

# A (DIS)CONTINUOUS FINITE ELEMENT MODEL FOR GENERALIZED 2D VORTICITY DYNAMICS

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**Abstract.** *A mixed continuous and discontinuous Galerkin finite element discretization has been constructed for a generalized vorticity-streamfunction formulation in two spatial dimensions. This formulation consists of a hyperbolic (potential) vorticity equation and a linear elliptic equation for a (transport) streamfunction. The advantages of this finite-element model are the allowance of complex shaped domains and (fixed) mesh refinement, and a (spatial) discretization preserving energy and vorticity, while the discrete enstrophy is  $L^2$ -stable. Verification examples support our error estimates.*

*The method is fully described in Bernsen et al. (2005, 2006). To illustrate our method, we therefore focus here on finite-element simulations of curved critical layers in two-dimensional vortical flows using our (dis)continuous Galerkin finite element method.*

## 1 INTRODUCTION

In a recent journal publication (Bernsen et al., 2006; see also Bernsen et al., 2005), we present and verify a mixed continuous and discontinuous Galerkin finite element discretization for a generalized vorticity-streamfunction formulation in two spatial dimensions. This formulation consists of a hyperbolic (potential) vorticity equation and a linear elliptic equation for a (transport) streamfunction. The generalized formulation includes three systems in geophysical fluid dynamics: the incompressible Euler equations, the barotropic quasi-geostrophic equations and the rigid-lid equations (Bernsen et al., 2006). Multiple connected domains are considered with impenetrable and curved boundaries such that the circulation at each connected piece of boundary must be introduced. The generalized system is shown to globally conserve energy and weighted smooth functions of the vorticity. In particular, the weighted square vorticity or enstrophy is conserved.

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<sup>1</sup>The simulations presented here were developed in collaboration with Jacques Vanneste, School of Mathematics, University of Edinburgh, Scotland.

By construction, the spatial finite-element discretization is shown to conserve energy and is  $L^2$ -stable in the enstrophy norm. The method is verified by numerical experiments which support our error estimates in Bernsen et al. (2006), and illustrated here by a simulation of time-dependent two-dimensional vortical flow with separatrices in a complex shaped domain. Particular attention has been paid to match the continuous and discontinuous discretization. Hence, the implementation with a third-order Runge-Kutta time discretization conserves energy and is  $L^2$ -stable in the enstrophy norm for increasing time resolution in multiple connected curved domains. More advanced error estimates have been made and are found in Van der Vegt et al. (2006).

Instead of presenting the finite element method already fully available and published recently in Bernsen et al. (2005, 2006) in this paper, we illustrate our method here with simulations of curved critical layers in two-dimensional vortical flows<sup>2</sup>. In unstable two-dimensional shear flows, critical layers emerge where the velocity of the basic state matches the phase speed of the disturbance. Considering stationary perturbations with zero phase speed, localized nonlinear flow arises in a critical layer where the basic state velocity vanishes. Parallel critical layers in a channel configuration have been investigated both analytically and numerically (Brunet and Warn, 1990), and are paradigms of certain nonlinear instabilities in quasi-two-dimensional atmospheric and oceanic flows. Flows in nature are, however, nearly never parallel, so we investigate curved critical layers. We therefore identify the critical layer as the separatrix where particle trajectories in the basic state have infinite period, by analogy with the zero-velocity line in parallel shear flows. We will therefore present a preliminary simulation of the nonlinear flow in a curved critical layer. Stationary perturbations are introduced by imposing wavy corrugations on an upper wall originally coinciding with a streamline in the basic-state flow.

The outline of our paper is as follows: the equations of motion are presented in Section 2, the set-up of the simulation is discussed in Section 3, results are shown in Section 4, and we draw some conclusions in Section 5.

## 2 EQUATIONS OF MOTION

The generalized vorticity-streamfunction formulation in two dimensions consists of a hyperbolic equation for the (potential) vorticity,  $\xi = \xi(x, y, t)$ , and a linear elliptic equation for the streamfunction,  $\psi = \psi(x, y, t)$ , in a bounded domain  $\Omega$  as function of the horizontal coordinates  $x, y$  and time  $t$ . The system of equations is

$$\partial_t \xi / A + \nabla \cdot (\xi \vec{U}) = 0 \tag{1}$$

$$\vec{U} = \nabla^\perp \psi \tag{2}$$

$$\nabla \cdot (A \nabla \psi) - B \psi + D = \xi / A \tag{3}$$

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<sup>2</sup>In the corresponding oral presentation, we focus primarily on the numerical algorithm fully available in Bernsen et al. (2005, 2006).

with  $0 < A = A(x, y) < \infty$ ,  $B = B(x, y) \geq 0$  and  $D = D(x, y)$ . The gradient operator is defined by  $\nabla = [\partial_x, \partial_y]^T$  and the two-dimensional curl operator by  $\nabla^\perp = [-\partial_y, \partial_x]^T$ . The system (1) is completed with boundary conditions and initial conditions .

The particular domain  $\Omega$  chosen is periodic in the  $x$ -direction and bounded by walls below and above, see Fig. 1. It is therefore a multiple connected curved domain with impenetrable walls. The solid boundary with slip flow boundary conditions is denoted by  $\partial\Omega_D$ , on which

$$\vec{U} \cdot \hat{n} = 0 \quad (4)$$

holds with  $\hat{n} = [n_x, n_y]^T$  the outward unit vector normal to the boundary. The boundary  $\partial\Omega_D$  is partitioned into two separate simply connected subsets,  $\partial\Omega_{D_1}$  and  $\partial\Omega_{D_2}$ .

On each part  $\partial\Omega_{D_i}$  of the boundary with here two solid walls and hence  $i = 1, 2$ ,  $\psi$  is independent of  $x$  and  $y$  because  $\partial\psi/\partial\hat{\tau} = \nabla\psi \cdot \hat{\tau} = -\nabla^\perp\psi \cdot \hat{n} \stackrel{(1)}{=} -\vec{U} \cdot \hat{n} \stackrel{(4)}{=} 0$  with  $\hat{\tau} = [-n_y, n_x]^T$  the unit vector tangential to  $\partial\Omega$ . On these boundaries

$$\psi|_{\partial\Omega_{D_i}} = f_i(t) \quad (5)$$

is a function only depending on time. Consider the circulation  $\mathcal{C}_i$  around  $\partial\Omega_{D_i}$ , defined by

$$\mathcal{C}_i = \int_{\partial\Omega_{D_i}} \vec{u} \cdot \hat{\tau} \, d\Gamma = \int_{\partial\Omega_{D_i}} A \vec{U} \cdot \hat{\tau} \, d\Gamma \quad (6)$$

with  $d\Gamma$  a line element along  $\partial\Omega_D$ . A relevant boundary condition at  $\partial\Omega_{D_i}$  is

$$d\mathcal{C}_i/dt = 0, \quad (7)$$

whence the functions  $f_i(t)$  in (5) are only implicitly defined.

The generalized system (1) serves as model for the two-dimensional incompressible Euler equations, the barotropic quasi-geostrophic equations and the rigid-lid equations (Bernsen et al., 2006). Hereafter, we entirely focus attention on the two-dimensional incompressible Euler equations for which  $A = 1$  and  $B = D = 0$ .

### 3 SET-UP

Inviscid two-dimensional vortical dynamics, system (1) with  $A = 1$  and  $B = D = 0$ , is governed by:

$$\partial_t \xi + J(\psi, \xi) = 0 \quad \text{and} \quad \nabla^2 \psi = \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial y^2} \psi = \xi \quad (8)$$

with coordinates  $x$  and  $y$ , time  $t$ , and Jacobian  $J(A_1, A_2) = \partial_x A_1 \partial_y A_2 - \partial_x A_2 \partial_y A_1$ . We split the fields into a steady state and a perturbation

$$\xi = Q + \omega, \quad \text{and} \quad \psi = \Psi + \phi. \quad (9)$$

From (8), we find that steady flow satisfies  $J(\Psi, Q) = 0$ , whence,  $\Psi = \Psi(Q)$ . We consider the cosine vortex, see Fig. 1, with

$$\Psi(x, y) = \cos x + a \cos y. \tag{10}$$

The domain is periodic in the  $x$ -direction and bounded by walls below and above and the separatrix streamlines divide the open and closed streamlines. Thus, the vorticity  $Q = \nabla^2 \Psi = -\Psi$  and  $d\Psi/dQ = -1$ . A fluid parcel will take infinitely long to traverse along the separatrix, as does a fluid parcel at the critical line, where the velocity is zero, of the parallel shear flow.

The cosine vortex is chosen as an idealization of cat's eye structures in the atmosphere, which emerge in Fig. 2 from the horizontal distribution of nitric acid at approximately 21 km altitude, obtained from measurements taken by the CRISTA instrument on the Space Shuttle during 6 November 1994 (Figure provided courtesy of Dirk Offermann, Wuppertal University; cf. Offermann et al., 1999).

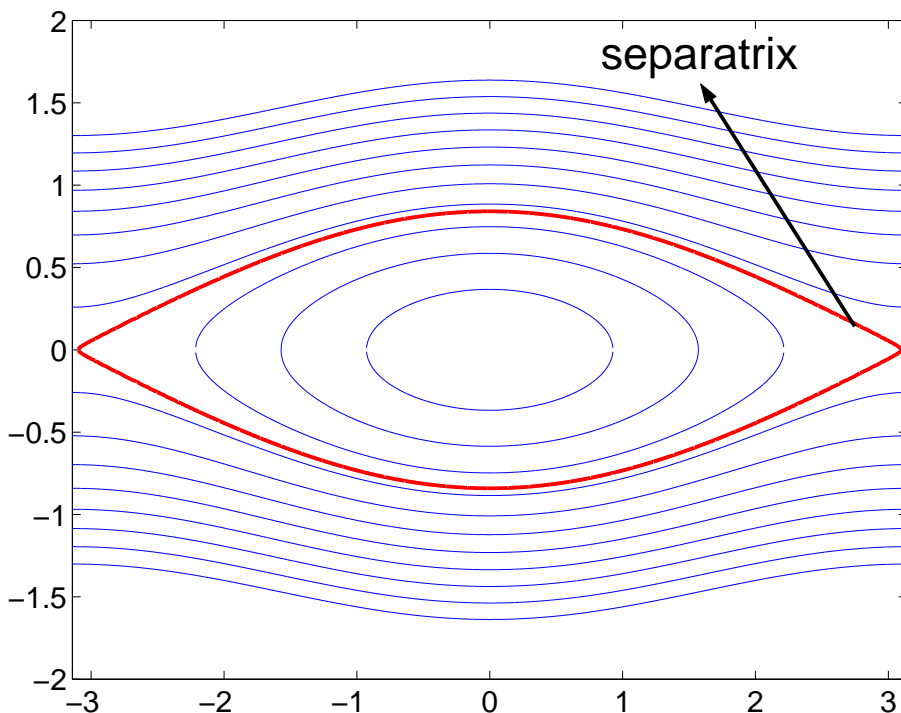


Figure 1: The streamlines of the cosine vortex are shown with solid walls coinciding with the last streamlines below and above, and periodic, straight walls parallel to the  $y$ -direction to the left and right.

An unperturbed cosine vortex with steady-state streamfunction is displayed on the left in Fig. 3. Walls reside where the streamfunction  $\Psi$  has a value of  $\Psi = c = 1.5$  with  $c < a - 1$ ,  $a = 6$  and  $c = 1.5$  in (10). The cosine vortex in a perturbed domain at time  $t = 0$  is displayed by using the same formula for the vorticity (Fig. 3 top right) and the

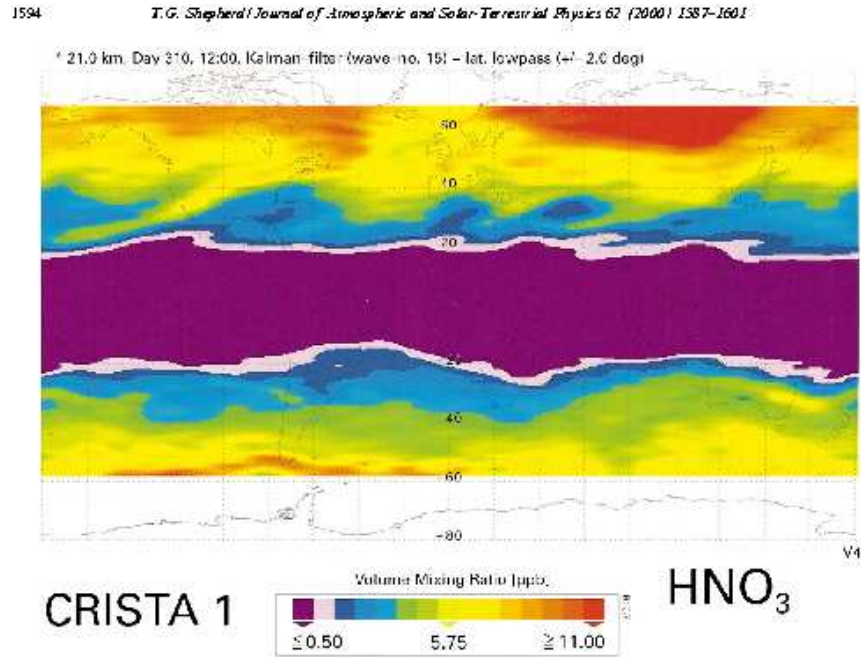


Figure 2: “Horizontal distribution of nitric acid at approximately 21 km altitude, obtained from measurements taken by the CRISTA instrument on the Space Shuttle during 6 November 1994. Figure provided courtesy of Dirk Offermann, Wuppertal University; it is similar to one for 7 November published as plate 6 in Offermann et al. (1999)” (from Shepherd, 2000, figure 5; courtesy of Ted Shepherd). The schematic on the left is a sketch of streamlines in a cat’s eye structure.

associated, calculated one for streamfunction (Fig. 3 bottom right) in which the axes have been removed for clarity but are as before in Fig. 2. The solid boundary at the top is changed in the perturbed case on the right by adding stationary wavy perturbations. We recall that the streamlines should coincide with the wall.

We identify these separatrices as possible regions of strong nonlinear behavior. Along separatrices, particle trajectories in the basic state have infinite period, as have trajectories along the zero-velocity line in parallel shear flows. Subsequently, we present preliminary nonlinear simulations of perturbed flows with separatrices. The flows are chosen to be nonlinearly, Arnol’d stable (in the pseudo-energy norm; see Holm et al., 1986), and stationary perturbations are introduced by imposing wavy corrugations on the domain boundary originally coinciding with a streamline in the basic-state flow (see the initial condition for the streamfunction and vorticity in the right part of Fig. 3). It turns out that the chosen cosine vortex in (10) and Fig. 2 is such a nonlinearly stable flow.

## 4 RESULTS

In Fig. 4, we show the evolution of the vorticity field at later times, starting with the initial condition from Fig. 3 (right). In the simulation, we aimed at nonlinear stability for

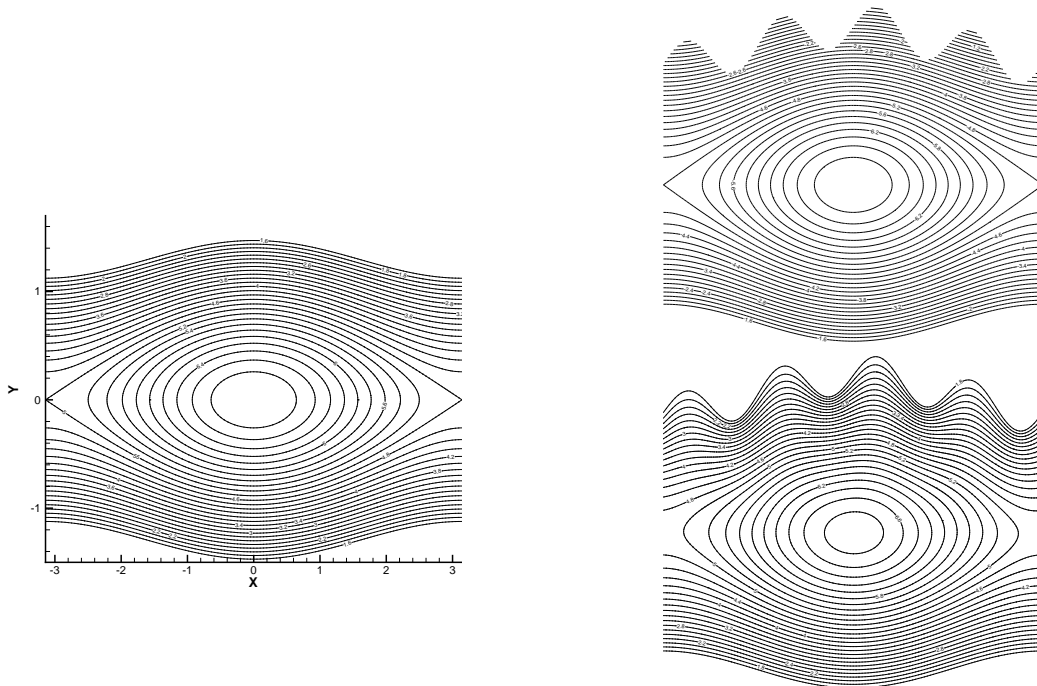


Figure 3: Streamlines of the unperturbed cosine vortex (right) and the vorticity and streamlines of the perturbed cosine vortex (left).

the unperturbed domain by suitably choosing the parameters using analytical methods (Holm et al., 1986; Shepherd, 1990). However, the simulations presented indicate that the nonlinear flow allows highly nontrivial dynamics. These dynamics might yield changes in the topology of the streamlines localized at the stagnation points and the separatrix, see Fig. 4 especially at times  $t = 1.5, 2, 2.5$ ; as in the parallel critical layer case. The simulated vorticity fields reveal signs of folding around the stagnation points in a manner similar to passive tracer advection; the vorticity, of course, is active.

What we observe is that nonlinear instability of the flow due to stationary perturbations leads to localized disturbances around the separatrix of the steady-state solution. It is important to note that these flows are strongly nonlinear in a region of phase space where weakly nonlinear theory with traveling-wave solutions is of no use. While we performed simulations for different resolutions, more convergence runs are required to fully resolve the fine structures around the separatrix (at longer times).

## 5 CONCLUSIONS

A finite-element simulation of curved critical layers in two-dimensional vortical flows in complex shaped domains has been presented to display the power of our (dis)continuous Galerkin finite element method recently developed in Bernsen et al. (2006). This particular simulation was motivated by the existence of cat's eye structures observed in the upper atmosphere (Fig. 2). The intricate dynamics in these cat's eye structures have

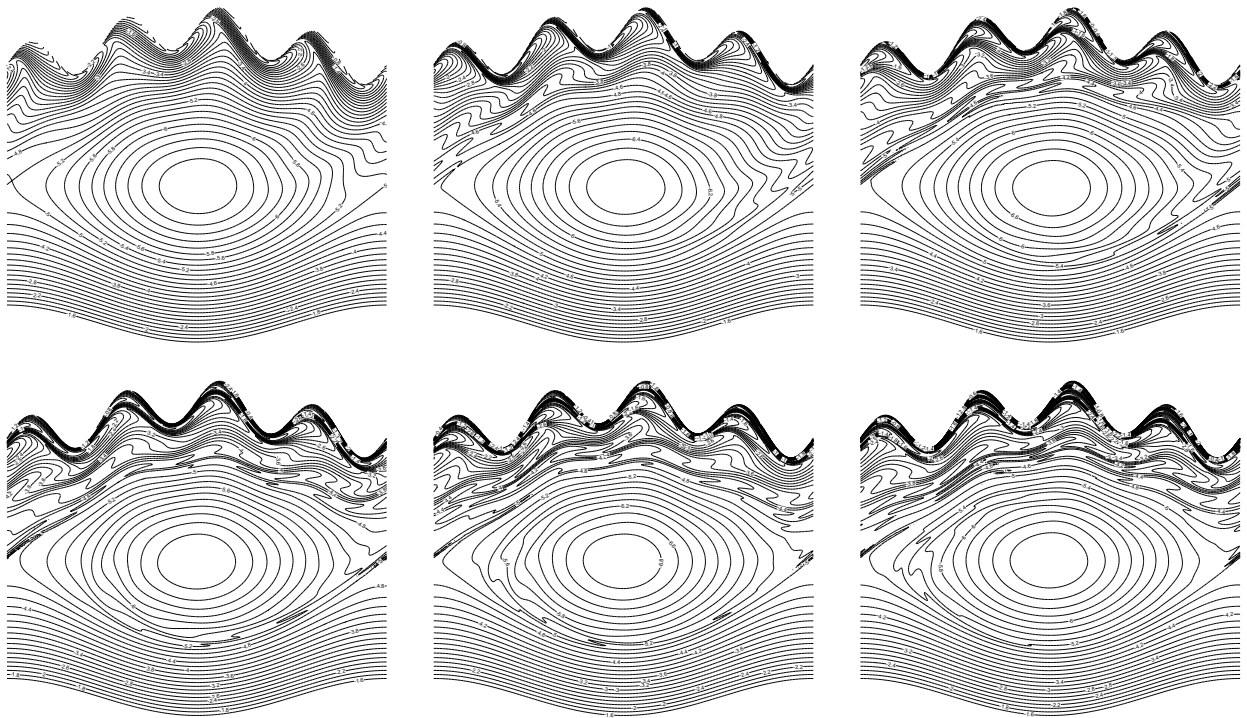


Figure 4: Vorticity at times  $t = 0.5, 1, 1.5, 2, 2.5, 3$  (top left through to bottom right). Axes have been removed for clarity but are as in Fig. 3.

been simulated, see Fig. 4, and show localized regions with strong mixing. This mixing is thought to be relevant to the chemistry in the upper atmosphere.

More numerical analysis is underway in three dimensions regarding a (dis)continuous Galerkin finite element discretization of incompressible flow in a vorticity and vector-streamfunction formulation, which also preserves energy and is an extension of the current two-dimensional formulation.

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