A Course in Complex Analysis

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Preface

This book is based on the lectures that I have given since the early 2000s at the University of Pennsylvania, Stony Brook University, and the City University of New York. It offers enough material for a year-long graduate-level course and serves as a preparation for further studies in complex analysis and beyond, especially Riemann surfaces, conformal geometry, and holomorphic dynamics.

The presentation is guided by a desire to highlight the topological underpinnings of complex analysis and to accentuate the geometric viewpoint whenever possible. This is evident from the following sample of special topics that are treated in the book: The dynamics of Möbius transformations, boundary behavior of Riemann maps à la Carathéodory, Hausdorff dimension and holomorphic removability, conformal metrics and Ahlfors's generalization of the Schwarz lemma, holomorphic (branched) covering maps, and the uniformization theorem for spherical domains. To remain loyal to the scope and spirit of the project, I have resisted the temptation to discuss Riemann surfaces.

The primary audience of the book are the beginning graduate students with a solid background in undergraduate analysis and topology. A basic knowledge of complex variables is helpful even though it is not formally assumed. Elementary measure theory (Lebesgue measure and integral, sets of measure zero, the dominated convergence theorem, etc.) shows up on a few occasions, but it is not a key prerequisite. Numerous worked-out examples, illustrations, short historical notes, and more than 360 problems have been incorporated throughout to make the text accessible for independent study by a strong and motivated student. Above all, I have strived to make the treatment of every topic appealing even to the more experienced readers with prior exposure to complex analysis.

The bulk of the book can be covered over two semesters, with enough remaining for reading projects if desired. A possible plan that I generally adhere to is to cover most of chapters 1-7 in the first semester followed by the "essential" material from chapters 8-13 in the second semester. Of course what is considered essential depends on one's taste and point of view, and the organization of topics allows quite a bit of flexibility in this respect. Occasionally I postpone a few items from chapters 1-7 until the second semester when students have developed more knowledge and skill. Examples are §6.3 on the boundary behavior of Riemann maps, or §4.4 and §4.5 on conformal metrics and the invariant form of the Schwarz lemma which can be presented ahead of Ahlfors's generalization in §11.4.

It would be hard to overstate how much this work has benefited from the existing literature on complex analysis. My sources have been gathered in the bibliography, but I must especially acknowledge the influence of the books by Walter Rudin [**Ru2**] (even though it famously lacks geometric flavor) and Reinhold Remmert [**Re**]; the latter also contains a wealth of historical remarks which have been helpful in several of my marginal notes. In writing a volume of this size I may have inadvertently misplaced an attribution or omitted a reference, in which case I would be grateful to be notified of the error.

This project was a long time in the making. It owes a great deal to my teachers and colleagues who have shared their insights into the subtleties of complex analysis and in doing so shaped my own view of the subject. I'd like to thank John Milnor for his guidance and inspiration, mathematical and otherwise, over the years. I'm also grateful to Christopher Bishop, Araceli Bonifant, Adam Epstein, Fredrick Gardiner, Linda Keen, Mikhail Lyubich, Bernard Maskit, Yair Minsky, and Dennis Sullivan. Finally, I'm indebted to my students, too many to name here, whose incisive comments and clever questions during my lectures have improved the presentation of this book.

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May 2020

"When I was a child I took pride in how many pages I read in an hour. In college I learned how foolish that was. When reading mathematics ten pages a day can be an extremely fast pace. Even one page a day can be quite fast. On the other hand, if you already understand something, you may get more by skimming than by reading every word. You need to be alert and suspicious; you need to question and think about what you're reading in your own way. . . . Don't be afraid to stop in midparagraph or midsentence when something surprises or puzzles you. Speed isn't the issue. Don't assume something is obvious just because an author treats it that way. What you work out on the side, even though it takes much more time, will have immensely more value than what you read straight through."

- William P. Thurston[1]

[1]From *Quantum* magazine's pilot issue 1, January 1990, page 7 ©National Science Teaching Association

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