A FAMILY OF "THREE ELEMENT" M.C. ESCHER PATTERNS

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ABSTRACT: In 1952, the Dutch artist M.C. Escher created his striking Notebook Drawing 85. It is a repeating Euclidean pattern of three animals, lizards, fish, and bats, representing the "three elements": earth, water, and air respectively. In this pattern, three of each animal meet head-to-head at the tails of other animals. He also realized a version of this pattern on a sphere, but he did not create a hyperbolic version of it. We fill in this "gap" by showing some hyperbolic versions of the pattern, and explain how they fit into a general family of "three element" patterns.

Keywords: M.C. Escher, hyperbolic geometry, mathematical art.

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1.INTRODUCTION

The Dutch artist M.C. Escher was known for his repeating patterns in the Euclidean plane. But he also created a few spherical patterns and some hyperbolic patterns. His Notebook Drawing Number 85 is one of his most striking Euclidean patterns. It is composed of lizards, fish, and bats, representing the "three elements": earth, water, and air respectively. We use the same animal motifs to create an entire family of "three element" patterns. Since Escher created the only possible Euclidean and spherical patterns in this family, we concentrate on hyperbolic three element patterns. Figure 1 shows one such three element hyperbolic pattern.

We first review some hyperbolic geometry, and specifically the Poincaré disk model. Then we discuss repeating patterns and their symmetries. Next, we will discuss families of repeating patterns. Each "three element" pattern can be classified by three parameters, the number of lizards that meet nose-to-nose, the number of fish that meet nose-to-nose, and the number of bats that meet head-to-head. We will show some examples of these patterns. For comparison, we also discuss a family of "Circle Limit I" patterns. Finally, we will summarize the results and indicate directions of future research.

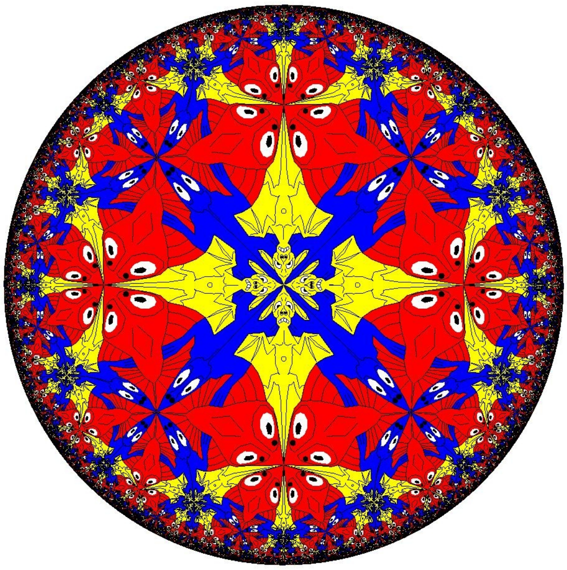

Figure 1: A hyperbolic "three element" pattern.

2.HYPERBOLIC GEOMETRY AND THE POINCARÉ DISK MODEL

Hyperbolic geometry is one of the three "classical" geometries, which also include (plane) Euclidean and spherical geometry. In hyperbolic geometry, given a line and a point not on it, there is more than one line through the point that does not meet the original line, unlike Euclidean geometry in which there is only on parallel. Of course there are no parallels at all in spherical geometry. In contrast to the latter two geometries, hyperbolic geometry has no smooth isometric (distance preserving) embedding in Euclidean 3-space, a fact proved by David Hilbert more than 100 years ago [3]. Thus hyperbolic geometry is much less familiar to most people, even mathematicians, and especially artists. So it is amazing that Escher was able to create hyperbolic patterns.

Since there is no "nice" embedding of hyperbolic geometry in the 3-space of our experience, we must rely on *models* of it – in which Euclidean constructions are given hyperbolic interpretations. One such model is the *Poincaré disk model.* The hyperbolic points of this model are the interior points of a Euclidean circle. Hyperbolic lines are represented by circular arcs within the bounding circle that are perpendicular to it (with diameters as a special case). Escher had long sought to represent an infinite repeating pattern within a finite area, so that the viewer could see the entire pattern, unlike Euclidean "wallpaper" patterns in which the viewer can only see a finite part of the infinite pattern. The Poincaré disk model has another feature that appealed to Escher, namely it is *conformal*, meaning that the hyperbolic measure of an angle is the same as its Euclidean measure, consequently copies of a motif of a repeating pattern retain their same approximate shape regardless of their size. Here we remark that equal hyperbolic distances are represented by ever smaller Euclidean distances as one approaches the bounding circle. In Figure 1, the backbones of the animals lie along hyperbolic lines, and each fish is the same hyperbolic size, and the same is true of the lizards and bats. Figure 2 also shows these features of the disk model in a rendition of Escher's first hyperbolic pattern, *Circle Limit I.* The backbones of the fish lie along hyperbolic lines, as do the trailing edges of the fins of both the black and white fish. All of the white fish are hyperbolically congruent, as are all of the black fish. However, the white fish are *not* congruent to the black fish, since the noses of the white fish (and thus the tails of the black fish) form 90-degree angles, whereas the noses of the black fish (and tails of the white fish) form 60-degree angles.

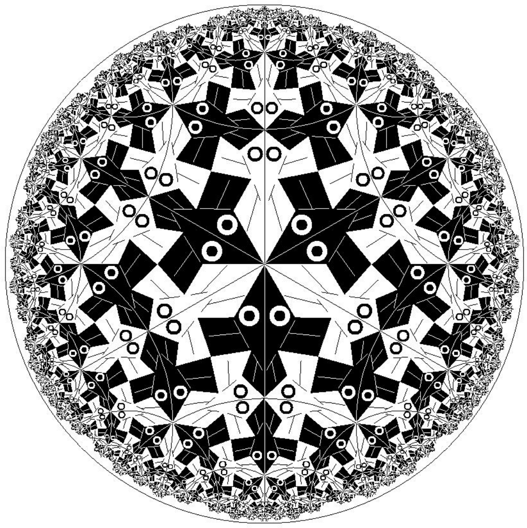

Figure 2: A rendition of Escher's hyperbolic pattern *Circle Limit I.*

3.REPEATING PATTERNS AND SYMMETRIES

A *repeating pattern* is a pattern composed of congruent copies of a basic sub-pattern or *motif.* This definition of a repeating pattern is equally valid in each of the three "classical" geometries. In Figure 1, the motif is formed from halves of each of three adjacent animals inside a triangle whose vertices are at the noses of a fish and lizard, and the top of the head of a bat. That motif is shown in Figure 3 below. Similarly, in Figure 2 the motif is formed from half a white fish together with an adjacent half of a black fish – such a motif is shown in Figure 4 below.

A *symmetry* of a repeating pattern is an isometry (distance-preserving transformation) that takes the pattern onto itself. So each copy of the motif goes onto another copy of the motif. In each of the three "classical" geometries, any symmetry can be formed from one, two, or three reflections. In the Poincaré disk model, reflection across a circular arc representing a hyperbolic line is just inversion with respect to that circular arc (with Euclidean reflections across diameters as special cases). Thus the animals' backbone lines form reflection lines of the pattern in both Figure 1 and Figure 2.

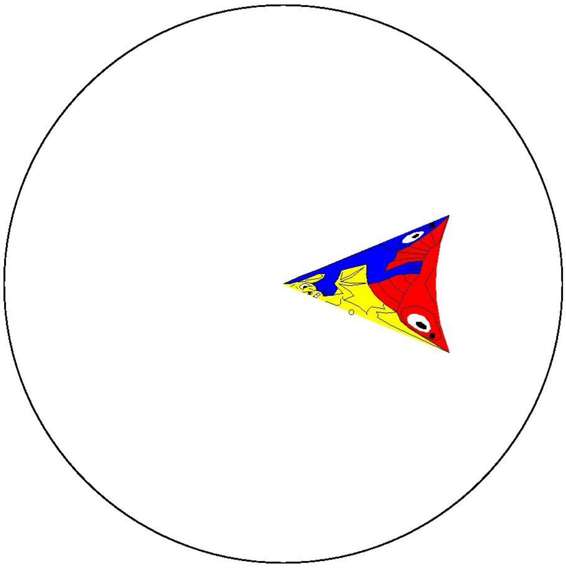

Figure 3: The motif for Figure 1.

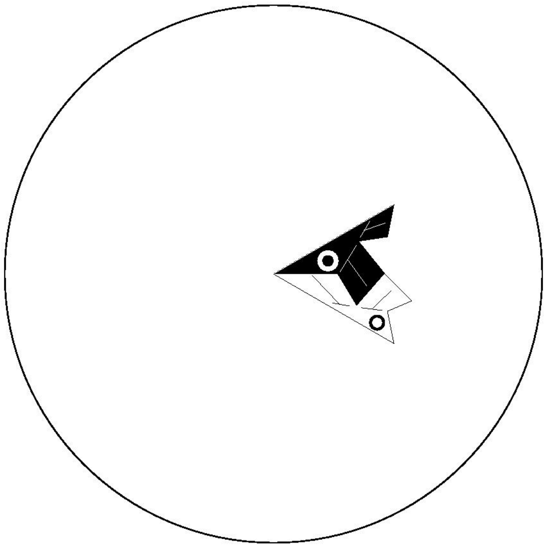

Figure 4: The motif for Figure 2.

In hyperbolic geometry, as in Euclidean geometry, successive reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection. Thus there are rotations about points where backbone lines cross in Figures 1 and 2. And since the trailing edges of the fish in Figure 2 also lie along hyperbolic lines and make a 90 degree angle, there is a 180 degree rotation about points where the fins meet. For more on hyperbolic geometry and its models, see [2].

4.THE FAMILY OF "THREE ELEMENT" PATTERNS

There is a 3-parameter family of "three element" patterns, denoted (p,q,r), in which p fish meet nose-to-nose, q lizards meet nose-to-nose, and r bats meet head-to-head. The pattern of Figure 1 is specified by (4,4,4) in this notation. Escher's Notebook Drawing Number 85 would be assigned (3,3,3). Figure 5 shows a (5,4,3) pattern.

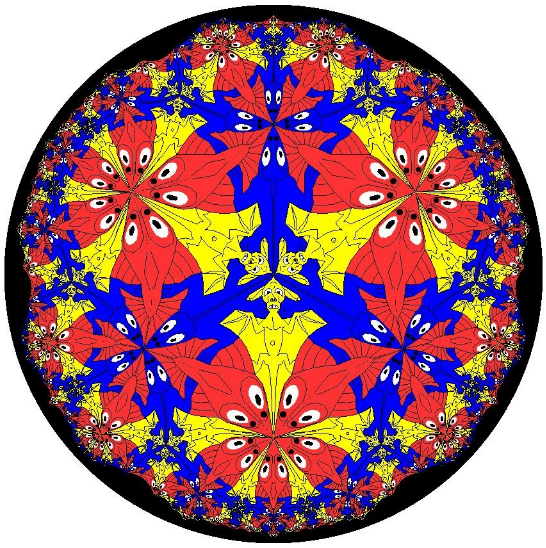

Figure 5: A (5,4,3) "three element" pattern.

Shortly after Escher finished Notebook Drawing 85 in 1952 he drew that pattern on a rhombic dodecahedron. In 1963 Escher and C.V.S. Roosevelt commissioned the Japanese netsuke artist Masatoshi to carve the "three element" pattern on an ivory sphere. Escher's Notebook Drawing Number 85, the rhombic dodecahedron, and the netsuke ball are shown on pages 184, 246, and 307 respectively in [5]. Later, in 1977 Doris Schattschneider and Wallace Walker placed the pattern on a regular octahedron as one of the polyhedra in M.C. Escher Kaleidocycles [4]. In each of these renditions two of each animal met head-to-head, so they were all (2,2,2) patterns.

The numbers *p, q*, and *r* determine the geometry: if 1/p + 1/q + 1/r > 1, the pattern is spherical, if 1/p + 1/q + 1/r = 1, the pattern is Euclidean, and if 1/p + 1/q + 1/r < 1, the pattern is hyperbolic. In the spherical case there is an infinite "dihedral" subfamily (2,2,*n*), (2,*n*,2), or (*n*,2,2) in which *n* animals meet head-to-head at the north and south poles of the sphere. The only other spherical solutions are (2,3,3), (2,3,4), and (2,3,5) (and permutations), corresponding to the regular tetrahedron, octahedron, and dodecahedron, respectively. In addition to the Notebook Drawing 85 pattern, there are three Euclidean solutions: (2,4,4), (4,2,4), and (4,4,2). Of course there are infinitely many hyperbolic solutions. Figure 6 shows a (4,5,3) pattern – here the roles of the fish and lizards have been interchanged from their roles in Figure 5.

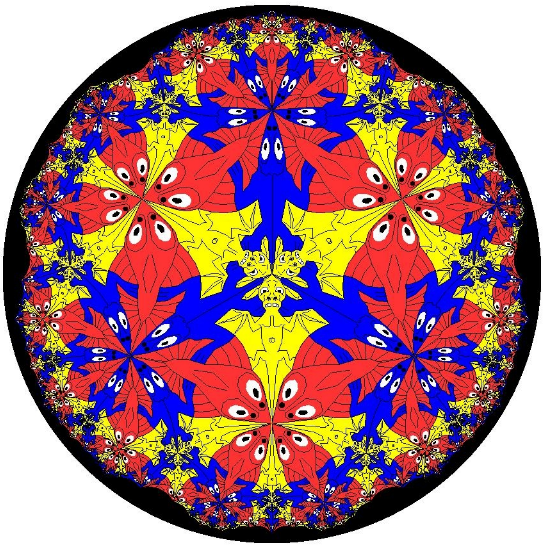

Figure 6: A (4,5,3) "three element" pattern.

Figure 7 shows a (5,3,4) pattern, which differs from Figure 1 in that five fish and three lizards meet head-to-head, instead of four each.

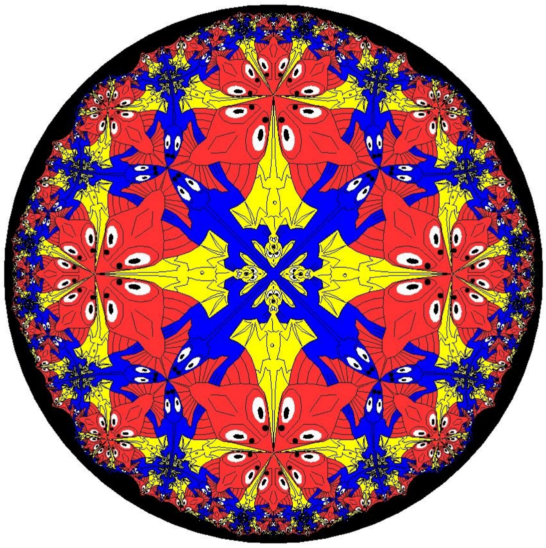

Figure 7: A (5,3,4) "three element" pattern.

Figure 8 shows a (4,4,3) pattern as an example in which two of the parameters are the same.

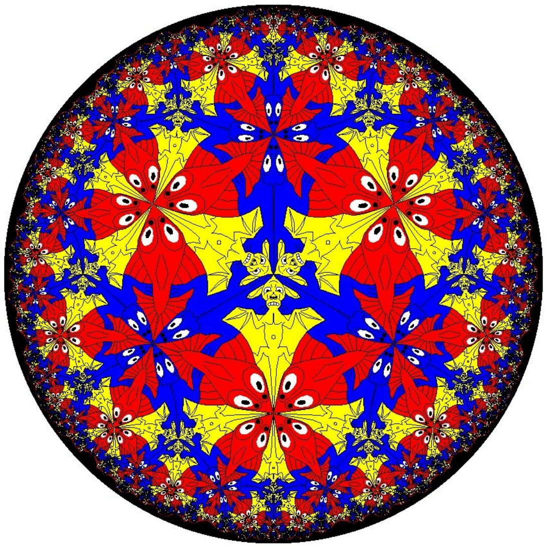

Figure 8: A (4,4,3) "three element" pattern.

Even though there are an infinite number of hyperbolic "three element" patterns, in those with large values of *p, q*, or *r* the animals are either strongly distorted or quickly pushed to the edge of the disk (or both). Figure 9 shows such a pattern, with *p* = *q* = 4, and *r* = 10. Consequently those patterns with small values of *p, q*, and *r* seem to be most aesthetically pleasing.

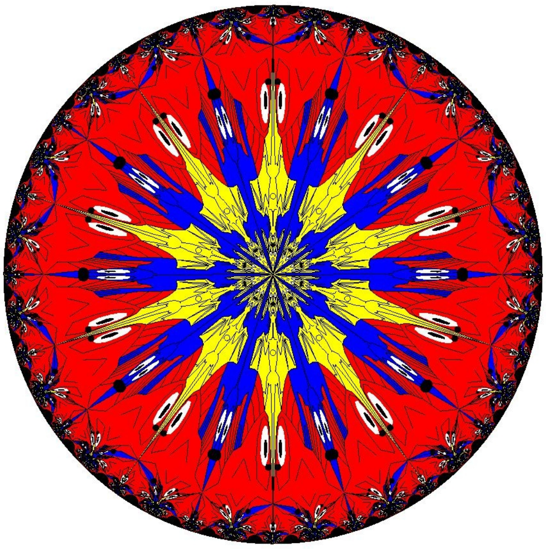

Figure 9: A (4,4,10) “three element” pattern.

5.THE FAMILY OF “CIRCLE LIMIT I ” PATTERNS

Figure 2 shows a rendition of Escher's *Circle Limit I* pattern. This pattern can also form the basis for a 2-parameter family of patterns. We let (*p,q*) denote the pattern in which *p* black fish and *q* white fish meet nose-to-nose. So *Circle Limit I* would be denoted (2,3). Figure 10 shows a (3,3) “Circle Limit I” pattern.

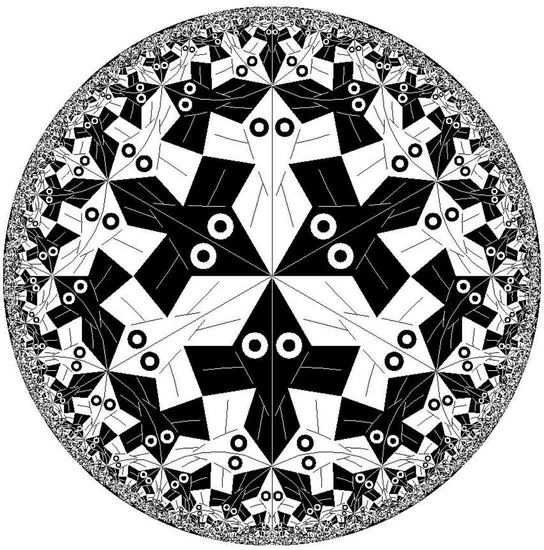

Figure 10: A (3,3) “Circle Limit I” pattern.

In the pattern of Figure 10, the black and white fish are the same hyperbolic size and shape, unlike *Circle Limit I.* In fact the Figure 10 pattern has 2-color (black/white) symmetry – a 90 degree rotation about the meeting point of trailing fin tips interchanges the black and white fish. Making the reasonable assumption that *p* and *q* are at least 2, there are no possible spherical “Circle Limit I” patterns and only one possible Euclidean pattern, (2,2). It seems that Escher did not create such a pattern. However he did design similar angular fish for pattern *A14* (shown on page 234 of [5]). He used the *A14* pattern in his print *Sphere Surface with Fish* (page 322 of [5]), which was completed about three or four months before *Circle Limit I.* If either *p* or *q* is greater than 2, the pattern is hyperbolic. For completeness Figure 11 shows a transformed version of the *Circle Limit I* pattern with the roles of the black and white fish reversed, so it is a (3,2) pattern.

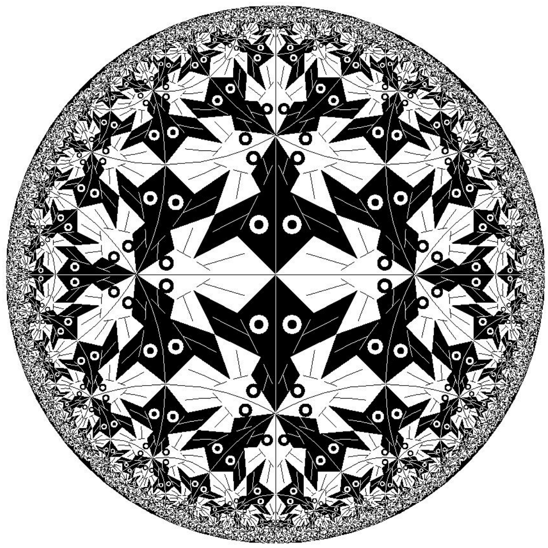

Figure 11: A (3,3) “Circle Limit I” pattern.

The same comment about large values of *p* and *q* applies to this family too – such values tend to produce distorted patterns. For more on creating hyperbolic patterns, see [1].

CONCLUSIONS AND FUTURE WORK

After setting the foundations for hyperbolic patterns and their symmetries, we investigated the 3-parameter family (p,q,r) of “three element” patterns, most of which are hyperbolic. This family was inspired by Escher's Euclidean Notebook Drawing Number 85, which would be denoted (3,3,3). Escher also realized this pattern on the sphere, in which two animals meet head-to-head. We have shown six more (hyperbolic) patterns in this family.

We also discussed the simpler 2-parameter family of “Circle Limit I” patterns, and showed two more patterns in that family. Viewing a pattern as a member of an entire family of patterns gives the original pattern more context, and may lead to more interesting new patterns, even in different geometries.

In the future, we would like to investigate more families of patterns by Escher and others. And we would like to create new patterns from such families.

ACKNOWLEDGMENTS

I would like to thank the director, Lisa Fitzpatrick, and the staff of the Visulalization and Digital Imaging Laboratory (VDIL) at the University of Minnesota Duluth for their support and the use of the VDIL facilities.

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