{\mathbf{\jmath}}$ out of the circle with equation $(x-1)^2 + y^2 = 1$,

(c) The flux $\mathcal{F}[\mathbf{F}, \mathcal{C}] := \int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{N}}_s \,\mathrm{d}s$ where $\mathbf{F} = (x^3 - 3xy^2)\,\hat{\mathbf{\imath}} + (y^3 - 3x^2y)\,\hat{\mathbf{\jmath}}$, and $\mathcal{C}$ is the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$,

(d) The flux of $\mathbf{F} = (x^3 - 3xy^2)\,\hat{\mathbf{\imath}} + (y^3 - 3x^2y)\,\hat{\mathbf{\jmath}}$ out of the region bounded by the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$.

(3) Compute the following flux surface integrals directly. Where possible verify the divergence theorem by checking equality of the surface integral and the relevant triple integral of divergence.

(a) The flux of the field $\mathbf{F} = xz^2\hat{\mathbf{\imath}} + yx^2\hat{\mathbf{\jmath}} + zy^2\hat{\mathbf{k}}$ out of the region bounded by the upper hemisphere of a radius a sphere together with the radius a disk in the plane $z = 0$,

(b) The flux of $\mathbf{F} = \mathbf{r}e^{-\mathbf{r}\cdot\mathbf{r}}$ out of the unit sphere,

(c) The flux of the field $(2x-1)\hat{\mathbf{\imath}} + (2y-1)\hat{\mathbf{\jmath}} + (2z-1)\hat{\mathbf{k}}$ through the unit cube $[0,1]^3$,

(d) the flux integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{A}$ where $\mathbf{F} = x^2\hat{\mathbf{\imath}} + y^2\hat{\mathbf{\jmath}} + z^2\hat{\mathbf{k}}$ and $\mathcal{S}$ is the surface bounding the region

$$\{(x,y,z) : x^2 + y^2 + z^2 \le 4, 1 \le x^2 + y^2\}.$$

(4) Derive coordinate expressions for 3D Gradient, Divergence and Curl using the integral definitions in each of the following coordinate systems: (a) rectangular, (b) cylindrical, and (c) spherical.

(5) Calculate the following line integrals. Where possible, by selecting appropriate parameterizations or capping surfaces, verify the conclusion of the Stokes-Kelvin theorem.

(a) The work of $\mathbf{F} = z\hat{\mathbf{\imath}} - y\hat{\mathbf{\jmath}} + x\hat{\mathbf{k}}$ along the curve of intersection of the plane $x + y - 2z = 0$ and the cylinder $x^2 + y^2 = 2$,

(b) $\displaystyle\oint_{\mathcal{C}} xz^2 \,\mathrm{d}x + yx^2 \,\mathrm{d}y + zy^2 \,\mathrm{d}z$ where $\mathcal{C}$ is the square with vertices $(1,1,1)$, $(-1,1,1)$, $(-1,1,-1)$ and $(1,1,-1)$ traversed in the order listed,

(c) $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{r}$ where $\mathbf{F} = zy\hat{\mathbf{\imath}} + xz\hat{\mathbf{\jmath}} + xy\hat{\mathbf{k}}$ and $\mathcal{C}$ is the curve consisting of the portion of the helix $\mathbf{r}(t) = \cos t\,\hat{\mathbf{\imath}} + \sin t\,\hat{\mathbf{\jmath}} + t\hat{\mathbf{k}}$ with $-\pi \le t \le \pi$ together with the line segment from

$\mathbf{r}(\pi)$ to $\mathbf{r}(-\pi)$. Verify with Stokes-Kelvin using the surface $\boldsymbol{\sigma}(u,v) = \big(u(1+\cos v) - 1\big)\,\hat{\mathbf{\imath}} + u\sin v\,\hat{\mathbf{\jmath}} + v\hat{\mathbf{k}}$, $0 \le u \le 1$, $-\pi \le v \le \pi$.

(6) Let $0 < a < b$ be constants, and recall that one may parameterize a torus by

$$\mathbf{r}(u,v) = \big(a\cos(u)+b\big)\,\hat{\mathbf{u}}_r(v) + a\sin(u)\,\hat{\mathbf{k}} = \big(a\cos(u)+b\big)\cos(v)\,\hat{\mathbf{\imath}} + \big(a\cos(u)+b\big)\sin(v)\,\hat{\mathbf{\jmath}} + a\sin(u)\,\hat{\mathbf{k}}\,.$$

(a) By constructing an appropriate vector field expressed in cylindrical coordinates, which restricts to a normal field along the torus, calculate the surface area of the torus via a triple integral over the region enclosed by the torus.

(b) Similarly, use a surface integral to calculate the volume enclosed by the torus.

(c) Construct coordinates and a frame for $\mathbb{R}^3$ well defined on the complement of the z-axis and the circle $r = b$, $z = 0$, such every point $\mathbf{r}$ not on either the z-axis or the circle $r = b$, $z = 0$ lies on a unique torus, and the frame vectors at such a position $\mathbf{r}$ consist of two tangent vectors to the torus containing $\mathbf{r}$ and an outward normal to the torus at $\mathbf{r}$.

(d) Use your choice of methods to calculate expressions for the gradient, divergence, and curl operators with respect to these toroidal coordinates on $\mathbb{R}^3$.

(e) Repeat the surface area and volume calculations in parts (a) and (b) using the expressions found in part (c).

(7) Recall that the vector projection operator $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ is in fact independent of the length of $\mathbf{u}$, and depends only on the line $\ell_{\mathbf{u}} = \{t\mathbf{u} : t \in \mathbb{R}\}$ which $\mathbf{u}$ spans. Given a surface $\mathcal{S}$, let $\ell_\nu(\mathbf{r})$ denote the normal line through the point with position $\mathbf{r}$ on $\mathcal{S}$, and for any vector $\mathbf{v}$ and point $\mathbf{r}$ on $\mathcal{S}$, write $\mathrm{proj}_{\nu(\mathbf{r})}(\mathbf{v})$ for the projection of $\mathbf{v}$ onto the line $\ell_\nu(\mathbf{r})$. Then define the *vector valued net flux* of a vector field $\mathbf{F}$ through a surface $\mathcal{S}$ to be the vector-valued surface integral

$$\iint_{\mathcal{S}} \mathrm{proj}_{\nu(\mathbf{r})}\mathbf{F}(\mathbf{r})\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,,$$

provided the necessary limiting vector exists. Similarly, define *gross scalar flux* to be

$$\iint_{\mathcal{S}} \|\mathrm{proj}_{\nu(\mathbf{r})}\mathbf{F}(\mathbf{r})\|\,\mathrm{d}\mathcal{A}_{\mathcal{S}}\,.$$

(a) Show by explicit examples justified through computation that for an orientable surface $\mathcal{S}$ it is possible for the vector valued net flux to be non-zero while the usual scalar flux is zero, and that for a different vector field with the same surface $\mathcal{S}$ the vector valued net flux may be zero while the scalar flux may be nonzero. What must hold (geometrically) about a particular vector field and surface pair for the *gross scalar flux* to be zero?

(b) Argue that the notions of vector valued net flux and gross flux are well defined for a non-orientable surface, meaning that the results are independent of any choices in local patches and normal vectors used in the process of computation.

(c) Set up integrals to compute the vector valued net flux and the gross flux of the constant vector field $\hat{\mathbf{k}}$ through the Möbius band given in example 2.29. Use a computer system and a preferred numerical method to approximate these integrals. Are the results what you might expect by symmetry considerations?

The above problem is unique (in that it is not inspired by problems I've seen or found anywhere else), and uniquely useless to most engineering, computer science, and other STEM majors.

(8) Let $\mathbf{F}$ be a $\mathcal{C}^1(\mathcal{D}, \mathbb{R}^3)$ vector field on some domain $\mathcal{D}$ and suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two simple closed curves in $\mathcal{D}$ which do not intersect each other. Show that if there exists a smooth oriented surface $\mathcal{S}$ whose boundary consists of these two curves, and such that $\nabla \times \mathbf{F} = 0$ at every point along $\mathcal{S}$ then

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}\,.$$

(9) Let f and g be scalar fields on a domain $\mathcal{D} \subseteq \mathbb{R}^3$ which are continuously differentiable to the second order, and suppose the surface $\mathcal{S}$ bounds a region $\mathcal{E}$ where $\mathcal{S}$ and $\mathcal{E}$ satisfy the hypotheses of the divergence theorem. Show the following:

(a) $\displaystyle\iiint_{\mathcal{E}} f\nabla^2 g \,\mathrm{d}\mathcal{V} = \oiint_{\mathcal{S}} f\nabla g \cdot \hat{\mathbf{n}} \,\mathrm{d}\mathcal{A}_{\mathcal{S}} - \iiint_{\mathcal{E}} \nabla f \cdot \nabla g \,\mathrm{d}\mathcal{V}$,

(b) $\displaystyle\iiint_{\mathcal{E}} f\nabla^2 g - g\nabla^2 f \,\mathrm{d}\mathcal{V} = \oiint_{\mathcal{S}} (f\nabla g - g\nabla f) \cdot \hat{\mathbf{n}} \,\mathrm{d}\mathcal{A}_{\mathcal{S}}$,

These are known as Green's first and second formulae respectively. He also has a third formula; you should look it up and try to prove it!

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