Solutions Manual

A Digital Signal Processing Primer

with Applications to Digital Audio and Computer Music

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Note to the Reader

Here are solutions to all the problems in my book, *A Digital Signal Processing Primer*, with the exception of a few open-ended projects.

As in the book, references to figures and equations are to the current chapter unless stated otherwise. The figures in this solutions manual are labeled Pn, where n is the problem number. There was no need to number the equations.

I am indebted to Dan Trueman for checking these solutions and suggesting clarifications and improvements. When Dan did that much appreciated work he was a graduate student in the music department at Princeton; he is now professor in that same department. The inevitable errors that remain are, of course, my own doing.

Chapter 1

Tuning Forks, Phasors

1. One simply defined solution is the set of polynomial signals of fixed degree. More precisely, the following set of signals is closed under time shift:

$$\mathcal{S}_\omega = \left\{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0\right\}$$

for a fixed degree n, all coefficients a_i, and all t. Thus, for example, the set of linear signals is closed under time shift.

We can also use the set of finite sums of one-sided real exponential signals:

$$\mathcal{S}_\omega = \left\{\sum_{i=1}^{n} a_i e^{\sigma_i t}\right\}$$

for all real numbers a_i, σ_i and, time $t \geq 0$.

A really grand example is the class of all finite sums of exponentially weighted sinusoidal signals:

$$\mathcal{S}_\omega = \left\{\sum_{i=1}^{n} a_i t^{k_i} e^{\sigma_i t} \sin(\omega_i t + \phi_i)\right\}$$

for all real numbers $a_i, \sigma_i, \omega_i, \phi_i$, time $t \geq 0$, and integer powers k_i. This is the set of all one-sided signals with rational Laplace transforms (when t is continuous) or z-transforms (when t is discrete). These are discussed in Section 9 of Chapter 9. Obvious variations can be obtained by restricting the time constants σ_i or the frequencies ω_i to finite sets.

A wiseacre student may suggest the class of *all* signals, a perfectly good answer. The empty set is also correct, vacuously.

2. The class of signals $\mathcal{S}_\omega$ defined in Problem 1 is not closed under multiplication. For example,

$$2\sin^2(\omega t) = 1 - \cos(2\omega t)$$

which includes a sinusoid at frequency 2ω and is therefore outside the class $\mathcal{S}_\omega$.

Neither is the class closed under division, as can be seen from the fact that $1/\sin(\omega t)$ has an infinite value at $\omega = 0$ and therefore cannot be a sinusoid.

3. By induction. Consider the sum of n elements of the class. The basis of the induction is the case $n = 2$, which is already established. To carry out the induction, assume that the class is closed under the addition of $n - 1$ elements, and consider the sum of n elements. The sum of the first $n-1$ elements is in the class by the induction hypothesis, and this sum plus the nth element is in the class by the $n = 2$ case.

4. If a countably infinite sum of members from the class $\mathcal{S}_\omega$ converges, it also belongs to the class. Just consider the real part of

$$\sum_{k=0}^{\infty} A_k e^{j(\omega t+\phi_k)} = e^{j\omega t} \sum_{k=0}^{\infty} A_k e^{j\phi_k}$$

The same is true for an uncountably infinite sum.

If the class of functions is allowed to contain finite sums of phasors with all frequencies that are integer multiples of some constant ω, then the class is no longer closed under addition of a countably infinite number of terms. As we'll see in Chapter 7, the Fourier series for a square wave provides a counterexample — it's an infinite sum of members of this class that converges to a function outside the class.

5. The two tuning forks may differ in pitch, and you may be able to hear a beat. I have three tuning forks designated as A 440 Hz, and the first two do not produce a beat when sounded together. The third, however, beats at about 2 sec against each of the first two. The pitch of the third is therefore either 0.5 Hz higher or lower than that of the first two.

The higher-pitched clang tones mentioned in the Notes are more likely to differ enough in pitch to produce a beat. If the two forks are held apart, you may be able to tell there are two sources by moving your head.

7. This simple experiment is an elegant demonstration of destructive interference. As the tuning fork is rotated 360°, four distinct nulls are observed, located symmetrically in the plane perpendicular to the axis of the tines, as shown in Fig. P7. An intuitive explanation is as follows (see, for example, J. Askill, *Physics of Musical Sounds*, Van Nostrand, New York, N.Y., 1979, p. 29): The tines vibrate in the horizontal direction, moving towards one another, and then away from each other. This radiates sound waves most strongly in the direction aligned with the x-axis (left-right in the figure), with a strength that diminishes as one approaches the direction aligned with the y-axis (up-down in the figure). The motion of the tines also radiates sound in the direction of y direction, by alternately compressing and rarefying the air between the tines. This y-axis radiation is 180° out of phase with the x-axis radiation, and also diminishes in strength as one approaches the x-axis. At some angle roughly halfway between 0° and 90°, the two radiation waves cancel destructively.

Mathematically, the radial velocity in polar coordinates (r, θ) can be derived from the velocity potential of two dipoles (see, for example, H.Lamb, *The Dynamical Theory of Sound*, Dover, New York, N.Y., 1960, p. 231). The x-axis radial velocity is of the form

$$A_x f(r) e^{j(\omega t - 2\pi r/\lambda)} \cos^2 \theta$$

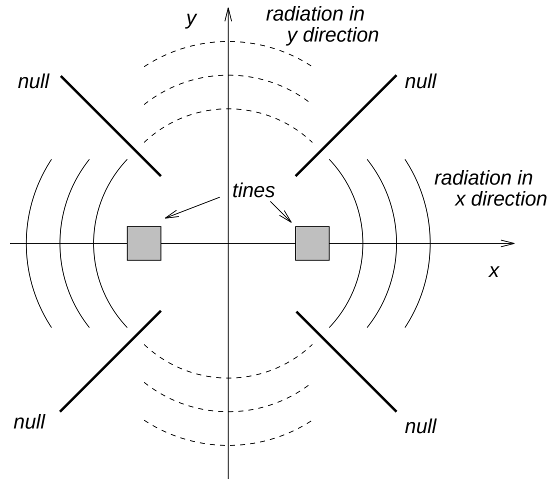

Fig. P7 Nulls in the radiation from a tuning fork. Shown is a plane perpendicular to the tines.

where ω is the fundamental frequency of the tuning fork in radians per sec, $\lambda = 2\pi c/\omega$ is the wavelength in m, and c is the speed of sound in air in m/sec. The y-axis radial velocity is approximately of the same form, with a different amplitude, say A_y, and rotated 90°:

$$A_y f(r) e^{j(\omega t - 2\pi r/\lambda)} \sin^2 \theta$$

The nulls in the total therefore occur when

$$\tan^2 \theta = A_x / A_y$$

Tuning forks seem to be designed to have $A_x \approx A_y$, so the nulls occur along axes at about 45° and 135°.

8. Three Lissajous figures are shown in Fig. P8, for the frequency ratios 4:1, 31:7, and 99:97. The simpler the ratio, the simpler the figure.

9. I generated this sound real-time in front of a class, and the result stopped me in my tracks. It's a good example of the advantage of thinking of sinusoids as phasors.

If we view the generated signal as a rotating phasor, its instantaneous frequency is the rate of change of its phase angle:

$$\omega = \frac{d}{dt}\left\{ \left(\omega_1 + \frac{t}{T}(\omega_2 - \omega_1) \right) \cdot t \right\} = \omega_1 + \frac{2t}{T}(\omega_2 - \omega_1)$$

The instantaneous frequency therefore starts at ω_1 and increases linearly to $2\omega_2 - \omega_1$ at $t = T$, not ω_2. If we continue at the fixed frequency ω_2, the instantaneous frequency will drop suddenly to ω_2.

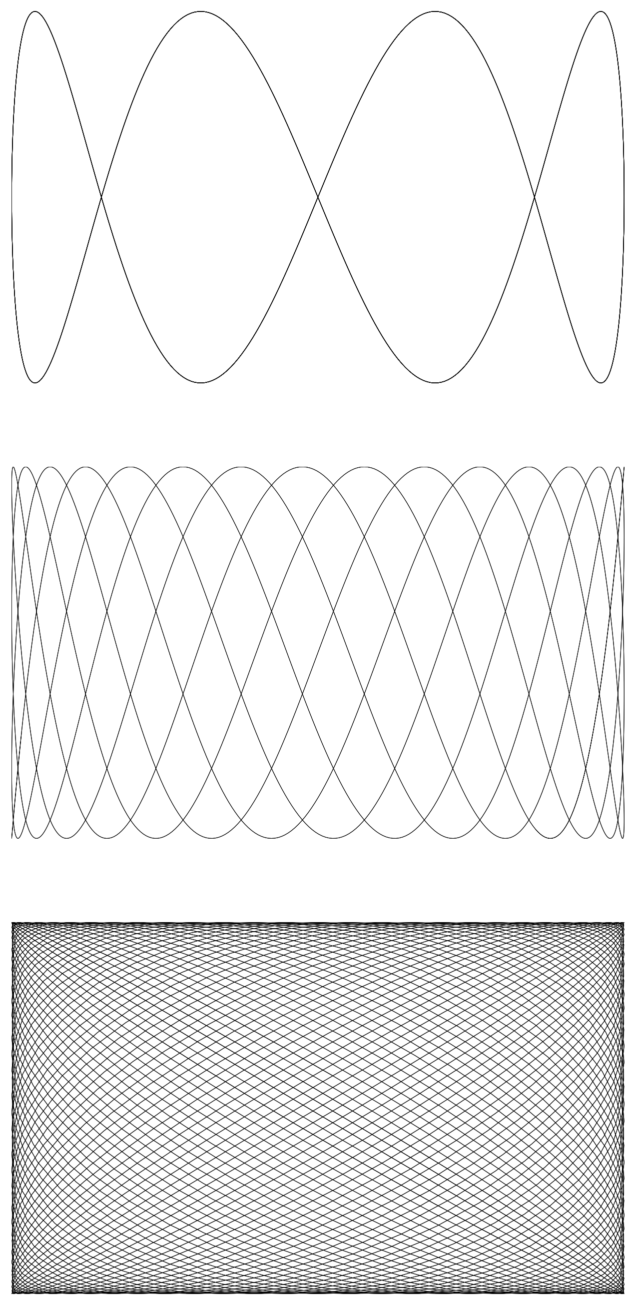

Fig. P8 Some Lissajous figures. The frequency ratios are (top) 4:1, (middle) 31:7, and (bottom) 99:97.

10. Denote by S the length of the vector labeled SUM in Fig. 9.3, and note that the vector a_2 makes an angle δt with the real axis. Then the law of cosines yields

$$S^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \delta t) = a_1^2 + a_2^2 + 2a_1a_2\cos(\delta t)$$

The length thus varies between $|a_1 - a_2|$ and $|a_1 + a_2|$, as we expect.

Alternatively, the squared-magnitude of Eq. 9.4 is the sum of the real part squared and the imaginary part squared,

$$S^2 = (a_1 + a_2\cos(\delta t))^2 + (a_2\sin(\delta t))^2$$

which simplifies to the same thing.

11. The actual waveform, Eq. 9.1, is the real part of the complex waveform, Eq. 9.2, and touches the envelope exactly when the complex waveform is real-valued — that is, the real part is equal to the magnitude when and only when the imaginary part is zero. Looking at it geometrically, this occurs when the tip of the complex vector crosses the real axis. The position of the tip in the complex plane is a continuous function of time, and moves between the upper half plane and the lower half plane. Hence, there is at least one instant of time when it lies precisely on the real line.

12. The actual waveform touches the envelope exactly when the imaginary part of the complex waveform, Eq. 9.2, is zero:

$$a_1\sin(\omega\tau) + a_2\sin((\omega+\delta)\tau) = 0$$

I don't know of any way to solve this equation analytically for τ in the general case, and I don't think there is any.

13. The angle of the complex signal in Eq. 9.2 is

$$\Theta(t) = \arctan\left(\frac{a_1\sin(\omega t) + a_2\sin((\omega+\delta)t)}{a_1\cos(\omega t) + a_2\cos((\omega+\delta)t)}\right) \quad \text{radians}$$

Differentiating and simplifying yields the following expression for the instantaneous frequency:

$$\frac{d\Theta(t)}{dt} = \omega + \delta a_2\left(\frac{a_2 + a_1\cos(\delta t)}{a_1^2 + a_2^2 + 2a_1a_2\cos(\delta t)}\right) \quad \text{radians per sec}$$

Assume without loss of generality that $a_1 \geq a_2 \geq 0$ and $\delta \geq 0$. Then differentiating this expression for instantaneous frequency shows that the maxima and minima occur when $\cos(\delta t) = \pm 1$ and are

$$\omega + \delta\frac{a_2}{a_1 + a_2} \quad \text{radians per sec}$$

and

$$\omega - \delta\frac{a_2}{a_1 - a_2} \quad \text{radians per sec}$$

respectively. The range is the difference, which when normalized by ω can be written

$$\frac{\Delta\omega}{\omega} = 2\left(\frac{\delta}{\omega}\right)\frac{(a_1/a_2)}{(a_1/a_2)^2 - 1}$$

This is proportional to the fractional difference in frequencies between the two beating sinusoids, and becomes infinite as their amplitudes approach each other.

14. The main point of this question is to get the student to think about the ambiguity in quadrant when using a one-argument arctan function. For example, arctan(1) can be either 45° or 225°. The following C program solves this problem by using the two-argument math library C function `atan2()`:

```
/* from rectangular to polar form */
#include <stdio.h>
#include <math.h>
main() {
double x, y, R, theta, pi;
printf("rectangular to polar\n");
pi = 4.*atan2(1., 1.);
printf("Please enter x, y for x + jy\n");
scanf("%lf %lf", &x, &y);
printf("%lf + j%lf =\n", x, y);
R = sqrt(x*x + y*y);
theta = (180./pi)*atan2(y, x);
printf("  %lf at %lf degrees\n", R, theta);
} /* main */
```

This problem doesn't come up in the conversion from polar to rectangular coordinates:

```
/* from polar to rectangular form */
#include <stdio.h>
#include <math.h>
main() {
double x, y, R, theta, pi;
pi = 4.*atan2(1., 1.);
printf("polar to rectangular\n");
printf("Please enter R, theta for R at theta");
printf(", theta in degrees\n");
scanf("%lf %lf", &R, &theta);
printf("%lf at %lf =\n", R, theta);
x = R*cos(theta*pi/180.);
y = R*sin(theta*pi/180.);
printf("  %lf + j%lf\n", x, y);
} /* main */
```

15. Write the complex exponential as a power series formally, and separate the real and imaginary terms:

$$e^{j\theta} = \sum_{k=0}^{\infty} \frac{(j\theta)^k}{k!} = \sum_{k=0,2,4,\ldots}^{\infty} \frac{(-1)^{k/2}\theta^k}{k!} + j \sum_{k=1,3,5,\ldots}^{\infty} \frac{(-1)^{(k-1)/2}\theta^k}{k!}$$

The sums on the right are the power series for $\cos\theta$ and $\sin\theta$, respectively.

16. The frequency of the clang tone varies from fork to fork. Lord Rayleigh (*The Theory of Sound*, see the Notes) reports that the forks examined by Helmholtz had clang tones with frequencies from 5.8 to 6.6 times that of the main pitch. It's easy to verify by ear that the clang tone dies out more rapidly than the fork's main pitch.

17. Convert to Hz by dividing by 2π; then take the reciprocal to find the period $2\pi/0.02 = 314$ sec. This checks measurement on the figure.

18. Take the simple case when two sine waves of different frequencies and equal amplitudes are used. The result is

$$\begin{aligned}
&(\sin(\omega_1 t) + \sin(\omega_2 t))^2 \\
&\quad = \sin^2(\omega_1 t) + \sin^2(\omega_2 t) + 2\sin(\omega_1 t)\sin(\omega_2 t) \\
&\quad = 1 - \frac{1}{2}\cos(2\omega_1 t) - \frac{1}{2}\cos(2\omega_2 t) \\
&\qquad + \cos((\omega_1 - \omega_2)t) - \cos((\omega_1 + \omega_2)t)
\end{aligned}$$

using standard trigonometric identities. Thus, the positive frequencies present are $\omega = 0, 2\omega_1, 2\omega_2$, and $|\,\omega_1 \pm \omega_2|$, with relative proportions in this case equal to 1, 1/2, 1/2, 1, and 1, respectively.

In the general case when the two sinusoids have arbitrary amplitudes and phase angles, only the same frequencies as above are possible. Carrying out the more general calculation,

$$\begin{aligned}
&(A_1\sin(\omega_1 t + \phi_1) + A_2\sin(\omega_2 t + \phi_2))^2 \\
&\quad = A_1^2\sin^2(\omega_1 t + \phi_1) + A_2^2\sin^2(\omega_2 t + \phi_2) + 2A_1A_2\sin(\omega_1 t + \phi_1)\sin(\omega_2 t + \phi_2) \\
&\quad = \frac{A_1^2}{2} + \frac{A_2^2}{2} - \frac{A_1^2}{2}\cos(2\omega_1 t + 2\phi_1) - \frac{A_2^2}{2}\cos(2\omega_2 t + 2\phi_2) \\
&\qquad + A_1A_2\cos((\omega_1 - \omega_2)t + \phi_1 - \phi_2) - A_2A_2\cos((\omega_1 + \omega_2)t + \phi_1 + \phi_2)
\end{aligned}$$

Chapter 2

Strings, Pipes, the Wave Equation

1. The E_4 string on my guitar (highest pitch) is 27.75 in long (0.70485 m), and its pitch as an open string is 329.6 Hz. The speed of sound on the string is therefore $c = 2L/T = 2Lf = 464.6$ m/sec. The E_2 string, two octaves below, yields one-fourth this value, 116.2 m/sec. Thus the speed of sound on the strings of a guitar spans the speed of sound in air, about 345 m/sec.

2. By Eq. 2.5, the square of the velocity of sound on a string is P/ρ, where P is the tension and ρ is the mass per unit length. Increasing the tension therefore increases the velocity of sound on the string. The fundamental pitch is proportional to the velocity of sound on the string, so the pitch also increases, as every guitarist knows.

3. Notice first that γ is dimensionless, being defined in Eq. 6.4 as a ratio of fractional changes. From Eq. 6.14,

$$\gamma = \frac{c^2 \rho_0}{p_0}$$

and from the *CRC Handbook of Chemistry and Physics*, 75th edition, (D. R. Lide, ed.), CRC Press, Cleveland, Ohio, 1994-1995,

$$\begin{aligned} c &= 340.29 \,\mathrm{m\,s^{-1}} \\ \rho_0 &= 1.2250 \,\mathrm{m^{-3}\,kg} \\ p_0 &= 1.01325 \times 10^5 \,\mathrm{N\,m^{-2}} = \mathrm{m^{-1}\,kg\,s^{-2}} \end{aligned}$$

for sea-level dry air at 15° C. The calculation yields the value $\gamma = 1.4000$. In his 1894 edition of *Theory of Sound* (see the Notes to Chapter 2), Lord Rayleigh calculates the value $\gamma = 1.410$ from thermodynamic considerations, independent of the speed of sound.

4. From Eq. 3.2, the general solution for the local deviation of the position of an air molecule is

$$\xi(x,t) = f(t - x/c) + g(t + x/c)$$

We have already seen in Section 7 that the tube being open at $x = 0$ leads to Eq. 7.4,

$$\xi(x, t) = f(t - x/c) + f(t + x/c)$$

Now the condition that the tube is open at $x = L$ is enforced by

$$\left.\frac{\partial \xi}{\partial x}\right|_{x=L} = 0$$

which leads to the same periodicity condition as in the case of the vibrating string fixed at two points:

$$f(t) = f(t + 2L/c)$$

Therefore, define the fundamental frequency $\omega_0 = \pi c/L$ radians per sec and substitute the proposed solution

$$\xi(x, t) = e^{jk\omega_0 t}\Xi(x)$$

where k is an integer, in the wave equation, Eq. 6.13, yielding

$$\frac{d^2\Xi}{dt^2} = -\left(\frac{k\pi}{L}\right)^2 \Xi(x)$$

Therefore,

$$\Xi(x) = \cos(k\pi x/L + \phi)$$

where we need to find the phase angle ϕ from the boundary conditions. Imposing the conditions $\Xi'(0) = \Xi'(L) = 0$ yields

$$\sin(\phi) = \sin(k\pi + \phi) = 0$$

so we can take $\phi = 0$, and solutions are of the form

$$e^{jk\omega_0 t}\cos(k\pi x/L)$$

where k is any integer.

Of course any linear combination of solutions will also be a solution, so the grand combination is

$$\sum_{k=1}^{\infty} c_k e^{jk\omega_0 t}\cos(k\pi x/L)$$

in analogy to Eq. 8.1 for a string fixed at two points. In the case of a tube open at both ends, the nodes and antinodes are interchanged, but the same frequencies are present.

5. From Problem 4, the fundamental frequency for the case when the straw is open at both ends is $\omega_0 = \pi c/L$ radians per sec. When we close one end of the straw, the fundamental frequency becomes $\omega_0 = \pi c/(2L)$ radians per sec (from Section 7), so the pitch is halved.

6 and **7.** I recommend Benade's discussion of a glockenspiel bar in his book, *Fundamentals of Musical Acoustics*, referenced in the Notes to Chapter 3. In the simplest mode a bar flexes between a U-shape and an inverted U-shape. The next higher mode has two humps instead of one, looking like a $\sim$. The higher modes are more complicated, and involve twisting motion of the bar.

Benade goes on to describe the vibration of rectangular plates, guitar plates, and drumheads. All these are two-dimensional boundary-value problems, and the modes of vibration can be astoundingly complex. For a real treat, see Mary Désirée Waller's *Chladni Figures: A Study in Symmetry*, G. Bell & Sons, London, 1961. She reproduced and extended the experiments of Ernst Florens Friedrich Chladni (1756–1827), who visualized the vibration of plates by sprinkling them with fine sand and bowing them. Waller had been asked by an itinerant ice-cream vendor why dry ice made his bicycle bell ring, and refined a method for exciting vibrations in plates that uses dry ice instead of a bow. She went on to devote a large part of her life to the exploration of what are called *Chladni figures*, and her book is what she herself calls "essentially a picture book" of her experimental results.

8. Putting your finger at the center of a string should suppress all harmonics except those that already have a node there. Figure 5.1 shows that the odd harmonics should then be suppressed, and the frequencies remaining should be the even multiples of ω_0. Thus the pitch of the string is effectively doubled, which makes sense because it can now be thought of as two strings, each having half the original length.

Benade discusses the experiment at the end of Chapter 8 of his book (see the Notes to Chapter 3), suggesting the "corner of a soft foam sponge" instead of a finger to absorb the vibrations of the odd harmonics. Benade suggests many similar experiments involving plucking, striking, and damping different vibrating systems at different points. It's an excellent way to gain insight about vibrational modes and their excitations.

Chapter 3

Sampling and Quantizing

1. One bel is 10 decibels, an amplitude ratio of $\sqrt{10}$, or a power ratio of 10.

2. (a) The signal is the rotating image of the hubcap pattern, and the sampling mechanism is the periodic shutter of the motion-picture camera, often 24 frames per second.

(b) In this case the signal is a spatial one — the periodic stripes in the tweed material. The spatial sampling is provided by the raster scan lines of the TV camera. The result is similar to the interference patterns between stripes of different spacings that are overlaid at different angles. When the TV anchor squirms, the change in perspective changes the apparent spatial frequency of the tweed, and the pattern shifts in complicated ways. The resulting spatial aliasing patterns are sometimes called *moiré patterns*.

(c) Some people have told me this effect occurs, but I've never observed it myself. The signal is clear — it's the same as in Part (a). If the effect is real, the sampling must occur somewhere in the observer's eye-brain vision system. This would give us a clue about how vision works, and would be very interesting indeed.

The reason I've specified the *sun* in this part is that artificial lighting is often powered by AC current, which can provide a 60 Hz variation in intensity. This is often quite noticeable in fluorescent lighting, and the effect is used to calibrate phonograph turntables. The idea is to draw radial stripes on a paper disk, which is then put on the moving turntable. The stripes are spaced so that they appear stationary in 60 Hz lighting when the turntable is moving at exactly the correct speed.

3. From Eq. 1.5, the frequency 330 Hz gets aliased to $f_0 - f_s = 330 - 300 = 30$ Hz in the baseband. This checks in Fig. 1.1, where the period of the aliased, sampled waveform is 10 periods at the sampling frequency of 300 Hz.

4. Figure P4 shows the first three terms in the Fourier series of a square wave. The period in this example is 225 sec.

5. Figure P5 shows four sampling periods in three repetition periods of the square wave. There are 40,000/30,000 = 4/3 samples per period, and the sampling pattern repeats every three periods of the square wave.

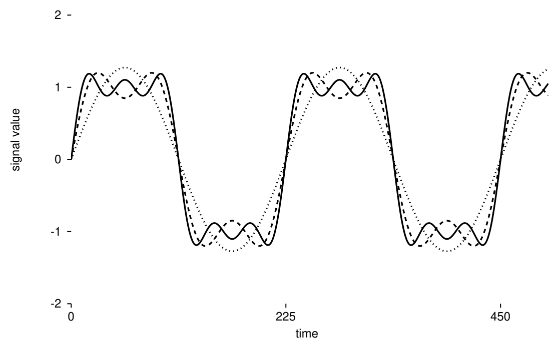

Fig. P4 The first three terms in the Fourier series of a square wave.

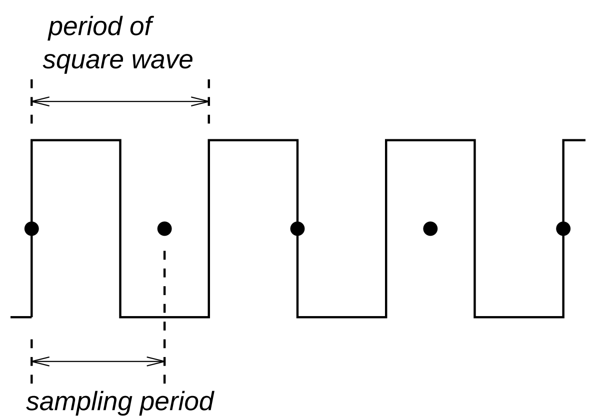

Fig. P5 Sampling a 30 kHz square wave at 40 kHz.

6. The 79th harmonic occurs at 55.3 kHz, which at a 40 kHz sampling rate aliases to 15.3 kHz in the baseband.

7. Let $f_s = 1/T$ Hz and $f_0 = 1/P$ Hz be the sampling frequency and repetition frequency of the original waveform, respectively. Then the frequencies present in the sampled signal are, by Eq. 1.5,

$$kf_0 + nf_s$$

where k and n are arbitrary integers.

If f_0 is an integer multiple of f_s, all signal frequencies are aliased after sampling to the zero frequency, and the conditions of the problem are satisfied trivially — no new frequencies appear in the baseband. If $f_0 = f_s/2$, the Nyquist frequency, all frequencies are aliased to zero or the Nyquist frequency, and again the conditions of the problem are satisfied trivially. If f_0 is neither an integer multiple of f_s nor equal to $f_s/2$, we can assume that $f_0 < f_s/2$, for if not, the frequency f_0 will itself certainly be aliased into the baseband $[-f_s/2, f_s/2]$.

Assume next that $2 < r = f_s/f_0$ is not an integer, and let $r = A + \epsilon$, where A is an integer and $0 < \epsilon < 1$. The frequencies present in the sampled signal are

$$f_0 \left(k + n\frac{f_s}{f_0} \right) = f_0 \left(k + nA + n\epsilon \right)$$

Choosing $n = 1$ and $k = -A$ shows that the frequency ϵf_0 is present after sampling, so the conditions of the problem are not satisfied.

In the final case $2 < r = f_s/f_0$ is an integer, say A. The frequencies present in the sampled signal are

$$f_0 \left(k + n\frac{f_s}{f_0} \right) = f_0 \left(k + nA \right)$$

so no new frequencies result from sampling.

To summarize, sampling introduces no new frequencies only when

- f_0 is an integer multiple of f_s;
- f_0 is equal to $f_s/2$;
- f_s is an integer multiple of f_0.

The third case is the only interesting one, and represents the situation when the sampling pattern is identical from period to period of the original waveform.

8. Shifting the square wave by $T/4$ results in a Fourier series with only cosines because the waveform becomes even. Only odd harmonics are present because the waveform is odd about the center of each half-period. Another way to see this is to replace t by $t + T/4$ in the Fourier series in Eq. 2.2.

(*Erratum in first printing:* The fifth paragraph in Section 2 should begin, "Next, observe that our square wave is even about the center of each half-period. That is, if we shift the signal so that it's centered at the center of a half-period, say at $t = T/4$, the resulting signal is even. Now sine waves at the odd harmonics have this property, but sine waves at the even harmonics are odd about the quarter-period point.")

Shifting by $T/2$ gets us back to a sine series with only odd harmonics, with the same symmetry properties as the original. In fact, the shift just inverts the sign of the signal.

9. As observed in the text, the sampling pattern drifts by 1/7 of a sample each period of the square wave, so it repeats only after 7 periods. The sampled waveform therefore has a period of $7T = 1/100$ sec, which corresponds to a fundamental frequency of 100 Hz. In general, the new period is the least common multiple of the period of the signal and the sampling period. This is equivalent to saying that the new frequency is the greatest common divisor of the fundamental frequency of the signal and the sampling frequency.

The spectral components in Fig. 2.2 would therefore be 100 Hz apart if all harmonics of the original signal were present. These are all frequencies of the form $k700 + n40000$ Hz for all integers k and n. But in fact the square wave allows only odd k, so we get all frequencies of the form $(2m + 1)700 + n40000 = 700 + m1400 + n40000$ Hz for all integers m and n.

This yields a spacing that is the greatest common divisor of 1400 Hz and 40000 Hz, or 200 Hz. The actual frequencies present after sampling are all the odd multiples of 100 Hz.

10. A periodic waveform becomes not periodic after sampling if and only if the signal's fundamental period and the sampling period have no least common multiple, or, in other words, when their ratio is irrational.

12. (a) First use $\cos(x+y) = \cos x \cos y - \sin x \sin y$ for $x = 2\theta$ and $y = \theta$:

$$\cos(3\theta) = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$$

Then use $\sin x \sin y = (1/2)(\cos(x-y) - \cos(x+y))$ for the second term to get

$$\cos(3\theta) = \cos(2\theta)\cos\theta - (1/2)\cos\theta + (1/2)\cos(3\theta)$$

Collecting the two terms in $\cos(3\theta)$ and replacing $\cos(2\theta)$ by $2\cos^2\theta - 1$ results in

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

(b) Repeating the strategy above for $\cos(n\theta)$ leads to

$$\cos(n\theta) = 2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta)$$

which shows that $\cos(n\theta)$ is a polynomial in $\cos\theta$ by induction.

Letting $x = \cos\theta$, the Chebyshev polynomial of order n is defined by

$$T_n(x) = \cos\left[n \arccos x\right]$$

We have shown above that

$$\begin{aligned} T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

and, in general,

$$T_n = 2xT_{n-1}(x) - T_{n-2}(x)$$

(c) To illustrate the general method, suppose we want to generate the signal

$$A\cos(\omega t) + B\cos(2\omega t) + C\cos(3\omega t)$$

Letting $x = \cos(\omega t)$, this becomes

$$Ax + B(2x^2 - 1) + C(4x^3 - 3x) = 4Cx^3 + 2Bx^2 + (A - 3C)x - B$$

This is a fixed polynomial in x, so we can generate the desired signal by first generating $x = \cos(\omega t)$ and then passing it through this instantaneous nonlinearity.

Chapter 4

Feedforward Filters

1. Figure P1 shows sketches of the three cases. Notice that the samples in Case (a) can be obtained by taking alternate samples in Case (b), and the samples in Case (b) can be obtained by taking alternate samples in Case (c). The answers are not unique, but depend on the relative phase between the sinusoid and the sampling.

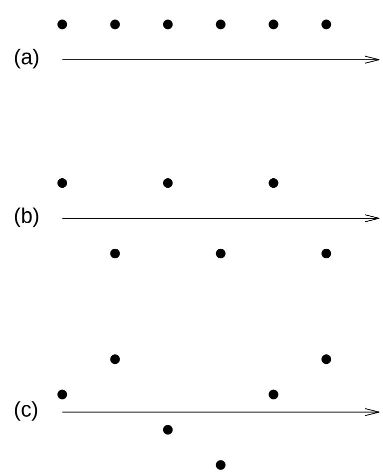

Fig. P1 Samples of a sinusoid with frequency (a) zero; (b) the Nyquist frequency; and (c) half the Nyquist frequency.

2. Reverse the sign of a_1, and use the filter

$$y_t = x_t - 0.99x_{t-\tau}$$

with the same delay as in the text example, $\tau = 167\mu\text{sec}$. The magnitude transfer function is given by Eq. 2.7 with $a_1 = -0.99$, and the notches now occur when $\omega\tau$ is an *even* multiple of π, or at $f = 0, 2/\tau, 4/\tau, \ldots$ Hz.

(*Erratum in first printing:* The left-hand side of Eq. 2.7 should be $|H(\omega)|$.)

3. In the general case, the transfer function corresponding to the filter equation

$$y_t = x_t + x_{t-1} + x_{t-2} + \cdots + x_{t-n+1}$$

is

$$\mathcal{H}(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-(n-1)}$$

This is a geometric series, with closed form

$$\mathcal{H}(z) = \frac{1 - z^{-n}}{1 - z^{-1}}$$

For the magnitude response replace z by $e^{j\omega}$:

$$H(\omega) = \frac{1 - e^{-jn\omega}}{1 - e^{-j\omega}}$$

Multiply the numerator by $e^{jn\omega/2}$, the denominator by $e^{j\omega/2}$, and take the magnitude:

$$|H(\omega)| = \left| \frac{e^{jn\omega/2} - e^{-jn\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right| = \left| \frac{\sin(n\omega/2)}{\sin(\omega/2)} \right|$$

The nulls occur at the zeros of the numerator that are not zeros of the denominator, which correspond to the frequencies $\omega = k2\pi/n$ radians per sample, for integer k not equal to integer multiples of n, $k = 1, 2, \ldots, n-1, n+1, \ldots$. The equivalent frequencies in terms of the sampling rate f_s are $f_s/n, 2f_s/n, \ldots, (n-1)f_s/n, (n+1)f_s/n, \ldots$. Major peaks occur precisely at integer multiples of the sampling frequency, where the zeros in the numerator and denominator cancel out. The other peaks occur close to (but not precisely) midway between nulls.

This is the frequency content of a rectangular window of length n, and is plotted for $n = 8$ and 64 in Fig. 3.1 of Chapter 10.

4. First write w_t and w_{t-1} in terms of x:

$$\begin{aligned} w_t &= a_0 x_t + a_1 x_{t-1} \\ w_{t-1} &= a_0 x_{t-1} + a_1 x_{t-2} \end{aligned}$$

Then substitute these values in

$$y_t = b_0 w_t + b_1 w_{t-1}$$

and collect terms to get

$$y_t = a_0 b_0 x_t + (a_0 b_1 + a_1 b_0) x_{t-1} + a_1 b_1 x_{t-2}$$

as required.

5. When we multiply polynomials, as in Eq. 5.11, we first use the distributive law, which justifies operations of the form $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. This holds true for signals when α is multiplication by a constant, because multiplying the sum of two signals by a constant is equivalent to multiplying each by the constant and then summing the signals. It's also clear that it holds true when the α represents a shift by any number of samples. Commutativity also holds; that is, the order of shifting and multiplication can be interchanged arbitrarily. Finally, we use the law $z^{m+n} = z^m z^n$ for integer shifts m and n. This is true because shifting by m samples and then by n samples is equivalent to shifting by $m + n$ samples.

6. Let the transfer function be $A(z) = \sum_i a_i z^{-i}$ and let z_0 be a complex zero. Then the transfer function evaluated at the conjugate point z_0^* is

$$A(z_0^*) = \sum_i a_i (z_0^*)^{-i} = \sum_i a_i \left(z_0^{-i}\right)^*$$

using the fact that complex conjugation and exponentiation commute. Taking the conjugate of this yields

$$(A(z_0^*))^* = \sum_i a_i^* z_0^{-i} = \sum_i a_i z_0^{-i} = 0$$

using the assumption that the a_i are real.

Because the a_i are real, replacing ω by $-\omega$ in the transfer function has the effect of complex-conjugating it. In other words, $H(-\omega) = H^*(\omega)$. Therefore, the magnitude is an even function of ω and the phase is an odd function of ω.

7. First calculate the quadratic factor that accounts for the zeros at $z = e^{j\pm\pi/6}$:

$$(z - e^{j\pi/6})(z - e^{-j\pi/6}) = z^2 - 2\cos(\pi/6)z + 1 = z^2 - \sqrt{3}z + 1$$

We want zeros at $z = 1$ and $z = e^{j\pm\pi/6}$, so the transfer function is

$$z^{-3}(z-1)(z^2 - z\sqrt{3} + 1) = 1 - (1+\sqrt{3})z^{-1} + (1+\sqrt{3})z^{-2} - z^{-3}$$

where we have introduced a delay of three samples to make the filter realizable.

Figure P7 shows the magnitude response of this filter, and the notches at the frequencies zero and $\pi/6$ radians per sample show clearly. The peak magnitude between those two frequencies is -25.45 dB and occurs at 0.298 radians per sample.

When the signal $x(n) = 1 + \cos(\pi t/6)$ is passed through this filter, the result is the signal

$$\begin{aligned} y(0) &= 2 \\ y(1) &= -3.598076 \\ y(2) &= 1.866025 \end{aligned}$$

and

$$y(t) = 0 \quad \text{for } t \geq 3$$

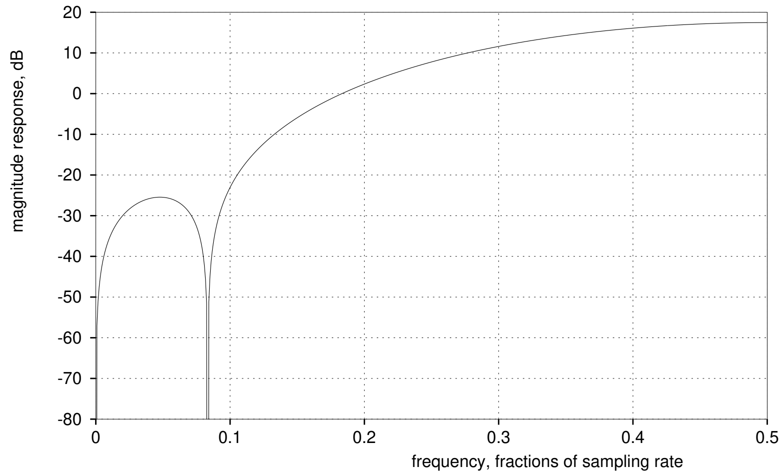

Fig. P7 The magnitude response of the filter in Problem 7.

and yes, the output does become exactly zero.

8. Assume the transfer function is of the following form:

$$\mathcal{H}(z) = az^m + \cdots + az^n$$

where the exponents are in ascending order. The problem stipulates that the coefficients are symmetric about their center. Multiply by $z^{-(m+n)/2}$:

$$z^{-(m+n)/2}\mathcal{H}(z) = az^{(m-n)/2} + \cdots + az^{-(m-n)/2}$$

This makes the exponents odd about the center, so when we let $z = e^{j\omega}$ to get the frequency response, pairs of terms symmetrically placed about the center combine to form cosines, and the right-hand side becomes a real-valued cosine series:

$$e^{-j(m+n)\omega/2}H(\omega) = \text{real-valued cosine series}$$

This shows that the phase of $H(\omega)$ is linear. When $m = 0$ and $n = N - 1$, the resultant delay is $(N - 1)/2$ samples.

9. Cases (a) and (b) are narrowband, having concentrated energy at DC and 1/100 the Nyquist frequency, respectively. In general, if a signal is narrowband about ω_0, the transfer function frequency response can be approximated by

$$H(\omega) \approx |H(\omega_0)|\, e^{j\phi(\omega_0)}$$

where $\phi(\omega)$ is the phase response. Thus, if we normalize by $|H(\omega_0)|$, the effect of the filter should be very close to an ideal delay. Figure P9 shows the results for these three cases, using the filter normalized at zero frequency, and therefore having the transfer function $0.5 + 0.5z^{-1}$. The first two cases illustrate the delay of about one-half sample.

10. Multiply out the right-hand side:

$$(z - e^{j\pi/3})(z - e^{-j\pi/3}) = z^2 - 2\cos(\pi/3)z + 1 = z^2 - z + 1$$

11. The transfer function is $1 - R^L z^{-L}$ and its magnitude has minima and maxima when $z^L = +1$ and -1, respectively. The values achieved at minima and maxima are therefore $1 - R^L$ and $1 + R^L$, respectively.

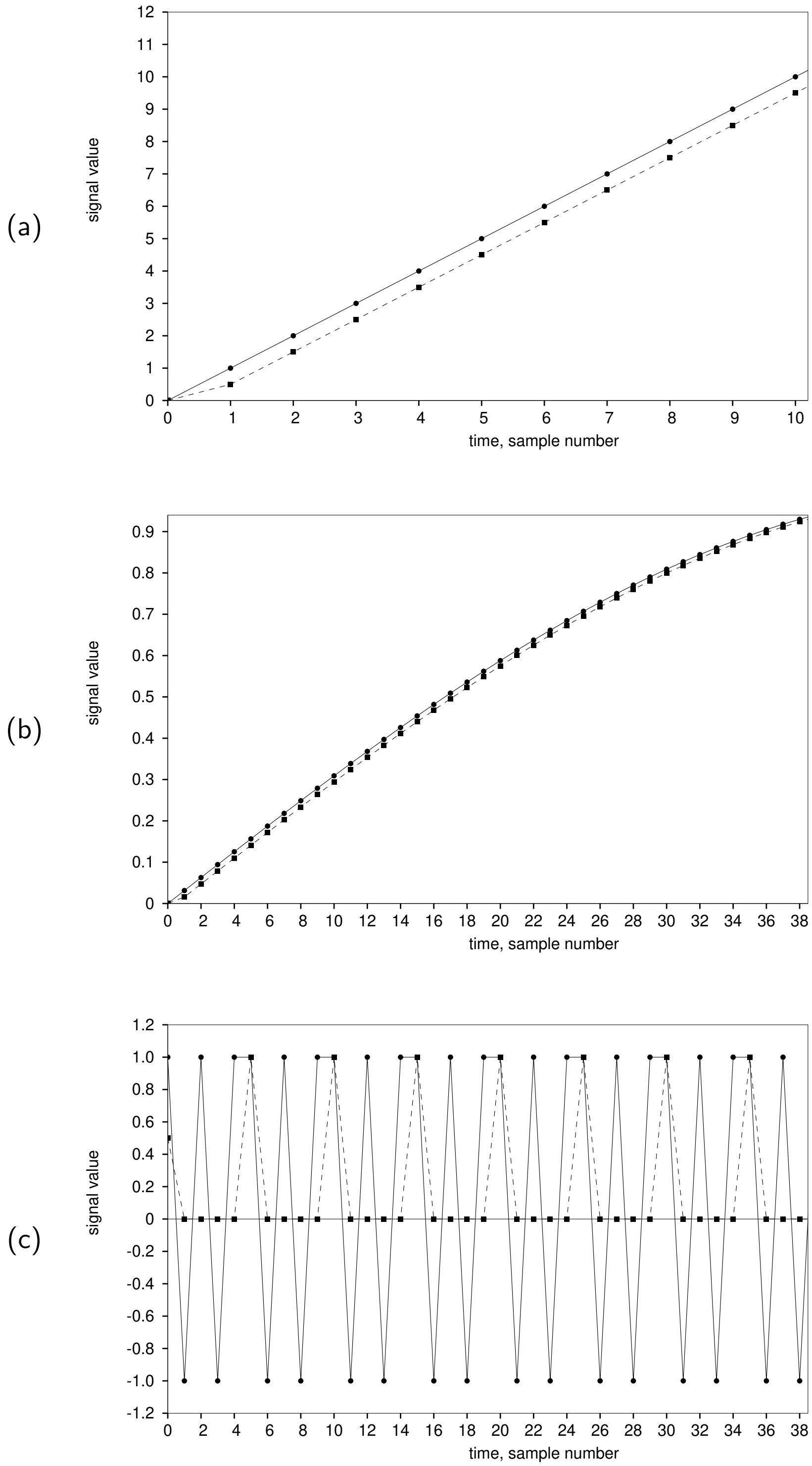

Fig. P9 The results of filtering three signals with a simple feedforward filter. In Cases (a) and (b) the signals are narrowband and the result is close to a pure delay of one-half sample; in Case (c) the signal is not narrowband, and the result has no such easy interpretation.

Chapter 5

Feedback Filters

1. Expand the exponential in a power series:

$$\begin{aligned} e^{-B/2} &= 1 - B/2 + (1/2)(B/2)^2 - \cdots \\ &\approx 1 - B/2 \end{aligned}$$

The origin of the exponential form is that monument of electrical engineering: the tuned RLC circuit. It's the precise counterpart of the digital reson discussed in this chapter. In a classic treatment of circuit theory, D. F. Tuttle, Jr. (*Circuits*, McGraw-Hill, New York, N.Y., 1977) devotes an entire chapter of 116 pages just to this circuit. It turns out that the impulse response is a damped sinusoid (no surprise), and the damping factor is

$$e^{-(\Delta\omega/2)t}$$

where $\Delta\omega$ is the bandwidth in radians per sec. To correspond to the digital reson, set this equal to R^t, yielding

$$R = e^{-\Delta\omega/2}$$

as desired.

2. Start with Eq. 5.1, the inverse-square of the magnitude response:

$$1/|H(\phi)|^2 = \left| \left(e^{j\phi} - Re^{j\theta}\right) \left(e^{j\phi} - Re^{-j\theta}\right) \right|^2$$

Multiply out the binomials and write this as the sum of the squares of the real and imaginary parts:

$$\begin{gathered} \left[\cos(2\phi) - R\cos(\phi+\theta) - R\cos(\phi-\theta) + R^2\right]^2 \\ + \left[\sin(2\phi) - R\sin(\phi+\theta) - R\sin(\phi-\theta)\right]^2 \end{gathered}$$

Replace $\cos(\phi \pm \theta)$ and $\sin(\phi \pm \theta)$, using the usual identities:

$$\left[\cos(2\phi) - 2R\cos\phi\cos\theta + R^2\right]^2 + \left[\sin(2\phi) - 2R\sin\phi\cos\theta\right]^2$$

Expand the squares and collect terms, using the identity $\cos\phi\cos(2\phi)+\sin\phi\sin(2\phi) = \cos\phi$:

$$1 + 4R^2\cos^2\theta + R^4 - 4R\cos\phi\cos\theta + 2R^2\cos(2\phi) - 4R^3\cos\phi\cos\theta$$

Replace $\cos(2\phi)$ by $2\cos^2\phi - 1$ and use the resulting $-2R^2$ to form the perfect square $\left(1 - R^2\right)^2$:

$$\begin{aligned}&\left(1 - R^2\right)^2 + 4R^2\cos^2\theta - 4R\cos\phi\cos\theta + 4R^2\cos^2\phi - 4R^3\cos\phi\cos\theta \\ &\quad = \left(1 - R^2\right)^2 + 4R^2\cos^2\theta - 4R(R^2+1)\cos\phi\cos\theta + 4R^2\cos^2\phi\end{aligned}$$

which is Eq. 5.2.

Differentiate with respect to $\cos\phi$ and set to zero:

$$-4R(R^2+1)\cos\theta + 8R^2\cos\phi = 0$$

or

$$\cos\phi = \frac{1+R^2}{2R}\cos\theta$$

which is Eq. 5.3.

3. Solving Part (c) first will help us with Parts (a) and (b). Let's assume that we are dealing with a case when θ_0 is close to θ, so let $\theta_0 = \theta + \delta$, where δ is small. Approximate $\cos\theta_0$ by its first-order Taylor series:

$$\cos\theta_0 = \cos\theta - \delta\sin\theta + \text{ higher-order terms in } \delta$$

and substitute in Eq. 5.3 derived in Problem 2:

$$\cos\theta - \delta\sin\theta = \frac{1+R^2}{2R}\cos\theta$$

Rearranging gives us an approximation for δ

$$\delta = -\frac{(1-R)^2}{2R\tan\theta}$$

From this we see that when θ is between 0 and $\pi/2$, δ is negative, so θ_0 is shifted to lower frequencies. When θ is between $\pi/2$ and π, δ is positive, so θ_0 is shifted upward, towards the Nyquist frequency. In other words, when θ is in the baseband, θ_0 is always shifted away from the center frequency, $\pi/2$, which is half the Nyquist frequency. (We can also see this directly from Eq. 5.3, because the factor $(1 + R^2)/2R$ is greater than 1 when R is less than 1, as we'll prove in the solution to Problem 5.)

This approximate formula for δ also tells us that the discrepancy between θ_0 and θ is greatest when R is small (bandwidth large) and θ is near 0 or π radians per sample.

4. Setting $\phi = \theta$ in Eq. 5.2 gives us

$$\begin{aligned} 1/|H(\theta)|^2 &= (1-R^2)^2 + 8R^2\cos^2\theta - 4R(R^2+1)\cos^2\theta \\ &= (1-R)^2\left(1+R^2-2R\cos(2\theta)\right) \end{aligned}$$

as required.

To compare this with the gain factor normalizing at the peak resonant frequency, divide this by the square of the gain factor in Eq. 5.5:

$$\frac{(1-R)^2(1+R^2-2R\cos(2\theta))}{(1-R^2)^2\sin^2\theta}$$

Replacing $(1-R^2)^2$ by $(1-R)^2(1+R)^2$ in the denominator enables us to cancel factors $(1-R)^2$, and then letting $R \to 1$ shows that this ratio approaches one.

5. Assume that $0 < R < 1$, and note that

$$(1-R^2) = 1 - 2R + R^2 > 0$$

from which it follows that

$$\frac{1+R^2}{2R} > 1$$

Thus, if we solve for $\cos\theta$ in Eq. 5.3, it will be determined by a factor less than one times $\cos\psi$, and there will always be a solution for θ. On the other hand, there are ranges of R and $\cos\theta$ where the equation yields $\cos\psi > 1$, and this means there is no true peak frequency ψ.

6. In computer music, we might want to make a note by using a reson to filter wideband noise (from a random number generator, for example). If the bandwidth of the reson is a small fraction of an octave, the resulting note will have a definite pitch. A good measure of how well the pitch will be perceived is the bandwidth of the reson relative to its center frequency — in other words, the normalized bandwidth $Q = \Delta\omega/\omega_0$, where $\Delta\omega$ is the bandwidth and ω_0 is the center frequency, both in radians per sample. To use a reson in this way over a range of pitches we usually hold the Q constant, which means making the bandwidth proportional to the center frequency. Constant-Q bandpass filters of all sorts are commonly used in many other applications, such as audio equalizers and spectrum analyzers.

On the other hand, if we are using a reson (or any bandpass filter) to analyze or synthesize in a narrow range, it may not be worth the trouble to vary the bandwidth as the center frequency changes. It's also very convenient in many cases to have the bandwidths of a filter bank equal, rather than being geometrically scaled with center frequency. The output energy of each filter in the filter bank is then proportional to the energy of the signal at that frequency, and there is no need to scale.

Finally, there are many situations where we naturally want to vary the bandwidth of a note produced with a reson, or vary the selectivity of a reson used for analysis.

7. (a) The transfer function of reson_R is

$$\mathcal{H}(z) = \frac{1 - Rz^{-2}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}}$$

(b) The corresponding input-output equation is

$$y_t = x_t - Rx_{t-2} + 2R\cos\theta y_{t-1} - R^2 y_{t-2}$$

(c) To find the magnitude response at the frequency corresponding to θ radians per sample, write $\mathcal{H}(z)$ explicitly in terms of its poles:

$$\mathcal{H}(z) = \frac{1 - Rz^{-2}}{(1 - Re^{j\theta}z^{-1})\,(1 - Re^{-j\theta}z^{-1})}$$

and set $z = e^{j\theta}$:

$$H(\omega) = \frac{1 - Re^{-2j\theta}}{(1 - R)\,(1 - Re^{-2j\theta})} = \frac{1}{1 - R}$$

8. (a) The transfer function of reson_z is given in Eq. 7.1:

$$\mathcal{H}(z) = \frac{1 - z^{-2}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}}$$

Let $y = \cos\psi$ and $x = \cos\theta$, for convenience. Then the magnitude-square of the numerator is (after some algebra)

$$|N(\omega)|^2 = 4\left(1 - y^2\right)$$

and, after further simplification, the magnitude-square of the denominator is

$$|D(\omega)|^2 = (1 - R^2)^2 + 4R^2(x^2 + y^2) - 4Rxy(1 + R^2)$$

Setting $d(|D|^2/|N|^2)/dy = 0$ yields (again, after some algebra) the quadratic equation in y

$$y^2 - y\left[\frac{(1 + R^2)^2 + 4R^2x^2}{2Rx(1 + R^2)}\right] + 1 = 0$$

which factors into

$$\left(y - \frac{1 + R^2}{2Rx}\right)\left(y - \frac{2Rx}{1 + R^2}\right) = 0$$

The two roots are reciprocal, and the first is always greater than 1, so choose the second, which results in

$$\cos\psi = \frac{2R}{1 + R^2}\cos\theta$$

Isn't it interesting that the ratio $\cos\psi/\cos\theta$ is the reciprocal of that for reson (*cf.* Eq. 5.3)?

(b) The gain of reson_z at its true peak frequency can now be obtained by substituting $y = 2R/\left(1+R^2\right)$ into the square-magnitude of the transfer function. Using the expressions for $|N|^2$ and $|D|^2$, we get (more algebra!)

$$\frac{|N|^2}{|D|^2} = \frac{4}{(1-R^2)^2}$$

which is independent of θ, as advertized.

(c) Reson_z not only has a peak gain independent of center frequency but also requires one fewer multiplication per sample.

9. A feedback filter can turn out to have an impulse response of finite duration if all the poles are canceled by zeros. Here's the transfer function for a simple example:

$$\mathcal{H}(z) = \frac{(1-z^{-1})(1-0.5z^{-1})}{1-0.5z^{-1}} = \frac{1-1.5z^{-1}+0.5z^{-2}}{1-0.5z^{-1}}$$

which corresponds to the input-output equation

$$y_t = x_t - 1.5x_{t-1} + 0.5x_{t-2} + 0.5y_{t-1}$$

which is a feedback filter. But the impulse response is zero for $t \geq 2$. This becomes clear when you cancel the pole and zero at $z = 0.5$; the transfer function is also simply $\mathcal{H}(z) = 1 - z^{-1}$.

The only way the impulse response can be of infinite duration without feedback is to have an infinite number of feedforward terms in a feedforward filter.

10. What's missing from the proof is showing that the cumulative effect due to a single pole and all the input samples at an arbitrary time is bounded. To do this we need to assume something about the input. Here's a standard argument.

Suppose the magnitude of the input signal x_k is bounded by some constant X_{max}, and consider the response y_t at an arbitrary time t due to a pole p of a filter:

$$y_t = \sum_{k=0}^{t} x_k p^{t-k}$$

This comes directly from Eq. 2.6, with each impulse δ_{t-k} replaced by the response to that impulse, p^{t-k}. Take the magnitude of y_t and use the fact that the magnitude of a sum is no larger than the sum of magnitudes:

$$\begin{aligned} |y_t| &= \left|\sum_{k=0}^{t} x_k p^{t-k}\right| \\ &\leq \sum_{k=0}^{t} |x_k|\,|p|^{t-k} \end{aligned}$$

Next, use the assumption that $|x_k| \leq X_{max}$, and bring X_{max} outside the summation:

$$|y_t| \leq X_{max} \sum_{k=0}^{t} |p|^{t-k} \leq \frac{X_{max}}{1 - |p|}$$

where we assume $|p| \leq 1$ and bound the finite geometric series by the infinite geometric series. This shows that the output signal is bounded in magnitude when the pole p is less than 1 in magnitude, assuming also that the input signal is bounded in magnitude — which fills the gap in the proof.

Chapter 6

Comb and String Filters

1. Let

$$x_t = \begin{cases} R^t & \text{if } t = 0 \bmod L \\ 0 & \text{otherwise} \end{cases}$$

for all t, both negative and nonnegative. The output of the inverse comb filter, by Eq. 1.1, is

$$y_t = x_t - R^L x_{t-L} = 0$$

for all t, so the output w_t of the following comb filter is also zero for all t. Thus, in this example $w_t \neq x_t$.

The apparent difficulty in using transfer functions and z-transforms here stems from the fact that the z-transform of two-sided signals like this particular x_t have two parts, one for negative t and one for nonnegative t, and the two transforms have no common region of convergence. This jumps way ahead — beyond of the scope of the book, in fact.

3. By Eq. 3.4, the lowpass filter in the feedback loop has magnitude transfer function $\cos(\omega/2)$ and a delay of $L+1/2$ samples. After k samples, a signal makes $k/(L+1/2)$ trips around the feedback loop, and therefore a phasor component at frequency ω radians per sample has its magnitude multiplied by

$$|\cos(\omega/2)|^{k/(L+1/2)}$$

assuming $R = 1$. The magnitude of the contribution of every frequency is decreased with each round-trip, and so the overall plucked-string filter is stable.

4. (a) To find the time constant k in samples for general R, set

$$|R\cos(\omega/2)|^{k/(L+1/2)} = e^{-1}$$

and solve for k by taking the natural logarithm of both sides:

$$k \approx \frac{L+1/2}{-\ln\left(R\,|\cos(\omega/2)|\right)} \quad \text{samples}$$

(b) If the frequency of a resonance is f Hz, the frequency in radians per sample is $\omega = (f/f_s)2\pi$, where f_s is the sampling frequency in Hz. The fundamental frequency of the plucked-string filter is $f_0 = f_s/(L+1/2)$ Hz, so the time constant estimate in Part (a) can be written

$$k \approx \frac{f_s/f_0}{-\ln\left(R\left|\cos(\pi f/f_s)\right|\right)} \quad \text{samples}$$

Let $\tau_0 = 1/f_0$, the fundamental period in sec; $T_s = 1/f_s$, the sampling interval in sec; and $f = nf_0$, so we are dealing with the nth partial. Then the time constant estimate becomes

$$\tau = kT_s \approx \frac{\tau_0}{-\ln\left(R\left|\cos\left(\pi n/(L+1/2)\right)\right|\right)} \quad \text{sec}$$

5. The filter equation is

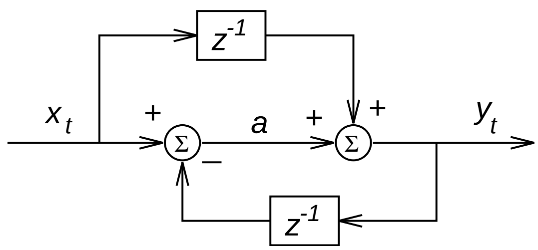

Fig. P5 Signal flowgraph for an allpass filter that uses one multiplication.

$$y_t = a(x_t - y_{t-1}) + x_{t-1}$$

and the corresponding signal flowgraph is shown in Fig. P5.

6. The low-frequency delay δ of an allpass filter is, by Eq. 7.9, $(1-a)/(1+a)$. When this is negative, $a > 1$, and the filter is unstable.

7. Letting $x = \omega_0/2$, $y = \omega_0\delta/2$, and taking the tangent of both sides of Eq. 8.1 yields

$$\tan y = \left(\frac{1-a}{1+a}\right)\tan x$$

Solve for a:

$$\begin{aligned} a &= \frac{\tan x - \tan y}{\tan x + \tan y} \\ &= \frac{\sin x\cos y - \sin y\cos x}{\sin x\cos y + \sin y\cos x} \\ &= \frac{\sin(x-y)}{\sin(x+y)} \end{aligned}$$

which is Eq. 8.2.

8. The proof is perfectly analogous to the one given in Section 7 for the one-pole, one-zero case. Multiply numerator and denominator of the transfer function by $z^{n/2}$:

$$\mathcal{H}(z) = \frac{a_0 z^{-n/2} + a_1 z^{-n/2+1} + \cdots + a_n z^{n/2}}{a_0 z^{n/2} + a_1 z^{n/2-1} + \cdots + a_n z^{-n/2}}$$

When $z = e^{j\omega}$, the numerator is the complex conjugate of the denominator, so the magnitude response can be written in the form

$$|H(\omega)| = \left| \frac{re^{j\psi(\omega)}}{re^{-j\psi(\omega)}} \right| = 1$$

which shows that the filter is allpass.

9. (a) To show that the second-order allpass filter is stable when $\mu > 1$, we need to show that in that range, its poles, the roots of

$$z^2 + bz + a = 0$$

are inside the unit circle in the z-plane, where

$$b = 2\frac{2-\mu}{1+\mu} \quad \text{and} \quad a = \frac{(2-\mu)(1-\mu)}{(2+\mu)(1+\mu)}$$

The discriminant of this quadratic equation is

$$\frac{b^2}{4} - a = \frac{3(2-\mu)}{(1+\mu)^2\,(2+\mu)}$$

Therefore, the roots occur in a complex pair when $\mu > 2$. In this case the coefficient a is the squared radius ρ of the complex roots, and

$$\rho^2 = \left(\frac{\mu-2}{\mu+2}\right)\left(\frac{\mu-1}{\mu+1}\right) < 1 \quad \text{for} \quad \mu \geq 2$$

When $\mu = 2$, the roots are double at $z = 0$. We have left to consider the case $1 < \mu < 2$, when the roots are real and opposite in sign, since the discriminant is positive and the product of roots, a, is negative. The center of gravity of the roots is

$$-\frac{b}{2} = \frac{\mu-2}{\mu+1}$$

which moves from $z = -1/2$ to $z = 0$ as μ moves from 1 to 2. The roots are obtained by adding and subtracting to $-b/2$ the square root of the discriminant:

$$\sqrt{\frac{b^2}{4} - a} = \frac{\sqrt{3}}{1+\mu}\sqrt{\frac{2-\mu}{2+\mu}}$$

When $\mu = 1$, this square root takes the value $1/2$, and decreases monotonically to 0 as μ increases to 2. Thus the roots satisfy

$$-1 < -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a} < 1 \quad \text{for} \quad 1 < \mu < 2$$

which is what we set out to show.

(b) Figure P9 shows a comparison of computed delay for the first- and second-order allpass filters. The second-order filter provides a uniformly better approximation to constant delay at all frequencies. Whether it's worth using in the plucked-string instrument depends on how important it is to have accurate approximations to the pitch of the higher harmonics.

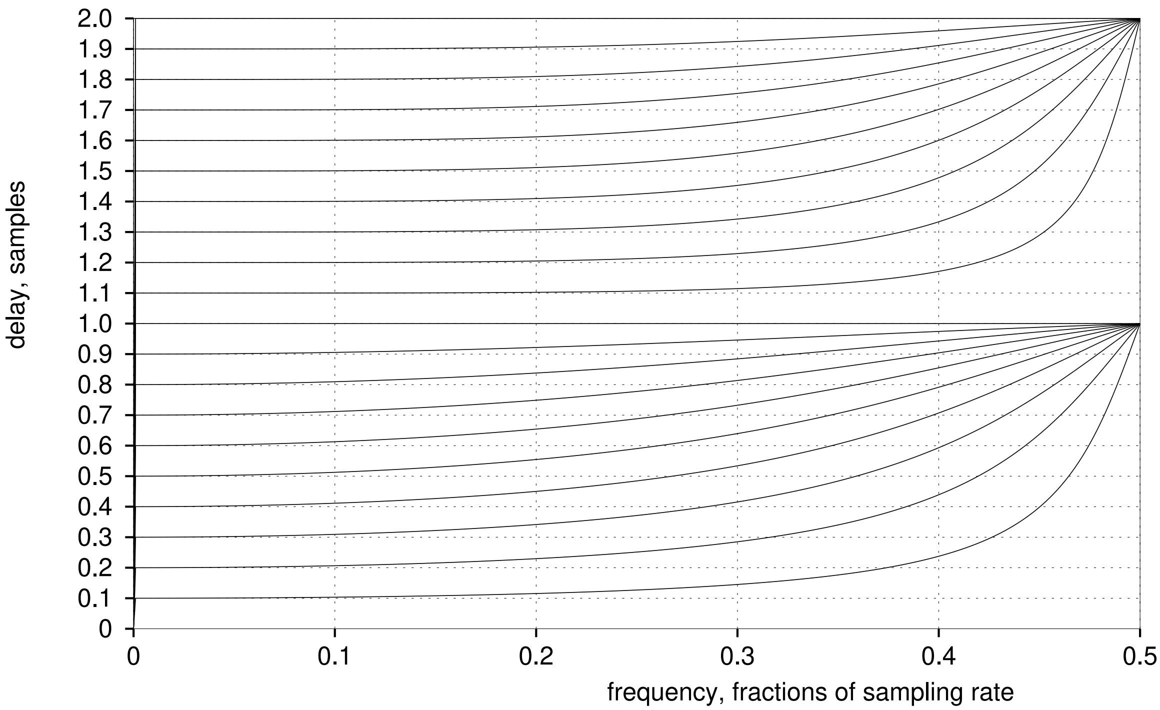

Fig. P9 Computed delay vs. frequency for the first- and second-order allpass filters. The bottom ten curves show the actual delay in the first-order case when the desired delay $= 0.1, 0.2, \ldots, 1.0$, and the top ten curves show the actual delay in the second-order case when the desired delay $= 1.1, 1.2, \ldots, 2.0$.

10. The transfer function of an allpass filter with the specified poles and zeros can be written

$$\mathcal{H}(z) = \left(\frac{z^{-1} + ae^{-j\omega_c}}{1 + ae^{j\omega_c} z^{-1}} \right) \left(\frac{z^{-1} + ae^{j\omega_c}}{1 + ae^{-j\omega_c} z^{-1}} \right)$$

For the poles and zeros to occur at angles $\pm\omega_c$, choose $a < 0$. The first factor then corresponds to a pole and zero at angle ω_c, and the second factor corresponds to a pole and zero at angle $-\omega_c$. Notice that the result in Problem 8 confirms that this is allpass because it can be put in the form

$$\mathcal{H}(z) = \frac{z^{-2} + (2a\cos\omega_c)z^{-1} + a^2}{1 + (2a\cos\omega_c)z^{-1} + a^2 z^{-2}}$$

To examine the phase response in the range of frequencies near ω_c, let $\omega = \omega_c + \epsilon$ and $z = e^{j\omega}$ in first form above:

$$H(\omega) = \left(\frac{e^{-j\epsilon} + a}{1 + ae^{-j\epsilon}}\right) G(\omega)$$

The factor $G(\omega)$ corresponds to the pole and zero at $-\omega_c$ and is allpass. Let's assume that ω_c is sufficiently far from zero frequency so that $G(\omega)$ is essentially constant in the neighborhood of ω_c. The factor $G(\omega)$ then contributes a constant phase shift to the total result, say the angle ϕ_0. The first factor is identical in form to Eq. 7.2, and therefore contributes the phase given by Eqs. 7.7 and 7.8. The total phase of the two-pole, two-zero allpass at frequency $\omega_c + \epsilon$ is therefore

$$-2\arctan\left(\frac{1-a}{1+a}\tan(\epsilon/2)\right) + \phi_0 \approx -\frac{1-a}{1+a}\epsilon + \phi_0$$

which is what we wanted to show.

11. As the signal circulates in the feedback loop, its low-frequency components are decreased relative to its high-frequency components, the opposite of what happens when the filter is lowpass. Thus, the spectrum of the output signal evolves in the opposite way: the high frequencies are sustained for a long time and the low frequencies decay fast. To me it sounds metallic, somewhat reminiscent of the clang tone of a tuning fork.

12. (a) When $b = 1$ the filter equation reduces to

$$y_t = \frac{1}{2}y_{t-L} + \frac{1}{2}y_{t-L-1}$$

which corresponds to the transfer function

$$\mathcal{H}(z) = \frac{1}{1 - \frac{1}{2}\left(1 + z^{-1}\right) z^{-L}}$$

This differs from Eq. 5.3 for the plucked-string filter only in having the lowpass filter in the feedback path instead of the feedforward path, and having the original comb pole radius $R = 1$ (see the signal flowgraph in Fig. P12).

(b) Setting $b = 0$ reverses the sign of the feedback terms, thus shifting the poles so that only odd harmonics are present. This sounds tube-like by analogy with the vibrating column of air in a half-open tube (see Section 2.7).

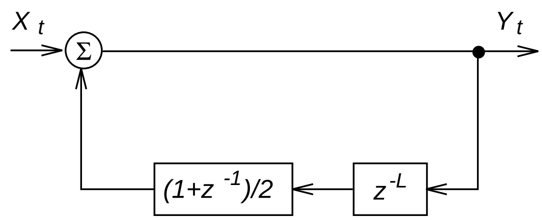

Fig. P12 Signal flowgraph for the case b = 1 in the Karplus-Strong drum-like instrument.

(c) When $b = 1/2$, the filter equation is

$$y_t = \frac{1}{2}y_{t-L} + \frac{1}{2}y_{t-L-1}$$

Following the approximate analysis of Karplus and Strong, square this:

$$y_t^2 = \frac{1}{4}y_{t-L}^2 + \frac{1}{4}y_{t-L-1}^2 + \frac{1}{2}y_{t-L}y_{t-L-1}$$

and take the expected value term by term. The expected value of the last term on the right-hand side is approximately zero, since y_{t-L} and y_{t-L-1} are close to being uncorrelated. Denoting the expected value of a signal by $Y_t = \mathrm{E}[y_t^2]$, we get

$$Y_t \approx \frac{1}{4}Y_{t-L} + \frac{1}{4}Y_{t-L-1}$$

The two terms on the right-hand side are approximately equal, assuming that the amplitude of the filter output decays reasonably slowly. We can therefore write the following approximate difference equation for Y_t:

$$Y_t \approx \frac{1}{2}Y_{t-L}$$

which implies that

$$Y_t \approx Y_0 2^{-t/L}$$

showing, heuristically at least, that the mean-square value of the filter's impulse response does decay exponentially.

13. Plucking a string at a particular point suppresses excitation of the harmonics that have nodes there, the opposite effect of putting your finger there while the string is vibrating (see Problem 8 of Chapter 2). Thus, as Jaffe and Smith point out, plucking a string $1/k$th of the way along its length suppresses every kth harmonic. This corresponds to filtering the excitation signal with the inverse comb filter with equation

$$y_t = x_t - x_{t-N/k}$$

which has the transfer function

$$1 - z^{-N/k}$$

and therefore has zeros at integer multiples of $k2\pi/N$ radians per sample.

Chapter 7

Periodic Sounds

1. *Method 1:* In the two-dimensional case we can prove the result using complex numbers. This is a good way to give the class some practice. Let the vector $v = v_x + jv_y$ and let $w = w_x + jw_y$. Then

$$< v, w >= \Re\{vw^*\} = v_x w_x + v_y w_y$$

But if we write v and w in polar form as $R_v e^{j\theta_v}$ and $R_w e^{j\theta_w}$, respectively,

$$< v, w >= \Re\{vw^*\} = R_v R_w \cos(\theta_v - \theta_w)$$

which is the desired result.

Method 2: Here's a more geometric approach, using the law of cosines. Define the vector $c = v - w$; that is, c is the vector with tail at the end of w and head at the end of v. Denote the lengths of v, w, and c by R_v, R_w, and R_c, respectively. Then the law of cosines states that

$$R_c^2 = R_v^2 + R_w^2 - 2R_v R_w \cos(\theta_v - \theta_w)$$

so

$$R_v R_w \cos(\theta_v - \theta_w) = \frac{1}{2}\left(R_v^2 + R_w^2 - R_c^2\right)$$

Using

$$\begin{aligned} R_v^2 &= v_x^2 + v_y^2 \\ R_w^2 &= w_x^2 + w_y^2 \end{aligned}$$

and

$$R_c^2 = (v_x - w_x)^2 + (v_y - w_y)^2$$

yields

$$R_v R_w \cos(\theta_v - \theta_w) = v_x w_x + v_y w_y$$

as required.

2. Method 2 in the previous solution works in any number of dimensions with very little change. In three dimensions, for example, define c in the same way, apply the law of cosines in the plane determined by v and w, and use

$$R_c^2 = (v_x - w_x)^2 + (v_y - w_y)^2 + (v_z - w_z)^2$$

where z is the third dimension.

3. First note that $c_n = 0$ when $n = 0$, because the two integrals in Eq. 3.1 cancel out. Evaluating the integrals in Eq. 3.1 when $n \neq 0$ yields

$$\begin{aligned} c_n &= \frac{1}{T}\left[\int_0^{T/2} e^{-jn\omega_0 t} dt - \int_{T/2}^{T} e^{-jn\omega_0 t} dt\right] \\ &= \frac{1}{T}\left[\left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right|_0^{T/2} - \left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right|_{T/2}^{T}\right] \\ &= \frac{1}{-jn2\pi}\left[e^{-jn\pi} - 1 - e^{-jn2\pi} + e^{-jn\pi}\right] \end{aligned}$$

where we used the fact that $\omega_0 T = 2\pi$. This is zero when $n \neq 0$ is even. When n is odd, the bracketed expression becomes -4, and so

$$c_n = \frac{-4}{-jn2\pi} = -\frac{2j}{n\pi}$$

verifying Eq. 3.2.

4. In general, we can state: Any symmetry of the waveform must be respected by every nonzero term in its Fourier series. This is a consequence of the orthogonality of the basis sinusoids. No basis element can be expressed as a linear combination of other basis elements, and therefore no term that destroys a symmetry can be canceled out by a sum of others in the series.

As another example, the triangle wave in Fig. 4.1 is even, and odd about every quarter-period point. Therefore, its Fourier series contains only cosine terms, and only the ones that are odd about the quarter-period points, the odd harmonics.

See also Section 2 and Problem 8 of Chapter 3 for similar observations. That symmetry implies that certain coefficients are zero can also be argued directly from their explicit formula in terms of the waveform, the projection in Eq. 2.5.

5. The nth coefficient is, from Eq. 2.5,

$$c_n = \frac{1}{T}\int_0^{\alpha} e^{-jn\omega_s t} dt$$

When $n = 0$, $c_n = \alpha/T$. When $n \neq 0$,

$$c_n = \left.\frac{e^{-jn\omega_s t}}{-jn\omega_s T}\right|_0^{\alpha} = \frac{1 - e^{-jn\omega_s \alpha}}{jn\omega_s T} = \frac{1 - e^{-jn\omega_s \alpha}}{jn2\pi}$$

where we used the definition $\omega_s = 2\pi/T$ in the last step. The magnitude of c_n for $n > 0$ is therefore

$$|c_n| = \left| \frac{e^{jn\omega_s\alpha/2} - e^{-jn\omega_s\alpha/2}}{2\pi n} \right| = \left| \frac{\sin(n\omega_s\alpha/2)}{\pi n} \right| = \left| \frac{\sin(n\pi\alpha/T)}{\pi n} \right|$$

In the case $\alpha = T/2$, $c_0 = 1/2$, the average value of the waveform; $c_n = 0$ when n is even; and $c_n = -j/(\pi n)$ when n is odd. This checks the Fourier series for the square wave, Eq. 3.2, since this case corresponds to the square wave in Fig. 3.1 multiplied by 1/2 and shifted upwards by adding 1/2.

6. Figure P6 shows the cases $\alpha/T = 0.5, 0.25$, and 0.1. What is actually shown is $\sin(n\pi\alpha/T)/(\pi n)$, which does not include a complex exponential factor, but which retains the sign. This makes it easier to see how the coefficients behave. For large α/T, the coefficients become small fast; and for small α/T, the coefficients decay slowly. This is an example of a general phenomenon we note at several points: A narrow waveform has a broad spectrum, and vice-versa.

7. When $\omega t = m2\pi$, an integer multiple of 2π, the sum in Eq. 7.1 becomes simply the sum of the constant 1, and is therefore equal to $2N + 1$.

When $\omega t \neq m2\pi$,

$$\begin{aligned} \sum_{n=-N}^{N} e^{jn\omega t} &= e^{-jN\omega t} \sum_{n=0}^{2N} e^{jn\omega t} = e^{-jN\omega t} \frac{1 - e^{j(2N+1)\omega t}}{1 - e^{j\omega t}} \\ &= \frac{e^{-j(2N+1)\omega t/2} - e^{j(2N+1)\omega t/2}}{e^{-j\omega t/2} - e^{j\omega t/2}} = \frac{\sin((2N+1)\omega t/2)}{\sin(\omega t/2)} \end{aligned}$$

as desired.

8. In this case, P is even, we choose $N = P/2$, and $\omega = 2\pi/P = \pi/N$. Write the left-hand side of Eq. 7.2 as

$$1 + 2 \sum_{n=1}^{P/2-1} \cos(n2\pi t/P) + 2(-1)^t$$

Then subtract $(-1)^t$ from both sides, so Eq. 7.2 becomes

$$1 + 2 \sum_{n=1}^{P/2-1} \cos(n2\pi t/P) + (-1)^t = \begin{cases} \frac{\sin((P+1)\pi t/P)}{\sin(\pi t/P)} - (-1)^t & \text{if } t \neq 0 \bmod P \\ P + 1 - (-1)^t & \text{if } t = 0 \bmod P \end{cases}$$

Expanding the sine in the numerator yields

$$\begin{aligned} \frac{\sin((P+1)\pi t/P)}{\sin(\pi t/P)} &= \frac{\sin(\pi t)\cos(\pi t/P) + \cos(\pi t)\sin(\pi t/P)}{\sin(\pi t/P)} \\ &= \cos(\pi t) = (-1)^t \end{aligned}$$

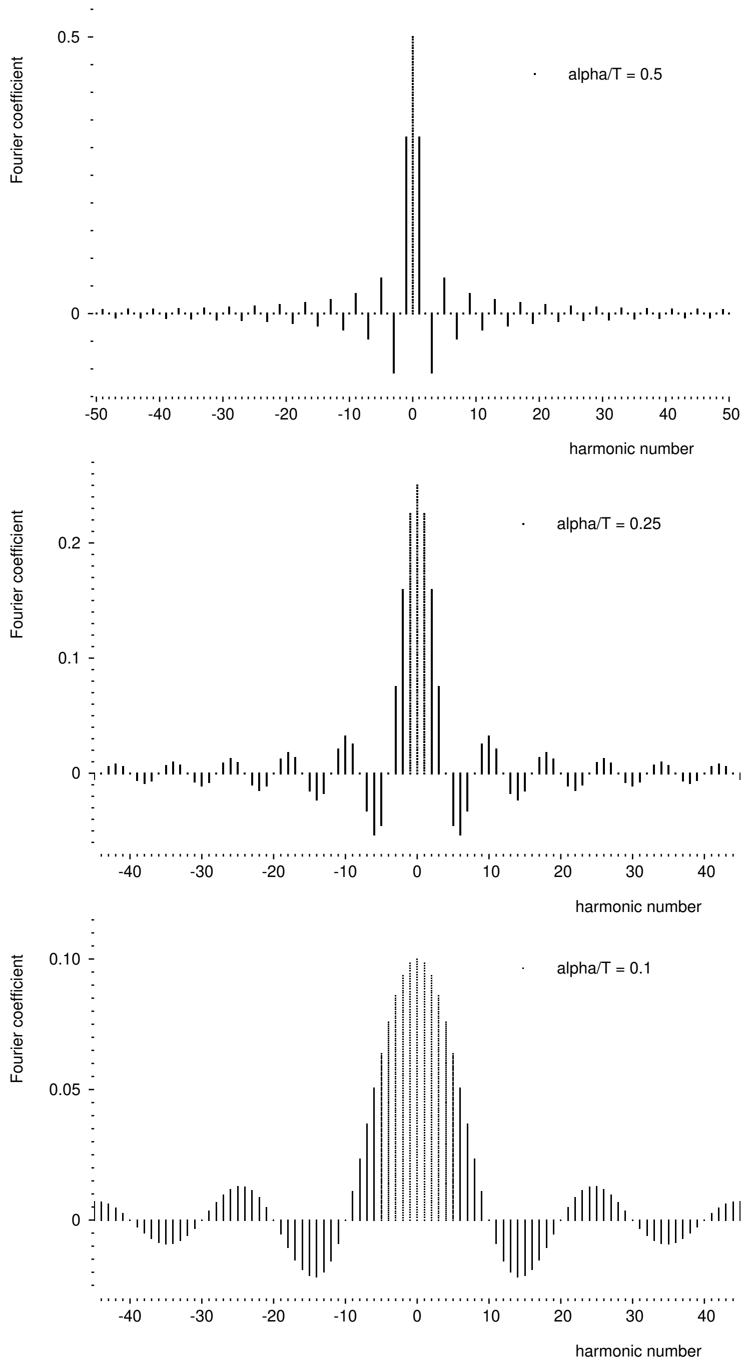

Fig. P6 The Fourier coefficients of the rectangular wave in Problem 7, for $\alpha/\mathsf{T} = 0.1, 0.25,$ and 0.5. A complex exponential factor is not included, but the sign is retained.

and using the fact that t is even in the second case, we finally get

$$1 + 2 \sum_{n=1}^{P/2-1} \cos(n2\pi t/P) + (-1)^t = \begin{cases} 0 & \text{if } t \neq 0 \bmod P \\ P & \text{if } t = 0 \bmod P \end{cases}$$

which is Eq. 7.4.

9. Dr. Godfrey Winham implemented a buzz generator in FORTRAN in the late sixties. Here is Prof. Paul Lansky's C translation, which he still uses for linear-predictive coding synthesis. It's in the form of a function and requires a 1024-point sine table `f`.

```
#define  ABS(x) ((x < 0) ? (-x) : (x))
#define EPS .1e-06
float buzz(amp,si,hn,f,phs)
float amp,si,hn,*f,*phs;
{
register j,k;
float q,d,h2n,h2np1;
j = *phs;
k = (j+1) % 1024;
h2n = 2. * hn;
h2np1 = h2n + 1.;
q = (int)((*phs - (float)j) * h2np1)/h2np1;
d = *(f+j);
d += (*(f+k)-d)*q;
if(ABS(d) < EPS) q = amp;
          else {
    k = (long)(h2np1 * *phs) % 1024;
    q = amp * (*(f+k)/d - 1.)/h2n;
    }
*phs += si;
while(*phs >= 1024.)
*phs -= 1024.;
return(q);
}
```

As mentioned in the problem statement, the critical point is the choice of the parameter `EPS`. When the magnitude of the denominator in Eq. 9.1 is smaller than this value, the division is skipped and the computed value is set equal to `amp`.

10. Taking the conjugate of Eq. 1.10, the inner product in the complex case, shows that

$$< v, w >^* = < w, v >$$

By the way, sometimes the inner product is defined with the first term conjugated instead of the second.

Chapter 8

The Discrete Fourier Transform and FFT

1. Let $q = m - n \neq 0$, and write the left-hand side of Eq. 2.4 as the closed form of a geometric series:

$$\sum_{t=0}^{N-1} e^{jtq2\pi/N} = \frac{1 - e^{jq2\pi}}{1 - e^{jq2\pi/N}}$$

The denominator of the right-hand side is not zero, because m and n range between 0 and $N-1$, and therefore $-N < q < N$. But the numerator *is* zero, which establishes the orthogonality result.

2. To avoid the duplicate use of indices, use the dummy variable s instead of t in Eq. 2.6, and substitute in Eq. 2.5:

$$x_t = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{s=0}^{N-1} x_s e^{jk(t-s)2\pi/N}$$

Interchanging the order of the summations then yields

$$x_t = \frac{1}{N} \sum_{s=0}^{N-1} x_s \sum_{k=0}^{N-1} e^{jk(t-s)2\pi/N}$$

The result in Problem 1 shows that the summation on k equals N if $s = t$ and equals 0 otherwise, thus getting us back to x_t, as we wanted.

3. The fact that the inverse DFT exists shows that the matrix $\boldsymbol{F}$ is nonsingular. From Eq. 2.5, the matrix representation of the inverse DFT is

$$\left[\boldsymbol{F}^{-1}\right]_{kt} = \frac{1}{N} e^{j2\pi kt/N}$$

where this notation indicates the element in row k and column t of the matrix $\boldsymbol{F}^{-1}$.

I should have warned the reader that calculating the determinant of $\boldsymbol{F}$ is not a simple problem. Finding its magnitude is relatively easy, but finding its phase angle is tricky.

Here's how to find the magnitude of the determinant of $\boldsymbol{F}$. The square of the determinant of $\boldsymbol{F}$ is the determinant of $\boldsymbol{R} = \boldsymbol{F}^2$, and $\boldsymbol{R}$ has the very simple form derived in Problem 15. It's N times a permutation matrix, and so

$$|\det(\boldsymbol{R})| = N^N$$

Therefore,

$$|\det(\boldsymbol{F})| = N^{N/2}$$

As mentioned above, it turns out that finding the phase angle of $\det(\boldsymbol{F})$ is much harder than finding its magnitude. I'll give the method that, as far as I know, uses the most elementary techniques. It's from the paper "Is Computing with the Finite Fourier Transform Pure or Applied Mathematics?" by L. Auslander and R. Tolimieri, *Bulletin of the Amer. Math. Soc.*, vol. 1, no. 6, Nov. 1979, pp. 847-897, and I thank Prof. Bradley Dickinson for pointing out this derivation.

The kth row of $\boldsymbol{F}$ is a fixed vector of numbers raised to the kth power, $0 \le k < N$, and that qualifies $\det(\boldsymbol{F})$ as a Vandermonde determinant, which has the following closed form:

$$\det(\boldsymbol{F}) = \prod_{0 \le k < t < N} (W^t - W^k)$$

where $W = e^{-j2\pi/N}$. (See, for example, G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra*, third edition, Macmillan, New York, N.Y., 1965.) Letting $U = e^{-j\pi/N}$, this can be written

$$\begin{aligned} \det(\boldsymbol{F}) &= \prod_{0 \le k < t < N} U^{t+k} \left(U^{t-k} - U^{-(t-k)} \right) \\ &= \prod_{0 \le k < t < N} U^{t+k} \prod_{0 \le k < t < N} (-2j) \sin((t-k)\pi/N) \end{aligned}$$

The first product can be evaluated using closed forms for the sums in the exponent (see, for example, L. B. W. Jolley, *Summation of Series*, second edition, Dover, New York, N.Y., 1961):

$$\sum_{0 \le k < t < N} (t+k) = \sum_{t=1}^{N-1} \sum_{k=0}^{t-1} (t+k) = 2N \left(\frac{N-1}{2} \right)^2$$

Because $t > k$, the sines in the second product are all positive. Putting all the powers of $-j$ together yields

$$\det(\boldsymbol{F}) = (-j)^{(3N-2)(N-1)/2} N^{N/2}$$

where we've used the magnitude derived above. Finally, use the facts that $j^2 = -1$ and $j^4 = 1$ to rewrite this in the beautiful form

$$\det(\boldsymbol{F}) = j^{2+3+\ldots+N} N^{N/2}$$

It turns out that the eigenvalues of $\boldsymbol{F}$ are, aside from a factor of $N^{1/2}$, $1, j^2, j^3, \ldots, j^N$, and had we known that to begin with, the determinant would have followed immediately as their product. But finding the eigenvalues is even harder than finding the determinant (see, for example, M. L. Mehta, "Eigenvalues and Eigenvectors of the Finite Fourier Transform," *J. Math. Phys.*, vol. 28, no. 4, April 1987, pp. 781-785).

4. (a) The signal $x(t)$ is bandlimited as well as periodic and so can be written as the finite Fourier series

$$x(t) = \sum_{k=-N/2}^{N/2} c_k e^{jk2\pi t/T}$$

The sum incorporates frequencies between $-1/(2T_s)$ and $1/(2T_s)$ Hz, as specified in the problem statement. To put this in a form closer to a DFT, replace k by $m - N/2$, and evaluate at the times $t = nT_s$ for $n = 0, \ldots, N$:

$$x(nT_s) = e^{-jn\pi} \sum_{m=0}^{N} c_{m-N/2} e^{jmn2\pi/N}, \quad \text{for } n = 0, \ldots, N$$

This is an $(N+1)$-point DFT and therefore relates the first $(N+1)$ samples of $x(t)$ to its Fourier coefficients c_k in the usual way. Notice, however, that we must always choose $x(0) = x(NT_s)$, because $x(t)$ is assumed to be periodic with period $T = NT_s$. This puts one linear constraint on the c_k, and can be attributed intuitively to the fact that we are including frequencies that are actually on the boundary $1/(2T_s)$ Hz.

(b) If $x(t)$ is not bandlimited, frequencies above $1/(2T_s)$ Hz will be aliased to their images in the baseband, the frequencies between $-1/(2T_s)$ and $1/(2T_s)$ Hz. This will be reflected in the values of the Fourier coefficients, which will then become aliased versions of the original infinite set.

5. The number of stages is the largest integer not less than $\log N$, denoted by $\lceil \log N \rceil$, the *ceiling* of $\log N$.

6. I adapted the following C program from the FORTRAN FFT program of Cooley, Lewis, and Welch, referenced in the Notes. It was used to generate Figs. 9.1 and 9.2.

```
/* in-place complex fft; adapted from the FORTRAN of
Cooley, Lewis, and Welch; from Rabiner & Gold (1975)*/

#include <stdio.h>
#include <strings.h>
#include <math.h>

main() {
FILE *fopen(), *fdata;
#define dB(x) (((fabs(x))>(0.0))?(20.*log10(fabs(x))):(-1000.))
```

```
#define verbose 0     /* 2 = print input and output */
#define N 1024        /* length of transform */
#define M  10         /* log N */
#define M_PI       3.14159265358979323846
int i, j, k, L;       /* indexes */
int LE, LE1, ip;
int NV2, NM1;
double ar[N], ai[N];  /* array of points */
double t;             /* temp */
double Ur, Ui, Wr, Wi, Tr, Ti;
double Ur_old;
double freq_test;
double mag;

fdata = fopen("data", "w");
NV2 = N/2;
NM1 = N-1;

/* generate original signal */
freq_test = 133.;
for (i=0; i<N; i++)
 { ar[i] = cos(freq_test*2.*M_PI*((double)i/(double)N));
   ai[i] = sin(freq_test*2.*M_PI*((double)i/(double)N)); }

if(verbose >= 2)
{ printf("input:\n");
  printf("real: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ar[i]);
  printf("\nimag: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ai[i]);
  printf("\n\n"); }

/* shuffle */
j = 1;
for (i=1; i<=NM1; i++)
 { if(i<j)             /* swap a[i] and a[j] */
    { t = ar[j-1];
      ar[j-1] = ar[i-1];
      ar[i-1] = t;
      t = ai[j-1];
      ai[j-1] = ai[i-1];
      ai[i-1] = t; }

   k = NV2;            /* bit-reversed counter */
   while(k<j)
    { j -= k;
```

```
      k /= 2; }

   j += k; }

if(verbose >= 2)
{ printf("shuffled input:\n");
  printf("real: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ar[i]);
  printf("\nimag: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ai[i]);
  printf("\n\n"); }

LE = 1.;
for (L=1; L<=M; L++)    /* stage L */
 { LE1 = LE;            /* LE1 = LE/2 */
   LE  *= 2;            /* LE  = 2^L */
   Ur = 1.0;
   Ui = 0.;
   Wr = cos(M_PI/(double)LE1);
   Wi = -sin(M_PI/(double)LE1);
 /* Cooley, Lewis, and Welch have "+" here */
   for (j=1; j<=LE1; j++)

    { for (i=j; i<=N; i+=LE)  /* butterfly */
       { ip = i+LE1;
         Tr = ar[ip-1]*Ur-ai[ip-1]*Ui;
         Ti = ar[ip-1]*Ui+ai[ip-1]*Ur;
         ar[ip-1] = ar[i-1] - Tr;
         ai[ip-1] = ai[i-1] - Ti;
         ar[i-1]  = ar[i-1] + Tr;
         ai[i-1]  = ai[i-1] + Ti; }
 /* end of butterfly */

      Ur_old = Ur;
      Ur = Ur_old*Wr-Ui*Wi;
      Ui = Ur_old*Wi+Ui*Wr; } /* end of j loop */

 } /* end of stage L */

if (verbose>=2)
 { printf("output:\n");
   printf("real: ");
   for(i=0; i<N; i++)
    printf("%6.4lf ", ar[i]);
   printf("\nimag: ");
   for(i=0; i<N; i++)
```

```
    printf("%6.4lf ", ai[i]);
   printf("\n"); }

for(i=0; i<N; i++)
 { mag = sqrt(ar[i]*ar[i]+ai[i]*ai[i]);
fprintf(fdata,"%6.4lf %le %lf\n",(double)i/(double)N, mag, dB(mag));}

} /* main */
```

---

10. Let $F(N)$ be the number of function calls in the recursive implementation of the decimation-in-time FFT, and assume N is an integer power of 2. When the algorithm is called with parameter N, it does two calls of the same algorithm with parameter $N/2$. Therefore,

$$F(N) = 2F(N/2) + 2$$

Repeatedly substituting for $F(N/2)$ results in the telescoped series

$$F(N) = 2\,(2\,(2\,(\cdots + 1) + 1) + 1)$$

where the nesting goes down until it reaches $F(1) = 0$. The sum therefore ranges from 2 to N, doubling each time:

$$F(N) = 2 + 4 + 8 + \cdots + N = 2(1 + 2 + 4 + \cdots + N/2) = 2(N - 1)$$

where we've used the fact that $1 + 2 + 4 + \cdots + N/2$ is the binary expansion of $(N - 1)$.

11. Here's an explicitly recursive decimation-in-time FFT in C. Note that it can be made more efficient by computing the complex exponentials immediately before the butterfly by repeated multiplication, as in the solution to Problem 6, rather than from scratch.

```
/* fft, implemented as direct recursion */

#include <stdio.h>
#include <strings.h>
#include <math.h>
#define M_PI       3.14159265358979323846
#define N 4096        /* length of transform */
#define dB(x) (((fabs(x))>(0.0))?(20.*log10(fabs(x))):(-1000.))
double ar[N], ai[N];  /* array of points to be transformed */

main() {
FILE *fopen(), *fdata;
#define verbose 0     /* 2 = print input and output */
int i;       /* index */
double freq_test;
```

```
double mag;
fdata = fopen("data", "w");

/* generate original signal */
freq_test = (double)N/(double)8;
for (i=0; i<N; i++)
 { ar[i] = cos(freq_test*2.*M_PI*((double)i/(double)N));
   ai[i] = sin(freq_test*2.*M_PI*((double)i/(double)N)); }

if(verbose >= 2)
{ printf("input:\n");
  printf("real: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ar[i]);
  printf("\nimag: ");
  for (i=0; i<N; i++)
   printf("%6.4lf ", ai[i]);
  printf("\n\n"); }

fft(0, N-1);

if (verbose>=2)
 { printf("output:\n");
   printf("real: ");
   for(i=0; i<N; i++)
    printf("%6.4lf ", ar[i]);
   printf("\nimag: ");
   for(i=0; i<N; i++)
    printf("%6.4lf ", ai[i]);
   printf("\n"); }

for(i=0; i<N; i++)
 { mag = sqrt(ar[i]*ar[i]+ai[i]*ai[i]);
fprintf(fdata,"%6.4lf %lf %lf\n",(double)i/(double)N, mag, dB(mag));}

} /* main */

fft(k, m)    /* fft of array from k to m */
int k, m;
{
double tr[N], ti[N];   /* temp, for shuffle */
double Wr, Wi, Tr, Ti;
int i, n, n2;

if(verbose>=2)
 printf("k= %d   m= %d\n", k, m);

n = m-k+1;            /* number of points this transform */
```

```
if(n==1) return;
n2 = n/2;

if(verbose>=2){
printf("before shuffle:\n");
for(i=k; i<=m; i++)
 printf("ar= %le  ai= %le\n", ar[i], ai[i]); }

for(i=k; i<k+n2; i++){    /* put even points in first half */
 tr[i] = ar[k+2*(i-k)];   /* odd points in second half */
 ti[i] = ai[k+2*(i-k)];
 tr[i+n2] = ar[k+2*(i-k)+1];
 ti[i+n2] = ai[k+2*(i-k)+1]; }

for(i=k; i<=m; i++){
 ar[i] = tr[i];
 ai[i] = ti[i]; }

if(verbose>=2){
printf("after shuffle:\n");
for(i=k; i<=m; i++)
 printf("ar= %le  ai= %le\n", ar[i], ai[i]); }

fft(k, k+n2-1);
fft(k+n2, m);

for(i=k; i<k+n2; i++){
 Wr = cos(2.*M_PI*(double)(i-k)/(double)n);
 Wi = -sin(2.*M_PI*(double)(i-k)/(double)n);
 Tr = ar[i+n2]*Wr-ai[i+n2]*Wi;
 Ti = ar[i+n2]*Wi+ai[i+n2]*Wr;

 tr[i] = ar[i] + Tr;       /* butterfly */
 ti[i] = ai[i] + Ti;
 tr[i+n2]  = ar[i] - Tr;
 ti[i+n2]  = ai[i] - Ti;
 ar[i] = tr[i];
 ai[i] = ti[i];
 ar[i+n2] = tr[i+n2];
 ai[i+n2] = ti[i+n2]; }
}
```

---

13. This bug is a beginner's mistake. The variable `Ur` is changed by the first statement and therefore the value used in the second statement is not what it should be.

14. All the numerical values needed are contained in a quarter of a period, so we need to store only $N/8$ numbers. Of course, some work is required to reduce the argument to the proper point within the first quarter, and to take the reduction into account in the sign of the result.

15. (a) The element in row k and column s of the product matrix $\boldsymbol{R} = \boldsymbol{F}^2$ is

$$[\boldsymbol{R}]_{ks} = \sum_{t=0}^{N-1} [\boldsymbol{F}]_{kt}[\boldsymbol{F}]_{ts} = \sum_{t=0}^{N-1} e^{-jt(k+s)2\pi/N} = \begin{cases} N & \text{if } k+s = 0 \bmod N \\ 0 & \text{otherwise} \end{cases}$$

Thus, the matrix $\boldsymbol{R}$ looks like

$$N \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) When forming the product $\boldsymbol{R}^2$, the kth row times the kth column yields N^2, and the kth row times any other column yields zero. Thus, $\boldsymbol{R}^2 = N^2\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Multiplying this by $\boldsymbol{R}^{-1}$ shows that $\boldsymbol{R}^{-1} = (1/N^2)\boldsymbol{R}$. We can also conclude that $\boldsymbol{F}^4 = \boldsymbol{R}^2 = N^2\boldsymbol{I}$.

(*Erratum in first printing:* As we see, the desired result is $\boldsymbol{R}^2 = N^2\boldsymbol{I}$.)

(c) Left-multiply the definition $\boldsymbol{F}^2 = \boldsymbol{R}$ by $\boldsymbol{F}^{-1}$, and then right-multiply by $\boldsymbol{R}^{-1}$. The result expresses the inverse DFT as $\boldsymbol{F}^{-1} = \boldsymbol{F}\boldsymbol{R}^{-1}$. Thus the inverse FFT can be computed by first multiplying by the matrix $\boldsymbol{R}^{-1}$, and then applying the forward FFT. From Part (b), multiplying by $\boldsymbol{R}^{-1}$ is equivalent to scaling by $1/N^2$ and then multiplying by $\boldsymbol{R}$; and multiplying by $\boldsymbol{R}$ is equivalent to a simple permutation of the elements of the array to be transformed.

17. Evaluating the DFT, Eq. 2.6, for the phasor at frequency $(m + 1/2)2\pi/N$ radians per sec results in

$$X_k = \sum_{t=0}^{N-1} e^{jt(m-k+1/2)2\pi/N}$$

By symmetry, the minimum amplitude occurs at the points closest to the point on the frequency circle opposite the phasor frequency, the frequency points $k = m + N/2$ and $k = m + N/2 + 1$. It doesn't matter which one we look at, so consider the transform at the first,

$$X_{m+N/2} = \sum_{t=0}^{N-1} e^{jt(-N/2+1/2)2\pi/N} = \sum_{t=0}^{N-1} e^{-jt((N-1)/N)\pi}$$

This is the usual finite geometric series, with closed form

$$\frac{1 - e^{-j\pi N}e^{j\pi}}{1 - e^{-j\pi}e^{j\pi/N}} = \frac{2}{1 + e^{j\pi/N}}$$

assuming that N is a power of 2, and hence even. For large N, the complex exponential in the denominator is very close to 1, making the value of $X_{m+N/2}$ very close to 1, which is what we wanted to show.

Chapter 9

The z-transform and Convolution

1. At low frequencies,

$$z^{-1} = e^{-j\omega} \approx 1 - j\omega$$

and the frequency response is

$$\frac{1}{1 - z^{-1}} \approx \frac{1}{j\omega}$$

This matches the variation of the frequency content of a square wave, whose nth harmonic is proportional to $1/n$ (see Eq. 3.5 in Chapter 7, for example).

On the other hand, near the Nyquist frequency, where $z = -1$,

$$\frac{1}{1 - z^{-1}} \approx \frac{1}{2}$$

and this does not bear any relationship to the frequency content of the square wave.

2. The z-transform of the signal g_k is

$$\mathcal{G}(z) = \sum_{\text{all } k} g_k z^{-k} = \sum_{k \text{ even}} f_{k/2} z^{-k} = \sum_{\text{all } k} f_k z^{-2k} = \mathcal{F}(z^2)$$

where $\mathcal{F}(z)$ is the z-transform of the original signal f_k. When z travels around the frequency circle once, the spectrum of the signal f_k traces out the frequency content of the signal g_k twice. We have effectively squeezed two copies of the spectrum of f_k into the baseband. The way this idea is used in a CD player for oversampled d-to-a conversion is explained in Section 1 of Chapter 14.

3. The z-transform of f_k is

$$\mathcal{F}(z) = \sum_{k=0}^{\infty} z^{-k}/k! = e^{1/z}$$

since this is precisely the power series for the exponential function.

4. The z-transform of the signal y_k is the transfer function of the feedback digital filter described by Eq. 6.1 in Chapter 5,

$$\mathcal{Y}(z) = \frac{1}{1-(2R\cos\theta)z^{-1}+R^2z^{-2}}$$

as in Eq. 4.2 of Chapter 5.

5. I wrote a simple double-precision C program that implements reson with $R = 1$ and a unit impulse initial condition. The reson's output is compared to the theoretical impulse response, as given in Problem 4 above, computed using the library sine function. Figure P5 shows the difference between the reson output and the theoretical response for the two values $\theta = 0.0625$ and 0.06, in fractions of the sampling rate. The error grows in magnitude in both cases, but remains centered on zero in the first case and drifts away from zero in the second case.

There are two causes of the error. First, there is the buildup of roundoff error. Second, there is the discrepancy between the pole angle that is actually achieved in the reson with the computed filter coefficient $2\cos(2\pi\theta)$, and the frequency that is used to compute the theoretical value. In the first example, it appears that the filter achieves a frequency very close to the direct calculation, and no drift is observable between the filter output and the directly computed sinusoid. In the second example, there is a drift, and in fact it dominates the error.

The figure shows the error for a million samples, which corresponds to more than 22 seconds at the standard audio sampling rate of 44,100 Hz. The maximum error in both cases is less in magnitude than one part in 10^9, and seems to grow no faster than linearly. Thus the method is a practical way to generate sinusoids, at least for times on the order of a minute at audio frequencies.

6. Below is a vertical slice down a typical column in the long division method applied to the simple z-transform $1/(1+az^{-1}+bz^{-2})$:

$$\begin{array}{c} \underline{y_t} \\ \cdots x_t \cdots \\ \vdots \\ \underline{by_{t-2}} \\ x_t - by_{t-2} \\ \underline{ay_{t-1}} \\ x_t - ay_{t-1} - by_{t-2} \end{array}$$

In the general case, the tth numerator term will appear where x_t appears in this column, and successive denominator terms will accumulate to add to the final line, which is precisely the output of the digital filter in the impulse-response method.

7. I'll do the work in algebraic rather than numerical form. Write Eq. 8.9 as

$$\frac{z^3}{(z-Re^{j\theta})(z-Re^{-j\theta})(z-1)} = \frac{Az}{z-1} + \frac{Bz}{z-Re^{j\theta}} + \frac{B^*}{z-Re^{-j\theta}}$$

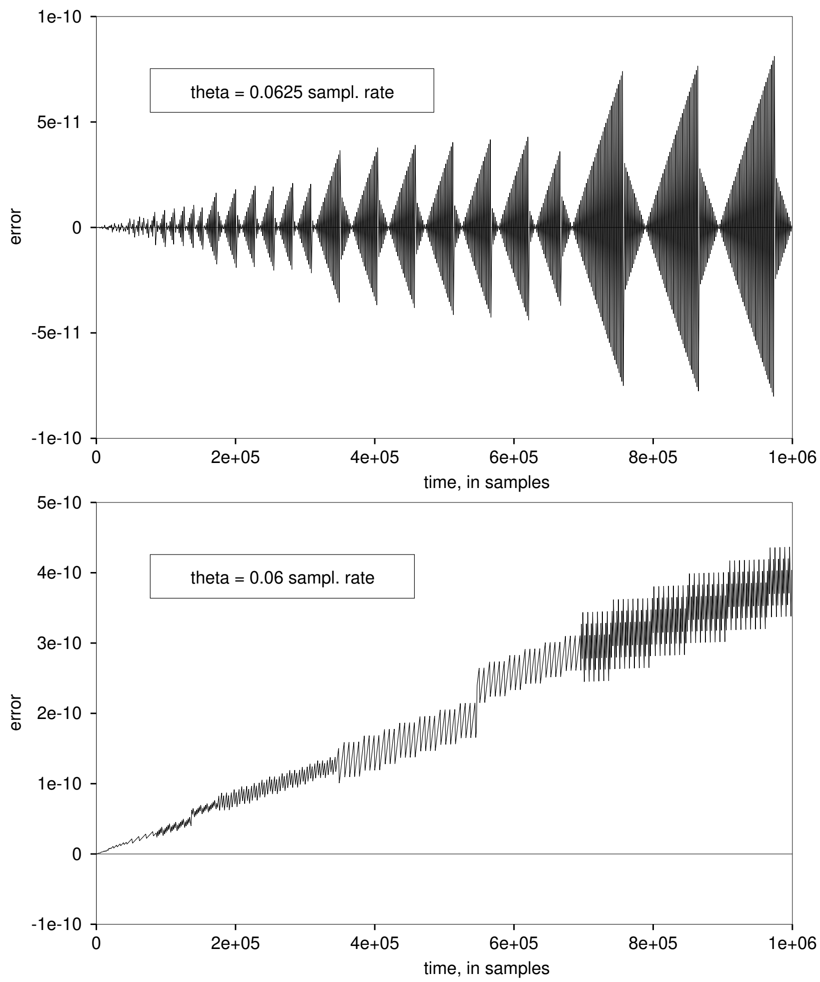

Fig. P5 Error between the impulse response of a reson with poles on the unit circle and the corresponding sinusoid directly computed. The top and bottom graphs show the cases with frequencies 0.0625 and 0.06 times the sampling rate, respectively.

Multiplying by $(z-1)$ and setting $z=1$ confirms the result in the text,

$$A = \frac{1}{1 - 2R\cos\theta + R^2}$$

Multiplying by $(z - Re^{j\theta})$ and setting $z = Re^{j\theta}$ results in

$$B = \frac{jRe^{2j\theta}(1 - Re^{-j\theta})}{2\sin\theta(R^2 - 2R\cos\theta + 1)}$$

From the first equation, the oscillatory part of the step response is therefore

$$BR^t e^{j\theta t} + B^* R^t e^{-j\theta t} = 2R^t \Re\left\{Be^{j\theta t}\right\}$$

The total step response then becomes, after some straightforward algebra,

$$\frac{1}{R^2 - 2R\cos\theta + 1}\left[1 + \frac{R^{t+1}}{\sin\theta}\left(R\sin((t+1)\theta) - \sin((t+2)\theta)\right)\right]$$

From this, the first two values of the step response are 1 and $1 + 2R\cos\theta$, which checks against long division of the full z-transform, Eq. 8.1 times $1/(1 - z^{-1})$.

8. The z-transform of a unit step signal is

$$\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}$$

Differentiating both sides yields

$$\sum_{k=1}^{\infty} (-k) z^{-k-1} = \frac{-z^{-2}}{(1 - z^{-1})^2}$$

Multiplying by $-z$ and extending the summation to $k = 0$, which doesn't affect the result, gives the desired z-transform, verifying Eq. 9.5:

$$\sum_{k=0}^{\infty} k z^{-k} = \frac{z^{-1}}{(1 - z^{-1})^2}$$

Repeated differentiating can be used to derive the z-transform of the signal $x_k = k^n$, which will have an $(n+1)$st-order pole at $z = 1$.

9. Differentiate the z-transform of the signal $y_k = k$, $k \geq 0$, Eq. 9.5:

$$\sum_{k=0}^{\infty} (-k^2) z^{-k-1} = \frac{-z^{-2} - z^{-3}}{(1 - z^{-1})^3}$$

Multiplying both sides by $-z$ then gives the answer,

$$\sum_{k=0}^{\infty} k^2 z^{-k} = \frac{z^{-1} + z^{-2}}{(1 - z^{-1})^3}$$

10. The idea is to simplify the algebra by setting $R = 1$ in a known result, deriving a new transform by differentiating, and then putting R back. The z-transform of the signal $\sin(k\theta)$, $k \geq 0$ is, setting $R = 1$ in the result in Table 5.1,

$$\sum_{k=0}^{\infty} \sin(k\theta) z^{-k} = \frac{(\sin\theta) z^{-1}}{1 - (2\cos\theta) z^{-1} + z^{-2}}$$

Differentiating yields, after the usual simplifications,

$$\sum_{k=0}^{\infty} k \sin(k\theta) z^{-k} = \frac{(\sin\theta)(z^{-1} - z^{-3})}{(1 - (2\cos\theta) z^{-1} + z^{-2})^2}$$

We can now weight the time signal at time t by R^t, and replace z^{-1} by Rz^{-1} in its transform, yielding the result

$$\sum_{k=0}^{\infty} k R^k \sin(k\theta) z^{-k} = \frac{(\sin\theta)(Rz^{-1} - R^3 z^{-3})}{(1 - (2\cos\theta) R z^{-1} + R^2 z^{-2})^2}$$

11. We want to find the coefficients A and B in the partial fraction expansion

$$\mathcal{G}(z) = \frac{A}{1 - pz^{-1}} + \frac{B}{(1 - pz^{-1})^2} + \text{ terms for other poles}$$

Since $\mathcal{G}(z)$ is assumed to have a double pole at $z = p$, write it in the form

$$\mathcal{G}(z) = \frac{\mathcal{H}(z)}{(1 - pz^{-1})^2} + \text{ terms for other poles}$$

and rewrite the partial fraction expansion as

$$\frac{z^2 \mathcal{H}(z)}{(z - p)^2} = \frac{Az}{z - p} + \frac{Bz^2}{(z - p)^2} + \text{ terms for other poles}$$

Multiplying by $(z - p)^2$ and setting $z = p$ yields

$$B = \mathcal{H}(p)$$

To find the coefficient A, multiply the original partial fraction expansion by $(1 - pz^{-1})^2$,

$$\mathcal{H}(z) = A(1 - pz^{-1}) + B + (1 - pz^{-1})^2 \, [\text{ terms for other poles }]$$

Differentiating with respect to z and setting $z = p$ now gives us

$$A = p\,\mathcal{H}'(p)$$

12. The form we use for partial fraction expansions, including the double-pole version in Problem 11, can represent ratios of polynomials in z^{-1} in which the numerator has degree (in z^{-1}) strictly smaller than the denominator. When this isn't the case, divide the denominator into the numerator, using the usual long-division method, until the remainder is a power of z^{-1} times a ratio that *is* in this form. Then expand this ratio in the usual way. As a simple example,

$$\frac{1 + z^{-1} + z^{-2}}{1 - z^{-1}} = 1 + 2z^{-1} + z^{-2}\frac{3}{1 - z^{-1}}$$

13. The result we're after is the sum of the two partial fraction terms obtained in Problem 11,

$$\frac{p\,\mathcal{H}'(p)}{1 - pz^{-1}} + \frac{\mathcal{H}(p)}{(1 - pz^{-1})^2} = \frac{p\,\mathcal{H}'(p)(1 - pz^{-1}) + \mathcal{H}(p)}{(1 - pz^{-1})^2}$$

where $\mathcal{H}(z)$ is the original ratio of polynomials without an assumed double pole at $z = p$.

To derive this in the way suggested, we'll assume there are poles at $z = p_1$ and $z = p_2$ instead of the double pole at $z = p$, and we'll let $p_2 \to p_1 = p$. With distinct poles, the two terms above are replaced by

$$\frac{\mathcal{H}(p_1)}{(1 - p_2 p_1^{-1})(1 - p_1 z^{-1})} + \frac{\mathcal{H}(p_2)}{(1 - p_1 p_2^{-1})(1 - p_2 z^{-1})}$$

Put this over the common denominator $(1-p_1 z^{-1})(1-p_2 z^{-1})$. The numerator then becomes

$$\frac{p_2\mathcal{H}(p_2) - p_1\mathcal{H}(p_1)}{p_2 - p_1} - z^{-1}p_1 p_2\left[\frac{\mathcal{H}(p_2) - \mathcal{H}(p_1)}{p_2 - p_1}\right]$$

A bit more simplification leads to

$$p_1\left[\frac{\mathcal{H}(p_2) - \mathcal{H}(p_1)}{p_2 - p_1}\right] + \mathcal{H}(p_2) - z^{-1}p_1 p_2\left[\frac{\mathcal{H}(p_2) - \mathcal{H}(p_1)}{p_2 - p_1}\right]$$

Letting $p_2 \to p_1 \to p$ now leads to the desired form, derived at the beginning of this problem.

14. Standard power series are a rich source of such z-transforms. For example,

$$\sum_{k=0,2,4,\ldots}^{\infty} (-1)^{k/2} z^{-k} = \cos(1/z)$$

or

$$1 + \frac{1}{2}z^{-2} + \frac{1\cdot 3}{2\cdot 4}z^{-4} + \cdots \;=\; \frac{1}{\sqrt{1 - z^{-2}}}$$

which is from L. B. W. Jolley, *Summation of Series*, second edition, Dover, New York, N.Y., 1961.

15. (a) The number of pairs of rabbits at the beginning of Month t is just the sum of those alive at the beginning of Month $t-1$, who survive, and those alive at the beginning of Month $t-2$, who give birth to new pairs of rabbits. Thus

$$r_t = r_{t-1} + r_{t-2}$$

(b) Interpret r_t as the output of a digital filter with the update equation

$$r_t = x_t + r_{t-1} + r_{t-2}$$

where x_t is an input signal equal to the unit pulse. The appropriate initial conditions with this input signal are $r_{-2} = r_{-1} = 0$, which yields $r_0 = 1$, $r_1 = 1$, $r_2 = 2$, and so on, as desired. This is a feedback filter, and is unstable, since its output grows without bound with this bounded input.

(c) The input signal x_t is a unit pulse, so $\mathcal{X}(z) = 1$. The update equation then gives us the z-transform

$$\mathcal{R}(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

(d) The poles are the roots of the denominator,

$$p_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

The plus sign yields a root outside the unit circle, as we expect for an unstable signal. The partial fraction expansion is

$$\mathcal{R}(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}}$$

where

$$A = \frac{p_1}{p_1 - p_2} = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$$

and

$$B = \frac{p_2}{p_2 - p_1} = -\frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)$$

(e) The inverse z-transform of this partial fraction expansion gives us the following explicit expression for r_t:

$$r_t = Ap_1^t + Bp_2^t = \frac{1}{2^{t+1}\sqrt{5}}\left[\left(1 + \sqrt{5}\right)^{t+1} - \left(1 - \sqrt{5}\right)^{t+1}\right]$$

(f) Assuming no leap year, we want r_{365}, the number of pairs of rabbits at the beginning of Day 365, since we started on Day 0. The second term in the expression for r_t is negligible compared to the first, so

$$r_{365} \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{365} = 8.531 \times 10^{75}$$

a fair number of pairs of rabbits.

16. Substitute the definitions of $X(\omega)$ and $Y(\omega)$ in the definition of $\langle X, Y \rangle$:

$$\begin{aligned}
\langle X, Y \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) Y^*(\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} x_k e^{-jk\omega} \sum_{m=-\infty}^{\infty} y_m^* e^{jm\omega} d\omega \\
&= \sum_{k=-\infty}^{\infty} x_k \sum_{m=-\infty}^{\infty} y_m^* \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(k-m)\omega} d\omega
\end{aligned}$$

The integral is equal to one if $k = m$ and is zero otherwise, so we get

$$\langle X, Y \rangle = \sum_{k=-\infty}^{\infty} x_k y_k^* = \langle x, y \rangle$$

which we set out to prove.

To verify this for a simple example, choose $x_0 = 1$, $x_1 = 1$, $y_0 = 1$, $y_1 = -1$, and all other sample values zero. Then $\mathcal{X}(z) = 1 + z^{-1}$, $\mathcal{Y}(z) = 1 - z^{-1}$, and

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) Y^*(\omega) d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + e^{-j\omega})(1 - e^{j\omega}) d\omega \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} 2j \sin \omega d\omega \\
&= 0
\end{aligned}$$

which checks the time domain inner product

$$\langle x, y \rangle = \sum_{k=-\infty}^{\infty} x_k y_k^* = 0$$

When the two signals are equal, Parseval's theorem states that the integral of the square of a signal's magnitude transform is equal to the total energy in the signal.

Chapter 10

Using the FFT

1. We want to show by induction that

$$\sum_{t=0}^{n-1} x^t = \frac{1-x^n}{1-x}$$

Therefore, assume this formula holds for all $n \leq N$, and consider the case $n = N+1$:

$$\sum_{t=0}^{N} x^t = \sum_{t=0}^{N-1} x^t + x^N = \frac{1-x^N}{1-x} + x^N = \frac{1-x^{N+1}}{1-x}$$

which is what we wish to show.

2. L'Hôpital's rule states that the limit of the indeterminate ratio can be computed by taking the limit of the ratio of derivatives, if that limit exists. Thus, to find the limit of

$$\frac{1-x^n}{1-x}$$

as $x \to 1$, consider the limit of the ratio of derivatives:

$$\lim_{x \to 1} \frac{-nx^{n-1}}{-1} = n$$

3. Let's drop the magnitude signs in Eq. 2.4, so that we can deal with the differentiable function $S(\theta - \omega)$, where we define

$$S(\phi) = \frac{\sin(n\phi/2)}{\sin(\phi/2)}$$

as in Eq. 5.6. This is valid because we know from the plot in Fig. 2.1 that this function is positive in the vicinity of $\phi = 0$. We can try to verify that $S(\phi)$ has a peak precisely at the origin by differentiating and setting $\phi = 0$. This results in the indeterminate form 0/0.

Applying L'Hôpital's rule results in the same indeterminate form. Applying L'Hôpital's rule for a second time finally shows that the derivative is zero at $\phi = 0$; this point corresponds to $\omega = \theta$. To be rigorous, we should then check that the second derivative is negative at $\phi = 0$, but the shape of the curve is clear from the plot in Fig. 2.1.

Here's an easier argument: The function $S(\phi)$ is an even function about the origin, and is differentiable everywhere. Therefore, if we consider a neighborhood $\pm\epsilon$ about the origin, the function must either increase symmetrically on both sides, or decrease symmetrically on both sides. That is, the function must have either a local maximum or minimum at that point. Again, the fact that it has a local maximum can be shown easily by plotting the function in a small neighborhood of the origin.

4. For the z-transform of a finite stretch of a cosine wave, write the cosine as the sum of two phasors, and use the closed form in Problem 1:

$$\begin{aligned}\sum_{t=0}^{n-1} \cos(\theta t) z^{-t} &= \frac{1}{2}\sum_{t=0}^{n-1}\left(e^{j\theta t} + e^{-j\theta t}\right) z^{-t} \\ &= \frac{1}{2}\frac{1 - e^{jn\theta}z^{-n}}{1 - e^{j\theta}z^{-1}} + \frac{1}{2}\frac{1 - e^{-jn\theta}z^{-n}}{1 - e^{-j\theta}z^{-1}}\end{aligned}$$

In general, the frequency content does not peak precisely at the point $\omega = \theta$. The situation is analogous to that of the two-pole reson: the contribution from the negative-frequency component distorts the contribution from the positive-frequency component. To see this algebraically, find the frequency content by letting $z = e^{j\omega}$:

$$\begin{aligned}&\sum_{t=0}^{n-1} \cos(\theta t) e^{-j\omega t} \\ &\quad= \frac{1}{2}e^{-j(\omega-\theta)(n-1)/2} S(\omega - \theta) + \frac{1}{2}e^{-j(\omega+\theta)(n-1)/2} S(\omega + \theta) \\ &\quad= \frac{1}{2}e^{-j\omega(n-1)/2}\left[e^{j\theta(n-1)/2} S(\omega-\theta) + e^{-j\theta(n-1)/2} S(\omega+\theta)\right]\end{aligned}$$

where $S(\phi)$ is defined as in Problem 3. Take the squared magnitude of this, which eliminates the complex exponential in front. Then replace the complex exponentials that are left by sines and cosines, and use the sum of the squares of the real and imaginary parts:

$$\begin{aligned}&\left|\sum_{t=0}^{n-1} \cos(\theta t) e^{-j\omega t}\right|^2 \\ &\quad= \frac{1}{4}S^2(\omega-\theta) + \frac{1}{4}S^2(\omega+\theta) + \frac{1}{2}\cos(\theta(n-1)) S(\omega-\theta) S(\omega+\theta)\end{aligned}$$

We know from Problem 3 that the $S^2(\omega - \theta)$ term peaks at $\omega = \theta$, but what's left in general doesn't.

5. The closed form for a finite geometric series helps us once again. We normalize by the sum of the window weights, so we need to find

$$\begin{aligned}\sum_{t=0}^{n-1} h_t &= \sum_{t=0}^{n-1} (0.54 - 0.46\cos(2\pi t/(n-1))) \\ &= 0.54n - 0.46 \sum_{t=0}^{n-1} \cos(2\pi t/(n-1))\end{aligned}$$

Writing the cosine as the real part of a phasor, the sum of cosines can be written

$$\begin{aligned}\sum_{t=0}^{n-1} \cos(2\pi t/(n-1)) &= \Re\left\{\sum_{t=0}^{n-1} e^{j2\pi t/(n-1)}\right\} \\ &= \Re\left\{\frac{1-e^{j2\pi n/(n-1)}}{1-e^{j2\pi/(n-1)}}\right\}\end{aligned}$$

Multiplying numerator and denominator by the conjugate of the denominator makes the denominator real, and allows us to finish the job of taking the real part:

$$\begin{aligned}\sum_{t=0}^{n-1} \cos(2\pi t/(n-1)) &= \frac{2-\cos(2\pi n/(n-1)) - \cos(2\pi/(n-1))}{2-2\cos(2\pi/(n-1))} \\ &= 1\end{aligned}$$

which completes the calculation, showing that the sum of the weights of a Hamming window is $0.54n - 0.46$.

You can also derive this result by evaluating $\mathcal{H}(1) = H(0) = \sum_{t=0}^{n-1} h_t$ in Eq. 5.7. The shifted functions $S(\pm 2\pi/(n-1))$ evaluate to -1.

General comment on Problems 6–8: Notice that I use the symmetric definition of a Hamming window in Oppenheim and Schafer, *Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, N.J., 1975. The window weights have the value 0.08 at each end. This is slightly different from the definition in T. Saramäki's Chapter 4 of S. K. Mitra and J. F. Kaiser (eds.), *Handbook for Digital Signal Processing*, John Wiley, New York, N.Y., 1993, where the argument of the cosine is an integer multiple of $2\pi/n$, instead of the $2\pi/(n-1)$ I use in Eq. 5.1. I adopt similarly symmetric definitions for the other windows, in Problems 6–8. This has a very small effect on the frequency content, noticeable in Fig. P6–8 only near the Nyquist frequency for the Hann and Blackman windows. To me, asymmetric window weights are difficult to defend philosophically, but to each his own; as Saramäki points out, the definitions "differ slightly in the literature."

6. The derivation of the closed form for the frequency content of the Hann window is identical to that for the Hamming window, except that both 0.54 and 0.46 are replaced by 0.5. The result is analogous to Eq. 5.7:

$$H(\omega) = e^{-j(n-1)\omega/2}\left[0.5S(\omega) + 0.25S(\omega - \frac{2\pi}{n-1}) + 0.25S(\omega + \frac{2\pi}{n-1})\right]$$

The sum of weights, for normalization, is $0.5n - 0.5$, using the same methods as in Problem 5.

Figure P6–8 shows the frequency content of this window for $n = 257$, together with the corresponding plots for the windows in Problems 7 and 8. Figure P6–8 (close-up) shows close-ups of the frequency content of these windows in the low-frequency region.

The Hann window's central lobe has a width essentially identical to that of the Hamming window, but its dominant side lobe is about 15 dB higher (for this value of n). In return, the Hann window rolls off much faster at high frequencies. It's evident from the plots that the Hamming window offers an amazingly good approximation to equiripple behavior, given its simplicity.

7. The closed form for the frequency content of the Blackman window can be derived in the same way as that for the Hamming and Hann windows, but in this case two more shifted versions of the function $S(\omega)$ appear. The complex exponential in front induces a factor of -1 for the versions shifted by $2\pi/(n-1)$, as before, but a factor of $+1$ for the versions shifted by $4\pi/(n-1)$. The result is

$$\begin{aligned} H(\omega) \quad = \quad & e^{-j(n-1)\omega/2} \Big[0.42S(\omega) + 0.25S(\omega - \frac{2\pi}{n-1}) \\ & + 0.25S(\omega + \frac{2\pi}{n-1}) + 0.04S(\omega - \frac{4\pi}{n-1}) + 0.04S(\omega + \frac{4\pi}{n-1}) \Big] \end{aligned}$$

For normalization, the sum of weights for the Blackman window is $0.42n - 0.42$, calculated in the same way as for the Hamming and Hann windows. The sum of cosines of the form $\cos(4\pi/(n-1))$ turns out to be the same as the sum of cosines of the form $\cos(2\pi/(n-1))$; both are simply one.

8. The Bartlett window doesn't have cosines in its definition, so the method used for finding the frequency content of the Hamming, Hann, and Blackman windows doesn't work. Instead, we'll calculate its z-transform directly. For convenience, we'll assume that n is odd for the Bartlett window, which means that the triangular waveform peaks at the single central point, $t = (n-1)/2$.

To find the z-transform, think of the triangle wave as the superposition of three linear functions: a ramp up with slope $+1$ starting at $t = 0$; a ramp down starting at $t = (n-1)/2$ with slope -2; and a ramp up with slope $+1$ starting at $t = n-1$. For $t \leq n-1$ these three ramps produce the triangle, and for $t > n-1$ these three ramps sum to precisely zero.

The z-transform of the first component ramp is, by Eq. 9.5 of Chapter 9,

$$\mathcal{Y}(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

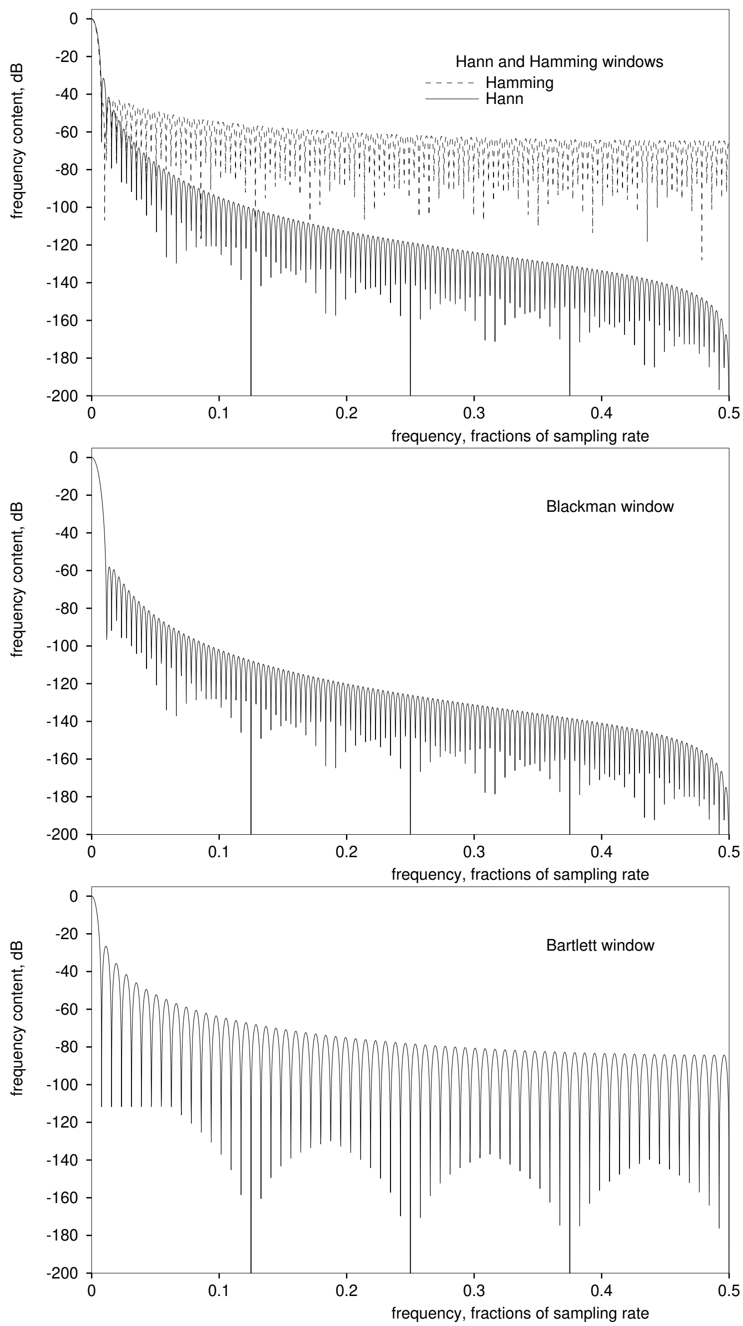

Fig. P6–8 The frequency content of the Hann, Blackman, and Bartlett windows, from top to bottom, for n = 257 points. For comparison, the frequency content of the Hamming window is plotted along with that for the Hann window.

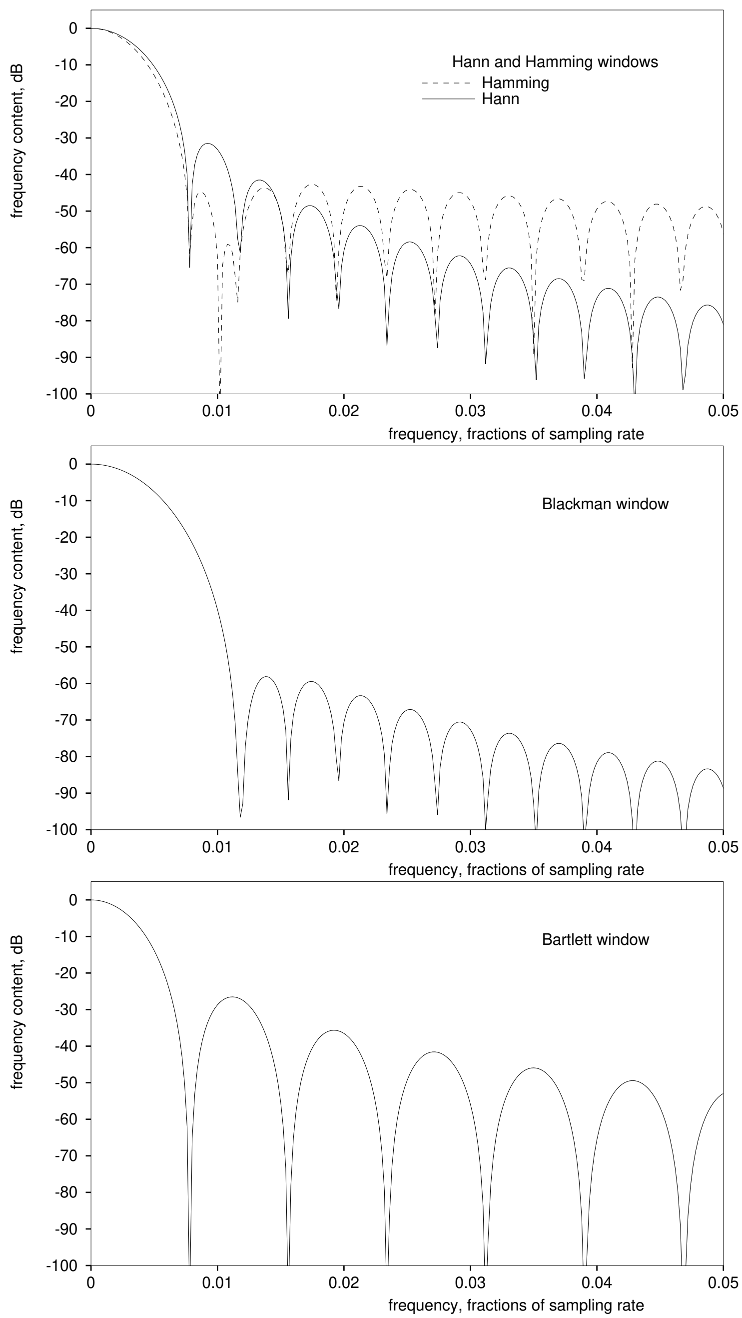

Fig. P6–8 (close-ups) The central lobe behavior of the Hann, Blackman, and Bartlett windows, from top to bottom, for n = 257 points. These are close-ups of the low-frequency region in Fig. P6–8.

The sum of the z-transforms of the three components just described is therefore

$$\begin{aligned}\mathcal{H}(z) &= \mathcal{Y}(z) - 2z^{-(n-1)/2}\mathcal{Y}(z) + z^{-(n-1)}\mathcal{Y}(z) \\ &= \frac{z^{-1} - 2z^{-(n+1)/2} + z^{-n}}{(1-z^{-1})^2} \\ &= z^{-1}\left(\frac{1-z^{-(n-1)/2}}{1-z^{-1}}\right)^2\end{aligned}$$

This is, aside from the delay factor of z^{-1}, the square of the z-transform of a rectangular window of length $(n-1)/2$ points. Thus the frequency content of the Bartlett window is

$$|H(\omega)| = \left|\frac{\sin((n-1)\omega/4)}{\sin(\omega/2)}\right|^2$$

To normalize the window, just divide by $H(0) = (n-1)/2$.

Of course, this is no accident: The frequency content of the Bartlett window is the square of the frequency content of an appropriate rectangular window because the convolution of a rectangle with itself is a triangle. See Problem 10 of Chapter 11.

9. Multiplying every other sample of a signal x_t by -1 is equivalent to multiplying it by $(-1)^t$. The z-transform of the resulting signal is therefore

$$\sum_{t=-\infty}^{\infty}(-1)^t x_t z^{-t} = \sum_{t=-\infty}^{\infty} x_t(-z)^{-t} = \mathcal{X}(-z)$$

where $\mathcal{X}(z)$ is the z-transform of the original signal. The effect is thus to replace z by $-z$, which flips the frequency circle so that the points 0 and -1 in the z-plane trade places. In other words, the frequency content of the signal is now in reverse order as we trace out frequencies from 0 to the Nyquist frequency.

Another way of looking at this is that the signal is multiplied by $(-1)^t = e^{j\pi t}$, which is a phasor. Multiplying by this phasor shifts the frequency axis by π radians per sample, and therefore slides the negative frequencies up to the positive range, where they appear in reverse order.

10. The rectangular window has a relatively narrow central lobe, but poor side-lobe rejection. The signal in this case has a dominant pitch and is narrowband. Therefore, the pitch of the signal sometimes lines up well with the window central lobe and sometimes does not. As the signal sweeps in frequency, its pitch passes in and out of the central-lobe region. When the pitch lines up well with the central lobe, the energy is concentrated at the correct pitch. When the pitch falls outside the central lobe, the side lobes pick up a relatively large amount of energy.

To exaggerate the effect further, the spectrogram program has a kind of automatic gain control, which normalizes the plot so the overall intensity remains constant as time progresses. So when the central-lobe signal fades out, the side-lobe signal is boosted, further smearing the plot.

11. The first 9 periods in Fig. 7.6 take about 132 samples, which, at a sampling rate of 22,050 Hz, corresponds to a frequency of $22050/(132/9) = 1503$ Hz. The final 8 periods take about 63 samples, which corresponds to a frequency of $22050/(63/8) = 2800$ Hz. This is reasonably close to a doubling of frequency, and checks reasonably well with the spectrogram in Fig. 7.4.

Chapter 11

Aliasing and Imaging

1. Denote the continuous-time signal by $x(t)$ and the window by $w(t)$. Windowing the signal first results in the signal $x(t)w(t)$; sampling then results in $x(kT_s)w(kT_s)$, where k is the sample number and T_s is the sampling interval, as usual. Sampling the signal first yields the digital signal with samples $x(kT_s)$, which after windowing by the window with samples of $w(t)$ yields $x(kT_s)w(kT_s)$, the same result.

2. The domains omitted correspond to periodic signals, when the time domain is continuous and finite (circular), and the corresponding transform domain is discrete and infinite. This is the left-right reversal of the second row in Fig. 1.1.

This case is crucial for the development of signal processing theory — Chapter 7 is devoted to it. When time and frequency are reversed, it corresponds to digital signals and their z-transforms. But it doesn't come up directly in much practical signal processing because continuous-time signals are usually processed in a continuous, virtually infinite, stream. When a finite section is processed, it is almost always sampled. The only case I can think of when a finite segment of a continuous-time signal is processed is the tape loop.

3. There are six pairs of transforms, one for each entry in Fig. 1.1, and each relating a convolution to its product-form transform or inverse transform. The method of deriving these pairs is the same in all six cases:

- Write the conjectured form of the convolution.
- Replace the signals or transforms in the convolution by their transforms or inverse transforms.
- Move the summation or integration corresponding to the convolution to the inside. That is, do it first. That produces a δ-function, either discrete-time or continuous-time.
- Perform one of the two remaining summations or integrations, using the δ-function. The result is in the form of a transform or inverse transform, and is the desired result.

This may sound complicated, but it's actually very natural. We'll start at the upper left in Fig. 1.1, and proceed from left to right and top to bottom.

Continuous time, infinite extent, time-domain convolution: The convolution between the two signals $f(t)$ and $g(t)$ is

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Replacing the signals by their inverse transforms, $F(\phi)$ and $G(\omega)$, respectively, yields

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\phi)e^{j\phi\tau}d\phi \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{j\omega(t-\tau)}d\omega d\tau$$

Performing the outside integration first results in the integral

$$\int_{-\infty}^{\infty} e^{j(\phi-\omega)\tau}d\tau = 2\pi\delta(\phi-\omega)$$

as promised. To derive this, note that the forward Fourier transform of $\delta(t)$ is one. Therefore, the inverse Fourier transform of one is

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}d\omega$$

The next integration, on ϕ, gives

$$\int_{-\infty}^{\infty} F(\phi)\delta(\phi-\omega)d\phi = F(\omega)$$

and what's left is now

$$f(t) \otimes g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{j\omega t}d\omega$$

The right-hand side is the inverse Fourier transform of $F(\omega)G(\omega)$, so we have the desired transform pair

$$f(t) \otimes g(t) \leftrightarrow F(\omega) \cdot G(\omega)$$

Since the method is essentially the same in all cases, I'll give only the results below.

Continuous time, infinite extent, frequency-domain convolution:

$$F(\omega) \otimes G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\phi)G(\omega-\phi)d\phi \leftrightarrow f(t) \cdot g(t)$$

Discrete time, infinite extent, time-domain convolution:

$$f_t \otimes g_t = \sum_{k=-\infty}^{\infty} f_k g_{t-k} \leftrightarrow F(\omega) \cdot G(\omega)$$

Discrete time, infinite extent, frequency-domain convolution:

$$F(\omega) \otimes G(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\phi) G(\omega - \phi) d\phi \leftrightarrow f_t \cdot g_t$$

Discrete time, finite extent, time-domain convolution:

$$f_t \otimes g_t = \sum_{k=0}^{N-1} f_k g_{t-k} \leftrightarrow F_k \cdot G_k$$

Discrete time, finite extent, frequency-domain convolution:

$$F_k \otimes G_k = \frac{1}{N} \sum_{k=0}^{N-1} F_m G_{k-m} \leftrightarrow f_t \cdot g_t$$

4. For convenience, denote the constant $U(-1)$ by the constant a. We can assume that $a \neq 0$, because otherwise the property stated would imply the trivial result that $U(k)$ is zero for all k. Setting $k = 0$ in the property yields

$$aU(0) = a$$

which tells is that $U(0) = 1$. Replacing k by $k + 1$ in the property and dividing by a gives us the recurrence relation

$$U(k + 1) = a^{-1} U(k)$$

which shows that $U(k) = a^{-k} U(0) = a^{-k}$, which is the desired result with $c = a^{-1}$, for $k \geq 0$. The same argument works in the opposite direction to give the result for $k \leq 0$.

5. I'll do the z-transform case; the other cases are similar. Suppose the (real) signal x_k is odd. Then $x_0 = 0$ and the z-transform can be written

$$\mathcal{X}(z) = \sum_{k=-\infty}^{-1} x_k z^{-k} + \sum_{k=1}^{\infty} x_k z^{-k}$$

Now use the fact that $x_k = -x_{-k}$ and let $z = e^{j\omega}$ to rewrite this as

$$\begin{aligned} X(\omega) &= -\sum_{k=-\infty}^{-1} x_k e^{-jk\omega} + \sum_{k=1}^{\infty} x_k e^{-jk\omega} \\ &= \sum_{k=1}^{\infty} x_k \left(e^{jk\omega} - e^{-jk\omega} \right) \\ &= 2j \sum_{k=1}^{\infty} x_k \sin(k\omega) \end{aligned}$$

which is imaginary-valued.

6. (This derivation follows my Chapter 1 of S. K. Mitra and J. F. Kaiser (eds.), *Handbook for Digital Signal Processing*, John Wiley, New York, N.Y., 1993.) Start with the digital signal x_t, in general infinite in extent, with z-transform $\mathcal{X}(z)$. We sample its frequency content at frequencies $k2\pi/N$, for $k = 0, 1, 2, \ldots, N-1$, obtaining

$$X(k2\pi/N) = \sum_{n=-\infty}^{\infty} x_n e^{-jkn2\pi/N}, \quad \text{for } k = 0, 1, \ldots, N-1$$

The question comes down to this: what finite-duration signal has this DFT? Thus, we compute the inverse DFT of this, which we call the signal $\tilde{x}_t$:

$$\begin{aligned} \tilde{x}_t &= \frac{1}{N} \sum_{k=0}^{N-1} X(k2\pi/N) e^{jkt2\pi/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} e^{jkt2\pi/N} \sum_{n=-\infty}^{\infty} x_n e^{-jkn2\pi/N} \end{aligned}$$

Interchanging the order of summations results in

$$\tilde{x}_t = \frac{1}{N} \sum_{n=-\infty}^{\infty} x_n \sum_{k=0}^{N-1} e^{jk(t-n)2\pi/N}$$

The inside summation is our old friend the δ-function, but this time we need to take into account the fact that the index n extends over an infinite range. The resulting function of $t-n$ is therefore

$$\delta(t-n) = \begin{cases} N & \text{if } n = t \bmod N \\ 0 & \text{otherwise} \end{cases}$$

Hence, the sum picks up x_t, plus all values of x that are displaced by integer multiples of N. The aliased time function that results from sampling the z-transform is therefore

$$\tilde{x}_t = \sum_{m=-\infty}^{\infty} x_{t+mN}$$

7. The usual application of the DFT almost always starts with a finite-duration segment of a digital signal. The signal can therefore be considered zero outside this range, and there is no time aliasing.

8. The aliasing operation on the transform can be visualized as follows. Wrap a plot of the signal's frequency content around a cylinder with circumference 2π radians per sample. Then add up the values of the frequency content that occur at each point around the cylinder. The points that get added then correspond to the sum of values displaced by all integer multiples of 2π.

9. Using the impulse response in Eq. 6.3, we want to evaluate

$$\lim_{t\to 0} \frac{\sin(\pi t/T)}{\pi t/T}$$

L'Hôpital's rule prescribes taking the derivative of the numerator and the denominator, and evaluating the ratio at $t = 0$:

$$\left.\frac{(\pi/T)\cos(\pi t/T)}{\pi/T}\right|_{t=0} = 1$$

10. (a) The impulse response connects the value zero at $t = -T$ to the value one at $t = 0$ with a straight line; then it connects the value one at $t = 0$ to the value zero at $t = T$ with a straight line. At all other points it connects the value zero to zero, and hence is zero outside the range $-T \leq t \leq T$.

(b) Consider the convolution of the following rectangular pulse with itself:

$$h(t) = \begin{cases} 1/T & \text{if } -T/2 \leq t \leq T/2 \\ 0 & \text{otherwise} \end{cases}$$

When the two instances of the rectangle are separated by a distance greater than T, they don't overlap, and the convolution is zero. As they slide past each other, the overlapping area increases linearly from the point $t = -T$ until it reaches a maximum at $t = 0$, when they coincide. Then the overlapping area decreases symmetrically to zero at $t = T$. The maximum overlap is the area of the product when the two instances of the rectangle coincide, which is $T \cdot (1/T)^2 = 1/T$. The convolution is therefore the triangle described in Part (a), normalized so its area is one.

(c) The rectangular pulse in Part (b) is the impulse response of a zero-order hold, but centered at the time origin. Its frequency response is therefore, from Eq. 4.4,

$$H(\omega) = \frac{\sin(\omega T/2)}{\omega T/2}$$

The frequency response of the linear-point-connecting hold circuit is the square of this:

$$H^2(\omega) = \left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2$$

11. Equation 8.3 shows that when alternate samples are thrown away, the new spectrum is

$$\frac{1}{2}\left(X(\omega) + X(\omega - \omega_s)\right)$$

where $X(\omega)$ is the spectrum of the original signal, and the original sampling rate is $2\omega_s$. Therefore, frequency ω is confounded with the frequency that differs from it by *half* the sampling frequency, in this case, half of $2\omega_s$. For example, the Nyquist frequency appears identical to zero frequency if every other sample is discarded — which makes sense, since the samples of a sinusoid at the Nyquist frequency are of the form $(-1)^t$.

12. When only every kth sample is used, the new spectrum is

$$\frac{1}{k}\sum_{i=0}^{k-1} X(\omega - i\omega_s)$$

where the original spectrum is $X(\omega)$, and the original sampling rate is $k\omega_s$. This follows in exactly the same way as the result for $k = 2$, using the argument between Eqs. 8.1 and 8.3.

If Fig. 8.1 is redrawn for the case when the original sampling frequency is $k\omega_s$, the lowpass filter used to prepare for the sampling-rate reduction should still have a cutoff frequency at $\omega_s/2$. This is $1/k$ times the Nyquist frequency at the point when the filtering takes place. For example, when $k = 2$, the filter is half-band.

Chapter 12

Designing Feedforward Filters

1. The story is told nicely in D. M. Burton, *The History of Mathematics, an Introduction*, Allyn and Bacon, Inc., Boston, Mass., 1985. I'll skip the best part — the rivalry and infighting among the mathematicians of the time.

The problem can be formulated as follows: Express the roots of the general polynomial of given degree n using only addition, multiplication, subtraction, division, and extraction of radicals. About 1530, Nicolo Tartaglia provided a solution for the cubic case ($n = 3$). The solution for the quartic case ($n = 4$) was found by Ludovico Ferrari, about ten years later. The quintic case ($n = 5$) was the natural next target, but attempts for the next 250 years failed.

Finally, in 1799, Paolo Ruffino published a flawed but basically sound proof that the solution in the general quintic case is impossible. Niels Henrik Abel provided a rigorous proof in 1824, and the result is known as the Abel-Ruffino theorem.

2. When the feedforward filter length n is even, the transfer function doesn't have a center term, and the derivation of the cosine form needs to be modified a bit. Let's do the case when $n = 4$. The transfer function is

$$\mathcal{H}(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$

Factor out a power of z corresponding to the average of the first and last exponents:

$$\mathcal{H}(z) = z^{-3/2}\left[a_0 z^{3/2} + a_1 z^{1/2} + a_2 z^{-1/2} + a_3 z^{-3/2}\right]$$

Set $z = e^{j\omega}$ to get the frequency response,

$$H(\omega) = e^{-3j\omega/2}\left[a_0 e^{3j\omega/2} + a_1 e^{j\omega/2} + a_2 e^{-j\omega/2} + a_3 e^{-3j\omega/2}\right]$$

Finally, assume that the coefficients are symmetric, so that $a_0 = a_3$ and $a_1 = a_2$:

$$H(\omega) = e^{-3j\omega/2}\left[2a_0 \cos(3\omega/2) + 2a_1 \cos(\omega/2)\right]$$

The general form for even n, corresponding to Eq. 2.9 for odd n, is easy to write from this simple example as

$$\begin{aligned}\hat{H}(\omega) &= e^{((2m-1)/2)j\omega} H(\omega) \\ &= c_1 \cos(\omega/2) + c_2 \cos(3\omega/2) + \cdots + c_m \cos((2m-1)\omega/2)\end{aligned}$$

where now $m = n/2$, the number of free coefficients.

Feedforward filters with even-symmetry coefficients are often classified as *Type I* when n is odd, and *Type II* when n is even. (See T. Saramäki's Chapter 4 of S. K. Mitra and J. F. Kaiser (eds.), *Handbook for Digital Signal Processing*, John Wiley, New York, N.Y., 1993.)

3. When the coefficients of a feedforward filter are *odd* about their center, instead of even, the terms in the frequency response pair as

$$a_q e^{jr\omega} - a_q e^{-jr\omega} = 2a_q j \sin(r\omega)$$

producing a sine series instead of a cosine series. These filters are called *Type III* and *Type IV*, according to whether n is odd or even, respectively.

4. Except for the DC component (an average value of 1/2), the desired frequency response is odd about the half-band point, $\omega = \pi/2$ radians per sample, or 0.25 in fractions of the sampling rate (see Fig. 5.1). For the odd-length, even-symmetry filters considered in this chapter, the frequency response is of the form given in Eq. 2.9:

$$\hat{H} = c_0 + c_1 \cos\omega + c_2 \cos(2\omega) + \cdots + c_m \cos(m\omega)$$

The cosines in this series with arguments $2\omega, 4\omega, \ldots$ are even about the half-band point, and so must be absent in the optimal filter design. Thus, $c_2 = c_4 = \cdots = 0$. In the METEOR design for the example in Fig. 5.1, which used double precision, the corresponding coefficients turned out to be less than about 10^{-15} in magnitude.

5. This argument is actually very similar to the one in the text using the frequency response in Fig. 5.3, and is equally heuristic. To make it rigorous amounts to developing part of linear programming theory.

We want to find a feasible point whose minimum distance to a constraint is as great as possible. Start at an arbitrary point within the feasibility region shown in Fig. 4.1. Find out which constraint is closest, and move away from it, until another constraint becomes equally close. Then try to find a direction that will move us farther from both those constraints, and that will increase the distance from both constraints equally. If such a direction is found, move away from the first two constraints, until we become equally close to a third constraint. Continue this process until we are equally close to a maximum number of closest constraints. The constraints that we are closest to correspond to the equal ripples in the frequency response. The more coefficients in the filter, and hence the higher the dimensionality of the feasible region in Fig. 4.1, the greater the number of such equal ripples we can find.

6. The measure $20\log_{10}[(1+\delta)/(1-\delta)]$ is the ratio of the maximum to the minimum value of magnitude response in the passband, in dB. In other words, it is the "variation" in the passband, expressed in dB — which is a natural way to think about deviation from ideal. When δ is small, it's small and positive. In fact, for small δ, it is proportional to δ. To see this, expand $(1+\delta)/(1-\delta)$ in a power series:

$$\frac{1+\delta}{1-\delta} = 1 + 2\delta + \text{higher-order terms}$$

and note that

$$20\log_{10}\frac{1+\delta}{1-\delta} \approx 20\log_{10}(1+2\delta) = 20\frac{\log_e(1+2\delta)}{\log_e(10)} \approx 17.37\delta$$

for small δ, where we used the fact that $\log_e(1+x) \approx x$. On the other hand, when δ is small, the measure $20\log_{10}\delta$ is negative and large in magnitude.

7. There are 18 ripples in Fig. 5.1, counting the bandedges, two more than $m = (n+1)/2 = 16$.

8. Filter response curves like the one in Fig. 5.2 are typically plotted by evaluating the response on a grid of equally spaced points, and those grid points will miss the zeros in the stopband by small amounts. But the logarithms of the smallest numbers near each zero, and hence their decibel measures, will in general vary greatly. For example, we may get the value 10^{-4} nearest one zero, and 10^{-5} nearest another. These will show up on the plot as -80 dB and -100 dB, respectively — far apart in dB, even though both numbers are close to zero.

9. Consider only odd n in this solution; analogous statements can be made for the even n case. The critical property can be thought of as *monotonicity*: if $F(n)$ is "infeasible," then $F(k)$ is also "infeasible" for all $k < n$. Thus, if n^* is the smallest n for which $F(n)$ is "feasible," then it is "feasible" for all larger n, as well as "infeasible" for all smaller n. This ensures that if we find that $F(n)$ is "infeasible," we can eliminate all smaller n; and if we find that $F(n)$ is "feasible" we can eliminate all larger n.

The binary search algorithm works by keeping track of two values of n, say *left* and *right*, for which $F(\textit{left})$ is "infeasible," and $F(\textit{right})$ is "feasible." The midpoint is tested, repeatedly, until *left* and *right* are consecutive (consecutive odd numbers in this case).

10. The following is a fragment of C code that uses a circular array to do feedforward filtering. The pointer variable `now` is wrapped around after it's incremented when it exceeds `L-1`, as explained in Section 9. The only additional work to allow for the circularity of the buffer is the calculation of `index`, which retrieves the past values of the stored input samples. It's wrapped around when it falls below zero.

```
for(i=0; i<L; i++)    /* initialize array */
 array[i] = 0.;
```

```
now = L-1;             /* initialize pointer */
for(t=0; t<M; t++){    /* time loop */
 new_sample = x[t];    /* new point arrives */
 now++;
 if(now > L-1) now = now-L;  /* keep in range */
 array[now] = new_sample;
 y[t] = 0;
 for(i=0; i<L; i++){
  index = now-i;
  if(index<0) index += L;
  y[t] += a[i]*array[index]; }}
```

The circular array is initialized with zero values, reflecting an assumption that input values `x[t]` are zero for $-L+1 \leq t < 0$. When these input values are unknown, the first $L-1$ values of the output are not really meaningful. When the values are actually known, they can be used instead of zero to initialize the array.

11. Suppose the L coefficients are symmetric about their center, and that L is even, so that

$$a_i = a_{L-1-i}, \quad \text{for } i = 0, 1, \ldots, L/2-1$$

(When L is odd, there is a center coefficient, and it changes the indexing slightly.) The $L/2$ pairs of terms in the filtering equation of the form

$$a_i x_{t-i} + a_{L-1-i} x_{t-(L-1-i)}$$

can therefore be combined as

$$a_i \left(x_{t-i} + x_{t-(L-1-i)} \right)$$

thus saving $L/2$ multiplications per sample. (When L is odd, this saves $(L-1)/2$ multiplications per sample, which is why I say "almost halve" in the problem statement.)

The following is the corresponding piece of C code. The only difference between this and the code in the general, asymmetric, case is that past input values are extracted from the circular array two at a time, from the locations `index1` and `index2` given above. The loop accumulating the output value uses `a[i]` for $i = 0, 1, \ldots, L/2$, and the values of `a[i]` for $i = L/2+1, L/2+2, \ldots, L-1$ are known by symmetry.

```
for(i=0; i<L; i++)     /* initialize array */
 array[i] = 0.;
now = L-1;             /* initialize pointer */
for(t=0; t<M; t++){    /* time loop */
 new_sample = x[t];    /* new point arrives */
 now++;
 if(now > L-1) now = now-L;  /* keep in range */
 array[now] = new_sample;
```

```
y[t] = 0;
for(i=0; i<L/2; i++){ /* only the first L/2 a's are used */
 index1 = now-i;      /* find indexes for the two inputs */
 if(index1<0) index1 += L;
 index2 = now-(L-1-i);
 if(index2<0) index2 += L;
 y[t] += a[i]*(array[index1]+array[index2]); }}
```

Chapter 13

Designing Feedback Filters

1. If all the coefficients but one, say c, are kept constant, the frequency response of a feedforward filter is of the form

$$cf(\omega) + g(\omega)$$

and the problem asks for the behavior of

$$M(c) = \max_{\omega} |cf(\omega) + g(\omega)|$$

as c is varied. When $c = 0$, the value of M is just

$$M(0) = \max_{\omega} g(\omega) = g_{max}$$

Letting $f_{max} = \max_{\omega} f(\omega)$, we can also say that the value of M is bounded:

$$M(c) \leq |c| f_{max} + g_{max}$$

Thus, the curve $M(\omega)$ is upper bounded by the vee-shaped intersection of two straight lines that intersect at the point $c = 0$ and $M = g_{max}$. Intuitively, as c is moved in either direction from zero, the maximizing value of ω shifts gradually from being dominated by $g(\omega)$ to being dominated by $f(\omega)$. We therefore expect the curve to be reasonably well behaved, as well as bounded.

Figure P1(a) shows an example in which the coefficient c_{13} is varied in the feedforward half-band filter in Section 5 of Chapter 12. This is an equiripple design, which may explain the fact that the minimum frequency response occurs when $c_{13} = 0.03156$, the original value in the design. It seems that if the maximum value can be reduced by changing c_{13}, then the design algorithm would have reduced the ripple by doing so. The segments of the curve are very close to being straight lines, but are not exactly straight.

When we plot the error between the frequency response and some fixed, prespecified response, the same arguments apply, except the prespecified response must be included in the definition of the curve $M(c)$, where it gets absorbed into the function $g(\omega)$.

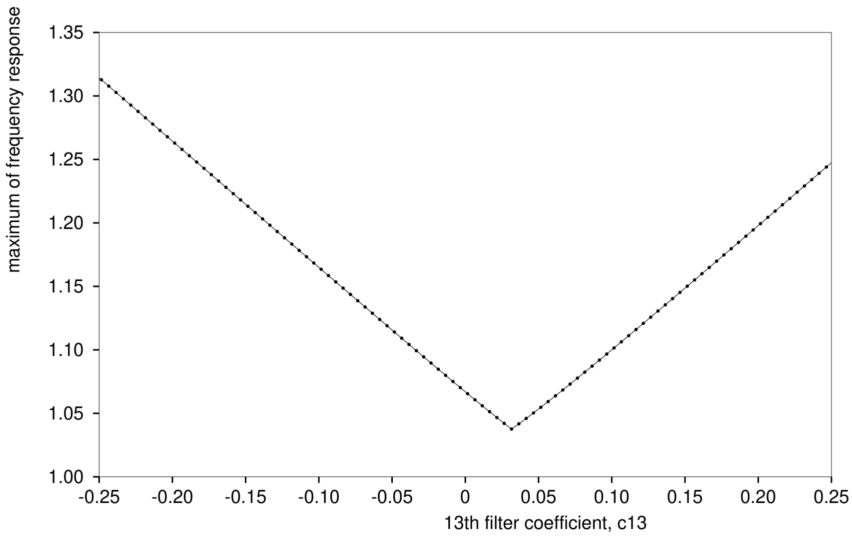

Fig. P1(a) The maximum value of the frequency response vs. the coefficient c_{13}, for the feedforward half-band filter example in Section 5 of Chapter 12.

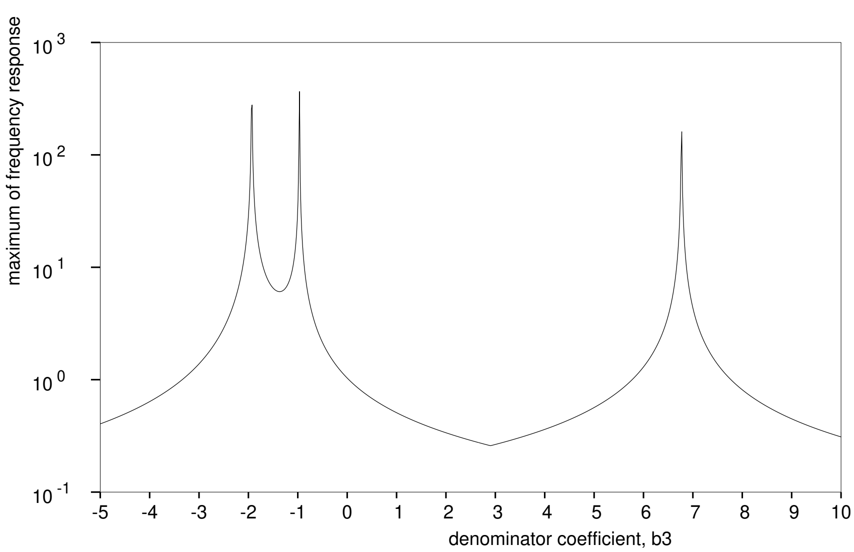

Fig. P1(b) The maximum value of the frequency response vs. the third coefficient, b_3, in the denominator of the feedback transfer function example in Section 8 of Chapter 9 — the step response of a reson.

By way of contrast, Fig. P1(b) shows the result of varying the third denominator coefficient in the z-transform in Section 8 of Chapter 9. This is the z-transform of the output of a reson when a step input is applied, but can also be interpreted as a filter transfer function. The curve of maximum frequency response vs. b_3 is much more wildly behaved than the one in the feedforward case, and actually has three points at which it becomes infinite. These points correspond to poles crossing the unit circle. These two figures illustrate why the design problem for feedforward filters is inherently easier than the design problem for feedback filters.

2. Multiplying the numerator and denominator of the transfer function in Eq. 2.1 by $z+1$ shows that

$$\mathcal{H}(z) = \frac{1}{2} + \frac{1}{2}z^{-1}$$

which is the simplest kind of feedforward filter considered in Chapter 4, as in Eq. 5.1 of that chapter.

3. Because the transfer function is symmetric in z and z^{-1}, every zero z_0 inside the unit circle has a corresponding zero z_0^{-1} outside the unit circle. Choosing the latter instead of the former is therefore equivalent to multiplying the transfer function by

$$\frac{z - z_0^{-1}}{z - z_0}$$

which we know is allpass, from Eq. 6.5 in Chapter 6. Choosing some of the zeros outside the unit circle, instead of inside, therefore has no effect on the magnitude of the resulting filter transfer function. But it does introduce extra negative phase, and therefore additional delay, which is why the design with all the zeros inside the unit circle is called *minimum phase*.

4. Setting $\mathcal{B}(1) = 1$ in Eq. 3.10 yields the following condition for the gain constant of the Butterworth filter:

$$A = 2^{-N}(1 + p_0^2) \cdots (1 + p_{N/2-1}^2)$$

where the pole parameters p_i are determined by Eqs. 3.5 and 3.8.

Similarly, setting $\mathcal{B}(1) = 1$ in Eq. 4.7 yields

$$A = 2^{-N}(1 - z_0) \cdots (1 - z_{N-1})$$

for the case with a general cutoff frequency. This also yields a real value for A, because the poles z_i occur in complex-conjugate pairs.

5. We have just the parameter N to adjust, so in general we can't hope to achieve two specified values simultaneously. Furthermore, N is an integer, so we can't even hope to meet one of the specifications precisely. What we *can* do is require that the magnitude response of the filter be at least as good as that implied by the specified points. Thus, if the specified values are A_1 in the passband, at $\omega_1 < \pi/2$, and A_2 in the stopband, at $\omega_2 > \pi/2$, we require that

$$|B(\omega_1)|^2 = \frac{1}{1 + \tan^{2N}(\omega_1/2)} \geq A_1^2$$

and

$$|B(\omega_2)|^2 = \frac{1}{1 + \tan^{2N}(\omega_2/2)} \leq A_2^2$$

Rearranging, we get the conditions

$$\tan^{2N}(\omega_1/2) \leq \frac{1}{A_1^2} - 1$$

and

$$\tan^{2N}(\omega_2/2) \geq \frac{1}{A_2^2} - 1$$

Because $\omega_1 < \pi/2$ — that is, ω_1 is in the passband — it follows that $\tan(\omega_1/2) < 1$. Similarly, $\tan(\omega_2/2) > 1$ because ω_2 is in the stopband. It is therefore possible to satisfy both of these conditions by choosing N sufficiently large. The smallest such N is the logical design choice. The conditions can be made explicit by taking logs, giving

$$N > \frac{1}{2} \cdot \frac{\log(1/A_1^2 - 1)}{\log(\tan(\omega_1/2))}$$

$$N > \frac{1}{2} \cdot \frac{\log(1/A_2^2 - 1)}{\log(\tan(\omega_2/2))}$$

Note that the first inequality gets turned around because we divide by a negative quantity. And since we're taking the ratios of logs, it doesn't matter what their base is.

6. The condition in Eq. 3.4 for the poles in the F-plane can be written

$$F^{2N} = (-1)^{N+1} = e^{j[(N+1)\pi + k2\pi]} \quad \text{for all integer } k$$

Taking the $2N$th root of both sides shows that the poles are therefore of unit magnitude at angles

$$\theta_k = \frac{\pi}{2} + \left(\frac{2k+1}{4N}\right) 2\pi$$

for any $2N$ consecutive values of the integer k, checking Eq. 3.5 and Fig. 3.1.

7. Equation 3.5 works for N odd as well as N even; the $2N$ pole angles in the F-plane are

$$\theta_k = \frac{\pi}{2} + \left(\frac{2k+1}{4N}\right) 2\pi, \quad \text{for } k = 0, 1, \cdots, 2N-1$$

When N is odd, there is a real pole at $F = -1$. It corresponds to the index $k = (N-1)/2$, when $\theta_{(N-1)/2} = \pi$.

The pole at $F = -1$ becomes a pole at $z = 0$, by Eq. 3.7. The transfer function when N is odd therefore has this left-over real pole when the complex pairs are combined, and Eq. 3.10 becomes

$$\mathcal{B}(z) = \frac{A(z+1)^N}{(z^2 + p_0^2)\cdots(z^2 + p_{(N-3)/2}^2)z}$$

To check, this accounts for $(N-1)/2$ pairs of complex poles, plus the real pole, a total of N poles. The real pole is indexed by $k = (N-1)/2$.

As mentioned above, the real pole ends up at $z = 0$ when the filter is half-band. When the cutoff frequency is ω_c radians per sample, Eq. 4.5 tells us that the real pole ends up at

$$z_{(N-1)/2} = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}$$

In the example illustrated in Fig. 4.1, $\omega_c = 2\pi/10$, and this formula gives the location of the real pole at $z = 0.5095$. That's where it would be in that figure if N were odd.

In the case when N is odd, the cascade implementation in Fig. 7.1 needs a first-order section to account for this lone real pole.

8. To convert a lowpass Butterworth filter to highpass, we set $z = -z$. This is equivalent to changing the sign of the coefficients c_i in Eq. 7.6, but leaving the coefficients d_i unaltered. (Of course, the zero locations should also be changed from $z = -1$ to $z = +1$.)

9. Assume for this problem that the numerator and denominator of the transfer function are both of degree $N = n = m - 1$ in z^{-1}, and that N is even. Assume further that the filter is otherwise completely arbitrary.

Implementing the filter in cascade form, as in Fig. 7.1, requires $N/2$ sections, and each section requires 4 multiplications and 4 additions. There's one more multiplication to account for the scale factor A, so the total is $2N + 1$ multiplications and $2N$ additions.

The direct form in Eq. 8.2 requires one multiplication for each numerator and denominator coefficient and one fewer addition, so the total is $2N + 1$ multiplications and $2N$ additions — exactly the same as the cascade form.

10. The magnitude frequency response of the analog Butterworth filter is, from Eq. 6.1,

$$\frac{1}{\sqrt{1 + \Omega^{2N}}}$$

Use the power series

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \text{higher-order terms}$$

to expand this in a power series around the point $\Omega = 0$:

$$\frac{1}{\sqrt{1 + \Omega^{2N}}} = 1 - \frac{\Omega^{2N}}{2} + \text{higher-order terms}$$

The first $2N-1$ derivatives of this vanish at $\Omega = 0$, and thus the magnitude frequency response is very flat. In fact, it is as flat as possible, given that the transfer function has N poles. This follows because this process of expanding the response always produces a power series of the form

$$1 + c_1\Omega^k + \text{higher-order terms}$$

where c_1 is a constant and $k \leq 2N$.

The same result holds in the digital filter case, because

$$\Omega = c_2\omega + \text{higher-order terms}$$

using Eq. 6.8 and the fact that $\tan x = x + \text{higher-order terms}$.

11. Different pairings of zeros and poles and different orderings of the sections in a cascade implementation can affect the roundoff error. It can also affect the maximum magnitude of signals in the sections, and this can affect the wordlength required in fixed-point implementations.

12. The Laplace transform of the unit step signal is, using Eq. 6.3,

$$\mathcal{X}(s) = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{\infty} = \frac{1}{s}$$

where we assume $\Re\{s\} > 0$, so that the value of the integrated function vanishes as $t \to \infty$. (It's necessary to establish a region of convergence for the Laplace transform, and for the z-transform as well, but I simplified the discussion in the book by ignoring this issue.) The bilinear transformation yields the z-transform

$$\mathcal{X}\left(\frac{z-1}{z+1}\right) = \frac{z+1}{z-1} = \frac{1+z^{-1}}{1-z^{-1}}$$

On the other hand, the sampled unit step signal has the z-transform

$$\frac{1}{1-z^{-1}}$$

from Eq. 5.1 in Chapter 9.

The Laplace transforms of an analog signal and its sampled version are related by the aliasing formula, Eq. 3.2 of Chapter 11.

13. To avoid dealing with special cases, assume in this problem that a straight line is a special case of a circle with infinite radius. Then there is a well-known mathematical result that includes the one posed in this problem: A bilinear transformation maps any circle to a circle. The bilinear transformation in Eq. 4.5 therefore maps the poles of the Butterworth filter, which lie on the unit circle in the F-plane, to a circle in the z-plane.

First, observe that a bilinear transformation can be decomposed into successive applications of the operations of

- multiplication by a constant scale factor;
- shift by a constant z_0 in the complex plane;
- inversion — replacing the complex variable z by $1/z$.

For example,

$$\frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$

so this bilinear transformation can be decomposed into a shift by 1, multiplication by -2, inversion, and shift by 1.

The first two operations clearly map any circle to a circle. The crux of the problem is to show that inversion does so also. Start with the general circle with center at the complex number z_0 and radius R:

$$|z - z_0|^2 = R^2$$

Replacing z by $w = 1/z$ yields

$$|1 - wz_0|^2 = R^2|w|^2$$

Letting

$$w = w_r + jw_i \ \text{ and } \ z_0 = z_r + jz_i$$

this becomes

$$|1 - w_r z_r + w_i z_i - j(w_r z_i + w_i z_r)|^2 = R^2(w_r^2 + w_i^2)$$

Expanding the magnitude square as the sum of real and imaginary parts squared, and collecting terms, results in

$$w_r^2 - 2w_r \frac{z_r}{|z_0|^2 - R^2} + w_i^2 - 2w_i \frac{z_i}{|z_0|^2 - R^2} = \frac{1}{R^2 - |z_0|^2}$$

and completing the square finally produces

$$\left(w_r - \frac{z_r}{|z_0|^2 - R^2}\right)^2 + \left(w_i - \frac{z_i}{|z_0|^2 - R^2}\right)^2 = \left(\frac{R}{|z_0|^2 - R^2}\right)^2$$

which is the equation of a circle.

To get this result we divided by $R^2 - |z_0|^2$. When this is zero — that is, when the circle passes through the origin, we can stop earlier in this derivation with the equation of a straight line through the origin:

$$2w_r z_r + 2w_i z_i = 1$$

14. Go back and look at the pole-zero plot for the elliptic filter used as an example in Section 8 of Chapter 5, Fig. 8.1 of that chapter. Observe that there is a pole for every peak in the passband and a zero for every notch in the stopbands. Taking into account that the frequency response is plotted for only half the baseband, corresponding to half the unit circle, it's clear that we should multiply these counts by two to get the total number of zeros and poles. Note that this works only when the poles and zeros are sufficiently close to the unit circle.

The frequency response of the elliptic filter in Fig. 6.2 has five peaks in the passband (the last one is scrunched up), and five notches in the passband, checking the fact that this is a 10-pole, 10-zero design.

15. The transformation is $s \to 1/s$, and you can see this in a number of ways.

The voltage across an inductor is Ldi/dt, where i is the current. In the Laplace transform domain, this corresponds to an impedance of sL. Similarly, the impedance of a capacitor is $1/(sC)$. Interchanging inductors and capacitors is therefore equivalent to replacing s by $1/s$, ignoring a scale factor.

If you're using the bilinear transformation in Eq. 6.7 to design a filter,

$$s = \frac{1}{\tan(\omega_c/2)} \cdot \frac{z-1}{z+1}$$

replacing z by $-z$ is mathematically equivalent to the transformation

$$s \to \frac{1}{\tan^2(\omega_c/2)} \cdot \frac{1}{s}$$

which, again, is inversion with a scale factor.

Intuitively, any transformation that interchanges the zero and infinite frequencies, and reverses the order of the frequencies in between, will act as a lowpass-to-highpass transformation. Inversion is the most natural, for the reasons given above.

Chapter 14

Audio and Musical Applications

1. It's just as easy to work this problem for the general case of an n-coefficient feedforward filter, and a signal padded with $L-1$ zeros between the original sample values.

Suppose the quarter-band feedforward filter has the transfer function

$$\sum_{k=0}^{n-1} a_k z^{-k}$$

At $t = 0$, the filtering operation uses only the terms involving $a_0, a_L, \ldots, a_{mL}$, where mL denotes the last nonzero coefficient with index $0 \bmod L$. At $t = 1$, the filter uses only $a_1, a_{L+1}, \ldots, a_{mL+1}$, where now $mL + 1$ denotes the last nonzero coefficient with index $1 \bmod L$. In general, if $t = i \bmod L$, the filter uses only coefficients $a_i, a_{L+i}, \ldots, a_{mL+i}$, where $mL + i$ denotes the last nonzero coefficient with index $i \bmod L$.

For example, consider the case $L = 3$ and $n = 8$. When $t = 0, 3, 6, \ldots$, the filter uses a_0, a_3, and a_6; when $t = 1, 4, 7, \ldots$, the filter uses a_1, a_4, and a_7; and when $t = 2, 5, 8, \ldots$, the filter uses a_2 and a_5. (There isn't any a_8.)

The filtering operation can be viewed as a *time-varying* one — the filter coefficients change cyclicly, depending on the value of $t \bmod L$. The filtering operations are

$$y_t = \sum_{k=i, L+i, \ldots}^{mL+i} a_k x_{t-k} \quad \text{for } t = i \bmod L$$

Thus, the filter does only $1/L$th the work that it would with general input.

2. Here's the input file to METEOR that produced Fig 1.3:

```
          96          96     smallest and largest length
c
neither left nor right: maximize distance from constraints
         500     number of grid points
```

```
limit spec
+     upper limit
geometric interpolation
not hugged spec
 0.00000E+00 0.11338E+00     band edges
 0.70795E+00 1.00000E+00     bounds
limit spec
-     lower limit
geometric interpolation
not hugged spec
 0.00000E+00 0.11338E+00     band edges
 0.50119E+00 0.70795E+00     bounds
limit spec
+     upper limit
geometric interpolation
not hugged spec
 0.13605E+00 5.00000E-01     band edges
 0.31623E-02 0.31623E-01     bounds
limit spec
-     lower limit
geometric interpolation
not hugged spec
 0.13605E+00 5.00000E-01     band edges
-0.31623E-02 -0.31623E-01     bounds
end
```

The input-preparation program FORMAT generates the following condensed version, which is a bit easier to read:

```
    type  sense  edge1    edge2   bound1   bound2  hugged?  interp
  1 limit   +  0.00000  0.11338  0.70795  1.00000     n        g
  2 limit   -  0.00000  0.11338  0.50119  0.70795     n        g
  3 limit   +  0.13605  0.50000  0.00316  0.03162     n        g
  4 limit   -  0.13605  0.50000 -0.00316 -0.03162     n        g
    OPTIMIZING, fixed length=  96
    COSINE model (symmetric coeffs.)
    501 grid points
```

There are four constraints of the *limit* type: an upper and lower limit in the passband and in the stopband. The bandedges *edge1* and *edge2* are the left and right edges of each band, in fractions of the sampling rate. The limiting values of the magnitude response are interpolated geometrically from the left to the right edge, which means linearly on a dB scale. Note that the specifications at the left and right edges of the stopband are −30 dB

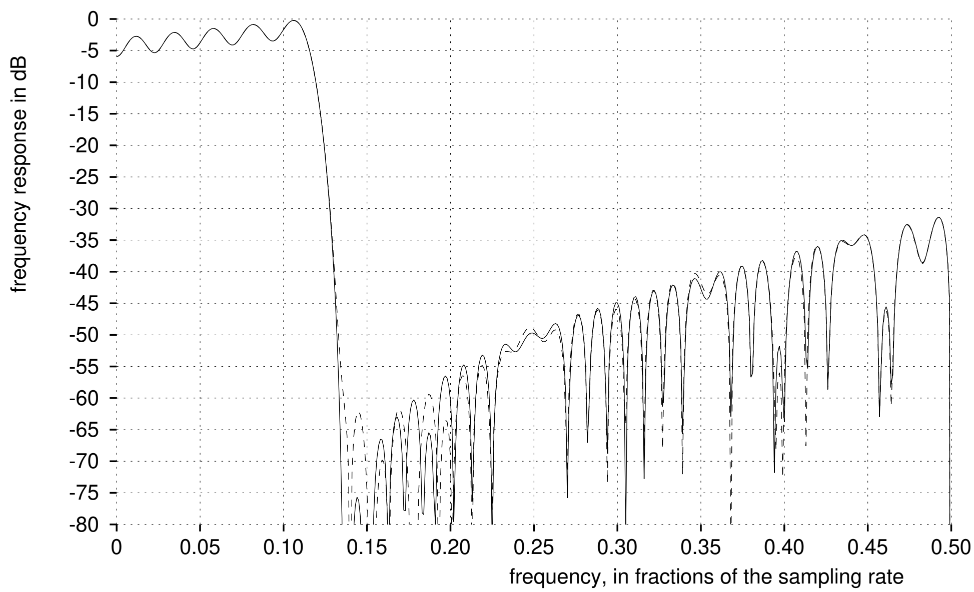

Fig. P2 Magnitude response of a quarter-band feedforward digital filter used to prepare for oversampled d-to-a conversion in a CD player. The solid line is the response of the infinite-precision design, and the dashed line is the response of the 12-bit version.

and -50 dB, respectively. Finally, *not hugged* means the design is moved as far as possible from a specification.

I'll interpret rounding off the coefficients to 12 bits as follows:

- multiply each coefficient by 2^{12};
- cast the result to type `int` (in C);
- divide by 2^{12}.

Think of this as shifting the decimal point of the binary representation 12 places to the right, chopping off the fractional part, and moving the decimal point back. The coefficients are all less than one in magnitude, so this just selects the first 12 binary bits.

Figure P2 shows the resulting magnitude response as a dashed line, along with the original (essentially) infinite-precision design as a solid line. As you might expect, the effects of using only 12 bits in the coefficients are most apparent where the magnitude response is smallest. But the result still meets the specifications.

3. Figure P3 illustrates the effect of a zero-order hold after oversampled d-to-a conversion. This would be Fig. 1.1(e). It is obtained by multiplying the signal spectrum after digital filtering at the new sampling rate by the frequency response of a zero-order hold, from Fig.

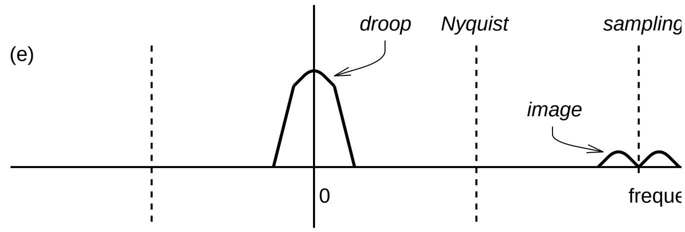

Fig. P3 The effect of a zero-order hold after oversampled d-to-a conversion, a candidate Fig. 1.1(e).

4.5 in Chapter 11. Two effects are important. First, the signal spectrum droops in the baseband, because of the decreasing magnitude response of the zero-order hold. Second, the signal spectrum image at the sampling frequency is multiplied by the region of the zero-order hold magnitude response that has a zero, producing two lobes, as in the example in Fig. 5.1 of Chapter 11. Note that this spurious image signal is centered at four times the original sampling rate, beyond the range of hearing.

4. The analog signal after oversampled d-to-a conversion must be amplified by an analog amplifier. Too much energy in that signal can saturate the electronics in that amplifier, even if it's not in the audible range.

5. The specification in the stopband of the pre-d-to-a filter slopes up on the ground that it is more important to do a good job at the lower edge of the stopband than at the upper edge, because the lower edge of the stopband is closer to the audible region.

6. It's no accident that the lowpass comb for reverb and the plucked-string filter have the same structure. They both arise from the same physical situation: Sound circulates and decays in a frequency-dependent way, faster at low frequencies than at high frequencies. In reverb, the frequency-dependent decay is caused by absorption of sound in air and on reflection; in the plucked string, it's caused by radiation. Of course, the time scales of the two phenomena are different.

Prof. Perry Cook pointed out to me that the one-pole feedback filter has a lowpass frequency response that is in some sense a more natural model for the physics in both applications. Certainly, the feedback filter works fine in the plucked-string filter, provided you adjust the loop gain to keep the filter stable. He also pointed out, however, that the frequency-independent delay of one-half a period makes it easier to predict the pitch of the plucked-string filter when the feedforward filter is used.

7. The property of being allpass can be thought of as follows: A transfer function $\mathcal{G}(z)$ is *allpass* if it maps the unit circle in the z-plane to the unit circle in the $\mathcal{G}$-plane. This is equivalent to the magnitude response being unity. (In the more general case of constant magnitude response, just normalize to one.)

Using this definition, it's easy to see that an allpass function of an allpass function is also allpass. Composing two allpass functions maps the unit circle to the unit circle, and then again to the unit circle. Now z^m is an allpass function, because its magnitude on the unit circle is one. Therefore, if $\mathcal{G}(z)$ is allpass, so is $\mathcal{G}(z^m)$. This is equivalent to replacing unit delays by delays of m samples.

8. Amplitude modulation results in a strictly bandlimited spectrum, which makes it easy to pack AM signals close together in the medium-wave band. The spacing of 10 kHz, however, means that the audio bandwidth is at most 5 kHz, and is somewhat less in practice because of the need for guard bands. This small bandwidth accounts for the low quality of AM audio, but leaves room for a hundred or so channels in the medium-wave band, which is about 1 MHz wide.

As it turns out, FM is more resistant to noise than AM for radio, and for this reason is better suited to the high-quality demands of commercial music broadcasting. It also spreads the spectrum around the carrier much more than AM, especially for large modulation indexes, as discussed in Section 4. It is therefore used with much larger bandwidth than commercial AM. To make this possible, the FM commercial band is in the 100 MHz region, and is about 20 MHz wide, allowing about 100 channels, 200 kHz apart, 20 times the AM spacing.

9. To be more realistic than Section 4, we'll assume that the carrier is real-valued, so it contains frequencies $\pm\omega_0$. The FM signal therefore contains all frequencies of the form

$$\pm\omega_0 \pm k\omega_m \quad \text{for all integer } k$$

where ω_m is the frequency of the modulating sinusoid. We're assuming that $\omega_0/\omega_m = N_1/N_2$, with N_1 and N_2 relatively prime, and this can be rewritten

$$\frac{\omega_m}{N_2}\left(\pm N_1 \pm kN_2\right) \quad \text{for all integer } k$$

All these frequencies are integer multiples of ω_m/N_2, which we'll call the fundamental frequency. We now consider the cases $N_2 = 1, 2, 3$ in turn.

When $N_2 = 1$, the fundamental frequency is the modulation frequency, ω_m, and all harmonics of the fundamental are present.

When $N_2 = 2$, the fundamental frequency is $\omega_m/2$. N_1 must be odd, since N_1 and N_2 are assumed to be relatively prime. Therefore, the FM signal contains all frequencies of the form

$$\frac{\omega_m}{2}\left(\pm N_1 \pm 2k\right)$$

Thus, all the odd harmonics of the fundamental are present, and all the even harmonics are missing.

When $N_2 = 3$, the fundamental frequency is $\omega_m/3$. In this case, $N_1 \neq 0 \bmod 3$ — again because N_1 and N_2 are relatively prime. This means that N_1 is of the form $3q + x$, where q is an integer, and $x = 1$ or 2. It follows that $-N_1 = -x \bmod 3$, and is therefore 1 if

$N_1 \bmod 3 = 2$, and 2 if $N_1 \bmod 3 = 1$. The FM signal now contains all frequencies of the form

$$\frac{\omega_m}{3}(\pm N_1 \pm 3k)$$

Therefore the FM signal contains all multiples of the fundamental except the ones that are 0 mod 3.

The results in this solution are from Chowning's paper, cited in the Notes. Note that when N_2 is even, the FM spectrum contains only odd-numbered harmonics of the fundamental, as Chowning points out, but not necessarily *all* the odd-numbered harmonics.

When the ratio N_1/N_2 is an irrational number, the spectral components present in the FM signal are not all integer multiples of a common fundamental frequency. Consider the example when $\omega_0/\omega_m = \sqrt{2}/7$. Then the frequencies present are

$$\pm\omega_0 \pm k\omega_m = \omega_m \left(\pm\frac{\sqrt{2}}{7} \pm k \right) \quad \text{for all integer } k$$

10. I'll use the notation $\mathcal{LP}[u]$ to denote the result of passing the signal u through the lowpass filter. Denote the (real-valued) input and output signals in Fig. 3.3 by x and y, respectively, and denote the output of the lowpass filter in the top branch by w. Then the output of the lowpass filter in the bottom branch is w^*, and

$$\begin{aligned} y &= 2\Re\left\{e^{j\omega_0 t} w\right\} \\ &= 2\Re\left\{e^{j\omega_0 t}\mathcal{LP}\left[x\cos(\omega_0 t) - jx\sin(\omega_0 t)\right]\right\} \\ &= 2\cos(\omega_0 t)\mathcal{LP}\left[x\cos(\omega_0 t)\right] + 2\sin(\omega_0 t)\mathcal{LP}\left[x\sin(\omega_0 t)\right] \end{aligned}$$

which is just twice the output of the figure shown in this problem.

11. A typical picture shows objects that take up more area than a picture element (a *pixel*), and so the value at a particular pixel is highly correlated with the values at neighboring pixels. This means the most straightforward pixel-by-pixel representation of a picture is redundant, and implies that we can compress the image.

A consequence of this redundancy is that the value directly beside or below a given pixel is highly correlated with it, and this can be exploited for run-length and line-to-line compression when we are using a scanline representation. Similarly, in television and movies, there is a high correlation between the value of a pixel at time t and its value at time $t + 1$, because objects tend to remain stationary or move slowly. This correlation can be exploited for frame-to-frame compression.

12. From audio bit-rate considerations, we have an hour of sound at 1.41 Mbit/sec, as estimated in Section 1, which yields about 5 Gbit. I state that the actual bit rate onto and off a CD is about three times this, so there should be a total of about 15 Gbit on a CD.