A First Course in Elementary Differential Equations: Problems and Solutions

Marcel B. Finan
Arkansas Tech University

Contents

1 Basic Terminology

Problem 1.1
A ball is thrown straight up from ground level and reaches its greatest height after 5 seconds. Find the initial velocity of the ball and the value of its maximum height above ground level.

Solution.
Let $y(t)$ be the height of the ball above ground level at time t seconds after it was thrown. We are given that $y(0) = 0$. We are also told that the ball reaches its maximum height after 5 seconds at which point the velocity is zero, i.e., $v(5) = 0$.
The body's position is governed by the differential equation $y''(t) = -32\ ft/sec$. So $y'(t) = v(t) = -32t + C_1$ for some constant C_1. Since $v(5) = 0$, solving the equation $-32(5) + C_1 = 0$ for C_1 we find $C_1 = 160$. Hence,

$$y'(t) = v(t) = -32t + 160.$$

Using this equation we have now that the initial velocity of the ball was $v(0) = 160\ ft/sec$. We still need to find the position of the ball at time 5 seconds (when the ball was at its greatest height). By integrating the previous equation we find

$$y(t) = -16t^2 + 160t + C_2.$$

Since the ball was thrown from ground level, we have that $y(0) = 0$, so $C_2 = 0$ and

$$y(t) = -16t^2 + 160t.$$

We were told that the maximum height was reached after five seconds, so the maximum height's value is given by

$$y(5) = -16(5)^2 + 160(5) = 400\ ft \blacksquare$$

Problem 1.2
Find the order of the following differential equations.
(a) $ty'' + y = t^3$
(b) $y' + y^2 = 2$
(c) $\sin y''' + 3t^2 y = 6t$

Solution.
(a) Since the highest derivative appearing in the equation is 2, the order of the equation is 2.
(b) Order is 1.
(c) Order is 3 ■

Problem 1.3
What is the order of the differential equation?
(a) $y'(t) - 1 = 0$.
(b) $y''(t) - 1 = 0$.
(c) $y''(t) - 2ty(t) = 0$.
(d) $y''(t)(y'(t))^{\frac{1}{2}} - \frac{t}{y(t)} = 0$.

Solution.
(a) First order.
(b) , (c), and (d) second order ■

Problem 1.4
In the equation
$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$$
identify the independent variable(s) and the dependent variable.

Solution.
u is the depedent variable whereas x and y are the independent variables ■

Problem 1.5
Classify the following equations as either ordinary or partial.
(a) $(y''')^4 + \frac{t^2}{(y')^2+4} = 0$.

(b) $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y-x}{y+x}$.

(c) $y'' - 4y = 0$.

Solution.
(a) ODE.
(b) PDE.
(c) ODE ■

Problem 1.6
Solve the equation $y'''(t) - 2 = 0$ by computing successive antiderivatives.

Solution.
Integrating for the first time we find $y''(t) = 2t + C_1$. Integrating the last equation we find $y'(t) = t^2 + C_1 t + C_2$. Integrating for a third and final time we get $y(t) = \frac{t^3}{3} + C_1\frac{t^2}{2} + C_2 t + C_3$ ■

Problem 1.7
Solve the initial-value problem

$$\frac{dy}{dt} = 3y(t), \;\; y(0) = 50.$$

What is the domain of the solution?

Solution.
The general solution is of the form $y(t) = Ce^{3t}$. Since $y(0) = 50$, we have $50 = Ce^{3\cdot 0}$, and so $C = 50$. The solution is $y(t) = 50e^{3t}$. The domain is the set of all real numbers ■

Problem 1.8
For what real value(s) of λ is $y = \cos \lambda t$ a solution of the equation $y'' + 9y = 0$?

Solution.
Finding the first and second derivatives, we find that $y'(t) = -\lambda \sin \lambda t$ and $y''(t) = -\lambda^2 \cos \lambda t$. By substitution, $\cos \lambda t$ is a solution if and only if $\lambda^2 - 9 = 0$. This equation has the real roots roots $\lambda = \pm 3$ ■

Problem 1.9
For what value(s) of m is $y = e^{mt}$ a solution of the equation $y'' + 3y' + 2y = 0$?

Solution.
Since $\frac{d}{dt}(e^{mt}) = me^{mt}$ and $\frac{d^2}{dt^2}(e^{mt}) = m^2 e^{mt}$ the requirement on m becomes $m^2 + 3m + 2 = 0$. Factoring the left-hand side to obtain $(m+2)(m+1) = 0$. Thus, $m = -2$ and $m = -1$ ■

Problem 1.10
Show that $y(t) = e^t$ is a solution to the differential equation

$$y'' - \left(2 + \frac{2}{t}\right) y' + \left(1 + \frac{2}{t}\right) y = 0.$$

Solution.
Substituting $y(t) = y'(t) = y''(t) = e^t$ into the equation we find

$$\begin{aligned} y'' - \left(2 + \frac{2}{t}\right)y' + \left(1 + \frac{2}{t}\right)y =& e^t - \left(2 + \frac{2}{t}\right)e^t + \left(1 + \frac{2}{t}\right)e^t \\ =& e^t - 2e^t - \frac{2}{t}e^t + e^t + \frac{2}{t}e^t = 0 \blacksquare \end{aligned}$$

Problem 1.11
Show that any function of the form $y(t) = C_1 \cos \omega t + C_2 \sin \omega t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0.$$

Solution.
Finding the first and the second derivatives of y we obtain

$$y'(t) = -C_1\omega \sin \omega t + C_2\omega \cos \omega t$$

and

$$y''(t) = -C_1\omega^2 \cos \omega t - C_2\omega^2 \sin \omega t$$

Substituting this into the equation to obtain

$$\begin{aligned} \frac{d^2y}{dt^2} + \omega^2 y =& - C_1\omega^2 \cos \omega t \quad -C_2\omega^2 \sin \omega t + \omega^2(C_1 \cos \omega t + C_2 \sin \omega t) \\ =& 0 \blacksquare \end{aligned}$$

Problem 1.12
Suppose $y(t) = 2e^{-4t}$ is the solution to the initial value problem $y' + ky = 0, y(0) = y_0$. Find the values of k and y_0.

Solution.
We have $y_0 = y(0) = 2$. The given function satisfies the equation $y' + ky = 0$, that is, $-8e^{-4t} + 2ke^{-4t} = 0$. Dividing through by $2e^{-4t}$ to obtain $-4 + k = 0$. Thus, $k = 4$ $\blacksquare$

Problem 1.13
Consider $t > 0$. For what value(s) of the constant n, if any, is $y(t) = t^n$ a solution to the differential equation

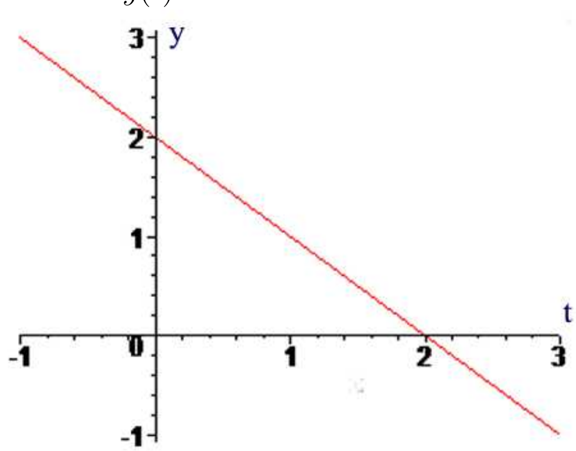
$$t^2y'' - 2ty' + 2y = 0?$$

Solution.
Since $t^2y'' - 2ty' + 2y = t^2(n(n-1)t^{n-2}) - 2t(nt^{n-1}) + 2t^n = 0$ we have $n(n-1) - 2n + 2 = 0$ or $n^2 - 3n + 2 = 0$. This last equation can be factored as $(n-1)(n-2) = 0$. Solving we find $n = 1$ or $n = 2$ ■

Problem 1.14
(a) Show that $y(t) = C_1e^{2t} + C_2e^{-2t}$ is a solution of the differential equation $y'' - 4y = 0$, where C_1 and C_2 are arbitrary constants.
(b) Find the solution satisfying $y(0) = 2$ and $y'(0) = 0$.
(c) Find the solution satisfying $y(0) = 2$ and $\lim_{t\to\infty} y(t) = 0$.

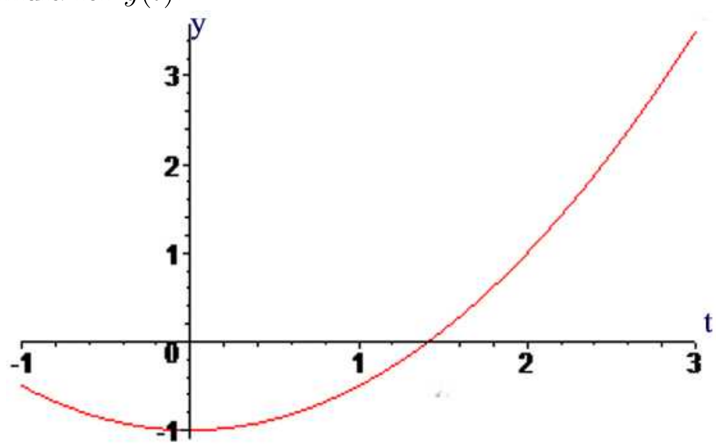
Solution.
(a) Finding the first and the second derivatives of $y(t)$ to obtain $y'(t) = 2C_1e^{2t} - 2C_2e^{-2t}$ and $y''(t) = 4C_1e^{2t} + 4C_2e^{-2t}$. Thus, $y'' - 4y = 4C_1e^{2t} + 4C_2e^{-2t} - 4(C_1e^{2t} + C_2e^{-2t}) = 0$
(b) The condition $y(0) = 2$ implies that $C_1 + C_2 = 2$. The condition $y'(0) = 0$ implies that $2C_1 - 2C_2 = 0$ or $C_1 = C_2$. But $C_1 + C_2 = 2$ and this implies that $C_1 = C_2 = 1$. In this case, the particular solution is $y(t) = e^{2t} + e^{-2t}$.
(c) The first condition implies that $C_1 + C_2 = 2$. The second condition implies that $C_1 = \lim_{t\to\infty} \frac{y(t) - C_2e^{-2t}}{e^{2t}} = 0$. Thus, $C_2 = 2$ and the particular solution is given by $y(t) = 2e^{-2t}$ ■

Problem 1.15
Suppose that the graph below is the particular solution to the initial value problem $y'(t) = m + 1, y(1) = y_0$. Determine the constants m and y_0 and then find the formula for $y(t)$.



Solution.
From the figure we see that $y_0 = y(1) = 1$. Since y is the slope of the line which is -1, we have $y'(t) = -1 = m + 1$. Solving for m we find $m = -2$. Hence, $y(t) = -t + 2$ ■

Problem 1.16
Suppose that the graph below is the particular solution to the initial value problem $y'(t) = mt, y(t_0) = -1$. Determine the constants m and t_0 and then find the formula for $y(t)$.



Solution.
From the graph we see that $y(0) = -1$ so that $t_0 = 0$. Also, by integration we see that $y = \frac{m}{2}t^2 + C$. From the figure we see that $C = -1$. Finally, $y(1) = -0.5$ implies $-\frac{1}{2} = \frac{m}{2} - 1$. Solving for m we find $m = 1$. Thus, $y(t) = \frac{t^2}{2} - 1$ ■

Problem 1.17
Show that $y(t) = e^{2t}$ is not a solution to the differential equation $y'' + 4y = 0$.

Solution.
Finding the second derivative and substituting into the equation we find

$$y'' + 4y = 4e^{2t} + 4e^{2t} = 8e^{2t} \neq 0$$

Thus, $y(t) = e^{2t}$ is not a solution to the given equation ■

Problem 1.18
At time $t = 0$ an object having mass m is released from rest at a height y_0 above the ground. Let g represent the constant gravitational acceleration. Derive an expression for the impact time (the time at which the object strikes the ground). What is the velocity with which the object strikes the ground?

Solution.
The motion satisfies the differential equation $y'' = -g$. Integrating twice and using the facts that $v(0) = 0$ and $y(0) = y_0$ we find

$$y(t) = -\frac{1}{2}gt^2 + y_0.$$

The object strikes the ground when $y(t) = 0$. Thus, $-\frac{1}{2}gt^2 + y_0 = 0$. Solving for t we find $t = \sqrt{\frac{2y_0}{g}}$. The velocity with which the object strikes the ground is $v(\sqrt{\frac{2y_0}{g}}) = -g(\sqrt{\frac{2y_0}{g}}) = -\sqrt{2gy_0}$ ■

Problem 1.19
At time $t = 0$, an object of mass m is released from rest at a height of 252 ft above the floor of an experimental chamber in which gravitational acceleration has been slightly modified. Assume (instead of the usual value of 32 ft/sec^2), that the acceleration has the form $32 - \epsilon \sin\left(\frac{\pi t}{4}\right)$ ft/sec^2, where ϵ is a constant. In addition, assume that the projectile strikes the ground exactly 4 sec after release. Can this information be used to determine the constant ϵ? If so, determine ϵ.

Solution.
The motion of the object satisfies the equation $y'' = -(32 - \epsilon \sin\left(\frac{\pi t}{4}\right))$. The velocity is given by $v(t) = -32t - (\frac{4}{\pi})\epsilon \cos\left(\frac{\pi t}{4}\right))$. The displacement function is given by

$$y(t) = -16t^2 - \left(\frac{4}{\pi}\right)^2 \epsilon \sin\left(\frac{\pi t}{4}\right) + 252.$$

The projectile strikes the ground at $t = 4$ sec. In this case $y(4) = 0$. Since $\sin \pi = 0$, ϵ cannot be determined from the given information ■

Problem 1.20
Consider the initial-value problem

$$y' + 3y = 6t + 5, \quad y(0) = 3.$$

(a) Show that $y = Ce^{-3t} + 2t + 1$ is a solution to the above differential equation.
(b) Find the value of C.

Solution.
(a) Substituting y and y' into the equation we find

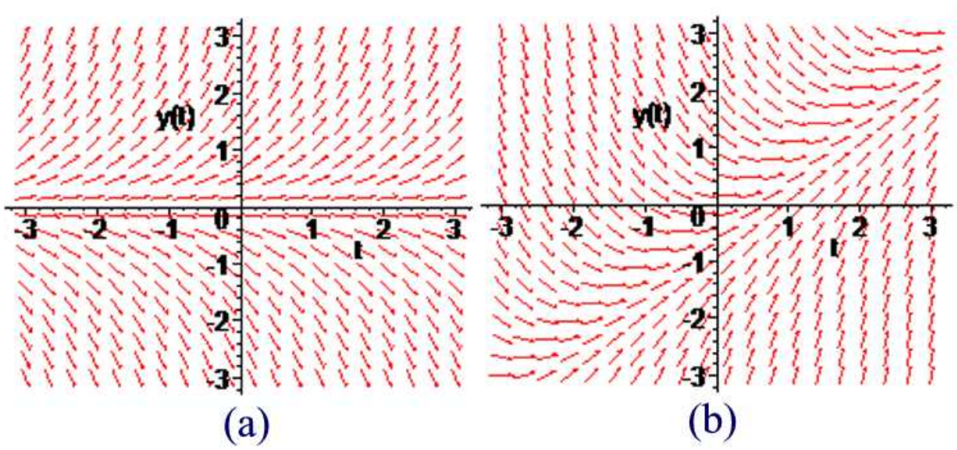
$$-3Ce^{-3t} + 2 + 3[Ce^{-3t} + 2t + 1] = -3Ce^{-3t} + 3Ce^{-3t} + 6t + 5 = 6t + 5.$$

(b) Since $y(0) = 3$ we have $C + 1 = 3$. Solving for C we find $C = 2$. Thus, the solution to the initial value problem is $y(t) = 2e^{-3t} + 2t + 1$ ■

2 Qualitative Analysis: Direction Field of $y' = f(t, y)$

Problem 2.1
Sketch the direction field for the differential equation in the window $-3 \leq t \leq 3, -3 \leq y \leq 3$.
(a) $y' = y$ (b) $y' = t - y$.

Solution.

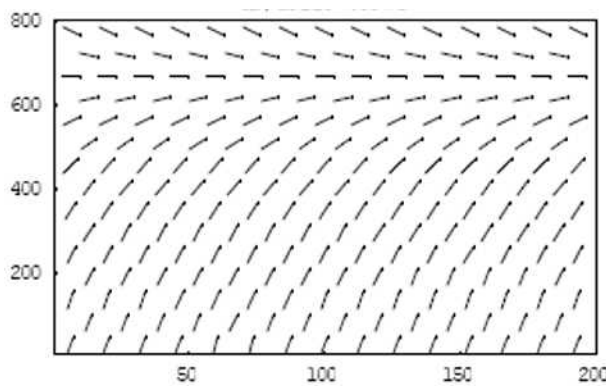


Problem 2.2
Sketch solution curves to the differential equation

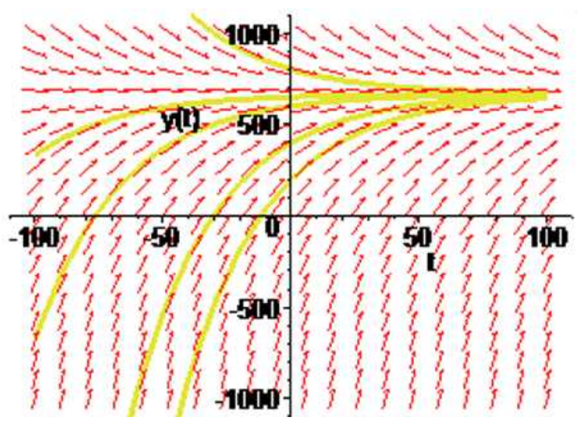
$$\frac{dy}{dt} = 20 - 0.03y$$

represented by the slope field below for the initial values

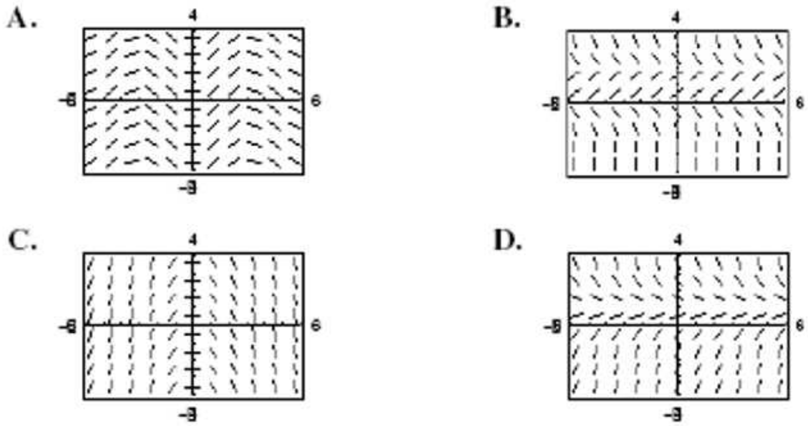
$$(t_0, y_0) = \{(0, 200), (0, 400), (0, 600), (0, 650), (0, 800)\}.$$



Solution.



Problem 2.3
Match each direction field with the equation that the slope field could represent. Each direction field is drawn in the portion of the ty-plane defined by $-6 \le t \le 6, -4 \le y \le 4$.
(a) $y' = -t$ (b) $y' = \sin t$ (c) $y' = 1 - y$ (d) $y' = y(2 - y)$.



Solution.
(A) $y' = \sin t$ (B) $y' = y(2-y)$ (c) $y' = -t$ (D) $y' = 1 - y$ ■

Problem 2.4
State whether or not the equation is autonomous.
(a) $y' = -t$ (b) $y' = \sin t$ (c) $y' = 1 - y$ (d) $y' = y(2-y)$.

Solution.
(a) No (b) No (c) Yes (d) Yes ■

Problem 2.5
Find the equations of the isoclines for the DE $y' = \frac{2y}{t}$.

Solution.
The isoclines have equations of the form $\frac{2y}{t} = c$ or $y = \frac{c}{2}t$ ■

Problem 2.6
Find all the equilibrium solutions of each of the autonomous differential equations below
(a) $y' = (y-1)(y-2)$.
(b) $y' = (y-1)(y-2)^2$.
(c) $y' = (y-1)(y-2)(y-3)$.

Solution.
(a) $y(t) \equiv 1, y(t) \equiv 2$
(b) $y(t) \equiv 1, y \equiv 2$
(c) $y \equiv 1, y \equiv 2, y \equiv 3$ ■

Problem 2.7
Find an autonomous differential equation with an equilibrium solution at $y = 1$ and satisfying $y' < 0$ for $-\infty < y < 1$ and $1 < y < \infty$.

Solution.
One answer is the differential equation: $y'' = -(y-1)^2$ ■

Problem 2.8
Find an autonomous differential equation with no equilibrium solutions and satisfying $y' > 0$.

Solution.
Consider the differential equation $y' = e^y$. Then $y' > 0$ for all y. Also, $e^y \neq 0$ for all y. That is, the DE does not have equilibrium solutions ■

Problem 2.9
Find an autonomous differential equation with equilibrium solutions $y = \frac{n}{2}$, where n is an integer.

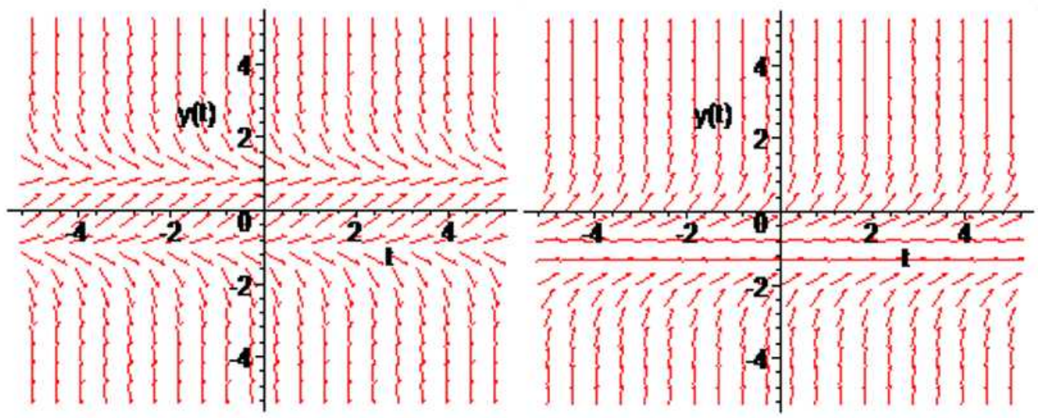
Solution.
One answer is the DE $y' = \sin(2\pi y)$ ■

Problem 2.10
Find an autonomous differential equation with equilibrium solutions $y = 0$ and $y = 2$ and satisfying the properties $y' > 0$ for $0 < y < 2$; $y' < 0$ for $y < 0$ or $y > 2$.

Solution.
An answer is $y' = y(2-y)$ ■

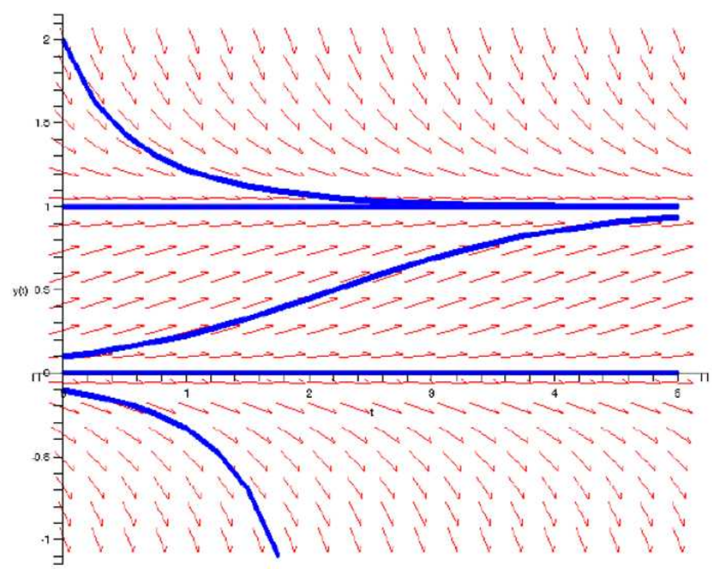
Problem 2.11
Classify whether the equilibrium solutions are stable, unstable, or neither.
(a) $y' = 1 - y^2$.
(b) $y' = (y+1)^2$.

Solution.
Using the direction fields shown below we find
(a) $y = 1$ stable, $y = -1$ unstable
(b) $y = -1$ is neither. This is a semi-stable equilibrium ■



(a) (b)

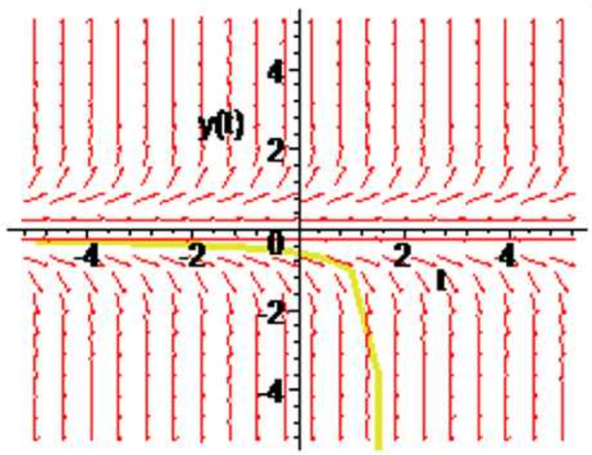
Problem 2.12
Consider the direction field below. Classify the equilibrium points, as asymptotically stable, semi-stable, or unstable.



Solution.
The equilibrium solution at $y = 1$ is asymptotically stable where as the equilibrium solution at $y = 0$ is unstable ■

Problem 2.13
Sketch the direction field of the equation $y' = y^3$. Sketch the solution satisfying the condition $y(1) = -1$. What is the domain of this solution?

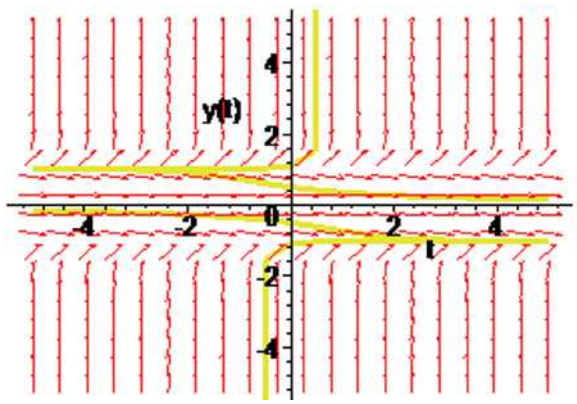
Solution.
As shown in the figure below, the domain of the solution is the interval $-\infty < t < 2$ ■



Problem 2.14
Find the equilibrium solutions and determine their stability

$$y' = y^2(y^2 - 1), \quad y(0) = y_0.$$

Solution.
The direction field is given below.



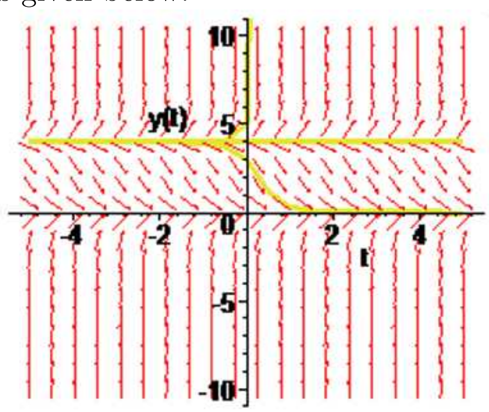
The equilibrium point $y = 1$ is unstable; $y = 0$ is semi-stable; $y = -1$ is asymptotically stable ■

Problem 2.15
Find the equilibrium solutions of the equation

$$y' = y^2 - 4y$$

then decide whether they are asymptotically stable, semi-stable, or unstable. What is the long-time behavior if $y(0) = 5?y(0) = 4?y(0) = 3?$

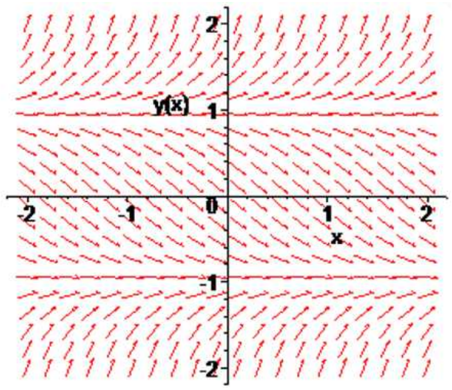
Solution.
The direction field is given below.



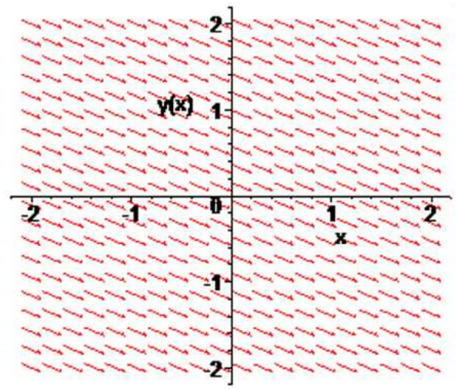
The equilibrium point $y = 4$ is unstable while $y = 0$ is asymptotically stable. If $y(0) = 5$ then $\lim_{t\to\infty} y(t) = \infty$. If $y(0) = 4$ then $\lim_{t\to\infty} y(t) = 4$. If $y(0) = 3$ then $\lim_{t\to\infty} y(t) = 0$ ■

Problem 2.16
Consider the six direction fields shown. Associate a direction field with each

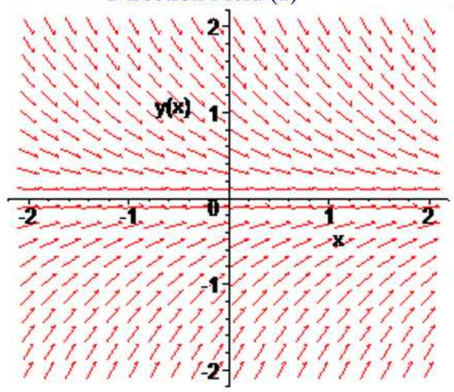
of the following differential equations.
(i) $y' = -y$ (ii) $y' = -t + 1$ (iii) $y' = y^2 - 1$ (iv) $y' = -\frac{1}{2}$ (v) $y' = y + t$
(vi) $y' = \frac{1}{y^2+1}$.



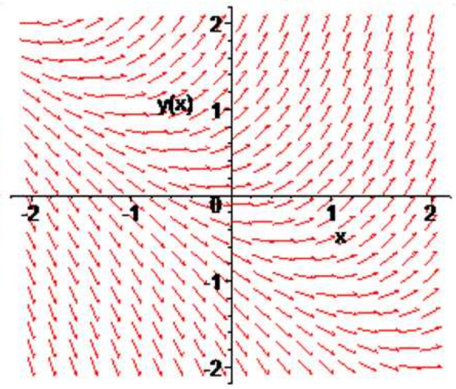
Direction Field (a)



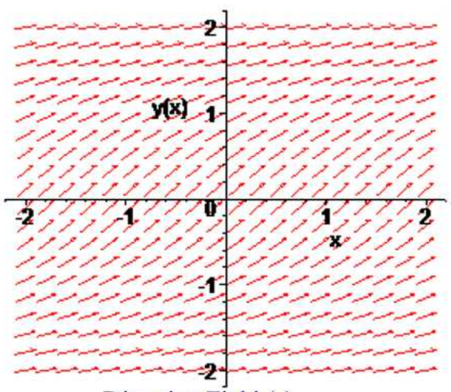
Direction Field (b)



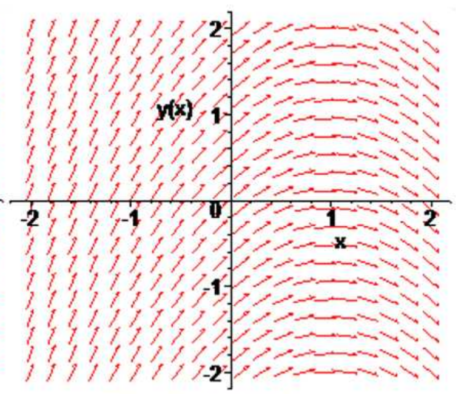
Direction Field (c)



Direction Field (d)



Direction Field (e)



Direction Field (f)

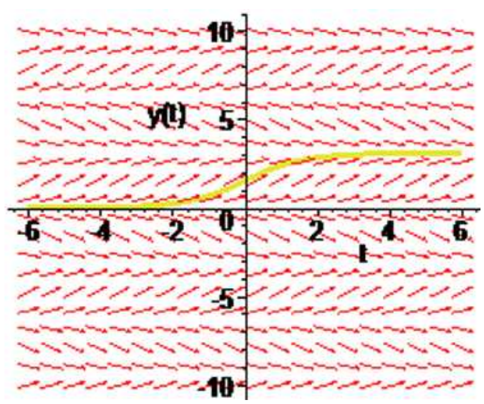
Solution.
(i) (c) (ii) (f) (iii) (a) (iv) (b) (v) (d) (vi) (e) ■

Problem 2.17
What is $\lim_{t\to\infty} y(t)$ for the initial-value problem

$$y' = \sin(y(t)),\ y(0) = \frac{\pi}{2}?$$

Solution.
According to the direction field shown below we conclude that

$$\lim_{t\to\infty} y(t) = \pi \ ■$$



Problem 2.18
The slope fields of $y' = 2 - y$ and $y' = \frac{t}{y}$ are shown in Figure 2.9(a) and Figure 2.9(b).
(a) On each slope field, sketch solution curves with initial conditions

$$\text{(i) } y(0) = 1 \quad \text{(ii) } y(1) = 0 \quad \text{(iii) } y(0) = 3.$$

(b) For each solution curve, what can you say about the long run behavior of y? That is, does $\lim_{t\to\infty} y$ exist? If so, what is its value?

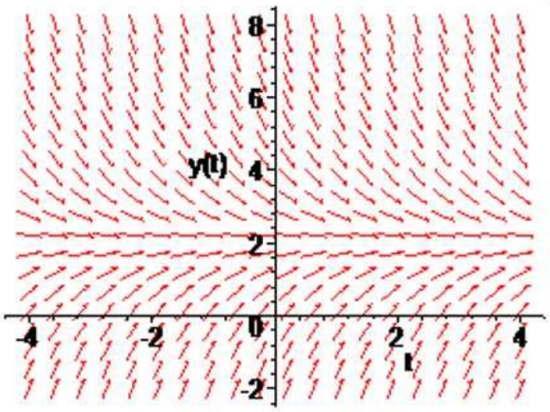


Figure 29(a)

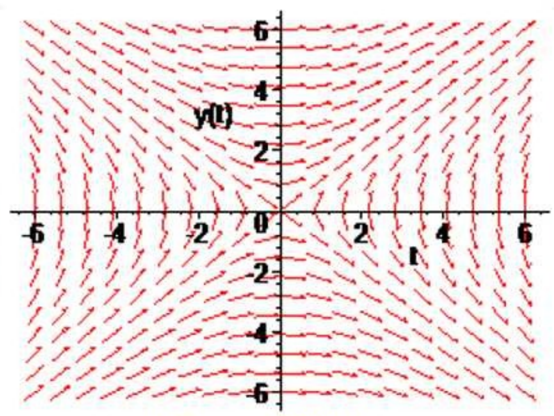
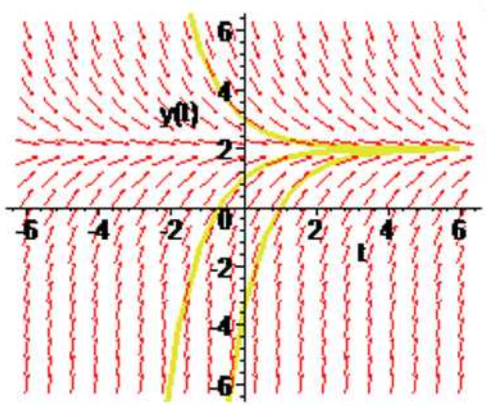
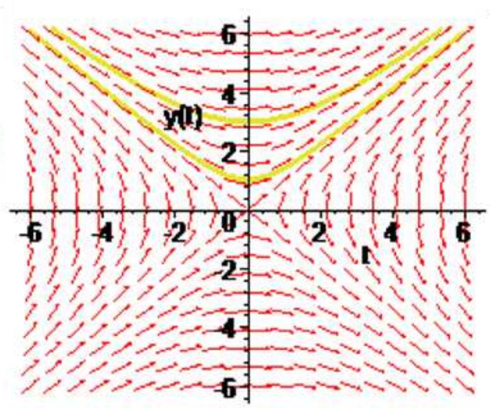


Figure 29(b)

Solution.
(a)



y' = 2 - y



y'=t/y

(b) See graphs in part (a) ■

Problem 2.19
The slope field for the equation $y' = t(y-1)$ is shown in Figure 2.10.
(a) Sketch the solutions passing through the points

$$\text{(i) } (0,1) \quad \text{(ii) } (0,-1) \quad \text{(iii) } (0,0).$$

(b) From your sketch, write down the equation of the solution with $y(0) = 1$.
(c) Check your solution to part (b) by substituting it into the differential

equation.

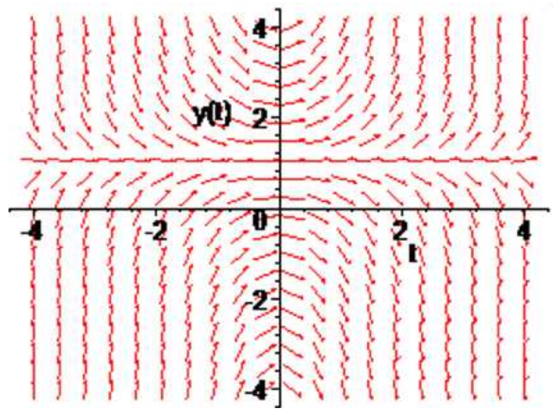
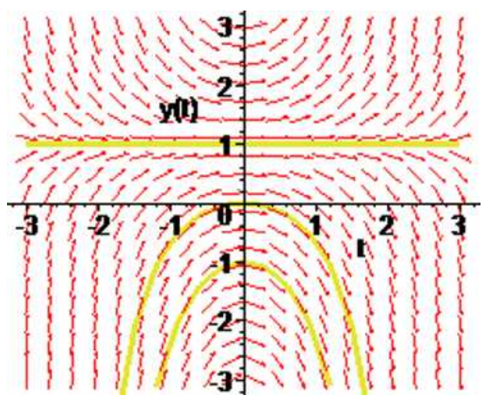


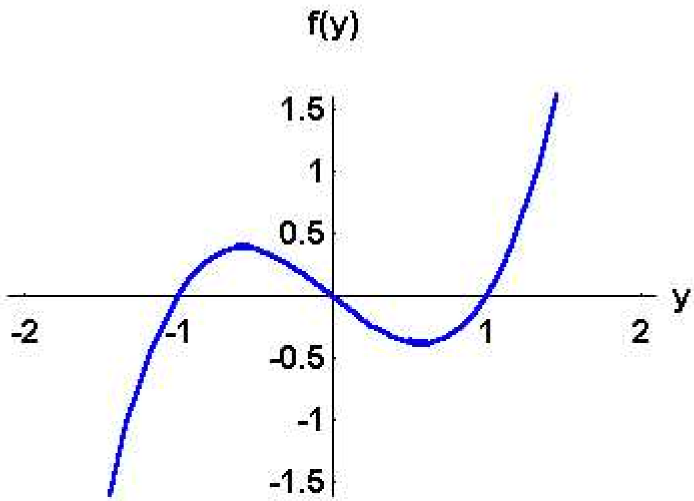
Figure 2.10

Solution.
(a)



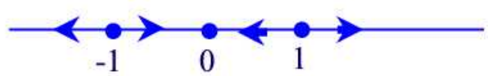
(b) $y(t) \equiv 1$ for all t.
(c) Since $y' = 0$ and $t(y-1) = 0$ when $y = 1$, $y' = t(y-1)$ is satisfied by $y(t) \equiv 1$ ■

Problem 2.20
Consider the autonomous differential equation $\frac{dy}{dt} = f(y)$ where the graph of $f(y)$ is

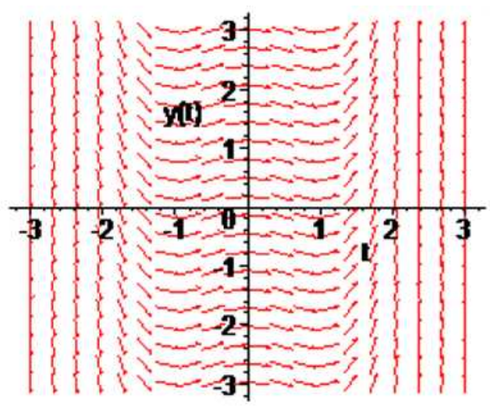


(a) Sketch the phase line.
(b) Sketch the Slope Field of this differential equation.
(c) Sketch the graph of the solution to the IVP $y' = f(y),\ y(0) = \frac{1}{2}$. Find $\lim_{t\to\infty} y(t)$.
(d) Sketch the graph of the solution to the IVP $y' = f(y),\ y(0) = -\frac{1}{2}$. Find $\lim_{t\to\infty} y(t)$.

Solution.
(a)

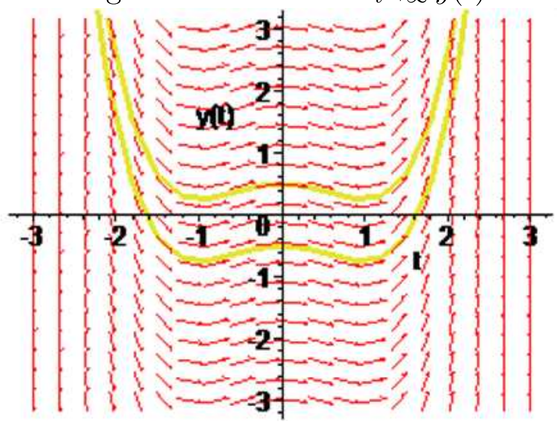


(b)



(c) We notice from the figure below that $\lim_{t\to\infty} y(t) = \infty$

(d) We notice from the figure below that $\lim_{t\to\infty} y(t) = \infty$ ■



3 Existence and Uniqueness of Solutions to First Order Linear IVP

Problem 3.1
Find $p(t)$ and y_0 so that the function $y(t) = 3e^{t^2}$ is the solution to the IVP $y' + p(t)y = 0, y(0) = y_0$.

Solution.
Since $y(t) = 3e^{t^2}$, we find $y(0) = y_0 = 3e^0 = 3$. On the other hand, $y(t)$ satisfies the equation $y' + p(t)y = 0$ or $6te^{t^2} + p(t)3e^{t^2} = 0$. Hence, $p(t) = -2t$ ■

Problem 3.2
For each of the initial conditions, determine the largest interval $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution.

$$y' + \frac{1}{t^2+1}y = \sin t.$$

(a) $y(0) = \pi$ (b) $y(\pi) = 0$.

Solution.
Here we have $p(t) = \frac{1}{t^2+1}$ and $g(t) = \sin t$.
(a) $(-\infty, \infty)$.
(b) $(-\infty, \infty)$ ■

Problem 3.3
For each of the initial conditions, determine the largest interval $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution

$$y' + \frac{t}{t^2-4}y = \frac{e^t}{t-3}.$$

(a) $y(5) = 2$ (b) $y(-\frac{3}{2}) = 1$ (c) $y(-6) = 2$.

Solution.
Notice that $p(t)$ and $g(t)$ are defined for all $t \neq -2, 2, 3$.
(a) $3 < t < \infty$.
(b) $-2 < t < 2$.
(c) $-\infty < t < -2$ ■

Problem 3.4
(a) For what values of the constant C and the exponent r is $y = Ct^r$ the solution of the IVP

$$2ty' - 6y = 0, y(-2) = 8?$$

(b) Determine the largest interval of the form $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution.
(c) What is the actual interval of existence for the solution found in part (a)?

Solution.
(a) Substitution leads to $2trCt^{r-1} - 6Ct^r = 0$. Divide through by Ct^r to obtain $2r - 6 = 0$ or $r = 3$. Now, since $y(-2) = 8$ we find $C(-2)^3 = 8$ or $C = -1$. Thus, $y(t) = -t^3$.
(b) Rewriting the equation in the form

$$y' - \frac{3}{t}y = 0$$

so that $p(t) = -\frac{3}{t}$ and $g(t) = 0$. The largest interval of the form $a < t < b$ that guarantees the existence of a unique solution is the interval $-\infty < t < 0$ since -2 is in that interval.
(c) By part (a) the actual interval of existence is the set of all real numbers ■

Problem 3.5
Solve the IVP

$$y' + 0.196y = 9.8, y(0) = 48.$$

Solution.
Let $p(t) = 0.196, g(t) = 9.8, t_0 = 0, y_0 = 48$ in Equation (3) to obtain (detailed left to the reader)

$$y(t) = 50 - 2e^{-0.196t} \blacksquare$$

Problem 3.6
Solve the IVP

$$y' + \frac{2}{t}y = 4t, y(1) = 2.$$

Solution.
Let $I(t) = e^{\int \frac{2}{t}ds} = t^2$. Then

$$\begin{aligned}(I(t)y)' =& 4tI(t)\\ I(t)y =& t^4 + C\\ y(t) =& t^2 + \frac{C}{t^2}.\end{aligned}$$

Since $y(1) = 2$ we find $C = 1$. Hence, the unique solution is $y(t) = t^2 + \frac{1}{t^2}$ ■

Problem 3.7
Let $w(t)$ be the unique solution to $w' + p(t)w = 0$ for all $a < t < b$ and $w(t_0) = w_0$. Show that either $w(t) \equiv 0$ for all $a < t < b$ or $w(t) \neq 0$ for all $a < t < b$ depending on whether $w_0 = 0$ or $w_0 \neq 0$. This result will be very useful when discussing Abel's Theorem (i.e., Theorem 16.3) in Section 16.

Solution.
By Equation (5), $w(t) = w(t_0)e^{\int_{t_0}^t p(s)ds}$. If $w_0 = 0$ then $w(t) \equiv 0$ for all $a < t < b$. If $w_0 \neq 0$ then $w(t) \neq 0$ for all $a < t < b$ ■

Problem 3.8
What information does the Existence and Uniqueness Theorem gives about the initial value problem $ty' = y + t^3 \cos t,\ y(1) = 1? y(-1) = 1?$

Solution.
The given differential equation can be written as

$$y' - \frac{y}{t} = t^2 \cos t.$$

We have $p(t) = \frac{-1}{t}$ which is continuous for all $t \neq 0$ whereas $g(t) = t^2 \cos t$ is continuous everyhwere. It follows that the interval of existence is $0 < t < \infty$ if $y(1) = 1$ and $-\infty < t < 0$ if $y(-1) = 1$ ■

Problem 3.9
Consider the following differential equation

$$(t-4)y' + 3y = \frac{1}{t^2 + 5t}.$$

Without solving, find the interval over which a unique solution is guaranteed for each of the following initial conditions:
(a) $y(-3) = 4$ (b) $y(1.5) = -2$ (c) $y(-6) = 0$ (d) $y(4.1) = 3$

Solution.
Rewriting the equation in the form

$$y' + \frac{3}{t-4}y = \frac{1}{(t-4)(t^2+5t)}$$

we find that $p(t)$ and $g(t)$ are continuous for all $t \neq -5, 0, 4$.
(a) $-5 < t < 0$.
(b) $0 < t < 4$.
(c) $-\infty < t < -5$.
(d) $4 < t < \infty$ ∎

Problem 3.10
Without solving the initial value problem, $(t-1)y' + (\ln t)y = \frac{2}{t-3}, y(t_0) = y_0$, state whether or not a unique solution is guaranteed to exist for the $y(t_0) = y_0$ listed below. If a unique solution is guaranteed, find the largest interval for which the solution satisfies the differential equation and the initial condition.
(a) $y(2) = 4$ (b) $y(0) = 0$ (c) $y(4) = 2$.

Solution.
Rewriting the equation in the form

$$y' + \frac{\ln t}{t-1}y = \frac{2}{(t-3)(t-1)}$$

we find that $p(t)$ and $g(t)$ are continuous on $(0,1) \cup (1,3) \cup (3,\infty)$
(a) $1 < t < 3$.
(b) No such solution.
(c) $3 < t < \infty$ ∎

Problem 3.11
(a) State precisely the theorem (hypothesis and conclusion) for existence and uniqueness of a first order initial value problem.
(b) Consider the equation $y' + t^2y = e^{t^3}$ with initial conditions $y(t_0) = y_0$. Briefly discuss if this has a solution, if it is unique and why.

Solution.
(a) If $p(t)$ and $g(t)$ are continuous functions in the open interval $I = (a,b)$ and t_0 a point inside I then the IVP

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution $y(t)$ defined on I.
(b) Since $p(t) = t^2$ and $g(t) = e^{t^3}$, the IVP has a unique solution for any choice of t_0 ■

Problem 3.12
Consider the initial value problem

$$y' + p(t)y = g(t), \quad y(3) = 1.$$

Suppose that $p(t)$ and/or $g(t)$ have discontinuities at $t = -2, t = 0$, and $t = 5$ but are continuous for all other values of t. What is the largest interval (a, b) on which the exitence and uniqueness theorem is applied.

Solution.
Because of the initial condition the largest interval of existence guaranteed by the existence and uniqueness theorem is $0 < t < 5$ ■

Problem 3.13
Determine α and y_0 so that the graph of the solution to the initial-value problem

$$y' + \alpha y = 0, \quad y(0) = y_0$$

passes through the points $(1, 4)$ and $(3, 1)$.

Solution.
The general solution is given by $y(t) = y(0)e^{-\alpha t}$. Since $y(1) = 4$ and $y(3) = 1$ we have

$$\frac{y(0)e^{-\alpha}}{y(0)e^{-3\alpha}} = 4$$

Solving for α we find $\alpha = \frac{\ln 4}{2} = \ln 2$. Thus, $y(t) = y(0)e^{-t \ln 2}$. Since $y(1) = 4$, we find $\frac{y(0)}{2} = 4$ so that $y_0 = 8$ ■

Problem 3.14
Match the following objects with the correct description. Every equation matches exactly one description.
(a) $y' = 3y - 5t$.
(b) $\frac{\partial y}{t} = \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2}$.
(c) $y' - y^2 = \sin t$.
(d) $y' + 3y = 0$.

(i) A partial differential equation
(ii) A homogeneous one-dimensional first order linear differential equation.
(iii) A nonlinear first order differential equation.
(iv) An nonhomogenous first order linear differential equation

Solution.
(a) (iv) (b) (i) (c) (iii) (d) (ii) ■

Problem 3.15
Consider the differential equation $y' = -t^2y + \sin y$. What is the order of this equation? Is it linear or nonlinear?

Solution.
A linear first order ordinary differential equation ■

Problem 3.16
Verify that $y(t) = e^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}$ is a solution of the differential equation $y' - 2ty = 1$.

Solution.
Finding the derivative we obtain

$$y' = 2te^{t^2}\int_0^t e^{-s^2}ds + e^{-t^2}\cdot e^{t^2} + 2te^{t^2}$$

Thus,

$$\begin{aligned} y' - 2ty =& 2te^{t^2}\int_0^t e^{-s^2}ds + 1 + 2te^{t^2} \\ & -2te^{t^2}\int_0^t e^{-s^2}ds - 2te^{t^2} = 1 \ \blacksquare \end{aligned}$$

Problem 3.17
Consider the initial value problem

$$y' = -\frac{y}{t} + 2, \quad y(1) = 2.$$

(a) Are the conditions of the Existence and Uniqueness theorem satisfied? Why or why not?
(b) Solve the IVP and state the domain of definition.

Solution.
(a) Since $p(t) = \frac{1}{t}$ and $g(t) = 2$, both functions are continuous for all $t \neq 0$. Since the initial condition is at $t = 1$, a unique solution on the interval $0 < t < \infty$ exists.
(b) We have

$$\begin{aligned} y' + \frac{y}{t} =& 2 \\ \left(e^{\int \frac{1}{t}dt} y\right)' =& 2e^{\int \frac{1}{t}dt} \\ (ty)' =& 2t \\ ty =& t^2 + C \\ y =& t + Ct^{-1}. \end{aligned}$$

Since $y(1) = 2$, $C = 1$. Thus, $y(t) = t + t^{-1}$. The domain of this function consists for all nonzero real numbers ■

Problem 3.18
Solve the differential equation $y'' + y' = e^t$ as follows. Let $z = y' + y$, find a differential equation for z, and find the general solution. Then using this general value of z, find y by solving the differential equation $y' + y = z$.

Solution.
The differential equation in terms of z is $z' = e^t$. Thus, $z(t) = e^t + C$. Thus, $y' + y = e^t + C$. We solve this equation as follows:

$$\begin{aligned} y' + y =& e^t + C \\ \left(e^{\int dt} y\right)' =& e^{2t} + Ce^t \\ e^t y =& \frac{1}{2}e^{2t} + Ce^t + C' \\ y =& \frac{1}{2}e^t + C'e^{-t} + C \ \blacksquare \end{aligned}$$

4 Solving First Order Linear Homogeneous DE

Problem 4.1
Solve the IVP

$$y' = -2ty, \ y(1) = 1.$$

Solution.
First we rearrange the equation to the form recognizable as first-order linear.

$$y' + 2ty = 0.$$

From this we see that $p(t) = 2t$ so that $\int 2tdt = t^2$. Thus, the general solution to the DE is $y(t) = Ce^{-t^2}$. But $y(1) = 1$ so that $C = e$. Hence, $y(t) = e^{1-t^2}$ ■

Problem 4.2
Solve the IVP

$$y' + e^t y = 0, \ y(0) = 2.$$

Solution.
Since $p(t) = e^t$, $\int e^t dt = e^t$ so that the general solution to the DE is $y(t) = Ce^{-e^t}$. But $y(0) = 2$ so that $C = 2e$. Hence, the unique solution is $y(t) = e^{2-e^t}$ ■

Problem 4.3
Consider the first order linear nonhomogeneous IVP

$$y' + p(t)y = \alpha p(t), \ y(t_0) = y_0.$$

(a) Show that the IVP can be reduced to a first order linear homogeneous IVP by the change of variable $z(t) = y(t) - \alpha$.
(b) Solve this initial value problem for $z(t)$ and use the solution to determine $y(t)$.

Solution.
(a) Note that the given DE can be written as $y' + p(t)(y - \alpha) = 0$. Since $z(t) = y(t) - \alpha$, we get the IVP

$$z' + p(t)z = 0, \quad z(t_0) = y(t_0) - \alpha.$$

(b)The general solution to the DE is $z(t) = (y_0 - \alpha)e^{-\int_{t_0}^t p(s)ds}$. Thus, $y(t) = (y_0 - \alpha)e^{-\int_{t_0}^t p(s)ds} + \alpha$ ■

Problem 4.4
Apply the results of the previous problem to solve the IVP

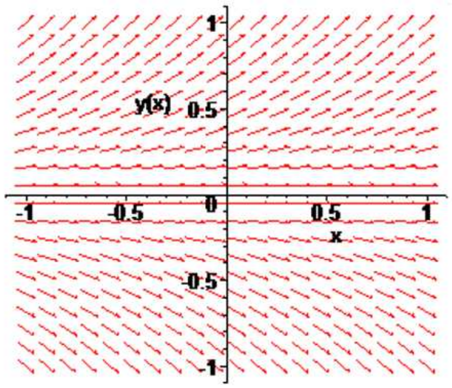
$$y' + 2ty = 6t, \quad y(0) = 4.$$

Solution.
Letting $z(t) = y(t) - 3$ the given IVP reduces to

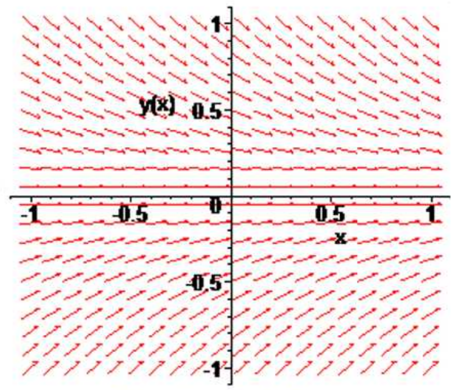
$$z' + 2tz = 0, z(0) = 1.$$

The unique solution to this IVP is $z(t) = e^{-t^2}$. Hence, $y(t) = e^{-t^2} + 3$ ■

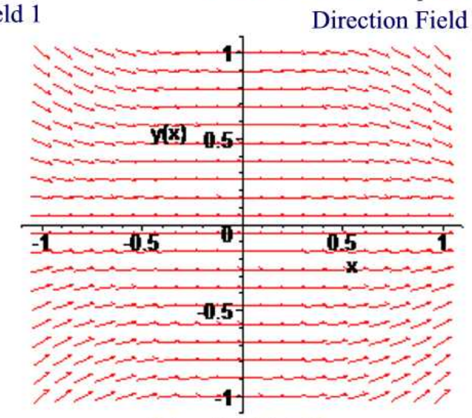
Problem 4.5
Consider the three direction fields shown below. Match each of the direction field with one of the following differential equations.
(a) $y' + y + 0$ (b) $y' + t^2y = 0$ (c) $y' - y = 0$.



Direction Field 1



Direction Field 2



Direction Field 3

Solution.
(a) Direction Field 2 (b) Direction Field 3 (c) Direction Field 1 ■

Problem 4.6
The unique solution to the IVP

$$ty' - \alpha y = 0, y(1) = y_0$$

goes through the points $(2, 1)$ and $(4, 4)$. Find the values of α and y_0.

Solution.
Rewriting the given IVP in the standard form

$$y' - \frac{\alpha}{t}y = 0, \quad y(1) = y_0$$

we find $p(t) = -\frac{\alpha}{t}$ and $\int -\frac{\alpha}{t}dt = -\alpha \ln|t| = \ln|t|^{-\alpha}$. Thus, the general solution to the DE is given by $y(t) = Ce^{-\ln|t|^{-\alpha}} = C|t|^{\alpha}$. But $y(2) = 1$ and $y(4) = 4$ so that $C2^{\alpha} = 1$ and $C4^{\alpha} = 4$. Taking the ratio of these last equations we find $2^{\alpha} = 4$ and thus $\alpha = 2$. From this we find $C = 2^{-\alpha} = 0.25$. Finally, $y_0 = y(1) = 0.25(1)^2 = 0.25$ ■

Problem 4.7
The table below lists values of t and $\ln[y(t)]$ where $y(t)$ is the unique solution to the IVP

$$y' + t^n y = 0, \quad y(0) = y_0.$$

t	1	2	3	4
$\ln[y(t)]$	-0.25	-4.00	-20.25	-64.00

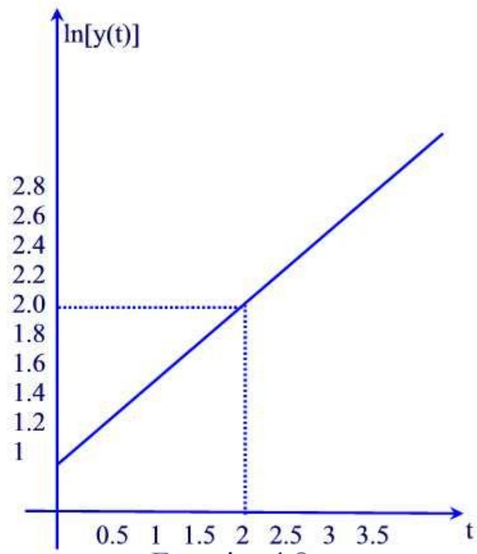
(a) Determine the values of n and y_0.
(b) What is $y(-1)$?

Solution.
(a) The general solution to the DE is $y(t) = Ce^{-\frac{t^{n+1}}{n+1}}$. Since $y(0) = y_0$, $C = y_0$ so that the unique solution is $y(t) = y_0 e^{-\frac{t^{n+1}}{n+1}}$. Thus, $\ln[y(t)] = \ln(y_0) - \frac{t^{n+1}}{n+1}$. Since $\ln y(1) = -\frac{1}{4}$ and $\ln y(2) = -4$ we find $\ln y_0 - \frac{1}{n+1} - \ln y_0 + \frac{2^{n+1}}{n+1} = 4 - \frac{1}{4} = \frac{15}{4}$. Thus, $\frac{2^{n+1}-1}{n+1} = \frac{15}{4}$. Using a calculator one finds $n = 3$. Finally, $\ln y_0 = -\frac{1}{4} + \frac{1}{n+1} = -\frac{1}{4} + \frac{1}{4} = 0$ so that $y_0 = 1$.
(b) $y(-1) = 1 \cdot e^{-\frac{(-1)^4}{4}} = \sqrt[4]{e^{-1}}$ ■

Problem 4.8
The figure below is the graph of $\ln[y(t)]$ versus $t, 0 \le t \le 4$, where $y(t)$ is the solution to the IVP

$$y' + p(t)y = 0, \quad y(0) = y_0.$$

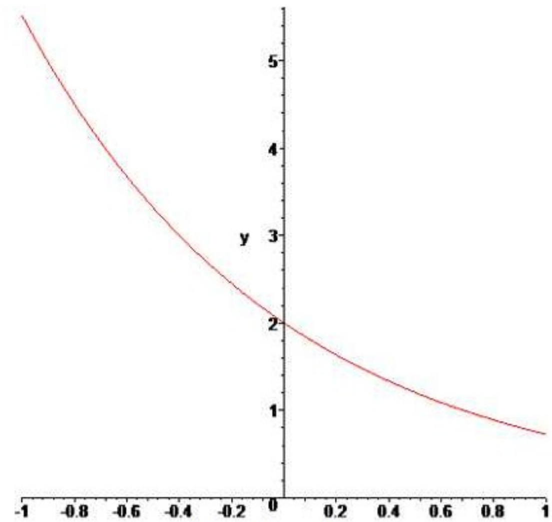
Determine $p(t)$ and y_0



Exercise 4.8

Solution.
From the figure we see that $\ln y_0 = \ln y(0) = 1$ so that $y_0 = e$. Also, $\ln y(t) = \frac{t}{2} + 1$. Thus, $p(t) = -\frac{d}{dt}(\ln y) = -\frac{1}{2}$ ∎

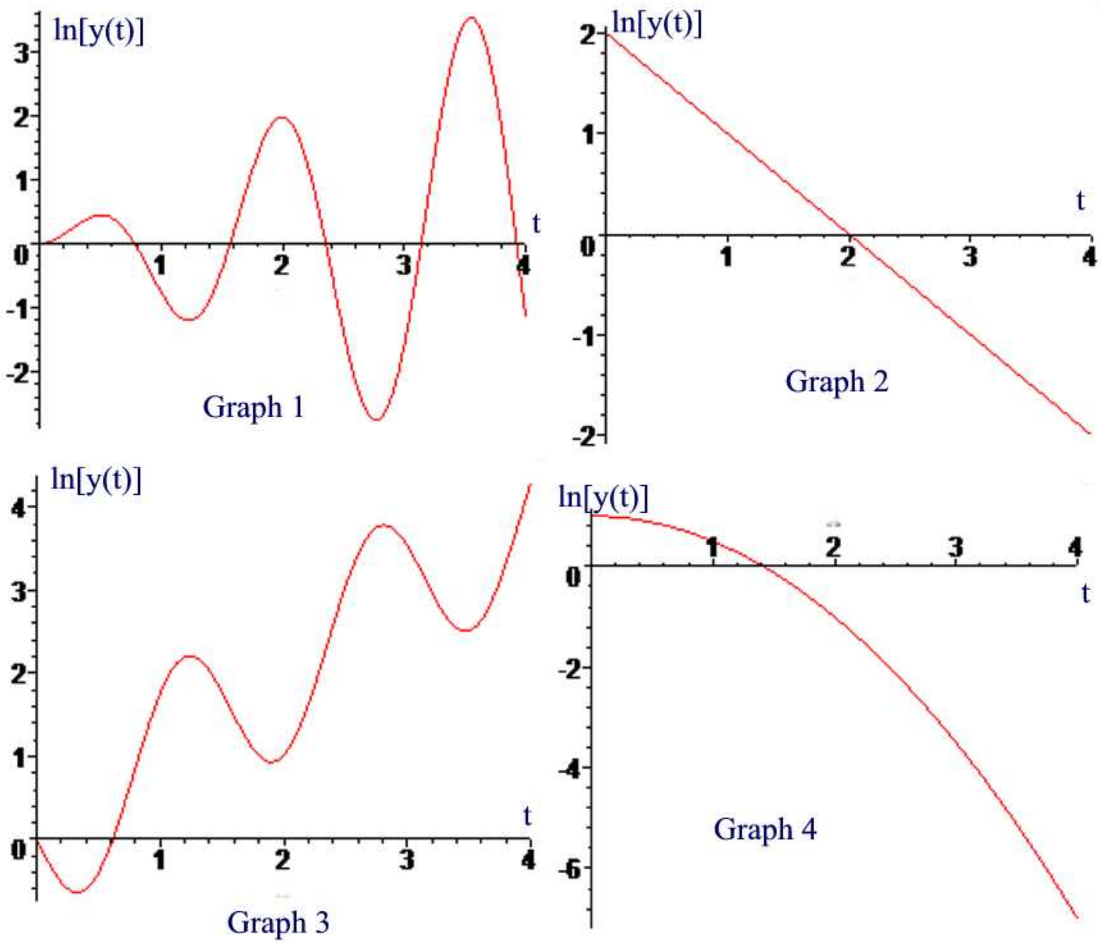
Problem 4.9
Given the initial value problem $y' + cy = 0, y(0) = y_0$. A portion of the graph of the solution is shown. Use the information contained in the graph to determine the constants c and y_0.



Exercise 4.9

Solution.
Solving the given differential equation we find $y(t) = y_0 e^{-ct}$. From the graph we have that $y(0) = 2$ so that $y_0 = 2$. Thus, $y(t) = 2e^{-ct}$. Again, using the graph we see that $y(-0.4) = 3$, that is $2e^{0.4c} = 3$. Solving for c we find $c = 2.5 \ln(1.5)$ ■

Problem 4.10
Given the four graphs of $\ln[y(t)]$ versus $0 \le t \le 4$, corresponding of the four differential equations (a)-(d). Match the graphs to the differential equations. For each match identify the initial condition, $y(0)$.
(a) $y' + y = 0$ (b) $y' - (\sin(4t) + 4t\cos(4t))y = 0$ (c) $y' + ty = 0$ (d) $y' - (1 - 4\cos(4t))y = 0$.



Solution.
(a) Solving the DE we find $y(t) = y_0 e^{-t}$. Taking the natural logarithm of both sides we find $\ln[y(t)] = -t + \ln y_0$. This DE corresponds to Graph # 2 with $y_0 = y(0) = e^{\ln y(0)} = e^2$.
(b) Solving the DE we find $y(t) = y_0 e^{t \sin(4t)}$. Taking the natural logarithm of both sides we find $\ln[y(t)] = t\sin(4t) + \ln y_0$. This DE corresponds to Graph # 1 with $y_0 = y(0) = e^{\ln y(0)} = 1$.
(c) Solving the DE we find $y(t) = y_0 e^{-\frac{t^2}{2}}$. Taking the natural logarithm of both sides we find $\ln[y(t)] = -\frac{t^2}{2} + \ln y_0$. This DE corresponds to Graph # 4 with $y_0 = y(0) = e^{\ln y(0)} = e$.
(d) Solving the DE we find $y(t) = y_0 e^{t - \sin(4t)}$. Taking the natural logarithm of both sides we find $\ln[y(t)] = t - \sin(4t) + \ln y_0$. This DE corresponds to

Graph # 3 with $y_0 = y(0) = e^{\ln y(0)} = 1$ ■

Problem 4.11
Consider the differential equation $y' + p(t)y = 0$. Find $p(t)$ so that $y = \frac{c}{t}$ is the general solution.

Solution.
Substituting in the equation we find

$$-\frac{c}{t^2} + \frac{c}{t}p(t) = 0.$$

Solving for $p(t)$ we find $p(t) = \frac{1}{t}$ ■

Problem 4.12
Consider the differential equation $y' + p(t)y = 0$. Find $p(t)$ so that $y = ct^3$ is the general solution.

Solution.
Substituting in the equation we find

$$3ct^2 + p(t)(ct^3) = 0.$$

Solving for $p(t)$ we find $p(t) = -\frac{3}{t}$ ■

Problem 4.13
Solve the initial-value problem: $y' - \frac{3}{t}y = 0,\ y(2) = 8.$

Solution.
From the previous problem, we see that $y(t) = ct^3$ is the general solution. Since $y(2) = 8,\ c2^3 = 8$ and thus $c = 1$. The unique solution to the initial-value problem is $y(t) = t^3$ ■

Problem 4.14
Solve the differential equation $y' - 2ty = 0.$

Solution.
Since $p(t) = -2t$, $y(t) = Ce^{\int 2tdt} = ce^{t^2}$ ■

Problem 4.15
Solve the initial-value problem $\frac{dP}{dt} - kP = 0, \quad P(0) = P_0.$

Solution.
The general solution to the differential equation is $P(t) = Ce^{kt}$. Since $P(0) = P_0$, $C = P_0$. Thus, $P(t) = P(0)e^{kt}$ ■

Problem 4.16
Find the value of t so that $P(t) = \frac{P_0}{2}$ where $P(t)$ is the solution to the initial-value problem $\frac{dP}{dt} = -kP, k > 0, P(0) = P_0$.

Solution.
From the previous problem, we have $P(t) = P_0e^{-kt}$. If $P(t) = \frac{P_0}{2}$ then $e^{-kt} = 0.5$. Solving for t we find $t = \frac{\ln 2}{k}$ ■

Problem 4.17
Find the function $f(t)$ that crosses the point $(0, 4)$ and whose slope satisfies $f'(t) = 2f(t)$.

Solution.
Solving the differential equation we find $f(t) = Ce^{2t}$. Since $f(0) = 4$ we find $C = 4$. Thus, $f(t) = 4e^{2t}$ ■

Problem 4.18
Find the general solution to the differential equation $y'' - 2y' = 0$.

Solution.
Let $z = y'$ so that $z' = y''$. Thus, $z' - 2z = 0$ and $y'(t) = z(t) = Ce^{2t}$. Hence, $y(t) = Ce^{2t} + C'$ ■

Problem 4.19
Consider the differential equation: $y' = 3y - 2$.
(a) Find the general solution y_h to the equation $y' = 3y$.
(b) Show that $y_p = \frac{2}{3}$ is a solution to $y' = 3y - 2$.
(c) Show that $y = y_h + y_p$ satisfies the given equation.
(d) Find the solution to the initial-value problem $y' = 3y - 2, \ y(0) = 2$.

Solution.
(a) $y_h(t) = Ce^{3t}$.
(b) $y'_p = 0$ and $3y_p - 2 = 3(\frac{2}{3}) - 2 = 0$ so that $y'_p = 3y_p - 2$.
(c) $y' = y'_h + y'_p = 3Ce^{3t}$ and $3y - 2 = 3Ce^{3t} + 2 - 2 = 3Ce^{3t}$.
(d) Since $y(t) = Ce^{3t} + \frac{2}{3}$ and $y(0) = 2$ we find $C + \frac{2}{3} = 2$ and $C = \frac{4}{3}$. Thus, $y(t) = \frac{4}{3}e^{3t} + \frac{2}{3}$ ■

Problem 4.20
Consider the differential equation $y'' = 3y' - 2$.
(a) Find the general solution y_h to the equation $y'' = 3y'$.
(b) Show that $y_p = \frac{2}{3}t$ is a solution to $y'' = 3y' - 2$.
(c) Show that $y = y_h + y_p$ satisfies the given equation.

Solution.
(a) Let $z = y'$. Then $z' = 3z$ and $z(t) = Ce^{3t}$. Thus, $y_h(t) = \int z(t)dt = Ce^{3t} + C'$.
(b) Since $y_p'' = 0$ and $3y_p' - 2 = 2 - 2 = 0$ we find $y_p'' = 3y_p' - 2$.
(c) Since $y'' = y_h'' + y_p'' = 9Ce^{3t}$ and $3y' - 2 = 9Ce^{3t} + 2 - 2 = 9Ce^{3t}$, y satisfies the differential equation ■

5 Solving First Order Linear Non Homogeneous DE: The Method of Integrating Factor

Problem 5.1
Solve the IVP: $y' + 2ty = t, \quad y(0) = 0.$

Solution.
Since $p(t) = 2t$, $\mu(t) = e^{\int 2tdt} = e^{t^2}$. Multiplying the given equation by e^{t^2} to obtain

$$\left(e^{t^2}y\right)' = te^{t^2}.$$

Integrating both sides with respect to t and using substitution on the right-hand integral to obtain

$$e^{t^2}y = \frac{1}{2}e^{t^2} + C.$$

Dividing the last equation by e^{t^2} to obtain

$$y(t) = Ce^{-t^2} + \frac{1}{2}.$$

Since $y(0) = 0$, $C = -\frac{1}{2}$. Thus, the unique solution to the IVP is given by

$$y = \frac{1}{2}(1 - e^{-t^2}) \blacksquare$$

Problem 5.2
Find the general solution: $y' + 3y = t + e^{-2t}$.

Solution.
Since $p(t) = 3$, the integrating factor is $\mu(t) = e^{3t}$. Thus, the general solution is

$$\begin{aligned}
y(t) =& e^{-3t}\int e^{3t}(t + e^{-2t})dt + Ce^{-3t} \\
=& e^{-3t}\int (te^{3t} + e^{t})dt + Ce^{-3t} \\
=& e^{-3t}\left(\frac{e^{3t}}{9}(3t-1) + e^{t}\right) + Ce^{-3t} \\
=& \frac{3t-1}{9} + e^{-2t} + Ce^{-3t} \blacksquare
\end{aligned}$$

Problem 5.3
Find the general solution: $y' + \frac{1}{t}y = 3\cos t,\ t > 0.$

Solution.
Since $p(t) = \frac{1}{t}$, the integrating factor is $\mu(t) = e^{\int \frac{dt}{t}} = e^{\ln t} = t$. Using the method of integrating factor we find

$$\begin{aligned} y(t) =& \frac{1}{t}\int 3t\cos t dt + \frac{C}{t} \\ =& \frac{3}{t}\left(t\sin t + \cos t\right) + \frac{C}{t} \\ =& 3\sin t + \frac{3\cos t}{t} + \frac{C}{t} \blacksquare \end{aligned}$$

Problem 5.4
Find the general solution: $y' + 2y = \cos(3t)$.

Solution.
We have $p(t) = 2$ so that $\mu(t) = e^{2t}$. Thus,

$$y(t) = e^{-2t}\int e^{2t}\cos(3t)dt + Ce^{-2t}$$

But

$$\begin{aligned} \int e^{2t}\cos(3t)dt =& \frac{e^{2t}}{3}\sin(3t) - \frac{2}{3}\int e^{2t}\sin(3t)dt \\ =& \frac{e^{2t}}{3}\sin(3t) - \frac{2}{3}\left(-\frac{e^{2t}}{3}\cos(3t) + \frac{2}{3}\int e^{2t}\cos(3t)dt\right) \\ \frac{13}{9}\int e^{2t}\cos(3t)dt =& \frac{e^{2t}}{9}(3\sin(3t) + 2\cos(3t)) \\ \int e^{2t}\cos(3t)dt =& \frac{e^{2t}}{13}(3\sin(3t) + 2\cos(3t)). \end{aligned}$$

Hence,

$$y(t) = \frac{1}{13}(3\sin(3t) + 2\cos(3t)) + Ce^{-2t} \blacksquare$$

Problem 5.5
Find the general solution: $y' + (\cos t)y = -3\cos t$.

Solution.
Since $p(t) = \cos t$, $\mu(t) = e^{\sin t}$. Thus,

$$\begin{aligned} y(t) =& e^{-\sin t}\int e^{\sin t}(-3\cos t)dt + Ce^{-\sin t} \\ =& -3e^{-\sin t}e^{\sin t} + Ce^{-\sin t} \\ =& Ce^{-\sin t} - 3 \blacksquare \end{aligned}$$

Problem 5.6
Given that the solution to the IVP $ty' + 4y = \alpha t^2, \ y(1) = -\frac{1}{3}$ exists on the interval $-\infty < t < \infty$. What is the value of the constant α?

Solution.
Solving this equation with the integrating factor method with $p(t) = \frac{4}{t}$ we find $\mu(t) = t^4$. Thus,

$$\begin{aligned} y =& \frac{1}{t^4}\int t^4(\alpha t)dt + \frac{C}{t^4} \\ =& \frac{\alpha}{6}t^2 + \frac{C}{t^4}. \end{aligned}$$

Since the solution is assumed to be defined for all t, we must have $C = 0$. On the other hand, since $y(1) = -\frac{1}{3}$ we find $\alpha = -2$ $\blacksquare$

Problem 5.7
Suppose that $y(t) = Ce^{-2t} + t + 1$ is the general solution to the equation $y' + p(t)y = g(t)$. Determine the functions $p(t)$ and $g(t)$.

Solution.
The integrating factor is $\mu(t) = e^{2t}$. Thus, $\int p(t)dt = 2t$ and this implies that $p(t) = 2$. On the other hand, the function $t + 1$ is the particular solution to the nonhomogeneous equation so that $(t + 1)' + 2(t + 1) = g(t)$. Hence, $g(t) = 2t + 3$ $\blacksquare$

Problem 5.8
Suppose that $y(t) = -2e^{-t} + e^t + \sin t$ is the unique solution to the IVP $y' + y = g(t), \ y(0) = y_0$. Determine the constant y_0 and the function $g(t)$.

Solution.
First, we find $y_0 : y_0 = y(0) = -2 + 1 + 0 = -1$. Next, we find $g(t) : g(t) = y' + y = (-2e^{-t} + e^t + \sin t)' + (-2e^{-t} + e^t + \sin t) = 2e^{-t} + e^t + \cos t - 2e^{-t} + e^t + \sin t = 2e^t + \cos t + \sin t$ ■

Problem 5.9
Find the value (if any) of the unique solution to the IVP $y' + (1 + \cos t)y = 1 + \cos t,\ y(0) = 3$ in the long run.

Solution.
The integrating factor is $\mu(t) = e^{\int (1+\cos t)dt} = e^{t+\sin t}$. Thus, the general solution is

$$\begin{aligned} y(t) =& e^{-(t+\sin t)} \int e^{t+\sin t}(1 + \cos t)dt + Ce^{-(t+\sin t)} \\ =& 1 + Ce^{-(t+\sin t)}. \end{aligned}$$

Since $y(0) = 3$, $C = 2$ and therefore $y(t) = 1 + 2e^{-(t+\sin t)}$. Finally,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (1 + 2e^{-\sin t}e^{-t}) = 1 \ ■$$

Problem 5.10
Find the solution to the IVP

$$y' + p(t)y = 2, \quad y(0) = 1$$

where

$$p(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ \frac{1}{t} & \text{if } 1 < t \le 2. \end{cases}$$

Solution.
First, we solve the IVP

$$y' = 2, \quad y(0) = 1, \quad 0 \le t \le 1.$$

The general solution is $y_1(t) = 2t + C$. Since $y(0) = 1$, $C = 1$. Hence, $y_1(t) = 2t + 1$ and $y_(1) = 3$.
Next, we solve the IVP

$$y' + \frac{1}{t}y = 2, \quad y(1) = 3, \quad 1 < t \le 2.$$

The integrating factor is $\mu(t) = t$ and the general solution is $y_2(t) = t + \frac{C}{t}$. Since $y_2(1) = 3$, $C = 2$. Thus,

$$y(t) = \begin{cases} 2t+1 & \text{if } 0 \le t \le 1 \\ t + \frac{2}{t} & \text{if } 1 < t \le 2 \end{cases} \blacksquare$$

Problem 5.11
Find the solution to the IVP

$$y' + (\sin t)y = g(t), \quad y(0) = 3$$

where

$$g(t) = \begin{cases} \sin t & \text{if } 0 \le t \le \pi \\ -\sin t & \text{if } \pi < t \le 2\pi. \end{cases}$$

Solution.
First, we solve the IVP

$$y' + \sin t y = \sin t, \quad y(0) = 3, \quad 0 \le t \le \pi.$$

The integrating factor is $\mu(t) = e^{-\cos t}$ and the general solution is $y_1(t) = 1 + Ce^{\cos t}$. Since $y_1(0) = 3$, $C = 2e^{-1}$. Hence, $y_1(t) = 1 + 2e^{\cos t - 1}$ and $y_1(\pi) = 1 + 2e^{-2}$.
Next, we solve the IVP

$$y' + \sin t y = -\sin t, \quad y(\pi) = 1 + 2e^{-2}, \quad \pi < t \le 2\pi.$$

The integrating factor is $\mu(t) = e^{-\cos t}$ and the general solution is $y_2(t) = -1 + Ce^{\cos t}$. Since $y_2(\pi) = 1 + 2e^{-2}$, $C = 2\left(\frac{1}{e} - e\right)$. Thus,

$$y(t) = \begin{cases} 1 + 2e^{\cos t - 1} & \text{if } 0 \le t \le \pi \\ -1 + 2\left(\frac{1}{e} - e\right)e^{\cos t} & \text{if } \pi < t \le 2\pi \end{cases} \blacksquare$$

Problem 5.12
Find the solution to the IVP

$$y' + y = g(t), \; t > 0, \quad y(0) = 3$$

where

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1. \end{cases}$$

Sketch an accurate graph of the solution, and discuss the long-term behavior of the solution. Is the solution differentiable on the interval $t > 0$? Explain your answer.

Solution.
First, we solve the IVP

$$y' + y = 1, \quad y(0) = 3, \quad 0 \le t \le 1.$$

The integrating factor is $\mu(t) = e^t$ and the general solution is $y_1(t) = 1+Ce^{-t}$. Since $y_1(0) = 3$, $C = 2$. Hence, $y_1(t) = 1 + 2e^{-t}$ and $y_1(1) = 1 + 2e^{-1}$.
Next, we solve the IVP

$$y' + y = 0, \quad y(1) = 1 + 2e^{-1}, \quad t > 1.$$

The integrating factor is $\mu(t) = e^t$ and the general solution is $y_2(t) = Ce^{-t}$. Since $y_2(1) = 1 + 2e^{-1}$, $C = 2 + e$. Thus,

$$y(t) = \begin{cases} 1 + 2e^{-t} & \text{if } 0 \le t \le 1 \\ (2+e)e^{-t} & \text{if } t > 1 \end{cases} \blacksquare$$

Problem 5.13
Find the solution to the IVP

$$y' + p(t)y = 0, \quad y(0) = 3$$

where

$$p(t) = \begin{cases} 2t - 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t \le 3 \\ -\frac{1}{t} & \text{if } 3 < t \le 4. \end{cases}$$

Solution.
First, we solve the IVP

$$y' + (2t-1)y = 0, \quad y(0) = 3, \quad 0 \le t \le 1.$$

The integrating factor is $\mu(t) = e^{t^2-t}$ and the general solution is $y_1(t) = Ce^{t-t^2}$. Since $y_1(0) = 3$, $C = 3$. Hence, $y_1(t) = 3e^{t-t^2}$ and $y_1(1) = 3$.
Next, we solve the IVP

$$y' = 0, \quad y(1) = 3, \quad 1 < t \le 3.$$

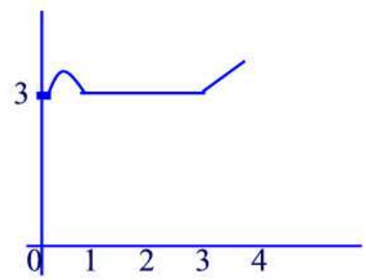
The general solution is $y_2(t) = C$. Since $y_2(1) = 3$, $C = 3$ and $y_2(t) \equiv 3$.
Next, we solve the IVP

$$y' - \frac{1}{t}y = 0, \quad y(3) = 3, \quad 3 < t \le 4.$$

The integrating factor is $\mu(t) = \frac{1}{t}$ and the general solution is $y_3(t) = Ct$. Since $y_3(3) = 3$, $C = 1$. Hence, $y_3(t) = t$. Hence,

$$y(t) = \begin{cases} 3e^{t-t^2} & \text{if } 0 \le t \le 1 \\ 3 & \text{if } 1 < t \le 3 \\ t & \text{if } 3 < t \le 4. \end{cases}$$

The graph of $y(t)$ is shown below



It follows that $\lim_{t\to\infty} y(t) = \infty$. The function $y(t)$ is not differentiable at $t = 1$ and $t = 3$ on the domain $t > 0$ ■

Problem 5.14
Solve $y' - \frac{1}{t}y = \sin t, \quad y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(t) = \int_0^t \frac{\sin s}{s} ds$.

Solution.
Since $p(t) = -\frac{1}{t}$, $\mu(t) = \frac{1}{t}$. Thus,

$$\begin{aligned} \left(\frac{1}{t}y\right)' =& \left(\int_0^t \frac{\sin s}{s} ds\right)' \\ \frac{1}{t}y(t) =& Si(t) + C \\ y(t) =& tSi(t) + Ct. \end{aligned}$$

Since $y(1) = 3$, $C = 3 - Si(1)$. Hence, $y(t) = tSi(t) + (3 - Si(1))t$ ■

Problem 5.15
Solve the initial-value problem $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$.

Solution.
Rewriting the equation in the form

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}.$$

Since $p(t) = \frac{2}{t}$, $\mu(t) = t^2$. The general solution is then given by

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}.$$

Since $y(1) = \frac{1}{2}$, $C = \frac{1}{12}$. Hence,

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2} \blacksquare$$

Problem 5.16
Solve the initial-value problem $y' + y = e^t y^2, y(0) = 1$ using the substitution $u(t) = \frac{1}{y(t)}$.

Solution.
Substituting into the equation we find

$$u' - u = -e^t, \quad u(0) = 1.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{-t}$ we find

$$u(t) = -te^t + Ce^t.$$

Since $u(0) = 1$, $C = 1$ and therefore $u(t) = -te^t + e^t$. Finally, we have

$$y(t) = (-te^t + e^t)^{-1} \blacksquare$$

Problem 5.17
Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$. Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Solution.
Since $p(t) = a$, $\mu(t) = e^{at}$. Suppose first that $a = \lambda$. Then

$$y' + ay = be^{-at}$$

and the corresponding general solution is

$$y(t) = bte^{-at} + Ce^{-at}.$$

Thus,

$$\begin{array}{rcl}\lim_{t\to\infty} y(t) & = & \lim_{t\to\infty}(\frac{bt}{e^{at}} + \frac{C}{e^{at}}) \\ & = & \lim_{t\to\infty}\frac{b}{ae^{at}} = 0.\end{array}$$

Now, suppose that $a \neq \lambda$ then

$$y(t) = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}.$$

Thus,

$$\lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 5.18
Solve the initial value problem $ty' = y + t, \quad y(1) = 7$.

Solution.
Rewriting the equation in the form

$$y' - \frac{1}{t}y = 1$$

we find $p(t) = -\frac{1}{t}$ and $\mu(t) = \frac{1}{t}$. Thus, the general solution is given by

$$y(t) = t\ln|t| + Ct.$$

But $y(1) = 7$ so that $C = 7$. Hence,

$$y(t) = t\ln|t| + 7t \blacksquare$$

Problem 5.19
Solve the differential equation $y' = -ay + b$ by using the susbtitution $w = -ay + b$ where a and b are constants with $a \neq 0$ and $y(t) \neq \frac{b}{a}$.

Solution.
Letting $w = -ay + b$ we find $w' + aw = 0$. Thus, $\frac{w'}{w} = -a$. Integrating both sides with respect to t we obtain $\ln|w(t)| = -at + C$. Thus, $w(t) = Ce^{-at}$. From this we find $y(t) = \frac{b-w}{a} = \frac{b}{a} + Ce^{-at}$ ■

Problem 5.20
Consider the following method of solving the equation

$$y' + p(t)y = g(t).$$

(a) Show that $y_h(t) = Ce^{-\int p(t)dt}$ is the general solution to the homogeneous equation $y' + p(t)y = 0$.
(b) Find a funcion $u(t)$ such that $y_p(t) = u(t)e^{-\int p(t)dt}$ is a solution to the nonhomogeneous equation.
This technique of finding a solution to the nonhomogeneous equation is known as the method of **variation of parameters**.

Solution.
(a) If $g(t) \equiv 0$ then $y' + p(t)y = 0$. Thus, $\left(e^{\int p(t)dt}y\right)' = 0$. Integrating with respect to t to obtain $e^{\int p(t)dt}y = C$. Hence, $y(t) = Ce^{-\int p(t)dt}$.
(b) Substituting y_p and its derivative in the equation we obtain

$$u'e^{-\int p(t)dt} - p(t)ue^{-\int p(t)dt} + p(t)ue^{-\int p(t)dt} = g(t).$$

Thus,

$$u'e^{-\int p(t)dt} = g(t)$$

and solving for $u'(t)$ we find

$$u'(t) = e^{\int p(t)dt}g(t).$$

Integrating this last equation we find

$$u(t) = \int e^{\int p(t)dt}g(t).$$

Hence,

$$y_p(t) = \left[\int e^{\int p(t)dt}g(t)\right]e^{-\int p(t)dt}$$ ■

6 Modeling with First Order Linear Differential Equations

Problem 6.1
Translating a value to the present is referred to as **discounting.** We call $(1+\frac{r}{n})^{-nt}$ the **discount factor**. What principal invested today will amount to $\$8,000$ in 4 years if it is invested at 8% compounded quarterly?

Solution.
The present value is found using the formula

$$P = B\left(1+\frac{r}{n}\right)^{-nt} = 8,000\left(1+\frac{0.08}{4}\right)^{-16} \approx \$5,827.57 \blacksquare$$

Problem 6.2
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

Solution.

$$\text{effective rate} = \left(1+\frac{0.05}{4}\right)^{4} - 1 \approx 0.051 = 5.1\% \blacksquare$$

Problem 6.3
Suppose you invested $1200 on January 1 of this year in an account at an annual rate of 6%, compounded monthly.
1. Set up (write down) the equation that models this problem.
2. Determine your account balance after 5 years.

Solution.
1. $B(t) = 1200\left(1+\frac{0.06}{12}\right)^{12t}$.

2. $B(5) = 1200\left(1+\frac{0.06}{12}\right)^{12(5)} \approx \$1618.62 \blacksquare$

Problem 6.4
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

Solution.
The effective rate corresponding to the first option is

$$\left(1+\frac{0.08}{4}\right)^4 - 1 \approx 8.24\%.$$

That of the second option

$$e^{0.0795} - 1 \approx 8.27\%.$$

Thus, we see that 7.95% compounded continuously is better than 8% compounded quarterly ■

Problem 6.5
An amount of \$2,000.00 is deposited in a bank paying an annual interest rate of 2.85 %, compounded continuously.
(a) Find the balance after 3 years.
(b) How long would it take for the money to double?

Solution.
Use the continuous compound interest formula, $B = Pe^{rt}$, with $P = 2000, r = 2.85/100 = 0.0285, t = 3$.
(a) Therefore,

$$B = 2000e^{0.0285(3)} \approx \$2178.52.$$

(b) Since the original investment is \$2,000.00, doubling means that the current balance is \$4,000.00. To find out how long it takes for this to happen (i.e. to find t), plug in $P = 2000, B = 4000$, and $r = 0.0285$ in the continuous compound interest formula, and solve for t. Doing this, one gets,

$$\begin{aligned}
2000e^{0.0285t} =& 4000\\
e^{0.0285t} =& 2\\
0.0285t =& \ln 2\\
t =& \frac{\ln 2}{0.0285} \approx 24.32 \text{ years} ■
\end{aligned}$$

Problem 6.6
Carbon-14 is a radioactive isotope of carbon that has a half life of 5600 years. It is used extensively in dating organic material that is tens of thousands of years old. What fraction of the original amount of Carbon-14 in a sample

would be present after 10,000 years?
Recall that the half life of a substance of a decaying material (or population) is the amount of time it takes for 50% of the original amount of substance (or material or population) to decay.

Solution.
Let $m(t)$ be the amount of C-14 present after t years. Since the problem is a decay problem, $m(t)$ satisfies the equation $m(t) = m(0)e^{kt}, k < 0$. Since the half life is given,we

$$\begin{aligned} e^{5600k} =& \frac{1}{2} \\ 5600k =& -\ln 2 \\ k =& -\frac{\ln 2}{5600} \approx -1.2 \times 10^{-4}. \end{aligned}$$

The fraction of the original amount left after 10,000 years is

$$\frac{m(10,000)}{m(0)} = e^{-1.2 \cdot 10^{-4} \cdot 10^4} \approx 0.3.$$

Hence, 30% of the original amount is left after 10,000 years ■

Problem 6.7
In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements iodine-131 whose half-life is 8 days and cesium-137 whose half life is 30 years. Predict how much of this material would remain over time.

Solution.
Let $m_I(t)$ be the amount of Iodine-13 after t days. Then $m_I(t) = m_I(0)e^{kt}$. Since the half-life of Iodine-13 is 8 days, we obtain $0.5 = e^{8k}$. Solving this equation for k we find $k = \frac{\ln 0.5}{8} \approx -0.08664$. Thus, $m_I(t) = m_I(0)e^{-0.08664t}$. Now, let $m_C(t)$ be the amount of Cesium-137 after t years. Then $m_C(t) = m_C(0)e^{kt}$. Since the half-life is 30 years, we have $e^{30t} = 0.5$. Solving for k we find $k = \frac{\ln 0.5}{30} \approx -0.02315$. Hence, $m_C(t) = m_C(0)e^{-0.02315t}$ ■

Problem 6.8
A team of archaeologists thinks they may have discovered Fred Flintsone's

fossilized bowling ball. But they want to determine whether the fossil is authentic before they report their discovery to ABC's "Nightline." (Otherwise they run the risk of showing up on "Hard Copy" instead.) Fortunately, one of the scientists is a graduate of ATU's Math 3163, so he calls upon his experience as follows:
The radioactive substance (Carbon 14) has a half-life of 5730 years. By measuring the amount of Carbon present in a fossil, scientists can estimate how old the fossil is.
Analysis of the "Flinstone bowling ball" determines that 15% of the radioactive substance has already decayed. How old is the fossil ?

Solution.
Let $m(t)$ denote the amount of the radioactive substance left after t years. Then $m(t) = m(0)e^{kt}, k < 0$. Since the half-life is 5730 years, we obtain $e^{5730k} = 0.5$. Solving for k we find $k = \frac{\ln 0.5}{5730} \approx -1.21 \times 10^{-4}$.
Now, since 15% decayed, $e^{-1.21\times10^{-4}t} = 0.85$. Solving for t we find $t = \frac{\ln 0.85}{-1.21\times10^{-4}} \approx 1343$ years ■

Problem 6.9
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Solution.
Since the problem involves exponential decay, if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50a^t$ with $0 < t < 1$. But $Q(13) = 25$. That is, $50a^{13} = 25$ or $a^{13} = 0.5$. Thus $a = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t$ ■

Problem 6.10
Statistics indicate that the world population since World War II has been growing at the rate of 1.9% per year. Further, United Nations records indicate that the world population in 1975 was (approximately) 4 billion. Assuming an exponential growth model,
(a) what will the population of the world be in the year 2000?
(b) When will the world population be 7 billion?

Solution.
(a) Let $P(t)$ be the world population t years after 1975. Then $P(t) = 4e^{0.19t}$.

In the year 2000, the value of t is 25. In this case, the world population is $P(25) = 4e^{0.19(25)} \approx 6.43$ billion.
(b) We want to find t that satisfies the equation $C(t) = 7$. That is, $4e^{0.19t} = 7$. Solving this equation for t we find $t = \frac{\ln 7/4}{0.19} \approx 29.5$ years ■

Problem 6.11
During the 1980s the population of a certain city went from 100,000 to 205,000. Populations by year are listed in the table below. $N(t)$ is the population (in thousands) at time t (in years).

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
N(t)	100	108	117	127	138	149	162	175	190	205

(a) Use your calculator (i.e. exponential regression) to show that the population satisfies an equation of the form $N(t) = n(0)e^{kt}$.
(b) Use the model to predict the population of the city in 1994.
(c) According to our model, when will the population reach 300 thousand?

Solution.
(a) Using an exponential regression procedure found in a calculator we find $N(t) = 99.8(1.08)^t$.
(b) $N(14) = 99.8(1.08)^{14} \approx 293.132$.
(c) We must solve the equation $300 = 99.8(1.08)^t$. Solving for t we find

$$\begin{aligned} 99.8(1.08)^t =& 300 \\ (1.08)^t =& \frac{300}{99.8} \\ t =& \frac{\ln \frac{300}{99.8}}{\ln 1.08} \approx 14.3. \end{aligned}$$

Thus, the population will surpass the 300,000 mark in the year 1995 ■

Problem 6.12
The population of fish in a pond is modeled by the differential equation

$$\frac{dN}{dt} = 480 - 4N$$

where time t is measured in years.
(a) Towards what number does the population of fish tend?
(b) If there are initially 10 fish in the pond, how long does it take for the number of fish to reach 90% of the eventual population?

Solution.
(a) Using the integrating factor method with $p(t) = 4$ and $\mu(t) = e^{4t}$ we find

$$\begin{aligned} N(t) =& e^{-4t}\int e^{4t}(480)dt + Ce^{-4t} \\ =& 120 + Ce^{-4t} \end{aligned}$$

So in the long run, $\lim_{t\to\infty} N(t) = 120$ fish.
(b) Since $N(0) = 10$, $10 = C + 120$ so that $C = -110$. Thus, $N(t) = -110e^{-4t} + 120$. Now, we are trying to find t such that $N(t) = 90\%(120) = 108$. That is, we must solve the equation $-110e^{-4t} + 120 = 108$. Solving for t we find $t = \frac{\ln \frac{12}{110}}{-0.4} \approx 0.554$ ■

Problem 6.13
The number of bacteria in a liquid culture is observed to grow at a rate proportional to the number of cells present. At the begining of the experiment there are 10,000 cells and after three hours there are 500,000. How many will there be after one day of growth if this unlimited growth continues? What is the doubling time of the bacteria, i.e. the amount of time it takes for the population to double in size?

Solution.
The population model satisfies the initial-value problem

$$\frac{dP}{dt} = kP, \quad P(0) = 10,000.$$

The solution to this IVP is

$$P(t) = 10,000e^{kt}.$$

Since $P(1) = 500,000$, $e^k = 50$ and therefore $k = \ln 50 \approx 3.912$. After one day the population is

$$p(24) = 10,000e^{(3.912)(24)} \approx 5.96 \times 10^{44}.$$

The doubling time is

$$t = \frac{\ln 2}{3.912} \approx 0.177\ hr \blacksquare$$

Problem 6.14
Bacteria is being cultured for the production of medication. Without harvesting the bacteria, the rate of change of the population is proportional to its current population, with a proportionality constant of 0.2 per hour. Also, the bacteria are being harvested at a rate of 1000 per hour. If there are initially 8000 bacteria in the culture, solve the initial value problem:

$$\frac{dN}{dt} = 0.2N - 1000, \ \ N(0) = 8000$$

for the number N of bacteria as a function of time and find the time it takes for the population to double its initial number.

Solution
Using the method of integrating factor we find

$$\begin{aligned}\left(e^{-0.2t}N\right)' =& - 1000e^{-0.2t}\\ e^{-0.2t}N(t) =& 5000e^{-.2t} + C\\ N(t) =& 5000 + Ce^{0.2t}\end{aligned}$$

But $N(0) = 8000$ so that $C = 3000$. Thus, $N(t) = 5000 + 3000e^{-0.2t}$. The doubling time is

$$t = \frac{\ln 2}{0.2} \approx 3.5 \ hours \ \blacksquare$$

Problem 6.15
A small lake supports a population of fish which, under normal circumstances, enjoys a natural birth process with birth rate $r > 0$. However, a fishing company has just discovered the lake and is now drawing fish out of the lake at a rate of h fish per day. A model capturing this situation is:

$$\frac{dP}{dt} = -h + rP, P(0) = P_0.$$

(a) Find the equilibrium level P_e of fish in the lake.
(b) Find $P(t)$ explicitly (i.e. solve the initial value problem.)

Solution.
(a) The equilibrium level occurs when $P_e = \frac{h}{r}$.

(b) Using the method of integrating factor we find

$$\begin{aligned}\left(e^{-rt}P\right)' =& -he^{-rt}\\ e^{-rt}P =& \frac{h}{r}e^{-rt}+C\\ P(t) =& P_e+Ce^{rt}.\end{aligned}$$

But $P(0)=P_0$ so that $C=P_0-P_e$. Hence, $P(t)=P_e+(p_0-P_e)e^{rt}$ ∎

Problem 6.16
The population of mosquitoes in a certain area increases at a rate proportional to the current population and, in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, etc.) eat 20,000 mosquitoes per day. Determine the population of mosquitoes in the area at any time.

Solution.
Since the doubling time is 1, we have

$$k=\ln 2\approx 0.693.$$

The model is given by the differential equation

$$\frac{dP}{dt}=0.693P-20000, \quad P(0)=200,000.$$

Solving this IVP problem we find

$$\begin{aligned}\left(e^{-0.693t}P\right)' =& -20000e^{-0.693t}\\ e^{-0.693t}P(t) =& 28860e^{-0.693t}+C\\ P(t) =& 28860+Ce^{0.693t}.\end{aligned}$$

But $P(0)=200000$ so that $C=171140$. Thus,

$$P(t)=28860+171140e^{0.693t}$$ ∎

Problem 6.17
At the time of the 1990 census the city of Renton, WA had a population of 8000 people. The last (2000) census revealed that the population of Renton was 12000 people. The city planners do not wish to limit growth until the population reaches 18000. Assuming the rate of change of the population is proportional to the population, when will this occur?

Solution.
The population at time t is given by the formula $P(t) = 8000e^{kt}$. But $P(10) = 12000$ so that $e^{10k} = 1.5$. Thus, $k = \frac{\ln 1.5}{10} \approx 0.04$. Thus, $P(t) = 8000e^{0.04t}$. If $P(t) = 18000$ then $e^{0.04t} = 2.25$ so that

$$t = \frac{\ln 2.25}{0.04} \approx 20.27 years \blacksquare$$

Problem 6.18
If initially there are 50 grams of a radioactive substance and after 3 days there are only 10 grams remaining, what percentage of the original amount remains after 4 days?

Solution.
The formula for the quantity of radioactive susbtance after t days is given by $m(t) = 50e^{-kt}$. Since $m(3) = 10$, we have $k = \frac{\ln 5}{3} \approx 0.207$. Hence, $m(t) = 50e^{-0.207t}$. The percentage of the original amount remaining after 4 days is

$$\frac{50 - P(4)}{50} = 1 - e^{-0.828} \approx .563 = 56.3\% \blacksquare$$

Problem 6.19
The half-life of radioactive cobalt is 5.27 years. A sample of radioactive cobalt weighing 100 kilograms is buried in a nuclear waste storage facility. After 200 years, how much cobalt will remain in the sample? (Give the answer in exact form, involving a fractional power of 2.)

Solution.
The mass of radiactive Cobalt after t years is given by $m(t) = 100e^{-kt}$. Since the half-life is 5.27 years we find $\frac{1}{2} = e^{-5.27k}$. Solving for k we find $k = \frac{\ln 2}{5.27}$. Finally, $P(200) = 100e^{-\frac{\ln 2}{5.27}200} = 2^{-\frac{200}{5.27}}$ $\blacksquare$

7 Additional Applications: Mixing Problems and Cooling Problems

Problem 7.1
Consider a tank with volume 100 liters containing a salt solution. Suppose a solution with 2kg/liter of salt flows into the tank at a rate of 5 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 5 liters/min. If initially there is 20 kg of salt in the tank, how much salt will be in the tank as a function of time?

Solution.
Let $y(t)$ denote the amount of salt in kg in the tank after t minutes. We use a fundamental property of rates:

$$\text{Total Rate} = \text{ Rate in } - \text{ Rate out.}$$

To find the rate in we use

$$5\frac{liters}{min}\cdot 2\frac{kg}{liter} = 10\frac{kg}{min}.$$

The rate at which salt leaves the tank is equal to the rate of flow of solution out of the tank times the concentration of salt in the solution. Thus, the rate out is

$$\frac{5\ liters}{min}\cdot\left(\frac{y}{100}\right)\frac{kg}{liter} = \left(\frac{y}{20}\right)\frac{kg}{min}.$$

Notice that the volume is always constant at 100 since the inflow rate and the outflow rate are the same.
The initial value problem for the amount of salt is

$$\left\{\begin{array}{ccc} y' & = & 10 - \frac{y}{20} \\ y(0) & = & 20. \end{array}\right.$$

Using the method of integrating factor we find the general solution

$$y(t) = 200 - Ce^{-0.05t}.$$

But $y(0) = 20$ so that $C = 180$. Hence, the amount of salt in the tank after t minutes is given by the formula

$$y(t) = 200 - 180e^{-0.05t} \blacksquare$$

Problem 7.2
A tank initially contains 50 gal of pure water. A solution containing 2 lb/gal of salt is pumped into the tank at 3 gal/min. The mixture is stirred constantly and flows out at the same rate of 3 gal/min.
(a) What initial-value problem is satisfied by the amount of salt $y(t)$ in the tank at time t?
(b) What is the actual amount of salt in the tank at time t?
(c) How much salt is in the tank at after 20 minutes?
(d) How much salt in in the tank after a long time?

Solution.
(a) $y' = 6 - \frac{3y}{50}, \quad y(0) = 0.$
(b) By using the method of integrating factor one finds $y(t) = 100(1-e^{-0.06t})$.
(c) $y(20) = 100(1 - e^{-0.06(20)} \approx 69.9\ lb.$
(d) $\lim_{t\to\infty} y(t) = 100\ lb$ ■

Problem 7.3
Brine containing 1 lb/gal of salt is poured at 1 gal/min into a tank that initially contained 100 gal of fresh water. The stirred mixture is drained off at 2 gal/min.
(a) what initial value problem is satisfied by the amount of salt in it?
(b) What is the formula for this amount of salt?

Solution.
Since the inflow rate is different from the outflow rate, we have

$$V(t) = 100 + \int_0^t (1-2)ds = 100 - t.$$

(a) $y' = 1 - \frac{2y}{100-t}, \quad y(0) = 0, \quad 0 \le t < 100.$
(b) $y(t) = -0.01(100-t)^2 + 100 - t$ ■

Problem 7.4
Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred, and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine solution entering the tank is 0.1 Kg/L, determine when the concentration of salt will reach 0.05 Kg/L.

Solution.
Let the amount of salt in the tank at time t be $y(t)$. We can determine the concentration of the salt in the tank by dividing $y(t)$ by the volume of solution in the tank at time t. Since the input and output flow rates are equal, the volume of the solution in the tank remains constant at 1000 L. We first compute the input rate

$$\text{input rate} = \tfrac{6\ L}{min} \times \tfrac{0.1\ Kg}{L} = \tfrac{0.6\ Kg}{min}.$$

The output rate will be the product of output flow rate and the concentration of salt in the outgoing solution. Since we have assumed that the solution is kept well stirred, we can assume that the concentration of salt in any part of the tank at time t is $y(t) = 1000\ Kg/L$, the volume of the solution in the tank being 1000 L. Hence the output rate of salt is

$$\text{output rate} = \tfrac{6\ L}{min} \times \tfrac{y(t)}{1000} = \tfrac{3y(t)}{500}\ Kg/min.$$

Also, since the tank initially contains pure water, we can set $y(0) = 0$. We can now model the problem as an initial-value problem

$$y' = 0.6 - \frac{3y(t)}{500}, \quad y(0) = 0.$$

This equation is linear, and we can solve it using the method of integrating factor, and use the initial condition to get

$$y(t) = 100(1 - e^{-\frac{3t}{500}}).$$

Thus the concentration of salt in the tank at time t is given by

$$\frac{y(t)}{1000} = 0.1(1 - e^{-\frac{3t}{500}})\ Kg/L.$$

In order to find out at what time the concentration becomes $0.05\ Kg/min$, we set

$$0.1(1 - e^{-\frac{3t}{500}}) = 0.05.$$

Solving this equation for t we find $t \approx 115.32\ min$ ■

Problem 7.5
A tank containing chocolate milk initially contains a mixture of 460 gallons of milk and 40 gallons of chocolate syrup. Milk is added to the tank at the

rate of 8 gallons per minute and syrup is added at a rate of 2 gallons per minute. At the same time, chocolate milk is withdrawn at the rate of 10 gallons per minute. Assuming perfect mixing of milk and syrup:
(a) Write up an initial value problem for the amount of syrup in the tank.
(b) Determine how much syrup will be in the tank over a long time.
(c) Determine how much syrup will be in the tank after 10 minutes.

Solution.
(a) Let $y(t)$ be the number of gallons of syrup in the tank at time t. Then the initial-value problem is given by

$$\frac{dy}{dt} = \text{input rate - output rate} = 2 - \frac{y}{50}, \quad y(0) = 40.$$

(b) Using the method of integrating factor we find $y(t) = 100 - 60e^{-0.02t}$. In the long run, $y(t)$ approaches 100 gallons.
(c) $y(10) = 100 - 60e^{-0.2} \approx 50.88$ gallons ■

Problem 7.6
A tank contains 100 L of water with 5kg of salt initially. An inlet pipe adds salt water with concentration of 2 kg/L at the constant rate of 10 L/min. The solution is well-stirred and is flowing out of the tank at the rate of 10 L/min. Give the IVP for the amount of salt y(t) in the tank at time t. Solve the IVP and determine y(2).

Solution.
The model is described by the initial-value problem

$$y' = 20 - 0.1y, \quad y(0) = 5.$$

Using the method of integrating factor we find $y(t) = 200 + Ce^{-0.1t}$. But $y(0) = 5$ so that $C = -195$. It follows that $y(t) = 200 - 195e^{-0.1t}$. Finally, $y(2) = 200 - 195e^{-0.2} \approx 40.35$ liters ■

Problem 7.7
A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at the rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank at $t \to \infty$.

Solution.
Let $y(t)$ be the amount of salt in the tank at any time t. Then the model is represented by the initial-value problem

$$y' = 2\gamma - \frac{y}{60}, \quad y(0) = 0.$$

Solving this differential equation by the method of integrating factor we find $y(t) = 120\gamma(1 - e^{-\frac{t}{60}})$. As $t \to \infty, y(t) \to 120\gamma$ ■

Problem 7.8
Consider a large tank holding 2,000 gallons of brine solution, initially containing 10 lbs of salt. At time t = 0, more brine solution begins to flow into the tank at the rate of 2 gal/min. The concentration of salt in the solution entering the tank is $3e^{-t}$ lbs/gal, i.e. varies in time. The solution inside the tank is well-stirred and is flowing out of the tank at the rate of 5 gal/min. Write down the initial value problem giving $y(t)$ = the amount of salt in the tank (in lbs.) at time t. Do not solve for $y(t)$.

Solution.
Since the rate in is different from the rate out, the volume of the solution at any time t is given by

$$V(t) = V_0 + \int_0^t (2-5)ds = 2000 - 3t.$$

The model is represented by the initial-value problem

$$y' = 6e^{-t} - \frac{5y(t)}{2000 - 3t}, \quad y(0) = 10 \blacksquare$$

Problem 7.9
As part of his summer job at a resturant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that, while refrigeration was essential to preserve the soup overnight, the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at $100°C$, and the fridge was not powerful enough to accomodate a big pot of soup if it was any warmer than $20°C$). Jim discovered that by cooling the pot in a sink full of cold water, (kept running, so that its temperature

was roughly constant at $5°C$) and stirring occasionally, he could bring the temperature of the soup to $60°C$ in ten minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time ?

Solution.
Let $H(t)$ = Temperature of the soup at time t (in min).
$H(0)$ = Initial Temperature of the soup = 100°. S = Ambient temperature (temp of water in sink) = $5°C$. Since $H(0) = C + S$ then $C = 100 - 5 = 95°C$. Thus,

$$H(t) = 95e^{-kt} + 5.$$

But we know that after 10 minutes, the soup cools to 60 degrees, so that $H(10) = 60$. Plugging into the last equation, we find that

$$\begin{aligned} 95e^{-10k} + 5 &= 60 \\ 95e^{-10k} &= 55 \\ e^{-10k} &= \tfrac{55}{95} \\ e^{10k} &= \tfrac{95}{55} \approx 1.73 \\ 10k &= \ln(1.73) \\ k &= \tfrac{\ln(1.73)}{10} \approx 0.054. \end{aligned}$$

Hence, the soup will cool according to the equation

$$H(t) = 95e^{-0.054t} + 5.$$

Let us determine how long it takes for the soup to be cool enough to put into the refrigerator. We need to wait until $H(t) = 20$, so at that time

$$20 = 95e^{-0.054t} + 5.$$

We solve this equation for t as follows:

$$\begin{aligned} 95e^{-0.054t} + 5 &= 20 \\ e^{-0.054t} &= \tfrac{15}{95} \\ t &= -\tfrac{1}{0.054}\ln\left(\tfrac{15}{95}\right) \\ &\approx 34.18. \end{aligned}$$

Thus, it will take a little over half an hour for Jim's soup to cool off enough to be put into the refrigerator ■

Problem 7.10 *(Determintating Time of Death)*
Police arrive at the scene of a murder at 12 am. They immediately take and record the body's temperature, which is $90°F$, and thoroughly inspect the area. By the time they finish the inspection, it is 1:30 am. They again take the temperature of the body, which has dropped to $87°F$, and have it sent to the morgue. The temperature at the crime scene has remained steady at $82°F$.

Solution.
Let $H(t)$ denote the temperature of the body at time t. We are given that $H(0) = 90°C$ and $H(1.5) = 87°C$. By Newton's Law of Cooling we have

$$\frac{dH}{dt} = k(H - 82).$$

Using the seperation of variables we find

$$H(t) = Ce^{kt} + 82.$$

Since $H(0) = 90$ we find $C + 82 = 90$ or $C = 8$. Since $H(1.5) = 87$ we have $8e^{1.5k} + 82 = 87$. Solving for k we find $k = \frac{\ln 5/8}{1.5} \approx -0.313336$. Hence, $H(t) = 8e^{-0.313336t} + 82$. The temperature of the body at the moment of death is $98.6°$. So we want to find t such that $H(t) = 98.6$. That is, $8e^{-0.313336t} + 82 = 98.6$. Solving this equation for t we find $t \approx -2hr20min$. So the crime occurred at 9:40 pm ■

Problem 7.11
Suppose you have just made a cup of tea with boiling water in a room where the temperature is $20°C$. Let $y(t)$ denote the temperature (in Celsius) of the tea at time t (in minutes).
(a) Write a differential equation that expresses Newton's Law of Cooling in this particular situation. What kind of differential equation is it?
(b) What is the initial condition?
(c) Substitute $u(t) = y(t) - 20$. What initial value problem does this new function $u(t)$ satisfy? What is the solution?
(d) Suppose it is known that the tea cools at a rate of $2°C$ per minute when its temperature is $70°C$. Write a formula for $y(t)$.
(e) What is the temperature of the tea a half an hour later?
(f) When will the tea have cooled to $37°C$?

Solution.
(a) The equation is $\frac{dy}{dt} = k(y - 20), k < 0$ and $y(0) = 100°C$. This is a first order linear differential equation.
(b) $y(0) = 100°$.
(c) If $u(t) = y(t) - 20$ then this will lead to the equation $\frac{du}{dt} = ku(t)$ with $u(0) = 80$. Solving this equation will give $u(t) = 80e^{kt}$.
(d) From part (c), $y(t) = 80e^{kt}+20$. Since $\frac{dy}{dt} = k(y-20)$ we find $k(70-20) = -2$. Thus, $k = -\frac{2}{50} = -0.04$. Hence, $y(t) = 80e^{-0.04t} + 20$.
(e) $y(30) = 80e^{-0.04(30)} + 20 \approx 44$°C.
(f) $80e^{-0.04t} + 20 = 37$ implies that $80e^{-0.04t} = 17$. Solving for t we find $t = \frac{\ln 1780}{-0.04} \approx 38.72$ minutes ■

Problem 7.12
Newton's Law of Heating is a corresponding principle which applies if an object is being warmed rather than cooled. The same formulas apply except the constant of proportionality is positive in the warming case. Use Newton's Law of Heating to solve the following problem: A chicken is removed from the refrigerator at a temperature of $40°F$ and placed in an oven kept at the constant temperaturre of $350°F$. After 10 minutes the temperature of the chicken is $70°F$. The chicken is considered cooked when its temperature reaches $180°F$. How long must it remain in the oven?

Solution.
Solving the differential equation

$$\frac{dH}{dt} = k(350 - H), \quad k > 0$$

we find

$$H(t) = Ce^{-kt} + 350.$$

But $H(0) = 40$ so that $C = -310$°F. Hence,

$$H(t) = 350 - 310e^{-kt}.$$

Now, we are given that $H(10) = 70$ so that $70 = 350 - 310e^{-10k}$. Solving for k we find $k \approx 0.0102$. Hence,

$$H(t) = 350 - 310e^{-0.0102t}.$$

Finally, we want to find the time so that $H(t) = 180$. That is, $180 = 350 - 310e^{-0.0102t}$. Solving this equation for t we find $t =\approx 59$minutes ■

Problem 7.13
A corpse is discovered at midnight and its body temperature is $84°F$. If the body temperature at death is $98°F$, the room temperature is constant at $66°F$, and the proportionality constant is .10 per hour, how many hours have passed since the time of death when the corpse is found?

Solution.
By Newton's Second Law of Cooling we have

$$\frac{dH}{dt} = 0.10(66 - H), \quad H(0) = 84.$$

Solving for H we find

$$H(t) = 66 + 18e^{-0.10t}.$$

The time of death is the solution to the equation $H(t) = 98$. Solving this equation for t we find $t \approx -5.75$ hours or 5hr45min. So the time of death is at 6:15 pm ■

Problem 7.14
A tank initially contains 100 gal of a salt-water solution containing $0.05 = \frac{1}{20}$ lb of salt for each gallon of water. At time $t = 0$, pure water [containing no salt] is poured into the tank at a flow rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank that allows salt-water solution to leave the tank at a flow rate of 3 gal per minute. What will be the salt content in the tank when precisely 50 gal of salt solution remain?

Solution.
Let $y(t)$ be the amount of salt in the tank at time t. Then $y(0) = 100 \times \frac{1}{20} = 5$ lbs of salt. Since

$$\frac{dy}{dt} = \text{rate in - rate out} = 2\frac{gal}{min} \times 0\frac{lb}{gal} - 3\frac{gal}{min} \times \frac{y}{100-t}\frac{lb}{gal}$$

the model is described by the initial-value problem

$$y' = \frac{3y}{t - 100}, \quad y(0) = 5.$$

Solving this equation for y we find $y(t) = C(t - 100)^3$. Since $y(0) = 5$ we find $C = \frac{1}{200{,}000}$. Thus,

$$y(t) = \frac{(t - 100)^3}{200,000}.$$

The tank is losing solution at the rate of 1 gal/min. Since there was 100 gal in the tank at the start, after 50 min there will be 50 gal in the tank. The amount of salt in the tank at that time will be

$$y(50) = \frac{50^3}{200,000} = 0.625 \; lb \;\blacksquare$$

Problem 7.15
A tank contains 200 gal of a 2 % solution of HCl. A 5 % solution of HCl is added at 5 gal/min. The well mixed solution is being drained at 5 gal/min. When does the concentration of HCl in the solution reach 4 %?

Solution.
Let $y(t)$ be the concentration of HCL in the tanl at time t. Then $y(t)$ satisfies the initial-value problem

$$y' = 5(.05) - 5 \cdot \frac{y}{200} = \frac{1}{4} - \frac{y}{40}, \; y(0) = 4.$$

Solving this equation by the method of integrating factor we find

$$y(t) = e^{-\frac{t}{4}} \int e^{\frac{t}{40}} \left(\frac{1}{4}\right) dt = 10 + Ce^{-\frac{t}{40}}.$$

Since $y(0) = 4$ we find $C = -6$. Thus, $y(t) = 10 - 6e^{-\frac{t}{40}}$. We want the value of t which gives a concentration of 4%, so

$$200(0.4) = 10 - 6e^{-\frac{t}{40}}.$$

Solving for t we find $t \approx 43.94 \;\blacksquare$

Problem 7.16
Suppose that the temperature of the cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of $200°F$ when freshly poured, and one minute later has cooled to $190°F$ in a room at $70°F$, determine when the coffee reaches a temperature of $150°F$.

Solution.
By Newton's Second Law of Cooling we have

$$\frac{dH}{dt} = k(70 - H), \quad H(0) = 200.$$

Solving for H we find

$$H(t) = 70 + Ce^{-kt}.$$

Since $H(0) = 200$ we find $C = 130$. Since $H(1) = 190$ we find $70 + 130e^{-k} = 190$ and solving for k we find $k \approx 0.08$. Thus, $H(t) = 70 + 130e^{-0.08t}$. Finally, we want to find t such that $H(t) = 150$ that is $70 + 130e^{-0.08t} = 150$. Solving for t we find $t \approx 6.07$ minutes ■

Problem 7.17
Suppose that at 1:00 pm one winter afternoon, there is a power failure at your condo in Nanaimo, and your heat does not work without electricity. When the power goes out, it is 68°F in your condo. At 10:00 pm, it is 57°F in your condo, and you notice it is 10°F outside (what a pity!).
(i) Assuming that the temperature, H, in your condo obeys Newton's Law of Cooling, write the differential equation satisfied by H and then solve the intial-value problem.
(ii) Estimate the temperature in your condo when you get up at 7:00 am the next morning.

Solution.
(i) By Newton's Law of Cooling we have

$$\frac{dH}{dt} = k(10 - H), \quad H(0) = 68.$$

Solving for H be the method of separation of variables we find

$$H(t) = 10 + Ce^{-kt}.$$

But $H(0) = 68$ so that $C = 58$. Hence, $H(t) = 10 + 58e^{-kt}$. Since $H(9) = 57$ we have $10 + 58e^{-9k} = 57$. Solving for k we find $k \approx 0.02337$.
(ii) At 7:00 am, $t = 18$ so that $H(18) = 10 + 58e^{-0.02337(18)} \approx 48°F$ ■

Problem 7.18
Johnny is in the basement watching over a tank with a capacity of 100 L. Originally, the tank is full of pure water. Water containing a salt at a concentration of 2 g/L is flowing into the tank at a rate of r L/minute, and the well mixed liquid in the tank is flowing out at the same rate.
(a) Write down and solve an initial value problem describing the quantity of salt in the mixture at time t in terms of r.
(b) If Johnny's mixture contains 10 g of salt after 50 minutes, what is r?

Solution.
(a) The equation sets up as:

$$\frac{dy}{dt} = 2r - \frac{y}{100}r.$$

Integrating factor is $e^{\frac{rt}{100}}$, so the equation becomes

$$e^{\frac{rt}{100}} = \int 2re^{\frac{rt}{100}}dt = 200e^{\frac{rt}{100}} + C.$$

Initial conditions give $0 = 200 + C$ so $C = -200$, and the formula for y is

$$y(t) = 200(1 - e^{-\frac{rt}{100}}).$$

(b) $y(50) = 10 = 200(1 - e^{-\frac{r}{2}})$, so after a little algebra, $r = -2\ln\frac{19}{20}$ ■

Problem 7.19
A brine tank holds 15000 gallons of continuously mixed liquid. Let $y(t)$ be the amount of salt (in pounds) in the tank at time t. Brine is flowing in and out at 150 gallons per hour, and the concentration of salt flowing is 1 pound per 10 gallons of water.
(a) Find the differential equation of $y(t)$ and find the solution assuming that there is no salt in the water at time t.
(b) What is the limiting amount of salt as $t \to \infty$?

Solution.
(a) The rate at which brine flows in is 150 gallons per hour, and the concentration of salt is 1 lb per 10 gallons of water or 0.1 lb per gallon, so salt is entering the tank at the rate of 15 pounds per hour.
The mixture flowing out at 150 gallons per hour, and the concentration is

$\frac{y(t)}{15000}$ pounds per gallon, so the rate at which salt is leaving the tank is $150\frac{y}{15000} = 0.01y(t)$ pounds per hour. The initial-value problem is therefore

$$y' = 15 - 0.01y, \quad y(0) = 0.$$

Solving this equation by the method of integrating factor we find

$$\begin{array}{rcl}(e^{0.01t}y)' &=& 15e^{0.01t} \\ e^{0.01t}y &=& 1500e^{0.01t} + C \\ y(t) &=& 1500 + Ce^{-0.01t}.\end{array}$$

But $y(0) = 0$ so that 1500. Hence, $y(t) = 1500(1 - e^{-0.01t})$.
(b)

$$\lim_{t\to\infty} y(t) = 155\ lb \blacksquare$$

Problem 7.20
A 10 gal. tank initially contains an effluent at a concentration of 1 lb/gal. Water with an increasing concentration given by $1 - e^{-t}$ lbs/gal of effluent flows into the tank at a rate of 5 gal/day and the mixture in the tank flows out at the same rate.
(a) Assuming that the salt distributes itself uniformly, construct a mathematical model of this flow process for the effluent content $y(t)$ of the tank.
(b) Solve the initial-value problem.
(c) What is the limiting value of the effuent content as $t \to \infty$?

Solution.
(a) The initial-value problem descibing this problem is

$$y' = 5(1 - e^{-t}) - 5\frac{y}{10}, \quad y(0) = 10.$$

(b) Using the method of integrating factor we find

$$\begin{array}{rcl}\left(e^{\frac{t}{2}}y\right)' &=& 5e^{\frac{t}{2}}(1 - e^{-t}) \\ e^{\frac{t}{2}}y &=& 10(e^{-\frac{t}{2}} + e^{\frac{t}{2}}) + C \\ y(t) &=& 10(1 + e^{-t}) + Ce^{-\frac{t}{2}}.\end{array}$$

But $y(0) = 10$ so that $C = -10$. Hence

$$y(t) = 10(1 + e^{-t}) - 10e^{-\frac{t}{2}}.$$

(c)

$$\lim_{t\to\infty} y(t) = 10 \blacksquare$$

8 Existence and Uniqueness of Solutions to the IVP $y' = f(t, y),\ y(t_0) = y_0$

Problem 8.1
Use Picard iterations to find the solution to the IVP

$$y' = y - t, \quad y(0) = 2.$$

Solution.
Finding the first six iterations we find

$$\begin{aligned}
y_0(t) =& 2\\
y_1(t) =& 2 + 2t - \frac{t^2}{2}\\
y_2(t) =& 2 + 2t + \frac{t^2}{2} - \frac{t^3}{6}\\
y_3(t) =& 2 + 2t + \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24}\\
y_4(t) =& 2 + 2t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120}\\
y_5(t) =& 2 + 2t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} - \frac{t^6}{720}\\
y_6(t) =& 2 + 2t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} - \frac{t^7}{5040}.
\end{aligned}$$

Notice that when the last term in the Picard approximation is dropped, what is left is a Taylor polynomial approximation which converges uniformly to $1 + t + e^t$. That is, the unique solution to the IVP is $y(t) = 1 + t + e^t$ ■

Problem 8.2
On what interval we expect unique solutions to

$$y' = \frac{y^2}{1 - t^2}, \quad y(0) = 0?$$

Solution.
We have

$$f(t, y) = \frac{y^2}{1 - t^2}$$

and

$$\frac{\partial f}{\partial y}(t,y) = \frac{2y}{1-t^2}.$$

These are both continuous functions as long as we avoid the lines $t = \pm 1$. The Existence and Uniqueness Theorem tells us that we can expect one and only one solution of

$$y' = \frac{y^2}{1-t^2}, \quad y(t_0) = y_0$$

as long as t_0 is in the set $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ ■

Problem 8.3
Consider the IVP

$$y' = \frac{1}{2}(-t + \sqrt{t^2 + 4y}), \ y(2) = -1.$$

(a) Show that $y(t) = 1 - t$ and $y(t) = -\frac{t^2}{4}$ are two solutions to the above IVP.
(b) Does this contradict Theorem 8.3?

Solution.
(a) You can verify that the two functions are solutions by substitution.
(b) Since $f(t,y) = \frac{1}{2}(-t + \sqrt{t^2 + 4y})$ and $f_y(t,y) = \frac{1}{\sqrt{t^2+4y}}$, these two functions are not continuous at $(2, -1)$. Thus, we can not apply Theorem 8.3 for this problem. ■

For the given initial value problem in Problems 8.4 - 8.8,
(a) Rewrite the differential equation, if necessary, to obtain the form

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Identify the function $f(t,y)$.
(b) Compute $\frac{\partial f}{\partial y}$. Determine where in the ty-plane both $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous.
(c) Determine the largest open rectangle in the ty-plane that contains the point (t_0, y_0) and in which the hypotheses of Theorem 8.3 are satisfied.

Problem 8.4

$$3y' + 2t\cos y = 1, \quad y(\frac{\pi}{2}) = -1.$$

Solution.
(a) $y' = \frac{1}{3}(1 - 2t\cos y) = f(t, y)$.
(b) $\frac{\partial f}{\partial y}(t, y) = \frac{2}{3}t\sin y$. The functions $f(t, y)$ and $f_y(t, y)$ are both continuous in the entire plane,

$$D = \{(t, y) : -\infty < t < \infty, \ -\infty < y < \infty\}.$$

(c) $R = \{(t, y) : -\infty < t < \infty, \ -\infty < y < \infty\}$ ■

Problem 8.5

$$3ty' + 2\cos y = 1, \quad y(\frac{\pi}{2}) = -1.$$

Solution.
(a) $y' = \frac{1}{3t}(1 - 2\cos y) = f(t, y)$.
(b) $f_y(t, y) = \frac{2}{3t}\sin y$. Both $f(t, y)$ and $f_y(t, y)$ are continuous in

$$D = \{(t, y) : -\infty < t < 0, \ 0 < t < \infty, \ -\infty < y < \infty\}.$$

(c) $R = \{(t, y) : 0 < t < \infty, \ -\infty < y < \infty\}$ ■

Problem 8.6

$$2t + (1 + y^3)y' = 0, \quad y(1) = 1.$$

Solution.
(a) $y' = -\frac{2t}{1+y^3} = f(t, y)$.
(b) $f_y(t, y) = \frac{6ty^2}{(1+y^3)^2}$. Both $f(t, y)$ and $f_y(t, y)$ are continuous in

$$D = \{(t, y) : -\infty < t < \infty, \ -\infty < y < -1, \ -1 < y < \infty\}.$$

(c) $R = \{(t, y) : -\infty < t < \infty, \ -1 < y < \infty\}$ ■

Problem 8.7

$$(y^2 - 9)y' + e^{-y} = t^2, \quad y(2) = 2.$$

Solution.
(a) $y' = \frac{t^2-e^{-y}}{y^2-9} = f(t,y)$.
(b) $f_y(t,y) = \frac{(y^2+2y-9)e^{-y}-2t^2y^2}{y^2-9}$. Both $f(t,y)$ and $f_y(t,y)$ are continuous in

$$D = \{(t,y) : -\infty < t < \infty, \ -\infty < y < -3, \ -3 < y < 3, \ 3 < y < \infty\}.$$

(c) $D = \{(t,y) : -\infty < t < \infty, \ -3 < y < 3\}$ ■

Problem 8.8

$$\cos y y' = 2 + \tan t, \quad y(0) = 0.$$

Solution.
(a) $y' = \frac{2+\tan t}{\cos y} = f(t,y)$.
(b) $f_y(t,y) = (2+\tan t)\sec y \tan y$. Both $f(t,y)$ and $f_y(t,y)$ are continuous in

$$D = \{(t,y) : t \neq (2n+1)\tfrac{\pi}{2}, \ y \neq (2m+1)\tfrac{\pi}{2}, \text{ where } n \text{ and } m \text{ are integers.}\}$$

(c) $R = \{(t,y) : -\frac{\pi}{2} < t < \frac{\pi}{2}, \ -\frac{pi}{2} < y < \frac{\pi}{2}\}$ ■

Problem 8.9
Give an example of an initial value problem for which the open rectangle

$$R = \{(t,y) : 0 < t < 4, -1 < y < 2\}$$

represents the largest region in the ty-plane where the hypotheses of Theorem 8.3 are satisfied.

Solution.
An example is

$$y' = \frac{1}{t(t-4)(y+1)(y-2)}, \ y(2) = 0 \ ■$$

Problem 8.10
Consider the initial value problem: $t^2y' - y^2 = 0, \quad y(1) = 1$.
(a) Determine the largest open rectangle in the ty-plane, containing the point $(t_0, y_0) = (1,1)$, in which the hypotheses of Theorem 8.3 are satisfied.
(b) A solution of the initial value problem is $y(t) = t$. This solution exists on $-\infty < t < \infty$. Does this fact contradicts Theorem 8.3? Explain your answer.

Solution.
We have $f(t,y) = \frac{y^2}{t^2},\ f_y(t,y) = \frac{2y}{t^2}$. So

$$R = \{(t,y) : 0 < t < \infty,\ -\infty < y < \infty\}.$$

(b) No. Theorem 8.3 is a local existence theorem and not a global one ■

Problem 8.11 *(Gronwall's Inequality)*
Let $u(t)$ and $h(t)$ be continuous functions defined on a closed interval $[a,b]$, with $h \geq 0$, let C be a non-negative constant, and suppose that

$$u(t) \leq C + \int_a^t u(s)h(s)ds \tag{1}$$

for all t in the interval. Show that

$$u(t) \leq Ce^{\int_a^t h(s)ds}$$

for all t in the interval.
Note in particular that if C = 0, then $u(t) \leq 0$ for all t.

Solution.
Let us write $U(t) = C + \int_a tu(s)h(s)ds$. By the Fundamental Theorem of Calculus and (1), U is differentiable and

$$U'(t) = u(t)h(t) \leq U(t)h(t). \tag{2}$$

Now if (2) were a differential equation rather than a differential inequality, we would solve it by multiplying by the integrating factor $\mu(t) = e^{-\int_a^t h(s)ds}$. In fact however, the same method works on the inequality; multiplying (2) by $\mu(t)$ and rearranging leads to $(\mu U)'(t) \leq 0$, and integrating this inequality yields

$$\mu(s)U(s)\big|_a^t = \mu(t)U(t) - C \leq 0$$

and hence

$$u(t) \leq U(t) \leq C[\mu(t)] - 1 \ ■$$

Problem 8.12
Find the first three Picard iterates of the solution of the initial-value problem

$$y' = \cos t,\ y(0) = 0$$

and then try to find the nth Picard iterates.

Solution.
Since $y_0(t) \equiv 0$, the next three Picard iterates are

$$\begin{aligned}
y_1(t) =&0 + \int_0^t \cos s ds = \sin t \\
y_2(t) =&0 + \int_0^t \cos s ds = \sin t \\
y_3(t) =&0 + \int_0^t \cos s ds = \sin t.
\end{aligned} \quad (3)$$

The nth iterates is given by

$$y_n(t) = \sin t.$$

Thus, $y_n(t) \to \sin t$ as $n \to \infty$ and for all t. Hence, $y(t) = \sin t$ is the solution to the initial-value problem ■

Problem 8.13
Set up the Picard iteration technique to solve the initial value problem $y' = y^2,\ y(0) = 1$ and do the first three iterations.

Solution.
(a) Since $y_0(t) \equiv 1$ we have

$$\begin{aligned}
y_1(t) =&1 + \int_0^t 1^2 ds = 1 + t \\
y_2(t) =&1 + \int_0^t (1+s)^2 ds = 1 + t + t^2 + \frac{t^3}{3} \\
y_3(t) =&1 + \int_0^t (1 + s + s^2 + \frac{s^3}{3} ds = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7 \ ■
\end{aligned}$$

Problem 8.14
Can we apply the basic existence and uniqueness theorem to the following problem ? Explain what (if anything) we can conclude, and why (or why not):

$$y' = \frac{y}{\sqrt{t}}, \quad y(0) = 2.$$

Solution.
Since $f(t,y) = \frac{y}{\sqrt{t}}$ and $f_y(t,y) = \frac{1}{\sqrt{t}}$, both functions are continuous in the region

$$D = \{(t,y) : 0 < t < \infty, \ -\infty < y < \infty\}.$$

Since $(0,2)$ is not in D, Theorem 8.3 can not be applied in this case ■

Problem 8.15
Consider the differential equation $y' = \frac{t-y}{t+y}$. For which of the following initial value conditions does Theorem 8.3 apply?
(a) $y(0) = 0$ (b) $y(1) = -1$ (c) $y(-1) = -1$.

Solution.
The function $f(t,y) = \frac{t-y}{t+y}$ is continuous everywhere except along the line $t + y = 0$. Since both $(0,0)$ and $(1,-1)$ lie on this line, we cannot conclude existence from Theorem 8.3. On the other hand, the point $(-1,-1)$ is not on that line so we can find a small rectangle around this point where Theorem 8.3 guarantees the existence of a solution. Furthermore, since $f_y(t,y) = -\frac{2t}{(t+y)^2}$ is continuous at $(-1,-1)$, the solution is unique ■

Problem 8.16
Does the initial value problem $y' = \frac{y}{t} + 2, \ y(0) = 1$ satisfy the conditions of Theorem 8.3?

Solution.
The equation is of the form $y' = f(t,y) = \frac{y}{t} + 2$. The function f is continuous outside the line $t = 0$. The initial value point is (0, 1), so there is no rectangle containing it in which f is continuous, and the conditions of Theorem 8.3 are not satisfied ■

Problem 8.17
Is it possible to find a function $f(t,y)$ that is continuous and has continuous partial derivatives such that the functions $y_1(t) = \cos t$ and $y_2(t) = 1 - \sin t$ are both solutions to the equation $y' = f(t,y)$ near $t = \frac{\pi}{2}$?

Solution.
Since f is continuous and has continuous partial derivatives in the entire ty-plane, the equation $y' = f(t,y)$ satisfies the conditions of Theorem 8.3. Notice that $y_1(\frac{\pi}{2}) = y_2(\frac{\pi}{2}) = 0$, so the curves $y_1(t) = \cos t$ and $y_2(t) = 1 - \sin t$

have a common point $(\frac{\pi}{2}, 0)$, so if they were both solutions of our equation, by the uniqueness theorem they would have to agree on any rectangle containing $(\frac{\pi}{2}, 0)$. Since they do not, they cannot both be solutions of the equation $y' = f(t, y)$ ■

Problem 8.18
Does the initial value problem $y' = y \sin y + t,\ y(0) = -1$ satisfy the conditions of Theorem 8.3?

Solution.
The equation is of the form $y' = f(t, y) = y \sin y + t$. the function f is continuous in the whole plane, and so is its partial derivative $f_y(t, y) = \sin y + y \cos y$. In particular, any rectangle around the initial value point will satisfy the conditions of Theorem 8.3 ■

Problem 8.19
The condition of continuity of $f(t, y)$ in Theorem 8.3 can be replaced by the so-called Lipschitz continuous: A function $f(t, y)$ is said to be **Lipschitz continuous** in y on a closed interval $[a, b]$ if there is a positive constant k such that $|f(t, y_1) - f(t, y_2)| \le k|y_1 - y_2|$ for all y_1, y_2 and $a \le t \le b$.
Show that the function $f(t, y) = 1 + t \sin ty$ is Lipschitz continuous in y for $0 \le t \le 2$. Hint: Use the Mean Value Theorem.

Solution.
Fix t between 0 and 2. Let y_1 and y_2 be two given number where f is defined and such that $y_1 < y_2$.. By the Mean Value Theorem, there is $y_1 < y* < y_2$ such that

$$f(t, y_1) - f(t, y_2) = f_y(t, y*)(y_1 - y_2).$$

But $f_y(t, y) = t^2 \cos(ty)$. Thus, $|f_y(t, y)| \le 4$ for all t and all y. Hence,

$$|f(t, y_1) - f(t, y_2)| \le 2|y_1 - y_2|.$$

This shows that f is Lipschitz continuous in y ■

Problem 8.20
Find the region R of the ty-plane where both

$$f(t, y) = \frac{1}{\sqrt{y - \sin t}}$$

and $\frac{\partial f}{\partial y}(t, y)$ are continuous.

Solution.
Since $f_y(t,y) = \frac{-1}{\sqrt{(y-\sin t)^3}}$, the functions f and f_y are defined in the ty-region

$$D = \{(t,y) : y - \sin t > 0\}.$$

Therefore there is a unique solution passing through every point which lies above the graph of $y = \sin t$ ■

9 Separable Differential Equations

Problem 9.1
Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

Solution.
At first, this equation may not appear separable, so we must simplify the right hand side until it is clear what to do.

$$\begin{aligned} y' =& te^{t^2 - \ln y^2} \\ =& te^{t^2} \cdot e^{\ln\left(\frac{1}{y^2}\right)} \\ =& te^{t^2} \cdot \frac{1}{y^2} \\ =& \frac{t}{y^2} e^{t^2}. \end{aligned}$$

Separating the variables and solving the equation we find

$$\begin{aligned} y^2 y' =& te^{t^2} \\ \frac{1}{3}\int (y^3)' dt =& \int te^{t^2} \\ \frac{1}{3} y^3 =& \frac{1}{2} e^{t^2} + C \\ y^3 =& \frac{3}{2} e^{t^2} + C \blacksquare \end{aligned}$$

Problem 9.2
Solve the (separable) differential equation

$$y' = \frac{t^2 y - 4y}{t+2}.$$

Solution.

Separating the variables and solving we find

$$\begin{aligned}
\frac{y'}{y} =& \frac{t^2-4}{t+2} = t-2\\
\int (\ln|y|)'dt =& \int (t-2)dt\\
\ln|y| =& \frac{t^2}{2} - 2t + C\\
y(t) =& Ce^{\frac{t^2}{2}-2t} \blacksquare
\end{aligned}$$

Problem 9.3
Solve the (separable) differential equation

$$ty' = 2(y-4).$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}
\frac{y'}{y-4} =& \frac{2}{t}\\
\int (\ln|y-4|)'dt =& \int \frac{2}{t}dt\\
\ln|y-4| =& \ln t^2 + C\\
\ln|\frac{y-4}{t^2}| =& C\\
y(t) =& Ct^2 + 4 \blacksquare
\end{aligned}$$

Problem 9.4
Solve the (separable) differential equation

$$y' = 2y(2-y).$$

Solution.
Separating the variables and solving (using partial fractions in the process)

we find

$$
\begin{aligned}
\frac{y'}{y(2-y)} =& 2\\
\frac{y'}{2y} + \frac{y'}{2(2-y)} =& 2\\
\frac{1}{2}\int (\ln|y|)'dt - \frac{1}{2}\int (\ln|2-y|)'dt =& \int 2dt\\
\ln\left|\frac{y}{2-y}\right| =& 4t + C\\
\left|\frac{y}{2-y}\right| =& Ce^{4t}\\
y(t) =& \frac{2Ce^{4t}}{1+Ce^{4t}} \blacksquare
\end{aligned}
$$

Problem 9.5
Solve the IVP

$$
y' = \frac{4\sin(2t)}{y}, \quad y(0) = 1.
$$

Solution.
Separating the variables and solving we find

$$
\begin{aligned}
yy' =& 4\sin(2t)\\
(y^2)' =& 8\sin(2t)\\
\int (y^2)'dt =& \int 8\sin(2t)dt\\
y^2 =& -4\cos(2t) + C\\
y(t) =& \pm\sqrt{C - 4\cos(2t)}.
\end{aligned}
$$

Since $y(0) = 1$ we find $C = 5$ and hence

$$
y(t) = \sqrt{5 - 4\cos(2t)} \blacksquare
$$

Problem 9.6
Solve the IVP:

$$
yy' = \sin t, \quad y(\frac{\pi}{2}) = -2.
$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}\int \left(\frac{y^2}{2}\right)' dt =& \int \sin t dt\\ \frac{y^2}{2} =& -\cos t + C\\ y^2 =& -2\cos t + C.\end{aligned}$$

Since $y(\frac{\pi}{2}) = -2$ we find $C = 4$. Thus, $y(t) = \pm\sqrt{(-2\cos t + 4)}$. Since $y(\frac{\pi}{2}) = -2$ we must have $y(t) = -\sqrt{(-2\cos t + 4)}$ ■

Problem 9.7
Solve the IVP:

$$\frac{y'}{y+1} = -1, \quad y(1) = 0.$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}(\ln(y+1))' =& -1\\ \ln(y+1) =& -t + C\\ y + 1 =& Ce^{-t}\\ y(t) =& Ce^{-t} - 1.\end{aligned}$$

Since $y(1) = 0$ we find $C = e$. Thus, $y(t) = e^{1-t} - 1$ ■

Problem 9.8
Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}\int y'y^{-3}dt =& \int tdt\\ \int \left(\frac{y^{-2}}{-2}\right)' dt =& \frac{t^2}{2} + C\\ -\frac{1}{2y^2} =& \frac{t^2}{2} + C\\ y^2 =& \frac{1}{-t^2 + C}.\end{aligned}$$

Since $y(0) = 2$ we find $C = \frac{1}{4}$. Thus, $y(t) = \pm\sqrt{\frac{4}{-4t^2+1}}$. Since $y(0) = 2$ we find $y(t) = \frac{2}{\sqrt{-4t^2+1}}$ ■

Problem 9.9
Solve the IVP:

$$y' = 1 + y^2, \quad y(\frac{\pi}{4}) = -1.$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}
\frac{y'}{1+y^2} =& 1\\
\arctan y =& t + C\\
y(t) =& \tan(t + C).
\end{aligned}$$

Since $y(\frac{\pi}{4}) = -1$ we find $C = -\frac{\pi}{2}$. Hence, $y(t) = \tan\left(t - \frac{\pi}{2}\right)$ ■

Problem 9.10
Solve the IVP:

$$y' = t - ty^2, \quad y(0) = \frac{1}{2}.$$

Solution.
Separating the variables and solving we find

$$\begin{aligned}
\frac{y'}{y^2-1} =& -t\\
\frac{y'}{y-1} - \frac{y'}{y+1} =& -2t\\
\ln\left|\frac{y-1}{y+1}\right| =& -t^2 + C\\
\frac{y-1}{y+1} =& Ce^{-t^2}\\
y(t) =& \frac{1+Ce^{-t^2}}{1-Ce^{-t^2}}.
\end{aligned}$$

Since $y(0) = \frac{1}{2}$ we find $C = -\frac{1}{3}$. Thus,

$$y(t) = \frac{3-e^{-t^2}}{3+e^{-t^2}} \blacksquare$$

Problem 9.11
Solve the IVP
$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0.$$

Solution.
Separating the variables and solving we find
$$\int (2y - \sin y)y' dt = \int (\sin t - t)dt$$
$$y^2 + \cos y = -\cos t - \frac{t^2}{2} + C.$$

Since $y(0) = 0$ we find $C = 2$. Thus,
$$y^2 + \cos y + \cos t + \frac{t^2}{2} = 2 \blacksquare$$

Problem 9.12
For what values of the constants α, y_0, and integer n is the function $y(t) = (4+t)^{-\frac{1}{2}}$ a solution of the initial value problem?
$$y' + \alpha y^n = 0, \quad y(0) = y_0.$$

Solution.
We have $y_0 = y(0) = (4+0)^{-\frac{1}{2}} = \frac{1}{2}$. Also, $y' = -\frac{1}{2}(4+t)^{-\frac{3}{2}} = -\frac{1}{2}y^3$. Thus,
$$y' + \frac{1}{2}y^3 = 0$$
so that $\alpha = \frac{1}{2}$ and $n = 3 \blacksquare$

Problem 9.13
State an initial value problem, with initial condition imposed at $t_0 = 2$, having implicit solution $y^3 + t^2 + \sin y = 4$.

Solution.
Differentiating both sides of the given equation we find
$$3y^2y' + \cos y + 2t = 0, \quad y(2) = 0 \blacksquare$$

Problem 9.14
Consider the initial value problem

$$y' = 2y^2, \quad y(0) = y_0.$$

For what value(s) of y_0 will the solution have a vertical asymptote at $t = 4$, where the t-interval of existence is $-\infty < t < 4$?

Solution.
Solving the differential equation by the method of separating the variables we find

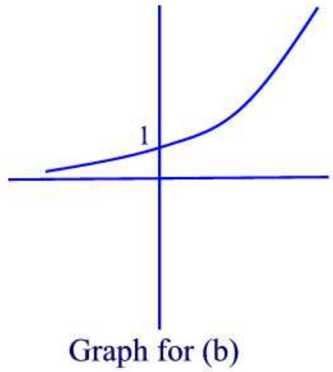
$$\begin{aligned}\frac{y'}{y^2} =&2\\ \int \frac{y'}{y^2}dt =& \int 2dt\\ -\frac{1}{y} =&2t + C\\ y(t) =&\frac{1}{C-2t}.\end{aligned}$$

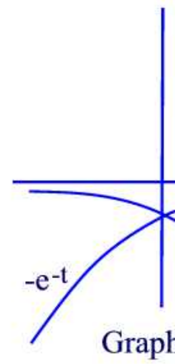
Since $y(0) = y_0$ we find $C = \frac{1}{y_0}$. Thus, $y(t) = \frac{y_0}{1-2y_0t}$. This function will have a vertical asymptot at $t = 4$ when $1 - 2y_0(4) = 0$ or $y_0 = \frac{1}{8}$ ■

Problem 9.15
Consider the differential equation $y' = |y|$.
(a) Is this differential equation linear or nonlinear? Is the differentiable equation separable?
(b) A student solves the two initial value problems $y' = |y|, \ y(0) = 1$ and $y' = y, \ y(0) = 1$ and then graphs the two solution curves on the interval $-1 \le t \le 1$. Sketch the two graphs.
(c) The student next solves the two initial value problems $y' = |y|, \ y(0) = -1$ and $y' = y, \ y(0) = -1$. Sketch the solution curves.

Solution.
(a) The equation is nonlinear and separable since $\frac{y'}{|y|} - 1 = 0$.
(b) Notice first that $y' \ge 0$. If $y \ge 0$ then $y' = y$. Solving this equation we find $y(t) = Ce^t$. But $y(0) = 1$ so that $y(t) = e^t$. If $y < 0$ then $y' = -y$. Solving this equation we find $y(t) = e^{-t}$. But for this one $y' < 0$. Thus, the

solution of the initial-value problem $y' = |y|,\ y(0) = 1$ coincides with that of the initial-value problem $y' = y,\ y(0) = 1$ and is given by $y(t) = e^t$ for all real numbers t.
(c) If $y(0) = -1$ then the solution to $y' = |y|,\ y(0) = -1$ is $y(t) = -e^{-t}$. The





solution to $y' = y,\ y(0) = -1$ is $y(t) = -e^t$ ■

Problem 9.16
Assume that $y \sin y - 3t + 3 = 0$ is an implicit solution of the initial value problem $y' = f(y),\quad y(1) = 0$. What is $f(y)$? What is an implicit solution to the initial value problem $y' = t^2 f(y),\quad y(1) = 0$?

Solution.
Taking the derivative of the given equation with respect to t we find

$$y' \sin y + yy' \cos y - 3 = 0.$$

Thus,

$$y' = \frac{3}{\sin y + y \cos y} = f(y).$$

If $y' = t^2 f(y)$ then

$$y' = \frac{3t^2}{\sin y + y \cos y}.$$

Solving this equation by the method of separation of variables we find

$$\begin{aligned} y' \sin y + yy' \cos y =& 3t^2 \\ (y \sin y)' =& 3t^2 \\ y \sin y =& t^3 + C. \end{aligned}$$

Since $y(1) = 0$ we find $C = -1$. Hence, the implicit solution is given by

$$y \sin y - t^3 + 1 = 0 \ ■$$

Problem 9.17
Find all the solutions to the differential equation $y' = \frac{2ty}{1+t}$.

Solution.
Separating the variables to obtain

$$\begin{aligned}\frac{y'}{y} =& \frac{2t}{1+t} = 2 - \frac{2}{t+1} \qquad (4)\\ \ln|y| =& 2t - \ln(t+1)^2 + C\\ \ln|(t+1)^2 y| =& 2t + C\\ (t+1)^2 y =& Ce^{2t}\\ y(t) =& \frac{Ce^{2t}}{(t+1)^2}. \blacksquare\end{aligned}$$

Problem 9.18
Solve the initial-value problem $y' = \cos^2 y \cos^2 t, \quad y(0) = \frac{\pi}{4}$.

Solution.
Solving by the method of separation of variables we find

$$\begin{aligned}\frac{y'}{\cos^2 y} =& \cos^2 t\\ \tan y =& \frac{t}{2} + \frac{1}{4}\sin 2t + C.\end{aligned}$$

Since $y(0) = \frac{\pi}{4}$ we find $C = 1$. Hence,

$$\tan y = \frac{t}{2} + \frac{1}{4}\sin 2t + 1 \blacksquare$$

Problem 9.19
Solve the initial-value problem $y' = e^{t+y}$, $y(0) = 0$ and determine the interval on which the solution $y(t)$ is defined.

Solution.
Separating the variable we obtain

$$y'e^{-y} = e^t.$$

Integrating both sides to obtain

$$e^{-y} = -e^t + C.$$

But $y(0) = 0$ so that $C = 2$. Hence, $e^{-y} = -e^t + 2$. Solving for y we find

$$y(t) = -\ln(2 - e^t).$$

This function is defined for $t < \ln 2$ ■

Problem 9.20
Solve the initial-value problem

$$y' = \frac{t^2}{e^{-y}} - \frac{e^y}{t^2}.$$

(a) State the name of the method you are using.
(b) Find the solution which satisfies the condition $y(1) = 1$.

Solution.
(a) Using the method of separation of variables we find

$$\begin{aligned} y'e^{-y} =& t^2 - \frac{1}{t^2} \\ e^{-y} =& -\frac{t^3}{3} - \frac{1}{t} + C. \end{aligned}$$

(b) Since $y(1) = 1$ we find $C = e^{-1} + \frac{4}{3}$. Thus, the unique solution is defined implicitly by the expression

$$e^{-y} + \frac{t^3}{3} + \frac{1}{t} = e^{-1} + \frac{4}{3} \blacksquare$$

10 Exact Differential Equations

Problem 10.1
Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ if $f(t,y) = y\ln y - e^{-ty}$.

Solution.

$$\frac{\partial f}{\partial t} = ye^{-ty}$$

$$\frac{\partial f}{\partial y} = \ln y + 1 + te^{-ty} \blacksquare$$

Problem 10.2
Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ if $f(t,y) = \ln ty + \frac{t^2+1}{y-5}$.

Solution.

$$\frac{\partial f}{\partial t} = \frac{1}{t} + \frac{2t}{y-5}$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} - \frac{t^2+1}{(y-5)^2} \blacksquare$$

Problem 10.3
Let $f(u,v) = 2u - 3uv$ where $u(t) = 2\cos t$ and $v(t) = 2\sin t$. Find $\frac{df}{dt}$.

Solution.
By the Chain Rule

$$\begin{aligned}\frac{df}{dt} =&\frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}\\ =&(2-3v)(-2\sin t) - 3u(2\cos t) = (2-6\sin t)(-2\sin t) - 6\cos t(2\cos t)\\ =&12\sin^2 t - 12\cos^2 t - 4\sin t\\ =&24\sin^2 t - 4\sin t - 12 \blacksquare\end{aligned}$$

In Problems 10.4 - 10.8, determine whether the given differential equation is exact. If the equation is exact, find an implicit solution and (where possible) an explicit solution.

Problem 10.4

$$yy' + 3t^2 - 2 = 0, \quad y(-1) = -2.$$

Solution.
We have $M(t,y) = 3t^2 - 2$ and $N(t,y) = y$. Thus, $\frac{\partial M}{\partial y}(t,y) = 0 = \frac{\partial N}{\partial t}(t,y)$ so that the equation is exact.

$$\frac{\partial H}{\partial t}(t,y) = 3t^2 - 2 \Longrightarrow H(t,y) = \int (3t^2-2)dt = t^3 - 2t + h(y).$$

But $\frac{\partial H}{\partial y}(t,y) = y$ so that $h'(y) = y$ and hence $h(y) = \frac{y^2}{2}$. Therefore

$$t^3 - 2t + \frac{y^2}{2} = C.$$

Since $y(-1) = 2$ we find $C = 3$. It follows

$$t^3 - 2t + \frac{y^2}{2} = 3.$$

Solving for y we find $y(t) = \pm\sqrt{4t - 2t^3 + 6}$. Since $y(-1) = 2$ we find $y(t) = -\sqrt{4t - 2t^3 + 6}$ ■

Problem 10.5

$$y' = (3t^2+1)(y^2+1), \quad y(0) = 1.$$

Since the equation is separable, it is exact. Integrating $\frac{\partial H}{\partial t}(t,y) = 3t^2 + 1$ with respect to t we find $H(t,y) = t^3 + t + h(y)$. But $\frac{\partial H}{\partial y}(t,y) = -(y^2+1)^{-1}$ which implies that $h'(y) = -(y^2+1)^{-1}$. Thus, $h(y) = -\arctan y$. Hence,

$$t^3 + t - \arctan y = C.$$

Since $y(0) = 1$ we find $C = -\frac{\pi}{4}$. It follows

$$t^3 + t - \arctan y = -\frac{\pi}{4}.$$

Solving for $y(t)$ we find

$$y(t) = \tan\left(t^3 + t + \frac{\pi}{4}\right) ■$$

Problem 10.6

$$(6t + y^3)y' + 3t^2y = 0, \quad y(1) = 2.$$

Solution.
We have $M(t,y) = 3t^2y$ and $N(t,y) = 6t + y^3$. Since $\frac{\partial M}{\partial y}(t,y) = 3t^2$ and $\frac{\partial N}{\partial t}(t,y) = 6$, the given differential equation is not exact ■

Problem 10.7

$$(e^{t+y} + 2y)y' + (e^{t+y} + 3t^2) = 0, \quad y(0) = 0.$$

Solution.
We have $M(t,y) = e^{t+y} + 3t^2$ and $N(t,y) = e^{t+y} + 2y$. Since $\frac{\partial M}{\partial y}(t,y) = e^{t+y} = \frac{\partial N}{\partial t}(t,y)$, the given differential equation is exact.

$$\frac{\partial H}{\partial t}(t,y) = e^{t+y} + 3t^2 \Longrightarrow H(t,y) = \int (e^{t+y} + 3t^2)dt = e^{t+y} + t^3 + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = e^{t+y} + 2y = h'(y) + e^{t+y} \Longrightarrow h'(y) = 2y \Longrightarrow h(y) = y^2.$$

Hence,

$$e^{t+y} + t^3 + y^2 = C.$$

Since $y(0) = 0$ we find $C = 1$. Therefore,

$$e^{t+y} + t^3 + y^2 = 1 \blacksquare$$

Problem 10.8

$$(\sin(t+y) + y\cos(t+y) + t + y)y' + (y\cos(t+y) + y + t) = 0, \quad y(1) = -1.$$

Solution.
We have $M(t,y) = y\cos(t+y) + t + y$ and $N(t,y) = \sin(t+y) + y\cos(t+y) + t + y$. Since $\frac{\partial M}{\partial y}(t,y) = \cos(t+y) - y\sin(t+y) + 1 = \frac{\partial N}{\partial t}(t,y)$, the differential equation is exact.
Now

$$\begin{aligned}\frac{\partial H}{\partial t}(t,y) =& y\cos(t+y) + t + y\\ H(t,y) =& \int (y\cos(t+y) + t + y)dt = y\sin(t+y) + \frac{t^2}{2} + yt + h(y).\end{aligned}$$

Also

$$\begin{aligned}\frac{\partial H}{\partial y}(t,y) =&\sin{(t+y)}+y\cos{(t+y)}+t+y\\ =&y\cos{(t+y)}+\sin{(t+y)}+t+h'(y)\\ h'(y) =&y\\ h(y) =&\frac{y^2}{2}.\end{aligned}$$

Hence,

$$y\sin{(t+y)}+\frac{t^2}{2}+ty+\frac{y^2}{2}=C.$$

Since $y(1)=-1$ we find $C=0$. Therefore,

$$y\sin{(t+y)}+\frac{t^2}{2}+ty+\frac{y^2}{2}=0 \blacksquare$$

Problem 10.9
For what values of the constants $m,n,$ and α (if any) is the following differential equation exact?

$$t^my^2y'+\alpha t^3y^n=0$$

Solution.
We have $M(t,y)=\alpha t^3y^n$ and $N(t,y)=t^my^2$. Thus, $\frac{\partial M}{\partial y}(t,y)=n\alpha t^3y^{n-1}$ and $\frac{\partial N}{\partial y}(t,y)=mt^{m-1}y^2$. For the differential equation to be exact we must have $\frac{\partial M}{\partial y}(t,y)=\frac{\partial N}{\partial t}(t,y)$, i.e.,

$$n\alpha t^3y^{n-1}=mt^{m-1}y^2.$$

This shows that $m-1=3$ so that $m=4$. Also, $n-1=2$ so that $n=3$. Finally, $3\alpha=4$ so that $\alpha=\frac{4}{3}$ $\blacksquare$

Problem 10.10
Assume that $N(t,y)y'+t^2+y^2\sin t=0$ is an exact differential equation. Determine the general form of $N(t,y)$.

Solution.
We have $M(t,y)=t^2+y^2\sin t$. Since the differential equation is exact then $\frac{\partial N}{\partial t}(t,y)=\frac{\partial M}{\partial y}(t,y)=2y\sin t$. Hence,

$$N(t,y)=\int 2y\sin t dt=-2y\cos t+h(y) \blacksquare$$

Problem 10.11
Assume that $t^3y + e^t + y^2 = 5$ is an implicit solution to the differential equation

$$N(t,y)y' + M(t,y) = 0, \quad y(0) = y_0.$$

Determine possible functions $M(t,y), N(t,y)$, and the possible value(s) for y_0

Solution.
Replacing y by y_0 and t by 0 to obtain $y_0 = \pm 2$. Differentiating the given equation with respect to t we find $3t^2y + e^t + (t^3 + 2y)y' = 0$. Thus, $M(t,y) = 3t^2 + e^t$ and $N(t,y) = t^3 + 2y$ ■

Problem 10.12
Assume that $y = -t - \sqrt{4 - t^2}$ is an explicit solution of the following initial value problem

$$(y + at)y' + (ay + bt) = 0, \quad y(0) = y_0.$$

Determine values for the constants a, b and y_0

Solution.
We have $y_0 = -0 - \sqrt{4 - 0^2} = -2$. Since $\frac{\partial N}{\partial t}(t,y) = \frac{\partial M}{\partial y}(t,y) = a$, the differential equation is exact. From this we have

$$\frac{\partial H}{\partial y}(t,y) = y + at \Longrightarrow H(t,y) = \frac{y^2}{2} + aty + h(t)$$

and

$$\frac{\partial H}{\partial t}(t,y) = ay + bt = ay + h'(t) \Longrightarrow h'(t) = bt \Longrightarrow h(t) = \frac{b}{2}t^2.$$

Hence,

$$\frac{y^2}{2} + aty + \frac{b}{2}t^2 = C.$$

Since $y(0) = -2$ we find $C = 2$. Therefore,

$$y^2 + 2aty + bt^2 = 4.$$

Solving this quadratic equation for y we find

$$y = \frac{-2at \pm \sqrt{4a^2t^2 - 4(bt^2 - 4)}}{2}.$$

Thus,

$$y(t) = -at \pm \sqrt{4a^2t^2 - 4(bt^2 - 4)}.$$

Since $y(0) = -2$ we find $y(t) = -at - \sqrt{t^2a^2 - bt^2 + 4}$. Finally, $a = 1, a^2 - b = -1,\ b = 2$ ■

Problem 10.13
Let k be a positive constant. Use the exactness criterion to determine whether or not the population equation $\frac{dP}{dt} = kP$ is exact. Do NOT try to solve the equation or carry out any furhter calculation.

Solution.
Rewriting the equation in the form $k - \frac{1}{P}\frac{dP}{dt} = 0$ we find that $M(t, P) = k$ and $N(t, P) = -\frac{1}{P}$. Since $\frac{\partial M}{\partial P}(t, P) = \frac{\partial N}{\partial t}(t, P) = 0$, the differential equation is exact ■

Problem 10.14
Consider the differential equation $(2t+3)+(2y-2)y' = 0$. Determine whether this equation is exact or not. If it is, solve it.

Solution.
We have $M(t, y) = 2t+3$ and $N(t, y) = 2y-2$. Since $\frac{\partial M}{\partial y}(t, y) = \frac{\partial N}{\partial t}(t, y) = 0$, the differential equation is exact. Now,

$$\frac{\partial H}{\partial t}(t, y) = 2t + 3 \Longrightarrow H(t, y) = \int (2t + 3)dt = t^2 + 3t + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t, y) = 2y - 2 = h'(y) \Longrightarrow h'(y) = 2y - 2 \Longrightarrow h(y) = y^2 - 2y.$$

Hence,

$$t^2 + 3t + y^2 - 2y = C \ ■$$

Problem 10.15
Consider the differential equation $(ye^{2ty} + t) + bte^{2ty}y' = 0$. Determine for which value of b this equation is exact, and then solve it with this value of b.

Solution.
We have $M(t,y) = ye^{2ty} + t$ and $N(t,y) = bte^{2ty}$. For the equation to be exact we must have $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y)$, that is,

$$e^{2ty} + 2tye^{2ty} = be^{2ty} + 2ybte^{2ty}.$$

Dividing through by e^{2ty} to obtain

$$1 + 2ty = b + 2byt = b(1 + 2ty).$$

This implies $b = 1$. Hence, the equation is

$$(ye^{2ty} + t) + te^{2ty}y' = 0.$$

Now,

$$\frac{\partial H}{\partial t}(t,y) = ye^{2ty} + t \Longrightarrow H(t,y) = \int (ye^{2ty} + t)dt = \frac{1}{2}e^{2ty} + \frac{t^2}{2} + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = te^{2ty} = te^{2ty} + h'(y) \Longrightarrow h'(y) = 0 \Longrightarrow h(y) = C.$$

Hence,

$$\frac{1}{2}e^{2ty} + \frac{t^2}{2} = C \blacksquare$$

Problem 10.16
Consider the differential equation $y + (2t - ye^y)y' = 0$. Check that this equation is not exact. Now multiply the equation by y. Check that the new equation is exact, and solve it.

Solution.
If we let $M(t,y) = y$ and $N(t,y) = 2t - ye^y$ we see that $\frac{\partial M}{\partial y}(t,y) = 1$ and $\frac{\partial N}{\partial t}(t,y) = 2$ so that the equation is not exact. If we multiply the given equation by y then $M(t,y) = y^2$ and $N(t,y) = 2ty - y^2e^y$. In this case, $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y) = 2y$ so that the equation is exact.
Now,

$$\frac{\partial H}{\partial t}(t,y) = y^2 \Longrightarrow H(t,y) = \int y^2 dt = ty^2 + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = 2ty - y^2e^y = 2ty + h'(y) \Longrightarrow h'(y) = -y^2e^y.$$

Using integration by parts twice we find

$$h(y) = -y^2e^y + 2ye^y - 2e^y.$$

Hence,

$$ty^2 - y^2e^y + 2ye^y - 2e^y = C \blacksquare$$

Problem 10.17
(a) Consider the differential equation

$$y' + p(t)y = g(t)$$

with $p(t) \neq 0$. Show that this equation is not exact.
(b) Let $\mu(t) = e^{\int p(t)dt}$. Show that the equation

$$\mu(t)(y' + p(t)y) = \mu(t)g(t)$$

is exact and solve it.

Solution.
(a) We have $M(t,y) = p(t)y - g(t)$ and $N(t,y) = 1$. Since $\frac{\partial M}{\partial y}(t,y) = p(t) \neq 0$ and $\frac{\partial N}{\partial t}(t,y) = 0$, the differential equation is not exact.
(b) Here, we have $M(t,y) = \mu(t)p(t)y - \mu(t)g(t)$ and $N(t,y) = \mu(t)$. Thus, $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y) = p(t)e^{\int p(t)dt}$. That is, the new differential equation is exact.
Now,

$$\begin{aligned}\frac{\partial H}{\partial t}(t,y) =& \mu(t)p(t)y - \mu(t)g(t)\\ H(t,y) =& \int(\mu(t)p(t)y - \mu(t)g(t))dt\\ =& \mu(t)y - \int \mu(t)g(t)dt + h(y).\end{aligned}$$

Also

$$\frac{\partial H}{\partial y}(t,y) = \mu(t) = \mu(t) + h'(y) \Longrightarrow h'(y) = 0 \Longrightarrow h(y) = C.$$

Hence,

$$\mu(t)y - \int \mu(t)g(t) = C$$

and so

$$y(t) = e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt + Ce^{-\int p(t)dt} \blacksquare$$

Problem 10.18
Use the method of the previous problem to solve the linear, first-order equation $y' - \frac{y}{t} = 1$, with initial condition $y(1) = 7$. First, check that this equation is not exact. Next, find $\mu(t)$. Multiply the equation by $\mu(t)$ and check that the new equation is exact. Solve it, using the method of exact equations.

Solution.
For the given equation we have $M(t, y) = 1 + \frac{y}{t}$ and $N(t, y) = -1$. Since $\frac{\partial M}{\partial y}(t, y) = \frac{1}{t}$ and $\frac{\partial N}{\partial t}(t, y) = 0$, the equation is not exact. Let $\mu(t) = e^{-\int \frac{dt}{t}} = \frac{1}{t}$. Multiply the given equation by $\mu(t)$ to obtain

$$(1 + \frac{y}{t})(\frac{1}{t}) - \frac{1}{t}y' = 0.$$

In this equation, $M(t, y) = (1 + \frac{y}{t})(\frac{1}{t})$ and $N(t, y) = -\frac{1}{t}$. Also, $\frac{\partial M}{\partial y}(t, y) = \frac{\partial N}{\partial t}(t, y) = \frac{1}{t^2}$ so that the new equation is exact. By the previous exercise the solution is given by

$$y(t) = t \int \frac{1}{t} dt + Ct = t \ln t + Ct.$$

Since $y(1) = 7$ we find $C = 7$. Hence, $y(t) = t \ln t + 7t$ $\blacksquare$

Problem 10.19
Put the following differential equation in the "Exact Differential Equation" form and find the general solution

$$y' = \frac{y^3 - 2ty}{t^2 - 3ty^2}$$

Solution.
Rewriting thid equation in the form

$$(y^3 - 2ty) + (3ty^2 - t^2)y' = 0$$

we find $M(t,y) = y^3 - 2ty$ and $N(t,y) = 3ty^2 - t^2$. Also, notice that $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y) = 3y^2 - 2t$. Now,

$$\frac{\partial H}{\partial t}(t,y) = y^3 - 2ty \Longrightarrow H(t,y) = \int (y^3 - 2ty)dt = ty^3 - t^2y + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = 3ty^2 - t^2 = 3ty^2 - t^2 + h'(y) \Longrightarrow h'(y) = 0 \Longrightarrow h(y) = C.$$

Hence,

$$ty^3 - t^2y = C \blacksquare$$

Problem 10.20
The following differential equations are exact. Solve them by that method.
(a) $(4t^3y + 4t + 4)y' = 8 - 4y - 6t^2y^2, \quad y(-1) = 1.$
(b) $(6 - 4y + 16t) + (10y - 4t + 2)y' = 0, \quad y(1) = 2.$

Solution.
(a) We have $M(t,y) = 6t^2y^2 + 4y - 8$ and $N(t,y) = 4t^3y + 4t + 4$. Notice that $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y) = 12t^2y + 4$. Now,

$$\frac{\partial H}{\partial t}(t,y) = 6t^2y^2 + 4y - 8 \Longrightarrow H(t,y) = \int (6t^2y^2 + 4y - 8)dt = 2t^3y^2 + 4ty - 8t + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = 4t^3y + 4t + 4 = 4t^3y + 4t + h'(y) \Longrightarrow h'(y) = 4 \Longrightarrow h(y) = 4y.$$

Hence,

$$2t^3y^2 + 4ty - 8t + 4y = C.$$

Since $y(-1) = 1$ we find $C = 6$. Hence, $2t^3y^2 + 4ty - 8t + 4y = 6$.
(b) We have $M(t,y) = 6 - 4y + 16t$ and $N(t,y) = 10y - 4t + 2$. Notice that $\frac{\partial M}{\partial y}(t,y) = \frac{\partial N}{\partial t}(t,y) = -4$. Now,

$$\frac{\partial H}{\partial t}(t,y) = 6 - 4y + 16t \Longrightarrow H(t,y) = \int (6 - 4y + 16t)dt = 6t - 4ty + 8t^2 + h(y).$$

Also

$$\frac{\partial H}{\partial y}(t,y) = 10y - 4t + 2 = -4t + h'(y) \Longrightarrow h'(y) = 10y + 2 \Longrightarrow h(y) = 5y^2 + 2y.$$

Hence,

$$6t - 4ty + 8t^2 + 5y^2 + 2y = C.$$

Since $y(1) = 2$ we find $C = 30$. Hence, $6t - 4ty + 8t^2 + 5y^2 + 2y = 30$ ■

11 Substitution Techniques: Bernoulli and Ricatti Equations

Problem 11.1
Solve the Bernoulli equation

$$y' = \frac{t^2 + 3y^2}{2ty}, \quad t > 0.$$

Solution.
The given equation can be written in the form

$$y' - \frac{3}{2t}y = \frac{1}{2}ty^{-1}.$$

Divide through by y^{-1} to obtain

$$yy' - \frac{3}{2t}y^2 = \frac{t}{2}.$$

Let $z = y^2$. Then the last equation becomes

$$z' - \frac{3}{t}z = t$$

and this is a linear first order differential equation.
To solve this equation, we use the integrating factor method. Let $\mu(t) = t^{-3}$. Then

$$z(t) = t^3 \int t^{-3}tdt + Ct^3 = -t^2 + Ct^3.$$

The general solution to the initial problem is implicitly defined by

$$y^2 = -t^2 + Ct^3 \blacksquare$$

Problem 11.2
Find the general solution of $y' + ty = te^{-t^2}y^{-3}$.

Solution.
Divide the given equation by y^{-3} to obtain

$$y^3y' + ty^4 = te^{-t^2}.$$

Let $z = y^4$ so that the previous equation becomes

$$z' + 4tz = 4te^{-t^2}.$$

The integrating factor is $\mu(t) = e^{2t^2}$. Thus,

$$z(t) = e^{-2t^2}\int e^{2t^2}4te^{-t^2}dt + Ce^{-2t^2} = 2e^{-t^2} + Ce^{-2t^2}.$$

Finally, the general solution to the original equation is defined implicitely by the equation

$$y^4 = 2e^{-t^2} + Ce^{-2t^2} \blacksquare$$

Problem 11.3
Solve the IVP $ty' + y = t^2y^2, \quad y(0.5) = 0.5.$

Solution.
Divide through by y^2 to obtain

$$ty^{-2}y' + y^{-1} = t^2.$$

Let $z = y^{-1}$ so that

$$z' - \frac{1}{t}z = -t.$$

Solving this equation by the integrating factor method with $\mu(t) = \frac{1}{t}$ we find

$$z(t) = t\int \frac{1}{t}\cdot(-t)dt + Ct = -t^2 + Ct = t(C-t).$$

Hence, $y(t) = \frac{1}{t(C-t)}$. But $y(\frac{1}{2}) = \frac{1}{2}$ so that $C = 4.5$. Thus,

$$y(t) = \frac{1}{t(4.5-t)} \blacksquare$$

Problem 11.4
Solve the IVP $y' - \frac{1}{t}y = -y^2, \quad y(1) = 1, \quad t > 0.$

Solution.
Divide through by y^2 to obtain

$$y^{-2}y' - \frac{1}{t}y^{-1} = -1.$$

So let $z = y^{-1}$. Thus,

$$z' + \frac{1}{t}z = 1, z(1) = 1.$$

Solving this equation using the integrating factor method with $\mu(t) = t$ we find

$$z(t) = \frac{1}{t}\int tdt + Ct^{-1} = \frac{t}{2} + Ct^{-1}.$$

Since $z(1) = 1$ we find $C = \frac{1}{2}$. Hence, $z = \frac{1}{2}(t + \frac{1}{t})$ and $y(t) = \frac{2t}{t^2+1}$ ■

Problem 11.5
Solve the IVP $y' = y(1-y), \quad y(0) = \frac{1}{2}$.

Solution.
Rewriting the given equation in the form $y' - y = -y^2$. Divide through by y^2 to obtain

$$y^{-2}y' - y^{-1} = -1.$$

Let $z = y^{-1}$. Then

$$z' + z = 1, z(0) = 2.$$

Solving this equation using the integrating factor method with $\mu(t) = e^t$ we obtain

$$z(t) = e^{-t}\int e^t dt + Ce^{-t} = 1 + Ce^{-t}.$$

But $z(0) = 2$ so that $C = 1$ and thus $z(t) = 1 + e^{-t}$. Finally, $y(t) = (1 + e^{-t})^{-1}$ ■

Problem 11.6
Solve the Bernoulli equation $y' + 3y = e^{3t}y^2$.

Solution.
Dividing by y^2 to obtain

$$y^2 y' + 3y^{-1} = e^{3t}.$$

Let $z = y^{-1}$. Then,

$$z' - 3z = -e^{3t}.$$

Solving this equation using the integrating factor method with $\mu(t) = e^{-3t}$ we find

$$z(t) = e^{3t}\int e^{-3t}(-e^{3t})dt + Ce^{3t} = -te^{3t} + Ce^{3t} = e^{3t}(C - t).$$

Finally, $y(t) = e^{-3t}(C - t)^{-1}$ ■

Problem 11.7
Solve $y' + y = ty^4$.

Solution.
Divide through by y^4 to obtain

$$y^{-4}y' + y^{-3} = t.$$

Let $z = y^{-3}$ so that

$$z' - 3z = -3t.$$

Solving this equation using the integrating factor method with $\mu(t) = e^{-3t}$ we find

$$z(t) = e^{3t}\int e^{-3t}(-3t)dt + Ce^{3t} = t + \frac{1}{3} + Ce^{3t}.$$

So $y(t) = (t + \frac{1}{3} + Ce^{3t})^{-\frac{1}{3}}$ ■

Problem 11.8
Solve the equation $y' = \sin(t+y)$ using the substitution $z = t + y$ and separable method.

Solution.
If $z = t + y$ then $z' = 1 + y'$. Thus, $z' - 1 = \sin z$. Seperating the variables we find

$$\frac{dz}{1 + \sin z} = dt.$$

But

$$\int \frac{dz}{1+\sin z} = \int \frac{1 - \sin z}{\cos^2 z} = \int (\sec^2 z - \sec z \tan z)dz = \tan z - \sec z + C.$$

Hence,

$$\tan z - \sec z = t + C$$

so

$$\tan(t+y) - \sec(t+y) = t + C \text{ ■}$$

Problem 11.9
Solve the IVP: $y' = 2 + 2y + y^2, \quad y(0) = 0$ using the method of separation of variables.

Solution.
Notice first that $2 + 2y + y^2 = 1 + (1 + y)^2$. Separating the variables we find

$$\frac{y'}{1 + (1 + y)^2} = 1.$$

Integrating both sides with respect to t to obtain

$$\arctan(1 + y) = t + C.$$

But $y(0) = 0$ so that $C = \frac{\pi}{4}$. Thus,

$$y(t) = \tan(t + \frac{\pi}{4}) - 1 \blacksquare$$

Problem 11.10
Solve the differential equation $y' = 1 + t^2 - y^2$ given that $y_1(t) = t$ is a particular solution.

Solution.
Let $\frac{1}{z} = y - t$. Then $-\frac{z'}{z} = y' - 1$. Substituting we find

$$-\frac{z'}{z^2} + 1 = 1 + t^2 - \left(\frac{1}{z} + t\right)^2.$$

Simplifying this last equation to obtain

$$z' - 2tz = 1.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{-t^2}$ we find

$$z(t) = e^{t^2} \int_0^t e^{-s^2} ds + Ce^{t^2}.$$

The general solution to the differential equation is

$$y(t) = (e^{t^2} \int_0^t e^{-s^2} ds + Ce^{t^2})^{-1} + t \blacksquare$$

Problem 11.11
Solve the differential equation $y' = 5 - t^2 + 2ty - y^2$ given that $y_1(t) = t - 2$ is a particular solution.

Solution.
Let $\frac{1}{z} = y - t + 2$. Then the given equation reduces to

$$z' + 4z = 1.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{4t}$ to obtain

$$z(t) = e^{-4t}\int e^{4t}dt + Ce^{-4t} = \frac{1}{4} + Ce^{-4t}.$$

Thus,

$$y(t) = (\frac{1}{4} + Ce^{-4t})^{-1} + t - 2 \blacksquare$$

Problem 11.12
Perform a change of variable that changes the Bernoulli equation $y' + y + y^2 = 0$ into a linear equation in the new variable. Do NOT try to solve the equation or proceed further than with any calculations.

Solution.
Dividing through by y^2 to obtain

$$y^{-2}y' + y^{-1} = -1.$$

Letting $z = y^{-1}$ to obtain

$$z' - z = 1 \blacksquare$$

Problem 11.13
Consider the equation

$$y' = \epsilon y - \sigma y^3,\ \epsilon > 0,\ \sigma > 0$$

(a) Use the Bernoulli transformation to change this nonlinear equation into a linear equation.
(b) Solve the resulting linear equation in part (a) and use the solution to find the solution of the given differential equation above.

Solution.
(a) Dividing by y^3 to obtain

$$y^{-3}y' - \epsilon y^{-2} = -\sigma.$$

Letting $z = y^{-2}$ to obtain

$$z' + 2\epsilon z = 2\sigma.$$

(b) Using the method of integrating factor with $\mu(t) = e^{2\epsilon t}$ we find

$$z(t) = e^{-2\epsilon t}\int e^{2\epsilon t}2\sigma dt + Ce^{-2\epsilon t} = \frac{\sigma}{2\epsilon} + Ce^{-2\epsilon t}.$$

Finally,

$$y(t) = (\frac{\sigma}{\epsilon} + Ce^{-2\epsilon t})^{-\frac{1}{2}} \blacksquare$$

Problem 11.14

Consider the differential equation

$$y' = f\left(\frac{y}{t}\right).$$

(a) Show that the substitution $z = \frac{y}{t}$ leads to a separable differential equation in z.

(b) Use the above method to solve the initial-value problem

$$y' = \frac{t+y}{t-y}, \quad y(1) = 0.$$

Solution.

(a) Letting $z = \frac{y}{t}$ then $y = tz$. Thus, $y' = z + tz'$. Hence,

$$tz' + z = f(z)$$

or

$$z' = \frac{f(z) - z}{t}$$

which is a separable differential equation.

(b) Note first that

$$\frac{t+y}{t-y} = \frac{1+\frac{y}{t}}{1-\frac{y}{t}}.$$

Letting $z = \frac{y}{t}$ we obtain

$$\begin{aligned} tz' + z =& \frac{1+z}{1-z} \\ tz' =& \frac{1+z^2}{1-z} \\ \frac{1-z}{1+z^2}z' =& \frac{1}{t} \\ \int \frac{z'}{1+z^2}dt - \int \frac{zz'}{1+z^2}dt =& \int \frac{dt}{t} \\ \arctan z - \frac{1}{2}\ln(1+z^2) =& \ln|t| + C \\ 2\arctan z =& \ln t^2(1+z^2) + C \\ 2\arctan\left(\frac{y}{t}\right) =& \ln t^2\left(1+\left(\frac{y}{t}\right)^2\right) + C. \end{aligned}$$

Since $y(1) = 0$ we find $C = 0$ ■

Problem 11.15
Solve: $y' + \frac{y}{3} = e^t y^4$.

Solution.
Divide through by y^4 to obtain $y^{-4}y' + \frac{1}{3}y^{-3} = e^t$. Letting $z = y^{-3}$ to obtain

$$z' - z = -3e^t.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{-t}$ we find

$$z(t) = e^t \int e^{-t}(-3e^t)dt + Ce^t = -3e^t + Ce^t.$$

Finally,

$$y(t) = (-3e^t + Ce^t)^{-\frac{1}{3}} \blacksquare$$

Problem 11.16
Solve: $ty' + y = ty^3$.

Solution.
Dividing through by y^3 to obtain $y^{-3}y' + y^{-2} = t$. Letting $z = y^{-2}$ to obtain

$$z' - 2z = -2t.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{-2t}$ we find

$$z(t) = e^{2t} \int e^{-2t}(-2t)dt + Ce^{2t} = -t - 1 + Ce^{2t}.$$

Finally,

$$y(t) = (-t - 1 + Ce^{2t})^{-\frac{1}{2}} \blacksquare$$

Problem 11.17
Solve: $y' + \frac{2}{t}y = -t^2y^2 \cos t$.

Solution.
Dividing through by y^2 to obtain $y^{-2}y' + \frac{2}{t}y^{-1} = -t^2 \cos t$. Letting $z = y^{-1}$ to obtain

$$z' - \frac{2}{t}z = t^2 \cos t.$$

Solving this equation by the method of integrating factor with $\mu(t) = \frac{1}{t^2}$ we find

$$z(t) = t^2 \int \cos t dt + Ct^2 = t^2 \sin t + Ct^2.$$

Finally,

$$y(t) = (t^2 \sin t + Ct^2)^{-1} \blacksquare$$

Problem 11.18
Solve: $ty' + y = t^2y^2 \ln t$.

Solution.
Dividing through by ty^2 to obtain $y^{-2}y' + \frac{1}{t}y^{-1} = t \ln t$. Letting $z = y^{-1}$ to obtain

$$z' - \frac{1}{t}z = -t \ln t.$$

Solving this equation by the method of integrating factor with $\mu(t) = \frac{1}{t}$ we find

$$z(t) = t \int (-\ln t)dt + Ct = -t^2 \ln t + t^2 + Ct.$$

Finally,

$$y(t) = (-t^2 \ln t + t^2 + Ct)^{-1} \blacksquare$$

Problem 11.19
Verify that $y_1(t) = 2$ is a particular solution to the Ricatti equation

$$y' = -2 - y + y^2,$$

and then find the general solution.

Solution.
Since $y_1' = 0$ and $-2 - y_1 + y_1^2 = -2 - 2 + 4 = 0$ we find $y_1' = -2 - y_1 + y_1^2$. Now, to solve the equation we let $\frac{1}{z} = y - 2$. Substituting this into the above equation to obtain

$$z' + 3z = -1.$$

Solving this equation by the method of integrating factor with $\mu(t) = e^{3t}$ we find

$$z(t) = e^{-3t}\int -e^{3t}dt + Ce^{-3t} = -\frac{1}{3} + Ce^{-3t}.$$

Finally,

$$y(t) = (-\frac{1}{3} + Ce^{-3t})^{-1} + 2 \blacksquare$$

Problem 11.20
Verify that $y_1(t) = \frac{2}{t}$ is a particular solution to the Ricatti equation

$$y' = -\frac{4}{t^2} - \frac{1}{t}y + y^2,$$

and then find the general solution.

Solution.
Since $y_1' = -\frac{2}{t^2}$ and $-\frac{4}{t^2} - \frac{1}{t}y_1 + y_1^2 = -\frac{2}{t^2}$ we find y_1 is a solution to the differential equation. Next, let $z^{-1} = y - \frac{2}{t}$ then substituting into the previous equation we find

$$z' + \frac{3}{t}z = -1.$$

Solving this equation by the method of integrating factor with $\mu(t) = t^3$ we find

$$z(t) = t^{-3}\int -t^3dt + Ct^{-3} = -\frac{t}{4} + Ct^{-3}.$$

Finally,

$$y(t) = (-\frac{t}{4} + Ct^{-3})^{-1} + \frac{2}{t} \blacksquare$$

12 Applications of First Order Nonlinear Equations: The Logistic Population Model

Problem 12.1
Find $\int \frac{dx}{(x-2)(3-x)}$.

Solution.
First, we would like to have

$$\frac{1}{(x-2)(3-x)} = \frac{A}{x-2} + \frac{B}{3-x}.$$

Multiplying both sides by $x-2$ and then setting $x=2$ we find $A=1$. Multiplying both sides by $3-x$ and setting $x=3$ we obtain $B=1$. Thus,

$$\int \frac{dx}{(x-2)(3-x)} = \int \frac{dx}{x-2} - \int \frac{dx}{x-3} = \ln\left|\frac{x-2}{x-3}\right| + C \blacksquare$$

Problem 12.2
Find A and B so that $\frac{2x+3}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$.

Solution.
Multiplying through by $x+3$ and then setting $x=-3$ we find $A=\frac{1}{2}$. Similarly, multiplying through by $x-3$ and setting $x=3$ we obtain $B=\frac{3}{2}$. Hence,

$$\frac{2x+3}{x^2-9} = \frac{1}{2}\left(\frac{1}{x+3} + \frac{3}{x-3}\right) \blacksquare$$

Problem 12.3
Write the partial fraction decomposition of $\frac{x+7}{x^2+x-6}$.

Solution.
Since $x^2+x-6) = (x-2)(x+3)$, we would like to find A and B such that

$$\frac{x+7}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}.$$

Multitplying through by $x-2$ and setting $x=2$ we find $A=\frac{9}{5}$. Next, multiply through by $x+3$ and set $x=-3$ to obtain $B=-\frac{4}{5}$. Hence,

$$\frac{x+7}{(x-2)(x+3)} = \frac{1}{5}\left(\frac{9}{x-2} - \frac{4}{x+3}\right) \blacksquare$$

Problem 12.4
An important feature of any logistic curve is related to its shape: *every logistic curve has a single inflection point which separates the curve into two equal regions of opposite concavity.* This inflection point is called the **point of diminishing returns.** Find the Coordinates of the Point of Diminishing Returns.

Solution.
Since
$$\frac{dP}{dt} = r(1 - \frac{P}{K})P$$
by the product rule we find
$$\frac{d^2P}{dt^2} = r\frac{dP}{dt}\left(1 - \frac{2P}{K}\right).$$
Since $\frac{dP}{dt} > 0$, we conclude that $\frac{d^2P}{dt^2} = 0$ at $P = \frac{K}{2}$.
To find t, we set $P = \frac{K}{2}$ and solve for t :
$$\begin{aligned}
\frac{K}{2} =& \frac{KP(0)}{P(0) + (K - P(0))e^{-rt}}\\
\frac{1}{2} =& \frac{P(0)}{P(0) + (K - P(0))e^{-rt}}\\
(K - P(0))e^{-rt} =& P(0)\\
e^{-rt} =& \frac{P(0)}{K - P(0)}\\
-rt =& \ln\left(\frac{P(0)}{K - P(0)}\right)\\
t =& -\frac{\ln\left(\frac{P(0)}{K-P(0)}\right)}{r}.
\end{aligned}$$
Thus, the coordinates of the diminishing point of returns are $\left(\frac{\ln\left(\frac{K-P(0)}{P(0)}\right)}{r}, \frac{K}{2}\right)$ ■

Problem 12.5
A population of roaches grows logistically in John's kitchen cabinet, feeding off 65 half-empty can of beef stew. There are 10 roaches initially, and the

carrying capacity of the cabinet is $K = 10000$. The population reaches its maximum growth rate in 4 days. Determine the logistic equation for the growth of the population Find the number of roaches in the cabinet after 10 days.

Solution.
We have

$$P(t) = \frac{KP(0)}{P(0) + (K - P(0))e^{-rt}}.$$

But $P(0) = 10$ and $K = 10000$ so that

$$P(t) = \frac{10,000}{1 + 999e^{-rt}}.$$

From the phase portrait of a logisitic model, we can see that the maximum growth rate occurs at the point of diminishing return, i.e., when $P = \frac{K}{2} = 5000$. Thus,

$$5000 = \frac{10,000}{1 + 999e^{-4r}}.$$

Solving for r we find $r = \frac{1}{4}\ln 999$. Hence

$$P(t) = \frac{10,000}{1 + 999e^{-\frac{1}{4}\ln 999t}}$$

and

$$P(10) \approx 9999.68 \blacksquare$$

Problem 12.6
The number of people $P(t)$ in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially $P(0) = 500$, and it is observed that $P(1) = 1000$. If it is predicted that the limiting number of people in the community who will see the advertisement is 50,000, determine $P(t)$ at any time.

Solution.
We have $K = 50,000$ and $P(0) = 500$ so that

$$P(t) = \frac{50,000}{1 + 99e^{-rt}}.$$

Since $P(1) = 1000$ we obtain $1 + 99e^{-r} = 50$ and solving for r we find $r = \ln\left(\frac{99}{49}\right)$. Thus,

$$P(t) = \frac{50,000}{1 + 99e^{\ln\left(\frac{49}{99}\right)t}} \blacksquare$$

Problem 12.7
The population $P(t)$ at any time in a suburb of a large city is governed by the inititial value problem

$$\frac{dP}{dt} = (10^{-1} - 10^{-7}P)P, \quad P(0) = 5000$$

where t is measured in months. What is the limiting value of the population? At what time will the population be one-half of this limiting value?

Solution.
Rewriting the given differential equation we get

$$\frac{dP}{dt} = 10^{-1}(1 - 10^{-6}P)P.$$

Thus, $K = 1,000,000$. The population will reach 500,000 when

$$t = \frac{\ln\left(\frac{K-P(0)}{P(0)}\right)}{r} = 10\ln 1999 \blacksquare$$

Problem 12.8
Let $P(t)$ represent the population of a colony, in millions of individuals. Suppose the colony starts with 0.1 million individuals and evolves according to the equation

$$\frac{dP}{dt} = 0.1\left(1 - \frac{P}{3}\right)P$$

with time being measured in years. How long will it take the population to reach 90% of its equilibrium value?

Solution.
We have $90\%(3) = 2.7$. We want to find t so that $P(t) = 2.7$ where

$$P(t) = \frac{3}{1 + 29e^{-0.1t}}.$$

Solving the equation

$$\frac{3}{1 + 29e^{-0.1t}} = 2.7$$

we find $t = 10\ln 261 \blacksquare$

Problem 12.9
Consider a population whose dynamics are described by the logistic equation with constant migration

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P + M,$$

where r, K, abd M are constants. Assume that K is a fixed positive constant and that we want to understand how the equilibrium solutions of this nonlinear autonomous equation depend upon the parameters r and M.
(a) Obtain the roots of the quadratic equation that define the equilibrium solution(s) of this differential equation. Note that for $M \neq 0$, the constants 0 and K are no longer equilibrium solutions. Does this make sense?
(b) For definiteness, set $K = 1$. Plot the equilibrium solutions obtained in (a) as functions of the ratio $\frac{M}{r}$. How many equilibrium populations exist for $\frac{M}{r} > 0$? How many exist for $-\frac{1}{4} < \frac{M}{r} \leq 0$?
(c) What happens when $\frac{M}{r} = -\frac{1}{4}$? What happens when $\frac{M}{r} < -\frac{K}{4} = -\frac{1}{4}$? Are these mathematical results consistent with what one would expect if migration rate out of the colony were sufficiently large relative to the colony's ability to gain size through reproduction?

Solution.
(a) We have

$$\begin{aligned} r\left(1 - \tfrac{P}{K}\right)P + M &= 0 \\ rP^2 - KrP - KM &= 0 \end{aligned}$$

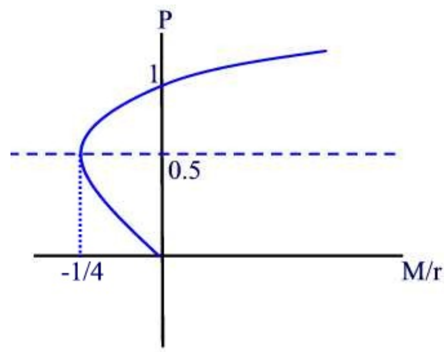
The solutions of this quadratic equation are

$$P = \frac{Kr \pm \sqrt{K^2r^2 + 4rKM}}{2r} = \frac{K \pm \sqrt{K^2 + \frac{4KM}{r}}}{2} \neq 0, K.$$

This makes sense since migration would alter equilibrium state.
(b) If $K = 1$ then

$$\left(P - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{M}{r}.$$

The graph is given below



For $\frac{M}{r} > 0$ there is one nonnegative equilibrium solution. For $-\frac{1}{4} < \frac{M}{r} \leq 0$ there are two equilibrium solutions.
(c) When $\frac{M}{r} = -\frac{1}{4}$ the two nonnegative equilibrium solutions reduce to a single equilibrium solution. When $\frac{M}{r} < -\frac{1}{4}$ there are no equilibrium solutions. This makes sense since if the migration out of the colony is too large relative to reproduction equilibrium could not be achieved ■

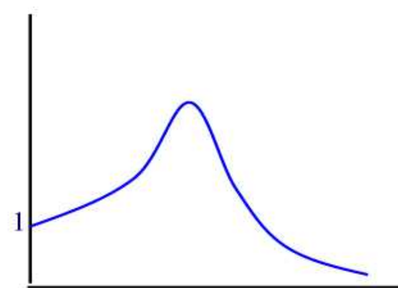
Problem 12.10
Let $P(t)$ represent the number of individuals who, at time t, are infected with a certain disease. Let N denote the total number of individuals in the population. Assume that the spread of the disease can be modeled by the initial value problem

$$\frac{dP}{dt} = k(N - P)P, \quad P(0) = P_0$$

At time $t = 0$, when 100,000 members of the population of 500,000 are known to be infected, medical authorities intervene with medical treatment. As a consequence of this intervenetion, the rate factor k is no longer constant but varies with time as $k(t) = 2e^{-t} - 1$, where time t is measured in months and $k(t)$ represents the rate of infection per month per 100,000 individuals.
Initially as the effects of medical intervention begin to take hold, $k(t)$ remaind positive and the disease continues to spread. Eventually, however, the effects of medical treatment cause $k(t)$ to become negative and the number of infected individuals then decreases.
(a) Solve the appropriate initial value problem for the number of infected individuals, $P(t)$, at time t and plot the solution.
(b) From your plot, estimate the maximum number of individuals that are at any time infected with the disease.

(c) How long does it take before the number of infected individuals is reduced to 50,000?

Solution.
(a) We have
$$\frac{dP}{dt} = (2e^{-t} - 1)(5 - P),\ P(0) = 1.$$
We solve this equation as follows
$$\begin{aligned}\frac{P'}{P-5} =& 1 - 2e^{-t}\\ \frac{P'}{P} - \frac{P'}{P-5} =& 10e^{-t} - 5\\ \int \frac{P'}{P}dt - \int \frac{P'}{P-5}dt =& \int (10e^{-t} - 5)dt\\ \ln\left|\frac{P}{P-5}\right| =& -10e^{-t} - 5t + C\\ \frac{P}{P-5} =& Ce^{-10e^{-t}-5t}.\end{aligned}$$
Since $P(0) = 1$ we find $C = -\frac{e^{10}}{4}$. Thus,
$$P(t) = \frac{5e^{-10e^{-t}-5t+10}}{4 + e^{-10e^{-t}-5t+10}}.$$



(b) Using a calculator we find $P_{max} \approx 2.7$ or 270,000 infected people.
(c) From the plot, we see that $t \approx 1.8$ months for $P < 0.5 = 50,000$ infected people ■

Problem 12.11
Consider a chemical reaction of the form $A + B \to C$, in which the rates of

change of the two chemical reactants, A and B, are described by the following two differential equations

$$A' = -kAB, \quad B' = -kAB$$

where k is a positive constant. Assume that 5 moles of reactant A and 2 moles of reactant B are present at the beginning of the reaction.
(a) Show that the difference $A(t) - B(t)$ remains constant in time. What the value of this constant?
(b) Use the observation made in (a) to derive an initial value problem for reactant A.
(c) It was observed, after the reaction had progressed for 1 sec, that 4 moles of reactant A remained. How much of reactants A and B will be left after 4 sec of reaction time?

Solution.
(a) Since

$$\frac{d}{dt}[A(t) - B(t)] = A'(t) - B'(t) = -kAB - (-kAB) = 0$$

we obtain $A(t) - B(t) = C$. Also, $C = A(0) - B(0) = 5 - 2 = 3$ moles. Hence, $A(t) - B(t) = 3$.
(b) From part (a), $B(t) = 3 + A(t)$ so $A' = -kAB = -kA(3 + A)$. Thus, A satisfies the initial-value problem

$$A' = -kA - 3kA^2, \ A(0) = 5.$$

(c) We solve the previous equation as follows

$$\begin{aligned}
\frac{A'}{A(A+3)} =& -k \\
\frac{A'}{A} - \frac{A'}{A+3} =& -3k \\
\int \frac{A'}{A}dt - \int \frac{A'}{A+3}dt =& \int -3kdt \\
\frac{\left|\frac{A}{A+3}\right|}{=} & -3kt + C \\
\frac{A}{A+3} =& Ce^{-3kt}.
\end{aligned}$$

Since $A(0) = 5$ we find $C = \frac{5}{8}$. Now, solving for A we find

$$A(t) = \frac{15}{5 - 2e^{-3kt}}.$$

But $A(1) = 4$ so that $k = \frac{1}{3}\ln\frac{8}{5}$. Finally,

$$A(4) = \frac{15}{5 - 2\left(\frac{5}{8}\right)^4} \approx 3.195 \text{ moles}$$

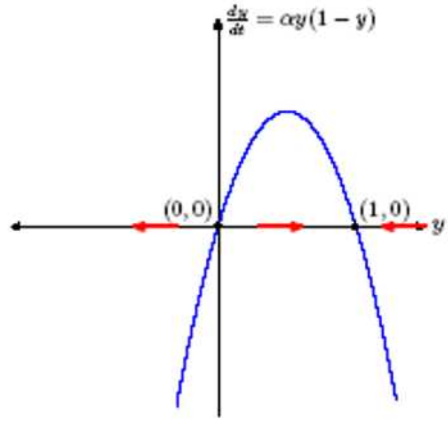
and $B(4) = A(4) - 3 = 0.195$ moles ■

Problem 12.12
Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of the infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population, and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so that the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

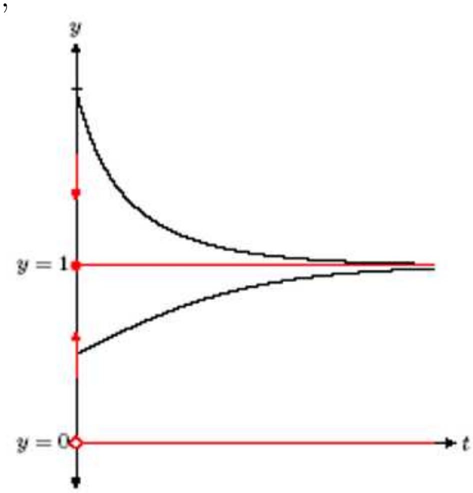
$$\frac{dy}{dt} = \alpha y(1-y), y(0) = y_0, \qquad (-21)$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.
(a) Find the equilibrium points for the given differential equation, and determine whether each is stable or unstable. That is, do a complete qualitative analysis on the equation, complete with a graph of $\frac{dy}{dt}$ versus y, and a sketch of possible solutions in the ty-plane.
(b) Solve the initial value problem and verify that the conclusion you reached in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

Solution.
(a) To find the equilibrium points, set the right hand side of equation (-21) equal to zero and solve for y to find the two values $y = 0$ and $y = 1$. The graph of $\frac{dy}{dt}$ is given next.



The graph indicates that $y' > 0$ for $0 < y < 1$, which means that y is increasing with time. This is indicated by an arrow on the phase line which points to the right in the direction of increasing y. Similarly, $y' < 0$ on $(-\infty, 0) \cup (1, \infty)$, so y is decreasing and the flow is to the left on the phase line on this set.
Thus, $y = 0$ is unstable, while $y = 1$ is a stable equilibrium point.
Turning the phase line vertical, then sketching the equilibrium solutions allows us to easily sketch a "portrait of the solutions. We limit our attention to the first quadrant,



where both y and t are positive. We conclude that regardless of initial condition, the entire population is eventually infected.

(b) The equation is separable.

$$\frac{y'}{y(1-y)} = \alpha.$$

Partial fraction decomposition reveals the following.

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

$$1 = (B-A)y + A.$$

Thus, $A = 1$ and $B = 1$. We can then write

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Thus,

$$\begin{aligned}
\int \frac{y'}{y}dt + \int \frac{y'}{1-y}dt =& \int \alpha dt \\
\ln\left|\frac{y}{1-y}\right| =& \alpha t + C \\
\frac{y}{1-y} =& Ce^{\alpha t} \\
y(t) =& \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}}.
\end{aligned}$$

Since $y(0) = y_0$ we find $C = \frac{y_0}{1-y_0}$. Hence,

$$y(t) = \frac{y_0 e^{\alpha t}}{(1-y_0) + y_0 e^{\alpha t}}.$$

Finally,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{y_0}{y_0 + (1-y_0)e^{-\alpha t}} = 1$$

as predicted by the qualitative analysis in part (a) ■

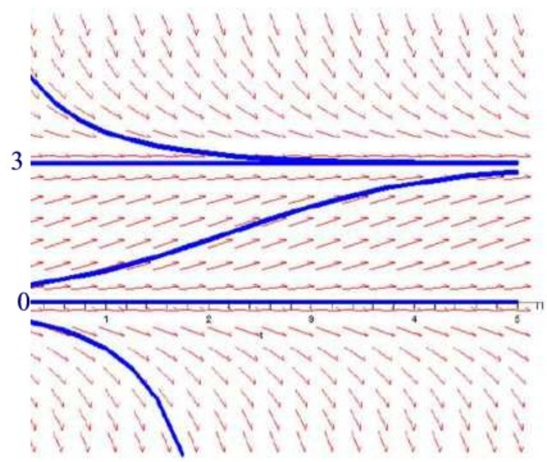
Problem 12.13

Suppose that a population can be modeled by the logistic equation

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{3}\right)$$

Use qualitative techniques to describe the population over time.

Solution.
We see from the direction field below that $P = 0$ is unstable whereas $P = 3$ is stable ■



Problem 12.14
Find the constants A and B so that

$$P(t) = \frac{e^{0.2t}}{A + Be^{0.2t}}$$

is the solution to the logistic model

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{200}\right), \quad P(0) = 150.$$

Solution.
Since $P(0) = 150$ we obtain $A + B = \frac{1}{150}$. Also, $K = 200 = \lim_{t\to\infty} P(t) = \frac{1}{B}$ so that $B = \frac{1}{200}$. Hence, $A = \frac{1}{150} - \frac{1}{200} = \frac{1}{600}$ ■

Problem 12.15
A restricted access lake is stocked with 400 fish. It is estimated that the lake will be able to hold 10,000 fish. The number of fish tripled in the first year. Assuming that the fish population follows a logistic model and that 10,000 is the limiting population, find the length of time needed for the fish population to reach 5000.

Solution.
We have $P(0) = 400$ and $K = 10,000$ so that

$$P(t) = \frac{10,000}{1 + 24e^{-rt}}.$$

But $P(1) = 1200$ so that

$$\frac{10,000}{1 + 24e^{-r}} = 1200.$$

Solving for r we find $r = \ln \frac{72}{22} \approx 1.186$ Hence,

$$P(t) = \frac{10,000}{1 + 24e^{-1.186t}}.$$

The population reaches $\frac{K}{2} = 5000$ when

$$t = \frac{\ln\left(\frac{K-P(0)}{P(0)}\right)}{r} = \frac{\ln 24}{\ln 7222} \approx 2.68.$$

Thus it takes 2.68 years for the fish population to reach 5000 ■

Problem 12.16
Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 50?

Solution.
We have $P(0) = 10$ and $K = 100$. Thus,

$$P(t) = \frac{100}{1 + 9e^{-rt}}.$$

Since $P(10) = 23$ we obtain

$$\frac{100}{1 + 9e^{-10r}} = 23.$$

Solving for r we find $r \approx .098891$. Thus,

$$P(t) = \frac{100}{1 + 9e^{-0.098891t}}.$$

Finally, we want to solve the equation

$$\frac{100}{1+9e^{-0.098891t}} = 50$$

Solving this equation for t we find $t \approx 22$ years ■

Problem 12.17
Show that $P(t) = \frac{800}{1+15e^{-1.6t}}$ satisfies the differential equation

$$\frac{dP}{dt} = 0.002P(800 - P).$$

Solution.
We have

$$\frac{dP}{dt} = 800(-1)(1+15e^{-1.6t})^{-2}(15)(-1.6)e^{-1.6t} = \frac{19,200e^{-1.6t}}{(1+15e^{-1.6t})^2}$$

and

$$0.002P(800-P) = 0.002\frac{800}{1+15e^{-1.6t}}\left(800 - \frac{800}{1+15e^{-1.6t}}\right) = \frac{19,200e^{-1.6t}}{(1+15e^{-1.6t})^2}.$$

Thus, $P(t)$ satisfies the given differential equation ■

Problem 12.18
A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate r and the time required for the population to reach three quarters of its carrying capacity.

Solution.
We have $K = 20,000$ and $P(0) = 1000$. Thus,

$$P(t) = \frac{20,000}{1+19e^{-rt}}.$$

Since $P(8) = 1200$ we obtain

$$\frac{20,000}{1+19e^{-8r}} = 1200.$$

Solving for r we find $r \approx .02411$. Now, we want to find t so that $P(t) = \frac{3}{4}(20,000) = 15,000$. That is,

$$\frac{20,000}{1+19e^{-0.02411t}} = 15,000.$$

Solving for t we find $t \approx 167.69$ hours ■

Problem 12.19
Let $P(t)$ be the population size for a bacteria colony at time t. The logistic model is that

$$\frac{dP}{dt} = kP(t)(M - P(t)),$$

where $k > 0$ and $M > 0$ are constants. Solve this equation when $k = 1$ and $M = 1000$ with $P(0) = 100$.

Solution.
We have

$$\frac{dP}{dt} = 0.001(1 - \frac{P}{1000})P.$$

Thus, $r = 0.001$ and $K = 1000$. The formula for $P(t)$ is then

$$P(t) = \frac{1000}{1+9e^{-0.001t}} \blacksquare$$

Problem 12.20
For the population model

$$P'(t) = 5P(t)(1000 - P(t))$$

with $P(0) = 100$ find the asymptotic population size $\lim_{t\to\infty} P(t)$.

Solution.
Rewriting the equation in the form

$$P' = 0.005P(1 - \frac{P}{1000}).$$

Since $r = 0.005, K = 1000$, and $P(0) = 100$ we find

$$P(t) = \frac{1000}{1+9e^{-.005t}}.$$

Thus,

$$\lim_{t\to\infty} P(t) = 1000 \blacksquare$$

13 Applications of First Order Nonlinear Equations: One-Dimensional Motion with Air Resistance

Problem 13.1
A parachutist whose mass is 75 kg drops from a helicopter hovering 2000 m above ground, and falls towards the ground under the influence of gravity. Assume that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $k = 30N - s/m$ when the chute is closed, and $k' = 90N - s/m$ when the chute is opened. If the chute doesn't open until the velocity of the parachutist reaches 20 m/s, after how many seconds will she reach the ground?

Solution.
We consider the two phases of her flight: a) when the chute is closed and b) when the chute is opened. In the first case, the velocity of the parachutist at any time t is given by

$$v(t) = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{k}{m}t}$$

with $m = 75, g = 9.81, k = 30,\ v_0 = 0$ so that

$$v(t) = -(\frac{75}{30})9.81 + (\frac{75}{30})(9.81)e^{-\frac{30}{75}t} = 24.525(e^{-\frac{2}{5}t} - 1).$$

We need to find at what time the chute opens. Let this time be denoted as T_1. Then, T_1 is obtained by solving

$$24.525(e^{-\frac{2}{5}T_1} - 1) = -20$$

and we find that $T_1 = 4.22$ sec. Also, when the chute opens, the parachutist is at a height of $2000 - y(T_1)$ from the ground, where

$$y(T_1) = -61.3125(e^{-\frac{2}{5}T_1} - 1) - 24.525T_1 \approx -53.52.$$

In the next phase of the flight down, the initial conditions are $v_0 = -20m/s$, and $y(0) = -53.52$. In this phase we have

$$v(t) = -(\frac{75}{90})9.81 + (-20 + (\frac{75}{90}9.81)e^{-\frac{90}{75}t}.$$

The equation of motion is

$$y(t) = -8.175t + 9.854e^{-\frac{90}{75}t} - 63.374.$$

Thus, the time T_2 that the parachutist takes to reach the ground from the moment the chute is opened can be obtained by solving

$$-2000 = -8.175T_2 + 9.854e^{-\frac{6}{5}T_2} - 63.374.$$

Solving this equation using a calculator we find $T_2 \approx 238.14$ sec. Thus the total time taken by the parachutist to reach ground is $T_1 + T_2 = 4.22 + 238.14 \approx 242$ seconds ■

Problem 13.2
An object of mass m is dropped from a high altitude. How long will it take the object to achieve a velocity equal to one-half of its terminal velocity if the drag force is assumed proportional to the velocity?

Solution.
Setting $v_0 = 0$ in the formula

$$v(t) = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{k}{m}t}$$

to obtain

$$v(t) = -\frac{mg}{k}(1 - e^{-\frac{k}{m}t}).$$

The terminal velocity is $v(t) = -\frac{mg}{k}$ and we want to find t such that

$$-\frac{mg}{2k} = -\frac{mg}{k}(1 - e^{-\frac{k}{m}t}).$$

This can be done as follows

$$\begin{aligned} 1 - e^{-\frac{k}{m}t} =& \frac{1}{2} \\ e^{-\frac{k}{m}t} =& \frac{1}{2} \\ -\frac{k}{m}t =& -\ln 2 \\ t =& \frac{m}{k}\ln 2 \ ■ \end{aligned}$$

Problem 13.3
An object of mass m is dropped from a high altitude. Assume the drag force is proportional to the square of the velocity with drag constant k. Find a formula for $v(t)$.

Solution.
We have

$$\begin{aligned}
mv' =& - mg + kv^2\\
v' =& \frac{k}{m}\left(v^2 - \frac{mg}{k}\right)\\
\frac{v'}{v^2 - \frac{mg}{k}} =& \frac{k}{m}\\
\sqrt{\frac{k}{mg}}\left(\frac{v'}{v - \sqrt{\frac{mg}{k}}} - \frac{v'}{v + \sqrt{\frac{mg}{k}}}\right) =& 2\frac{k}{m}\\
\frac{v'}{v - \sqrt{\frac{mg}{k}}} - \frac{v'}{v + \sqrt{\frac{mg}{k}}} =& 2\sqrt{\frac{kg}{m}}\\
\ln\left|\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}}\right| =& 2\sqrt{\frac{kg}{m}} + C\\
\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}} =& e^{2\sqrt{\frac{kg}{m}}}\\
v(t) =& -\sqrt{\frac{mg}{k}}\left(\frac{1 - e^{-2\sqrt{\frac{kg}{m}}t}}{1 + e^{2\sqrt{\frac{kg}{m}}t}}\right).
\end{aligned}$$

Note that $C = 0$ since $v(0) = 0$ ■

Problem 13.4
Assume that the action of a parachute can be modeled as a drag force proportional to the square of the velocity. What drag constant k of the parachute is needed for a 200 lb person to achieve a teminal velocity of 10 mph?

Solution.
From the previous problem we find that the terminal velocity is

$$v(t) = -\sqrt{\frac{mg}{k}}.$$

Now,

$$10\ mph = 10\left(\tfrac{5280}{3600}\right) = 14.67 \text{ ft/sec}$$

Therefore,

$$\sqrt{\frac{200}{k}} = 14.67 \Longrightarrow k \approx 0.929\ \frac{lb \cdot sec^2}{ft^2}\ \blacksquare$$

Problem 13.5
A drag chute must be designed to reduce the speed of 3000-lb dragster from 220 mph to 50 mph in 4 seconds. Assume that the drag force is proportional to the velocity.
(a) What value of the drag coefficient k is needed to accomplish this?
(b) How far will the dragster travel in the 4-sec interval?

Solution.
(a) We have

$$220 \text{ mph} = 220\left(\tfrac{5280}{3600}\right) \approx 322.67 \text{ ft/sec}$$

$$50 \text{ mph} = 50\left(\tfrac{5280}{3600}\right) \approx 73.33 \text{ ft/sec}$$

Now,

$$\begin{aligned} mv' =& -kv \\ \frac{v'}{v} =& -\frac{k}{m} \\ v(t) =& v_0 e^{-\frac{k}{m}t} \\ v(t) =& 220e^{-\frac{k}{m}t} \end{aligned}$$

But

$$\begin{aligned} v(4) =& 50 \\ 220e^{-4\frac{k}{m}} =& 50 \\ e^{-4\frac{k}{m}} =& \frac{73.33}{322.67} \\ -4\frac{32k}{3000} =& \ln\left(\frac{50}{220}\right) \\ k =& -\frac{3000}{128}\ln\left(\frac{50}{220}\right) \approx 34.725\ lb \cdot sec/ft. \end{aligned}$$

(b) We have

$$\begin{aligned} y(t) = & \int_0^4 v(t)dt = v_0\left[-\frac{m}{k}e^{-\frac{k}{m}t}\right]_0^4 \\ = & \frac{3000}{32}322.67\frac{1}{34.725}\left(1-e^{-\frac{4(34.725)}{\frac{3000}{32}}}\right) \\ \approx & 673\ ft \blacksquare \end{aligned}$$

Problem 13.6
A projectile of mass m is launched vertically upward from ground level at time $t = 0$ with initial velocity v_0 and is acted upon by gravity and air resistance. Assume the drag force is proportional to velocity, with drag coefficient k.
(a) Derive an expression for the time t_m when the projectile achieves its maximum height.
(b) Derive an expression for the maximum height.

Solution.
(a) t_m is the solution to

$$-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}} = 0.$$

Solving this equation we find

$$\begin{aligned} \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}} = & \frac{mg}{k} \\ \frac{mg}{k}\left(\frac{k}{mg}v_0 + 1\right)e^{-\frac{k}{m}} = & \frac{mg}{k} \\ e^{\frac{k}{m}t} = & \frac{k}{mg}v_0 + 1 \\ \frac{k}{m}t = & \ln\left(\frac{k}{mg}v_0 + 1\right) \\ t_m = & \frac{m}{k}\ln\left(\frac{k}{mg}v_0 + 1\right) \end{aligned}$$

(b) We have

$$
\begin{aligned}
y(t_m) &= \int_0^{t_m} v(t)dt = \int_0^{t_m} (-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t})dt \\
&= \left[-\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}\right]_0^{t_m} \\
&= -\frac{mg}{k}t_m - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t_m} + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) \\
&= -\frac{mg}{k}t_m + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t_m}\right) \\
&= -\frac{m^2}{k^2}g\ln\left(\frac{k}{mg}v_0 + 1\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{\ln\frac{1}{\frac{k}{mg}v_0+1}}\right) \\
&= -\frac{m^2}{k^2}g\ln\left(\frac{k}{mg}v_0 + 1\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - \frac{1}{\frac{k}{mg}v_0 + 1}\right) \blacksquare
\end{aligned}
$$

Problem 13.7

A projectile is launched vertically upward from ground level with initial velocity v_0. Neglect air resistance. Show that the time it takes the projectile to reach its maximum height is equal to the time it takes to fall from this maximum height to the ground.

Solution.

Since air resistance is negligible, we have $mv' = -mg$. Solving for v we find $v(t) = -gt + v_0$. Integrating to obtain $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 = -\frac{1}{2}gt^2 + v_0t$ since $y(0) = 0$. The time it takes the projectile to reach its maximum height occurs when $v(t) = 0$ and is given by $t_m = \frac{v_0}{g}$. Next, we find the impact time. This is the time when $y(t) = 0$. Solving this equation for t we find

$$t = 2\frac{v_0}{g} = 2t_m \blacksquare$$

Problem 13.8

A 180-lb skydiver drops from a hot-air balloon. After 10 sec of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to the velocity. After an additional 4 sec, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-lb person will reach a terminal velocity of 10 mph.

(a) What is the speed of the skydiver immediately before the parachute is

opened?
(b)What is the parachutist impact velocity?
(c) At what altitude was the parachute opened?
(d) What is the ballon altitude?

Solution.
(a) For $0 \le t \le 10$, $v' = -g$ so that $v(t) = -gt + v_0 = -gt$. Thus, $v(10) = -320$ ft/sec.
(b) For $10 \le t \le 14$ the motion is described by the initial-value problem

$$mv' + kv = -mg, \quad y(14) = 0.$$

Solving we find
$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}.$$
Since the terminal velocity is 10 mph, we have

$$\frac{200}{k} = 10\left(\frac{5280}{3600}\right) \Longrightarrow k \approx 13.64.$$

Thus,

$$v(4) = -\frac{180}{13.64} + \left(-320 + \frac{180}{13.64}\right)e^{-\frac{(13.64)(32)(4)}{180}} \approx -13.22\ ft/sec.$$

(c) We have

$$\begin{aligned} y(t) =& -\int_0^4 v(t)dt = \left[\frac{mg}{k}t + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]_0^4 \\ =&\frac{180(4)}{13.64} + \frac{180/32}{13.64}\left(-320 + \frac{180}{13.64}\right)e^{-\frac{13.64(4)}{180/32}} \\ \approx&179.35\ ft. \end{aligned}$$

(d) The balloon's altitude is $\frac{1}{2}32(10)^2 + 179.35 = 1779.35$ feet ■

Problem 13.9
A body of mass m is moving with velocity v in a gravity-free laboratory (i.e. outer space). It is known that the body experiences resistance in its flight

proportional to the square root of its velocity. Consequently the motion of the body is governed by the initial-value problem

$$m\frac{dv}{dt} = -k\sqrt{v},\ v(0) = v_0$$

where k is a positive constant. Find $v(t)$. Does the body ultimately come to rest? If so, when does this happen?

Solution.
Solving for v we find

$$\begin{aligned}
v' =& -\frac{k}{m}\sqrt{v}\\
\frac{v'}{\sqrt{v}} =& -\frac{k}{m}\\
\sqrt{v} =& -2\frac{k}{m}t + C\\
\sqrt{v} =& -2\frac{k}{m}t + \sqrt{v_0}\\
v(t) =& \left(\sqrt{v_0} - 2\frac{k}{m}t\right)^2.
\end{aligned}$$

The body comes to rest when $t = \frac{m}{2k}\sqrt{v_0}$ ■

Problem 13.10
A mass m is thrown upward from ground level with initial velocity v_0. Assume that air resistance is proportional to velocity, the constant of proportionality being k. Show that the maximum height attained is

$$-\frac{m^2g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - \frac{1}{\frac{k}{mg}v_0 + 1}\right)$$

Solution.
This is just Problem 13.6(b) ■

Problem 13.11
A ball weighing 3/4 lb is thrown vertically upward from a point 6 ft above ground level with an initial velocity of 20ft/sec. As it rises it is acted upon by air resistance that is numerically equal to v/64 lbs where v is velocity (in ft/sec). How high will it rise?

Solution.
We have

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} \\ &= -0.75(64) + (20 + 0.75(64))e^{-\frac{1}{64(0.75/32)}} \\ &= -48 + 68e^{-\frac{2t}{3}}. \end{aligned}$$

The maximum height occurs when $v = 0$. Solving this equation for t we find $t \approx 0.522$ sec. Now, the displacement function is

$$y(t) = -48t - 102e^{-\frac{2t}{3}} + C.$$

But $y(0) = 6$ so that $C = 108$. Thus, The maximum height of the ball is

$$y(0.522) = -48(0.522) - 102e^{-\frac{2(0.522)}{3}} + 108 \approx 10.9\ ft \blacksquare$$

Problem 13.12
A parachutist weighs 160 lbs (with chute). The chute is released immediately after the jump from a height of 1000 ft. The force due to air resistence is proportional to velocity and is given by $F_R = -8v$. Find the time of impact.

Solution.
We have

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} \\ &= -\frac{160}{8} + \frac{160}{8}e^{-\frac{8}{5}t} \\ &= -20 + 20e^{-\frac{8t}{5}}. \end{aligned}$$

The position function is then

$$y(t) = -20t - 12.5e^{-\frac{8t}{5}} + 12.5.$$

The time of impact is the solution to the equation

$$-1000 = -20t - 12.5e^{-\frac{8t}{5}} + 12.5.$$

Solving this equation using a calculator we find $t \approx 50.6$ sec. That is, the parachutist hits the ground 50.6 seconds after jumping $\blacksquare$

Problem 13.13
A parachutist weighs 100 Kg (with chute). The chute is released 30 seconds after the jump from a height of 2000 m. The force due to air resistance is defined by $F_R = -kv$ where $k = 15$ when the chute was closed and $k = 100$ when the chute was open. Find
(a) the distance and velocity function during the time the chute was closed (i.e., $0 \leq t \leq 30$ seconds).
(b) the distance and velocity function during the time the chute was open (i.e., $t \geq 30$ seconds).
(c) the time of landing.
(d) the velocity of landing or the impact velocity.

Solution.
(a) For $0 \leq t \leq 30$, we have

$$\begin{aligned} v_1(t) =& -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t} \\ =& -\frac{100(9.81)}{15} + \frac{100(9.81)}{15} e^{-0.15t} \\ =& -65.4 + 65.4e^{-0.15t}. \end{aligned}$$

This the velocity with the time t starting the moment the parachutist jump. After 30 seconds, this reaches $v_0 = v_1(30) = -65.4 + 65.4e^{-4.5} \approx -64.67$. The distance fallen is

$$y_1(t) = -65.4t - 436e^{-0.15t} + 436.$$

So after 30 seconds it has fallen

$$y_1(30) = -65.4(30) - 436e^{-4.5} + 436 \approx -1530 \text{ meters}.$$

(b) For $t \geq 30$ we have

$$v_2(t) = -\frac{100(9.81)}{100} + (-64.67 + \frac{100(9.81)}{100})e^{-t} = -9.81 - 54.86e^{-t}.$$

This is the velocity starting with the time the chute was open. The distance fallen is

$$y_2(t) = \int_0^t v_2(t)dt + y_1(30) = -9.81t + 54.8e^{-t} - 54.86 - 1530 = -9.81t + 54.8e^{-t} - 1584.86.$$

(c) The time of impact is the solution to the equation $y_2(t) = -2000$. That is,

$$-9.81t + 54.8e^{-t} - 1584.86 = -2000.$$

Solving this numerically we find $t \approx 42.44$ sec.
(d) The impact velocity is $v_2(42.44) \approx -9.81$ m/sec ■

Problem 13.14
Solve the equation

$$m\frac{dv}{dt} = -kv(t) - mg$$

with initial condition $v(0) = 0$ when $k = 0.1$ and $m = 1$ kg.

Solution.
With the given values we have

$$v' = -0.1v - 9.8$$

Solving for v and using the fact that $v(0) = 0$ we find

$$\begin{aligned} v' =& -0.1v - 9.8 \\ \frac{v'}{v-98} =& -0.1 \\ \ln|v-98 =& -0.1t + C \\ v(t) =& Ce^{-0.1t} + 98 \\ v(t) =& 98(1 - e^{-0.1t}). \end{aligned}$$ ■

Problem 13.15
A rocket is launched at time $t = 0$ and its engine provides a constant thrust for 10 seconds. During this time the burning of the rocket fuel constantly decreases the mass of the rocket. The problem is to determine the velocity $v(t)$ of the rocket at time t during this initial 10 second interval. Denote by $m(t)$ the mass of the rocket at time t and by U the constant upward thrust (force) provided by the engine. Applying Newton's Law gives

$$\frac{d}{dt}(m(t)v(t)) = U - kv(t) - m(t)g$$

where an air resistance term is included in addition to the gravitational and thrust terms. Find a fomrula for $v(t)$.

Solution.
The given equation can be written in the form

$$v' + \left(\frac{m'(t)+k}{m(t)}\right)v = U - m(t)g.$$

Solving this equation by the method of integrating factor we find

$$v(t) = e^{-\int\left(\frac{m'(t)+k}{m(t)}\right)dt}\int e^{\int\left(\frac{m'(t)+k}{m(t)}\right)}(U-m(t)g)dt + Ce^{-\int\left(\frac{m'(t)+k}{m(t)}\right)dt} \blacksquare$$

Problem 13.16
If $m(t) = 11-t, U = 200$, and $k = 0$ the equation of motion of the rocket is

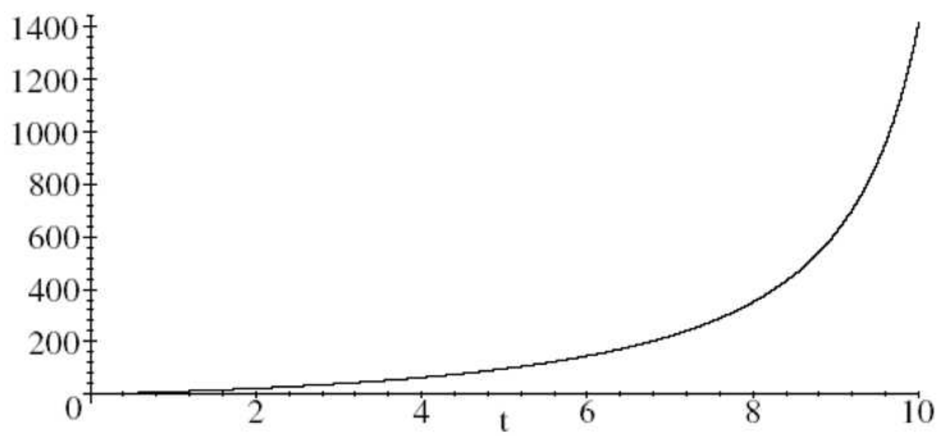
$$\frac{d}{dt}((11-t)v(t)) = 200 - (11-t)g.$$

Find $v(t)$ for $0 \le t \le 10$. Assume $v(0) = 0$. Make a graph of the velocity as a function of time.

Solution.
Since the right side does not depend on $v(t)$, the equation can be solved by simple integration giving

$$v(t) = \frac{200t + (g/2)(11-t)^2 - 121(g/2)}{11-t},\ 0 \le t \le 10.$$

The graph of v is given below $\blacksquare$



Problem 13.17
If $m(t) = 11 - t, U = 200$, and $k = 2$ the equation of motion of the rocket is

$$\frac{d}{dt}((11-t)v(t)) = 200 - 2v(t) - (11-t)g.$$

Find $v(t)$ for $0 \le t \le 10$. Assume $v(0) = 0$. Make a graph of the velocity as a function of time.

Solution.
Expanding the derivative on the left hand side and rearranging terms gives

$$v' + \frac{v}{11-t} = \frac{200}{11-t} - g.$$

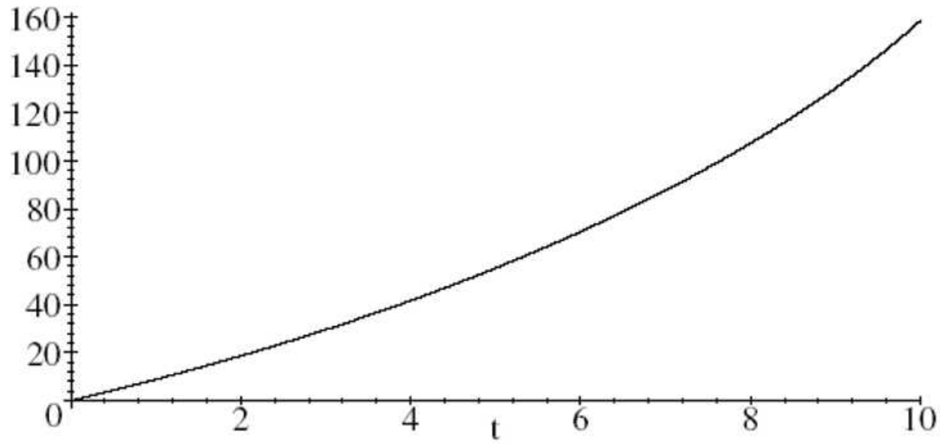
The integrating factor is thus $1/(11-t)$. Multiplying by the integrating factor gives

$$\left(\frac{v}{11-t}\right)' = \frac{200}{(11-t)^2} - \frac{g}{11-t}.$$

Integrating and solving we find

$$v(t) = 200 + (11-t)g\ln(11-t) - (\frac{200}{11} + g(\ln 11)(11-t).$$

The graph of v is given below ■



Problem 13.18
Using

$$v(t) = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{k}{m}t}$$

find the position function $y(t)$.

Solution.
By integrating $v(t)$ we find

$$\begin{aligned} y(t_m) =& \int_0^t v(s)ds = \int_0^t (-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}s})ds \\ =& \left[-\frac{mg}{k}s - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}s}\right]_0^t \\ =& -\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t_m} + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) \\ =& -\frac{mg}{k}t + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t}\right). \blacksquare \end{aligned}$$

Problem 13.19
An arrow is shot upward from the origin with an initial velocity of 300 ft/sec. Assume that there is no air resistance and use the model

$$m\frac{dv}{dt} = -mg$$

Find the velocity and position as a function of time. Find the ascent time, the descent time, maximum height, and the impact velocity.

Solution.
The velocity at time t is found as follows

$$\begin{aligned} v' =& -g \\ v(t) =& -gt + v(0) \\ v(t) =& -32t + 300. \end{aligned}$$

The position function is

$$y(t) = -16t^2 + 300t.$$

The maximum height occurs when $v = 0$. That is when $t = 9.375$ sec which is the ascent time. The maximum height is $y(9.375) = 1406.25$ ft. The impact time occurs when $y(t) = 0$ or $t = 18.75$ sec. The impact velocity is $v(18.75) = -300$ ft/sec. Notice that the ascent time is equal to descent time ■

Problem 13.20
An arrow is shot upward from the origin with an initial velocity of 300 ft/sec.

Assume that air resistance is proportional to the velocity, $F_R = 0.04mv$ and use the model

$$m\frac{dv}{dt} = -mg - kv$$

Find the velocity and position as a function of time, and plot the position function. Find the ascent time, the descent time, maximum height, and the impact velocity.

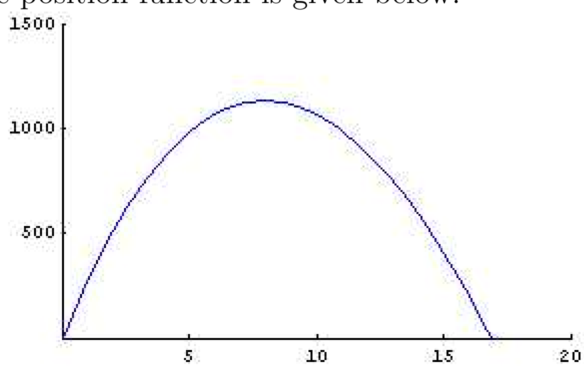
Solution.
The velocity at time t is found as follows

$$\begin{aligned} v' =& -g - 0.04v \\ v' + 0.04v =& -32 \\ \left(e^{0.04t}v\right)' =& -32e^{0.04t} \\ v(t) =& -800 + Ce^{-0.04t}. \end{aligned}$$

But $v(0) = 300$ so that $C = 1100$. Thus, $v(t) = 1100e^{-0.04t} - 800$.
The position function is

$$y(t) = -800t - 27500e^{-0.04t} + 27500.$$

The graph of the position function is given below.



The ascent is the solution to $v = 0$. That is $t \approx 7.96$ sec. The maximum height is $y(7.96) \approx 1130.93$ ft. The impact time occurs when $y(t) = 0$ or $t \approx 16.87$ sec. The impact velocity is $v(16.87) \approx -239.76$ ft/sec. Finally the descent time is 16.87 - 7.96 = 8.91 sec ■

14 One-Dimensional Dynamics: Velocity as Function of Position

In Problems 14.1 - 14.3, transform the equation into one having distance x as the independent variable. Determine the position x_f at which the object comes to rest.(If the object does not come to rest set $x_f = \infty$) Assume that $v = v_0$ when $x = 0$.

Problem 14.1

$$m\tfrac{dv}{dt} = -kx^2v.$$

Solution.
By the chain rule $\frac{dv}{dt} = v\frac{dv}{dx}$. Thus,

$$mv\frac{dv}{dx} = -kx^2v.$$

Solving this equation we find

$$\begin{aligned}\frac{dv}{dx} =& - kx^2\\ v(x) =& - \frac{k}{3}x^3 + C.\end{aligned}$$

But $v = v_0$ when $x = 0$ so that $C = v_0$. Thus, $v(x) = -\frac{k}{3}x^3 + v_0$. The object comes to rest when $v = 0$. In this case, $x_f^3 = \frac{3m}{k}v_0$ and therefore

$$x_f = \left(\frac{3m}{k}v_0\right)^{\frac{1}{2}} \blacksquare$$

Problem 14.2

$$m\tfrac{dv}{dt} = -kxv^2.$$

Solution.
By the chain rule $\frac{dv}{dt} = v\frac{dv}{dx}$. Thus,

$$mv\frac{dv}{dx} = -kxv^2.$$

Solving for $v(x)$ we find

$$\begin{aligned}\frac{1}{v}\frac{dv}{dx} =& -\frac{k}{m}x\\ \ln|v| =& -\frac{k}{2m}x^2 + C\\ v(x) =& Ce^{-\frac{k}{2m}x^2}\\ v(x) =& v_0e^{-\frac{k}{2m}x^2}.\end{aligned}$$

The object comes to rest when $v = 0$. This implies $x_f = \infty$ ■

Problem 14.3

$$m\frac{dv}{dt} = \frac{kv}{1+x}.$$

Solution.
By the chain rule $\frac{dv}{dt} = v\frac{dv}{dx}$. Thus,

$$mv\frac{dv}{dx} = \frac{kv}{1+x}.$$

Solving for $v(x)$ we find

$$\begin{aligned}\frac{dv}{dx} =& -\frac{k}{m}\frac{1}{1+x}\\ v(x) =& -\frac{k}{m}\ln(1+x) + C\\ v(x) =& -\frac{k}{m}\ln(1+x) + v_0.\end{aligned}$$

The object comes to rest when $v = 0$. This implies that $\ln(1+x) = \frac{mv_0}{k}$ so that $x_f = e^{\frac{mv_0}{k}} - 1$ ■

Problem 14.4
A boat having mass m is launched vertically with an initial velocity v_0. Assume the water exerts a drag force that is proportional to the square of the velocity. Determine the velocity of the boat when it is a distance d from the dock.

Solution.
We have $m\frac{dv}{dt} = -kv^2$. By the chain rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

Thus,

$$\begin{aligned}\frac{1}{v}\frac{dv}{dx} =& -\frac{k}{m}\\ \ln|v| =& -\frac{k}{m}x + C\\ v(x) =& Ce^{-\frac{k}{m}x}\\ v(x) =& v_0e^{-\frac{k}{m}x}.\end{aligned}$$

At a distance d from the dock the velocity is $v(d) = v_0e^{-\frac{k}{m}d}$ ■

Problem 14.5
We need to design a ballistics chamber to deccelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity. The coefficient k is given by $k(x) = k_0x$, where x_0 is a constant. If we use time as independent variable then Newton's second law of motion leads to the following differential equation

$$m\frac{dv}{dt} + k_0xv^2 = 0.$$

(a) Adopt distance x as the indepndent variable and rewrite the above differential equation as a first order equation in terms of the new independent variable.
(b) Determine the value k_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within d units of distance.

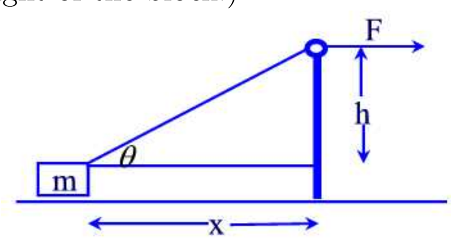
Solution.
(a) $mv\frac{dv}{dx} + k_0xv^2 = 0$, $v = v_0$ when $x = 0$.
(b) Solving the initial value problem in part (a) we find

$$\begin{aligned}\frac{dv}{dx} + \frac{k_0}{m}xv =&0\\ \frac{1}{v}\frac{dv}{dx} =& -\frac{k_0}{m}x\\ \ln|v| =& -\frac{k_0}{2m}x^2 + C\\ v(x) =&Ce^{-\frac{k_0}{2m}x^2}\\ v(x) =&v_0e^{-\frac{k_0}{2m}x^2}.\end{aligned}$$

When, $x = d,\ v = 0.01v_0$ so that $v_0e^{-\frac{k_0}{2m}d^2} = 0.01v_0$. Solving for k_0 we find

$$k_0 = \frac{2m\ln 100}{d^2} \blacksquare$$

Problem 14.6
A block of mass m is pulled over a frictionless (smooth) surface by a cable having a constant tension F (See Figure below). The block starts from rest at a horizontal distance D from the base of the pulley. Apply Newton's law of motion in the horizontal direction. What is the (horizontal) velocity of the block when $x = \frac{D}{3}$? (Assume the vertical component of the tensile force never exceeds the weight of the block.)



Solution.
By Newton's second law of motion

$$m\frac{dv}{dt} = -F\cos\theta.$$

But

$$\cos\theta = \frac{x}{\sqrt{x^2 + h^2}}$$

so that

$$m\frac{dv}{dt} = -\frac{Fx}{\sqrt{x^2+h^2}}.$$

By the chain rule we obtain

$$mv\frac{dv}{dx} = -\frac{Fx}{\sqrt{x^2+h^2}}.$$

Integrating we find

$$\frac{1}{2}mv^2 = -F\sqrt{x^2+h^2} + C.$$

But $v(D) = 0$ so that $C = F\sqrt{D^2+h^2}$. Hence,

$$v^2(x) = \frac{2}{m}F(\sqrt{D^2+h^2} - \sqrt{x^2+h^2}).$$

When $x = \frac{D}{3}$ we see that

$$v(x) = \left[\frac{2}{m}F(\sqrt{D^2+h^2} - \sqrt{x^2+h^2})\right]^{\frac{1}{2}} \blacksquare$$

15 Second Order Linear Differential Equations: Existence and Uniqueness Results

In Problems 15.1 - 15.6, determine the largest t-interval on which the existence and uniqueness theorem guarantees the existence of a unique solution.

Problem 15.1

$$y'' + y' + 3ty = \tan t,\ y(\pi) = 1,\ y'(\pi) = -1.$$

Solution.
In this equation $p(t) = 1,\ q(t) = 3t$ and $g(t) = \tan t$. All three functions are continuous for all $t \neq (2n+1)\frac{\pi}{2}$, where n is an integer. With $t_0 = \pi$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $\frac{\pi}{2} < t < \frac{3\pi}{2}$ ■

Problem 15.2

$$e^t y'' + \frac{1}{t^2-1}y = \frac{4}{t}, y(-2) = 1,\ y'(-2) = 2.$$

Solution.
In this equation $p(t) = 0,\ q(t) = \frac{1}{e^t(t^2-1)}$, and $g(t) = 4e^{-t}$. All three functions are continuous for all $t \neq -1, 0, 1$. With $t_0 = -2$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $-\infty < t < -1$ ■

Problem 15.3

$$ty'' + \frac{\sin 2t}{t^2-9}y' + 2y = 0,\ y(1) = 0, y'(1) = 1.$$

Solution.
In this equation $p(t) = \frac{\sin 2t}{t(t^2-9)},\ q(t) = \frac{2}{t}$, and $g(t) = 0$. All three functions are continuous for all $t \neq -3, 0, 3$. With $t_0 = 1$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $0 < t < 3$ ■

Problem 15.4

$$ty'' - (1+t)y' + y = t^2e^{2t}, y(-1) = 0, y'(-1) = 1.$$

Solution.
In this equation $p(t) = -\frac{1+t}{t},\ q(t) = \frac{1}{t}$, and $g(t) = te^{2t}$. All three functions are continuous for all $t \neq 0$. With $t_0 = -1$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $0 < t < \infty$ ■

Problem 15.5

$$(\sin^2 t)y'' - (2\sin t\cos t)y' + (\cos^2 t + 1)y = \sin^3 t,\ y(\tfrac{\pi}{4}) = 0,\ y'(\tfrac{\pi}{4}) = \sqrt{2}.$$

Solution.
In this equation $p(t) = -2\frac{\cos t}{\sin t},\ q(t) = \frac{\cos^2 t+1}{\sin^2 t}$, and $g(t) = \sin t$. All three functions are continuous for all $t \neq n\pi$, where n is an integer. With $t_0 = \frac{\pi}{4}$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $0 < t < \pi$ ■

Problem 15.6

$$t^2y'' + ty' + y = \sec(\ln t),\ y(\tfrac{\pi}{3}) = 0,\ y'(\tfrac{\pi}{3}) = -1.$$

Solution.
In this equation $p(t) = \frac{1}{t},\ q(t) = \frac{1}{t^2}$, and $g(t) = \frac{\sec(\ln t)}{t^2}$. All three functions are continuous for all $t > 0$ and $t \neq e^{(2n+1)\frac{\pi}{2}}$, where n is an integer. With $t_0 = \frac{\pi}{3}$ then the largest interval of existence guaranteed by the existence and uniqueness theorem is $0 < t < e^{\frac{\pi}{2}}$ ■

In Problems 15.7 - 15.9, give an example of an initial value problem of the form $y'' + p(t)y' + q(t)y = 0,\ y(t_0) = y_0,\ y'(t_0) = y'_0$ for which the given t-interval is the largest on which the existence and uniqueness theorem guarantees a unique solution.

Problem 15.7

$$-\infty < t < \infty.$$

Solution.
One such an answer is

$$y'' + y' + y = 0,\ y(0) = 0,\ y'(0) = 1 \blacksquare$$

Problem 15.8

$$3 < t < \infty.$$

Solution.
One such an answer is

$$y'' + \tfrac{1}{t-3}y' + y = 1,\ y(4) = 0,\ y'(4) = -1 \blacksquare$$

Problem 15.9

$$-1 < t < 5.$$

Solution.
One such an answer is

$$y'' + \tfrac{1}{t+1}y' + y = \tfrac{1}{t-5},\ y(0) = 1,\ y'(0) = 2 \blacksquare$$

Problem 15.10
Consider the initial value problem $t^2y'' - ty' + y = 0,\ y(1) = 1,\ y'(1) = 1.$
(a) What is the largest interval on which the existence and uniqueness theorem guarantees the existence of a unique solution?
(b) Show by direct substitution that the function $y(t) = t$ is the unique solution to this initial value problem. What is the interval on which this solution actually exists?
(c) Does this example contradict the assertion of Theorem 15.1? Explain.

Solution.
(a) Writing the equation in standard form to obtain

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$$

we see that the functions $p(t) = -\frac{1}{t}$ and $q(t) = \frac{1}{t^2}$ are continuous for all $t \neq 0$. Since $t_0 = 1$ then the largest t-interval according to the existence and uniqueness theorem is $0 < t < \infty$.
(b) If $y(t) = t$ then $y'(t) = 1$ and $y''(t) = 0$ so that $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0 - \frac{1}{t} + \frac{1}{t} = 0,\ y(1) = y'(1) = 1$. So $y(t) = t$ is a solution so that by the existence and uniqueness theoren it is the only solution. The t-interval for this solution is $-\infty < t < \infty$.
(c) No because the theorem is local existence theorem and not a global one ■

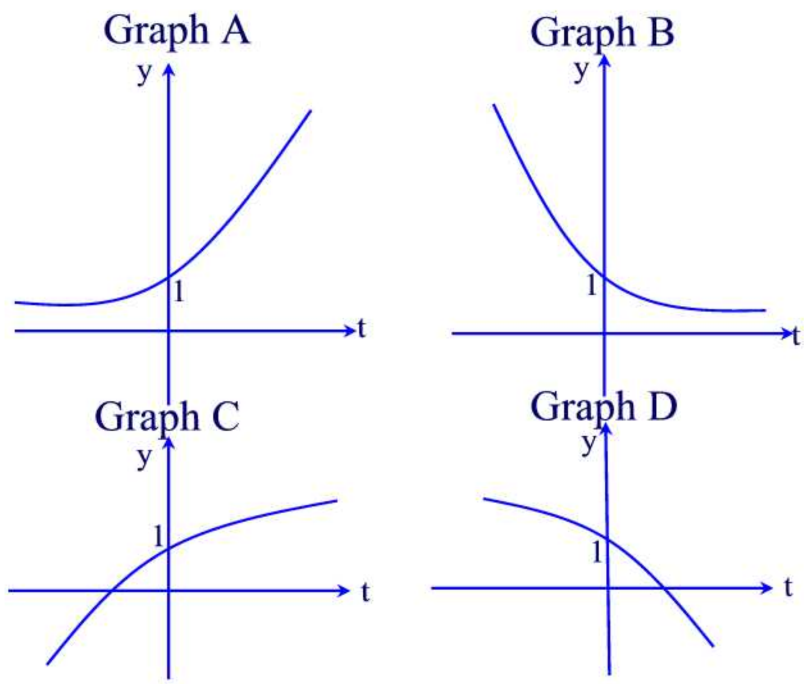
Problem 15.11
Is there a solution $y(t)$ to the initial value problem

$$y'' + 2y' + \frac{1}{t-3}y = 0,\ y(1) = 1,\ y'(1) = 2$$

such that $\lim_{t \to 0^+} y(t) = \infty$?

Solution.
Since $p(t) = 2,\ q(t) = \frac{1}{t-3}$, and $t_0 = 1$ we have according to the existence and uniqueness theorem the largest interval for which the solution $y(t)$ is defined is $-\infty < t < 3$. Since 0 is in that interval then the limit cannot hold ■

Problem 15.12
Consider the graphs shown. Each graph displays a portion of the solution of one of the four initial value problems given. Match each graph with the appropriate initial value problem.
(a) $y'' + y = 2 - \sin t,\ y(0) = 1, y'(0) = -1$.
(b) $y'' + y = -2t,\ y(0) = 1,\ y'(0) = -1$.
(c) $y'' - y = t^2,\ y(0) = y'(0) = 1$.
(d) $y'' - y = -2\cos t,\ y(0) = y'(0) = 1$.



Solution.
(a) B since $y'(0) < 0$ and $y''(0) = 1 > 0$.
(b) D since $y'(0) < 0$ and $y''(0) = -1 < 0$.
(c) A since $y'(0) > 0$ and $y''(0) = 1 > 0$.
(d) C since $y'(0) > 0$ and $y''(0) = -1 < 0$ ■

Problem 15.13
Determine the longest interval in which the initial-value problem

$$(t-3)y'' + ty' + (\ln|t|)y = 0, \; y(1) = 0, \; y'(1) = 1$$

is certain to have a unique solution.

Solution.
We have $p(t) = \frac{t}{t-3}$ and $q(t) = \frac{\ln|t|}{t-3}$. Both functions are continuous for all $t \neq 0, 3$. Since $t_0 = 1$ then the largest t-interval is $0 < t < 3$ ■

Problem 15.14
The existence and uniqueness theorem tells us that the initial-value problem

$$y'' + t^2 y = 0, y(0) = y'(0) = 0$$

define exactly one function $y(t)$. Using only the existence and uniqueness theorem, show that this function has the additional property $y(-t) = y(t)$.

Solution.
Let $Y(t) = y(-t)$. Then $Y'' + t^2Y = y'' + t^2y = 0,\ Y(0) = Y'(0) = 0$ so that $Y(t)$ is a solution to the given initial-value problem. By the existence and uniqueness theorem we must have $Y(t) = y(t)$, i.e., $y(-t) = y(t)$ for all real number t ■

Problem 15.15
By introducing a new variable z, write $y'' - 2y + 1 = 0$ as a system of two first order linear equations of the form $\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{b}$.

Solution.
By letting $z = y'$ we have

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \blacksquare$$

Problem 15.16
Write the differential equation $y'' + 4y' + 4y = 0$ as a first order system.

Solution.
By letting $z = y'$ we have

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \blacksquare$$

Problem 15.17
Using the substitutions $x_1 = y$ and $x_2 = y'$ write the differential equation $y'' + ky' + (t-1)y = 0$ as a first order system.

Solution.
By letting $x_1 = y$ and $x_2 = y'$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ t-1 & k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \blacksquare$$

Problem 15.18
Consider the 2-by-2 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(a) Find $-\int \mathbf{A}(t)dt$.
(b) Let $\mathbf{B} = -\int \mathbf{A}(t)dt$. Compute $\mathbf{B}^2, \mathbf{B}^3, \mathbf{B}^4$, and $\mathbf{B}^5$.
(c) Show that

$$e^B = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} & \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Solution.
(a) We have

$$-\int \mathbf{A}(t)dt = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$$

(b) We have

$$\mathbf{B} = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}, \mathbf{B}^2 = \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix}, \mathbf{B}^3 = \begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix}, \mathbf{B}^4 = \begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix}, \mathbf{B}^5 = \begin{bmatrix} 0 & t^5 \\ -t^5 & 0 \end{bmatrix}$$

(c) Follows from part (b) and the definitiion

$$e^{\mathbf{B}} = \sum_{n=0}^{\infty} \frac{\mathbf{B}^n}{n!} \blacksquare$$

Problem 15.19
Use the previous problem to solve the initial value problem

$$y'' + y = 0, \quad y(0) = 1, \; y'(0) = 0.$$

Solution.
The given equation can be written as a first order system

$$\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{0}$$

where $\mathbf{A}$ as defined in the previous problem. Solving this equation by the method of integrating factor we find

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus,

$$y(t) = c_1 \cos t + c_2 \sin t.$$

Since $y(0) = 1$ we find $c_1 = 1$. Also, $y'(0) = 0$ implies that $c_2 = 0$. Hence, the unique solution to the given initial-value problem is $y(t) = \cos t$ ■

Problem 15.20

Repeat the process of the previous two problems for solving the initial value problem

$$y'' - 2y' = 0,\ \ y(0) = 1,\ \ y'(0) = 2.$$

Solution.

The given equation can be written as a first order system

$$\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{0}$$

where

$$\mathbf{A} = \left[\begin{array}{cc} 0 & -1 \\ 0 & -2 \end{array}\right]$$

Thus,

$$-\int \mathbf{A}(t)dt = \left[\begin{array}{cc} 0 & t \\ 0 & 2t \end{array}\right]$$

Letting $\mathbf{B} = -\int \mathbf{A}(t)dt$ we find

$$\mathbf{B} = \left[\begin{array}{cc} 0 & t \\ 0 & 2t \end{array}\right], \mathbf{B}^2 = \left[\begin{array}{cc} 0 & 2t^2 \\ 0 & (2t)^2 \end{array}\right], \mathbf{B}^3 = \left[\begin{array}{cc} 0 & 2^2t^3 \\ 0 & (2t)^3 \end{array}\right], \mathbf{B}^4 = \left[\begin{array}{cc} 0 & 2^3t^4 \\ 0 & (2t)^4 \end{array}\right]$$

and for any positive integer n

$$\mathbf{B}^n = \left[\begin{array}{cc} 0 & 2^{n-1}t^n \\ 0 & (2t)^n \end{array}\right]$$

From this we find

$$e^{\mathbf{B}} = \sum_{n=0}^{\infty} \frac{\mathbf{B}}{n!} = \left[\begin{array}{cc} 1 & \sum_{n=0}^{\infty} 2^{n-1}t^n \\ 0 & \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \end{array}\right] = \left[\begin{array}{cc} 1 & \frac{e^{2t}}{2} - \frac{1}{2} \\ 0 & e^{2t} \end{array}\right]$$

Hence,

$$\mathbf{x}(t) = \left[\begin{array}{cc} 1 & \frac{e^{2t}}{2} - \frac{1}{2} \\ 0 & e^{2t} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]$$

From this we obtain

$$y(t) = c_1 + c_2 e^{2t}$$

Since $y(0) = 1$ then $c_1 + c_2 = 1$. Since $y'(0) = 2$ then $c_2 = 1$. Hence, $c_1 = 0$ and $y(t) = e^{2t}$ ■

16 The General Solution of Homogeneous Equations

In Problems 16.1-16.7, the $t-$interval of solution is $-\infty < t < \infty$ unless indicated otherwise.
(a) Determine whether the given functions are solutions to the differential equation.
(b) If both functions are solutions, calculate the Wronskian. Does this calculation show that the two functions form a fundamental set of solutions?
(c) If the two functions have been shown in (b) to form a fundamental set, construct the general solution and determine the unique solution satisfying the initial value problem.

Problem 16.1

$$y'' - 4y = 0, \; y_1(t) = e^{2t}, \; y_2(t) = 2e^{-2t}, \; y(0) = 1, \; y'(0) = -2.$$

Solution.
(a)

$$y_1'' - 4y_1 = 4e^{2t} - 4e^{2t} = 0$$

$$y_2'' - 4y_2 = 8e^{-2t} - 8e^{-2t} = 0$$

So both functions are solutions.
(b)

$$W(y_1(t), y_2(t)) = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix} = -8 \neq 0.$$

So $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) We have $y(t) = c_1e^{2t} + 2c_2e^{-2t}$ and $y'(t) = 2c_1e^{2t} - 4c_2e^{-2t}$. The initial conditions imply $c_1 + 2c_2 = 1$ and $2c_1 - 4c_2 = -2$. Solving we find $c_1 = 0$ and $c_2 = \frac{1}{2}$. Hence, $y(t) = e^{-2t}$ ■

Problem 16.2

$$y'' + y = 0, y_1(t) = \sin t \cos t, \; y_2(t) = \sin t, \; y(\tfrac{\pi}{2}) = 1, \; y'(\tfrac{\pi}{2}) = 1.$$

Solution.
(a)

$$y_1'' + y_1 = \cos^2 t - \sin^2 t + \sin t \cos t \neq 0$$

so y_1 is not a solution.

$$y_2'' + y_2 = \sin t + \sin t = 0$$

So y_2 is a solution ■

Problem 16.3

$$y'' - 4y' + 4y = 0,\ y_1(t) = e^{2t},\ y_2(t) = te^{2t},\ y(0) = 2,\ y'(0) = 0.$$

Solution.
(a)

$$y_1'' - 4y_1' + 4y_1 = 4e^{2t} - 8e^{2t} + 4e^{2t} = 0$$

$$y_2'' - 4y_2' + 4y_2 = 4e^{2t} + 4te^{2t} - 4e^{2t} - 8te^{2t} + 4te^{2t} = 0.$$

So both functions are solutions.
(b)

$$W(y_1(t), y_2(t)) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t} \neq 0.$$

So $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) We have $y(t) = c_1e^{2t} + c_2te^{2t}$ and $y'(t) = 2c_1e^{2t} + (c_2 + 2c_2t)e^{2t}$. The initial conditions imply $c_1 = 2$ and $c_2 = -4$. Hence, $y(t) = 2e^{2t} - 4te^{2t}$ ■

Problem 16.4

$$ty'' + y' = 0,\ y_1(t) = \ln t,\ y_2(t) = \ln 3t,\ y(3) = 0,\ y'(3) = 3,\ 0 < t < \infty.$$

Solution.
(a)

$$ty_1'' + y_1' = -\tfrac{t}{t^2} + \tfrac{1}{t} = 0$$

$$ty_2'' + y_2' = -\tfrac{t}{t^2} + \tfrac{1}{t} = 0.$$

So both functions are solutions.
(b)

$$W(y_1(t), y_2(t)) = \begin{vmatrix} \ln t & \ln(3t) \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix} = \frac{1}{t}\ln 3 \neq 0.$$

So $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) We have $y(t) = c_1 \ln t + c_2 \ln(3t)$ and $y'(t) = \frac{c_1}{t} + \frac{c_2}{t}$. The initial conditions imply $c_1 + 2c_2 = 0$ and $c_1 + c_2 = 9$. Solving we find $c_1 = 18$ and $c_2 = -9$. Hence, $y(t) = 18\ln t - 9\ln(3t),\ t > 0$ ■

Problem 16.5

$$t^2y'' - ty' - 3y = 0,\ y_1(t) = t^3,\ y_2(t) = -t^{-1},\ y(-1) = 0,\ y'(-1) = -2,\ -\infty < t < 0.$$

Solution.
(a)

$$t^2y_1'' - ty_1' - 3y_1 = t^2(6t) - t(3t^2) - 3t^3 = 0$$

$$t^2y_2'' - ty_2' - 3y_2 = t^2(-2t^{-3}) - t(t^{-2}) - 3(-t^{-1}) = 0.$$

So both functions are solutions.
(b)

$$W(y_1(t), y_2(t)) = \begin{vmatrix} t^3 & -t^{-1} \\ 3t^2 & t^{-2} \end{vmatrix} = 4t \neq 0,\ t < 0.$$

So $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) We have $y(t) = c_1t^3 + c_2t^{-1}$ and $y'(t) = 3c_1t^2 - c_2t^{-2}$. The initial conditions imply $-c_1 + c_2 = 0$ and $3c_1 + c_2 = -2$. Solving we find $c_1 = c_2 = -\frac{1}{2}$. Hence, $y(t) = \frac{1}{2}(t^{-1} - t^3),\ t > 0$ ■

Problem 16.6

$$y'' = 0,\ y_1(t) = t + 1, y_2(t) = -t + 2,\ y(1) = 4,\ y'(1) = -1.$$

Solution.
(a) Since $y_1'' = y_2'' = 0$, both functions are solutions.
(b)
$$W(y_1(t), y_2(t)) = \begin{vmatrix} t+1 & -t+2 \\ 1 & -1 \end{vmatrix} = -3 \neq 0.$$
So $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) We have $y(t) = c_1(t+1) + c_2(-t+2)$ and $y'(t) = c_1 - c_2$. The initial conditions imply $2c_1 + c_2 = 4$ and $c_1 - c_2 = 1$. Solving we find $c_1 = 1$ and $c_2 = 2$. Hence, $y(t) = -t + 5$ ■

Problem 16.7

$$4y'' + 4y' + y = 0,\ y_1(t) = e^{\frac{t}{2}},\ y_2(t) = te^{\frac{t}{2}},\ y(1) = 1,\ y'(1) = 0.$$

Solution.
(a)
$$4y_1'' + 4y_1' + y_1 = 4e^{\frac{t}{2}} \neq 0$$
so y_1 is not a solution.
$$4y_2'' + 4y_2' + y_2 = 8e^{\frac{t}{2}} + 4te^{\frac{t}{2}} \neq 0$$
so y_2 is not a solution ■

Problem 16.8
The functions $y_1(t) = t$ and $y_2(t) = t\ln t$ form a fundamental set of solutions to the differential equation
$$t^2y'' - ty' + y = 0, \quad 0 < t < \infty.$$
(a) Show that $y(t) = 2t + t\ln 3t$ is a solution to the differential equation.
(b) Find c_1 and c_2 such that $y(t) = c_1y_1(t) + c_2y_2(t)$

Solution.
(a) $t^2y'' - ty' + y = t^2t^{-1} - t(3 + \ln(3t)) + 2t + t\ln(3t) = 0$.
(b) We have
$$\begin{cases} c_1t + c_2t\ln t &= 2t + t\ln(3t) \\ c_1 + c_2(1 + \ln t) &= 3 + \ln(3t). \end{cases}$$
Using the elimination method we find $c_1 = 2\ln 3$ and $c_2 = 1$. Thus, $y(t) = (2 + \ln 3)t + t\ln t$ ■

Problem 16.9
The functions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are known to be solutions of $y'' + \alpha y' + \beta y = 0$, where α and β are constants. Determine α and β.

Solution.
Since $y_1'' + \alpha y_1' + \beta y_1 = 0$ we find $3\alpha + \beta = -9$. Since $y_2'' + \alpha y_2' + \beta y_2 = 0$ we find $-3\alpha + \beta = -9$. Hence, $\alpha = 0$ and $\beta = -9$ ■

Problem 16.10
The functions $y_1(t) = t$ and $y_2(t) = e^t$ are known to be solutions of $y'' + p(t)y' + q(t)y = 0$.
(a) Determine the functions $p(t)$ and $q(t)$.
(b) On what t-intervals are the functions $p(t)$ and $q(t)$ continuous?
(c) Compute the Wronskian of these two functions. On what t-intervals is the Wronskian nonzero?
(d) Are the observations in (b) and (c) consistent with Theorem 16.3?

Solution.
(a) Since $y_1'' + p(t)y_1' + q(t)y_1 = 0$ we find $p(t) + tq(t) = 0$. Since $y_2'' + p(t)y_2' + q(t)y_2 = 0$ we find $p(t)+q(t) = -1$. Solving for $p(t)$ and $q(t)$ we find $p(t) = \frac{-t}{t-1}$ and $q(t) = \frac{1}{t-1}$.
(b) Both $p(t)$ and $q(t)$ are continuous on $(-\infty, 1) \cup (1, \infty)$.
(c)

$$W(y_1(t), y_2(t)) = \left| \begin{array}{cc} t & e^t \\ 1 & e^t \end{array} \right| = e^t(t-1).$$

The Wronskian is nonzero for all $t \neq 1$.
(d) Yes. $W \neq 0$ on the two intervals on which p and q are both continuous ■

Problem 16.11
It is known that two solutions of $y''+ty'+2y = 0$ has a Wronskian $W(y_1(t), y_2(t))$ that satisfies $W(y_1(1), y_2(1)) = 4$. What is $W(y_1(2), y_2(2))$?

Solution.
From Abel's Theorem we have

$$W(y_1(t), y_2(t)) = W(y_1(1), y_2(1))e^{-\int_1^t sds} = 4e^{-\frac{t^2}{2}+\frac{1}{2}}.$$

Hence, $W(y_1(2), y_2(2)) = 4e^{1.5}$ ■

Problem 16.12
The pair of functions $\{y_1, y_2\}$ is known to form a fundamental set of solutions of $y''+\alpha y'+\beta y = 0$, where α and β are constants. One solution is $y_1(t) = e^{2t}$, and the Wronskian formed by these two solutions is $W(y_1(t), y_2(t)) = e^{-t}$. Determine the constants α and β.

Solution.
Since $y_1'' + \alpha y_1' + \beta y_1 = 0$ we find $2\alpha + \beta = -4$. Since $W(y_1(t), y_2(t)) = e^{-t}$ we find $W'(t) = -e^{-t}$. But $W' + pW = 0$ so that $-e^{-t} + pe^{-t} = 0$. Hence, $p(t) = 1 = \alpha$. Thus, $\beta = -4 - 2\alpha = -6$ ■

Problem 16.13
The Wronskian of a pair of solutions of $y'' + p(t)y' + 3y = 0$ is $W(t) = e^{-t^2}$. What is the coefficient function $p(t)$?

Solution.
Since $W' = -pW$ we find $-2te^{-t^2} = -p(t)e^{-t^2}$ so that $p(t) = 2t$ ■

Problem 16.14
Prove that if y_1 and y_2 have maxima or minima at the same point in an interval I, then they cannot be a fundamental set of solutions on that interval.

Solution.
Suppose for example that both functions have a same maximum at t_0. Then $y_1'(t_0) = y_2'(t_0) = 0$. But

$$W(y_1(t_0), y_2(t_0)) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) = 0.$$

Thus, $\{y_1, y_2\}$ is not a fundamental set ■

Problem 16.15
Without solving the equation, find the Wronskian of two solutions of Bessel's equation

$$t^2y'' + ty' + (t^2 - \mu^2)y = 0.$$

Solution.
By Abel's Theorem

$$W(y_1(t), y_2(t)) = W(y_1(t_0), y_2(t_0))e^{-\int_{t_0}^t \frac{ds}{s}} = t_0\frac{W(y_1(t_0), y_2(t_0))}{t}$$ ■

Problem 16.16
If $W(y_1, y_2) = t^2e^t$ and $y_1(t) = t$ then find $y_2(t)$.

Solution.
By the quotient rule
$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1^2} = e^t.$$
Thus, one possible answer is
$$y_2(t) = te^t \blacksquare$$

Problem 16.17
The functions t^2 and $1/t$ are solutions to a 2nd order, linear homogeneous ODE on $t > 0$. Verify whether or not the two solutions form a fundamental solution set.

Solution.
Finding the Wronskian
$$W(t^2, \frac{1}{t}) = \left| \begin{array}{cc} t^2 & t^{-1} \\ 2t & -t^{-2} \end{array} \right| = -3 \neq 0$$
so that $\{y_1, y_2\}$ is a fundamental set $\blacksquare$

Problem 16.18
Show that t^3 and t^4 can't both be solutions to a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous functions defined on the real numbers.

Solution.
Suppose that t^3 and t^4 are both solutions. Since $W(t) = t^6$ we find $W(1) = 1$ and so $\{y_1, y_2\}$ is a fundamental set. By Abel's Theorem, $W(t) \neq 0$ for all $-\infty < t < \infty$. But $W(0) = 0$, a contradiction. Hence, t^3 and t^4 can't be both solutions for the differential equation for $-\infty < t < \infty \blacksquare$

Problem 16.19
Suppose that $t^2 + 1$ is the Wronskian of two solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$. Find $p(t)$.

Solution.
Since $W' = -p(t)W$ we have $2t = -p(t)(t^2+1)$. Thus, $p(t) = -\frac{2t}{t^2+1}$ ■

Problem 16.20
Suppose that $y_1(t) = t$ is a solution to the differential equation

$$t^2y'' - (t+2)ty' + (t+2)y = 0.$$

Find a second solution y_2

Solution.
Rewriting the given equation in the form

$$y'' - (\frac{2}{t}+1)y' + (\frac{2}{t^2}+\frac{1}{t})y = 0.$$

Thus, $p(t) = -(\frac{2}{t}+1)$. But $W' + pW = 0$ so that

$$W' - (\frac{2}{t}+1)W = 0.$$

Using the method of integrating factor we find

$$W(t) = Ct^2e^t.$$

So we will look for a function $y_2(t)$ such that $W(t) = t^2e^t$. That is, a function satisfying the differential equation

$$ty_2' - y_2 = t^2e^t.$$

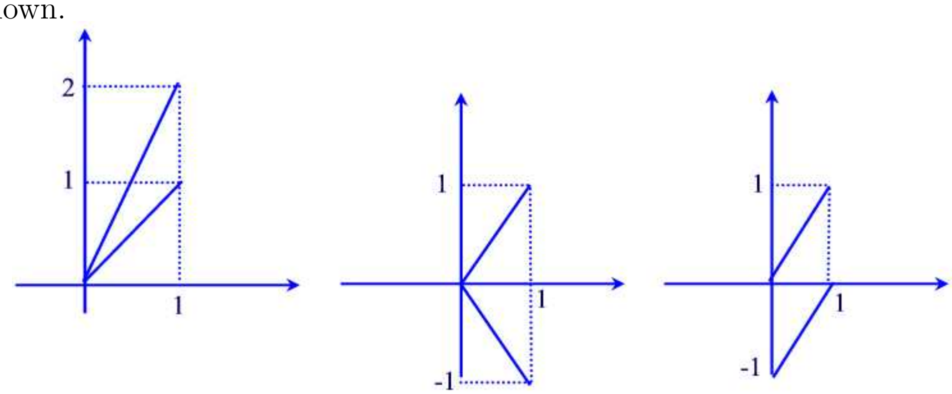
Solving this equation by the method of integrating factor we find $y_2(t) = te^t$ ■

17 Existence of Many Fundamental Sets

Problem 17.1
Do the given functions form a linearly independent set on the indicated interval?
(a) $y_1(t) = 2,\ y_2(t) = t^2,\ \ -\infty < t < \infty.$
(b) $y_1(t) = \ln t,\ y_2(t) = \ln t^2,\ 0 < t < \infty.$
(c) $y_1(t) = 2,\ y_2(t) = t,\ y_3(t) = -t^2,\ \ -\infty < t < \infty.$
(d) $y_1(t) = 2,\ y_2(t) = \sin^2 t,\ y_3(t) = 2\cos^2 t,\ \ -3 < t < 2.$

Solution.
(a) Suppose that $c_1(2) + c_2t^2 = 0$ for all $-\infty < t < \infty$. Letting $t = 0$ we find $c_1 = 0$. Letting $t = 1$ we find $c_2 = 0$. Hence, y_1 and y_2 are linearly independent.
(b) Since $y_2 = 2\ln t = 2y_1$, the functions y_1 and y_2 are linearly dependent.
(c) Suppose that $c_1(2) + c_2t - c_3t^2 = 0$ for all $-\infty < t < \infty$. Letting $t = 0$ we find $c_1 = 0$ so that $c_2t - c_3t^2 = 0$. Letting $t = 1$ we see that $c_2 = c_3$. In this case, $c_2(t^2 - t) = 0$. Letting $t = -1$ we find $c_2 = c_3 = 0$. Thus, y_1, y_2, and y_3 are linearly independent.
(d) Since $(-2)(1) + 2\sin^2 t + 2\cos^2 t = -2 + 2 = 0$, the functions y_1, y_2, y_3 are linearly dependent ■

Problem 17.2
Consider the graphs of the linear functions shown. In each case, determine if the functions form a linearly independent set of functions on the domain shown.



Solution.
(a) We have $f_1(t) = t$ and $f_2(t) = 2t$ so that $f_2(t) = 2f_1(t)$. Thus, $\{f_1, f_2\}$ is

linearly dependent.
(b) We have $f_1(t) = t$ and $f_2(t) = -t = -f_1(t)$. Thus, $\{f_1, f_2\}$ is linearly dependent.
(c) We have $f_1(t) = t$ and $f_2(t) = t - 1$. Suppose that $c_1(t) + c_2(t-1) = 0$ for all $0 \le t \le 1$. In particular if $t = 1$ then $c_1 = 0$. If $t = 0$ then $c_2 = 0$. Thus, $\{f_1, f_2\}$ is linearly independent ■

Problem 17.3
Consider the differential equation $y'' + 2ty' + t^2y = 0$ on the interval $-\infty < t < \infty$. Assuming that $y_1(t)$ and $y_2(t)$ are two solutions satisfying the given initial conditions. Answer the following two questions.
(a) Do the solutions form a fundamental set?
(b) Do the two solutions form a linearly independent set of functions on $-\infty < t < \infty$?
(i) $y_1(1) = 2,\ y_1'(1) = 2,\ y_2(1) = -1,\ y_2'(1) = -1.$
(ii) $y_1(-2) = 1,\ y_1'(-2) = 2,\ y_2(-2) = 0,\ y_2'(-2) = 1.$
(iii) $y_1(3) = 0,\ y_1'(3) = 0,\ y_2(3) = 1,\ y_2'(3) = 2.$

Solution.
(i) (a) Since $W(1) = 0$, $\{y_1, y_2\}$ is not a fundamental set. (b) Part (a) and Theorem 17.2 assert that the set $\{y_1, y_2\}$ is not linearly independent.
(ii) (a) Since $W(-2) = 1 \neq 0$, $\{y_1, y_2\}$ is a fundamental set. (b) Part (a) and Theorem 17.2 assert that $\{y_1, y_2\}$ is linearly independent set.
(iii) (a) Since $W(3) = 0$, $\{y_1, y_2\}$ is not a fundamental set. (b) Part (a) and Theorem 17.2 assert that the set $\{y_1, y_2\}$ is not linearly independent ■

Problem 17.4
The property of linear dependence or independence depends not only upon the rule defining the functions but also on the domain. To illustrate this fact, show that the pair of functions, $f_1(t) = t,\ f_2(t) = |t|$, is linearly dependent on the interval $0 < t < \infty$ but is linearly independent on the interval $-\infty < t < \infty$.

Solution.
For $0 < t < \infty$ we have $f_1(t) = f_2(t)$ so that $\{f_1, f_2\}$ is linearly dependent. Now, suppose that $c_1t + c_2|t| = 0$ for all $-\infty < t < \infty$. Letting $t = -1$ we find $c_1 = c_2$. Letting $t = 1$ we find $c_1 + c_2 = 0$. Hence, $c_1 = c_2 = 0$ so that $\{f_1, f_2\}$ is linearly independent ■

Problem 17.5
Suppose that $\{f_1, f_2\}$ is a linearly indepedent set. Suppose that a function $f_3(t)$ can be expressed as a linear combination of f_1 and f_2 in two different ways,i.e., $f_3(t) = a_1 f_1(t) + a_2 f_2(t)$ and $f_3(t) = b_1 f_1(t) + b_2 f_2(t)$. Show that $a_1 = b_1$ and $a_2 = b_2$.

Solution.
Since $a_1 f_1(t) + a_2 f_2(t) = b_1 f_1(t) + b_2 f_2(t)$ for all t we find $(a_1 - b_1) f_1(t) + (a_2 - b_2) f_2(t) = 0$ for all t. But $\{f_1, f_2\}$ is linearly independent so that $a_1 - b_1 = 0$ and $a_2 - b_2 = 0$. That is, $a_1 = b_1$ and $a_2 = b_2$ ■

Problem 17.6
Consider a set of functions containing the zero function. Can anything be said about whether they form a linearly dependent or linearly independent set? Explain.

Solution.
Consider a set like $\{0, f_1, f_2\}$. Then $1 \cdot 0 + 0 \cdot f_1(t) + 0 \cdot f_2(t) = 0$ for all t. This shows that $\{0, f_1, f_2\}$ is linearly dependent ■

In Problems 17.7 - 17.9, answer the following questions.
(a) Show that $y_1(t)$ and $y_2(t)$ are solutions to the given differential equation.
(b) Determine the initial conditions satisfied by each function at the specified t_0.
(c) Determine whether the functions form a fundamental set on $-\infty < t < \infty$.

Problem 17.7
$y'' - 4y = 0,\ y_1(t) = e^{2t},\ y_2(t) = e^{-2t},\ t_0 = 1.$

Solution.
(a) $y_1'' - 4y_1 = 4e^{2t} - 4e^{2t} = 0$; $y_2'' - 4y_2 = 4e^{-2t} - 4e^{-2t} = 0$.
(b) $y_1(1) = e^2; y_1'(1) = 2e^2; y_2(1) = e^{-2}; y_2'(1) = -2e^{-2}$.
(c)

$$W = \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = -6 \neq 0$$

$$W(1) = \begin{vmatrix} e^2 & e^{-2} \\ 2e^2 & -2e^{-2} \end{vmatrix} = -4 \neq 0$$

so that $\{y_1, y_2\}$ is a fundamental set ■

Problem 17.8
$y'' + 9y = 0,\ y_1(t) = \sin 3(t-1),\ y_2(t) = 2\cos 3(t-1),\ t_0 = 1.$

Solution.
(a) $y_1''+9y_1 = -9\sin 3(t-1)+9\sin 3(t-1) = 0;\ y_2''+9y_2 = -18\cos 3(t-1)+ 18\cos 3(t-1) = 0.$
(b) $y_1(1) = 0; y_1'(1) = 3; y_2(1) = 2; y_2'(1) = 0.$
(c)

$$W(1) = \left|\begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array}\right| = -6 \neq 0$$

so $\{y_1, y_2\}$ is a fundamental set ■

Problem 17.9
$y'' + 2y' - 3y = 0,\ y_1(t) = e^{-3t},\ y_2(t) = e^{-3(t-2)},\ t_0 = 2.$

Solution.
(a) $y_1'' + 2y_1' - 3y_1 = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0;\ y_2'' + 2y_2' - 3y_2 = 9e^{-3(t-2)} - 6e^{-3(t-2)} - 3e^{-3(t-2)} = 0.$
(b) $y_1(2) = e^{-6}; y_1'(2) = -3e^{-6}; y_2(2) = 1;\ y_2'(2) = -3.$
(c)

$$W(2) = \left|\begin{array}{cc} e^{-6} & 1 \\ -3e^{-6} & -3 \end{array}\right| = 0$$

so $\{y_1, y_2\}$ is not a fundamental set ■

In Problems 17.10 - 17.11, assume that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ on the t-interval of interest. Determine whether or not the functions $y_3(t)$ and $y_4(t)$, formed by the given linear combinations, also form a fundamental set of solutions on the same t-interval.

Problem 17.10
$y_3(t) = 2y_1(t) - y_2(t),\ y_4(t) = y_1(t) + y_2(t).$

Solution.
In matrix form we have

$$\left[\begin{array}{c} y_3 \\ y_4 \end{array}\right] = \left[\begin{array}{cc} 2 & -1 \\ 1 & 1 \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$$

Since

$$\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

the coefficient matrix is invertible and so $\{y_3, y_4\}$ is a fundamental set of solutions ■

Problem 17.11
$y_4(t) = 2y_1(t) - 2y_2(t),\ y_4(t) = y_1(t) - y_2(t).$

Solution.
In matrix form we have

$$\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Since

$$\begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = 0$$

the coefficient matrix is not invertible and so $\{y_3, y_4\}$ is not a fundamental set of solutions ■

In Problems 17.12 - 17.13, the sets $\{y_1, y_2\}$ and $\{y_3, y_4\}$ are both fundamental sets of solutions for the given differential equation on the indicated interval. Find a constant 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Problem 17.12
$t^2y'' - 3ty' + 3y = 0,\ 0 < t < \infty,\ y_1(t) = t,\ y_2(t) = t^3,\ y_3(t) = 2t - t^3,\ y_4(t) = t^3 + t.$

Solution.

$$\begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$ ■

Problem 17.13
$y'' - 4y' + 4y = 0, \ -\infty < t < \infty, \ y_1(t) = e^{2t}, \ y_2(t) = te^{2t}, \ y_3(t) = (2t-1)e^{2t}, \ y_4(t) = (t-3)e^{2t}$.

Solution.

$$\left[\begin{array}{c} y_3(t) \\ y_4(t) \end{array}\right] = \left[\begin{array}{cc} -1 & 2 \\ -3 & 1 \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] \blacksquare$$

Problem 17.14
Verify whether the functions $f_1(t) = t^2$, $f_2(t) = 2t^2 - 3t$, $f_3(t) = t$, and $f_4(t) = 1$ are linearly independent. Do not use Wronskian to solve this problem.

Solution.
Suppose $c_1t^2 + c_2(2t^2 - 3t) + c_3t + c_4(1) = 0$ for all t. Letting $t = 0$ we find $c_4 = 0$. Thus, $c_1t^2 + c_2(2t^2 - 3t) + c_3t = 0$ for all t. Letting $t = 1.5$ we obtain $3c_1 + 2c_3 = 0$. Letting $t = 1$ we obtain $c_1 - c_2 + c_3 = 0$. Letting $t = -1$ we obtain $c_1 + 5c_2 - c_3 = 0$. From these equalities we find $c_1 = -\frac{2}{3}c_3$ and $c_2 = \frac{1}{3}c_3$. So letting $c_3 = 1, c_1 = -\frac{2}{3}, c_2 = \frac{1}{3}$, and $c_4 = 0$ we find

$$-\frac{2}{3}t^2 + \frac{1}{3}(2t^2 - 3t) + t + 0(1) = 0$$

so $\{f_1, f_2, f_3, f_4\}$ is a linearly dependent set $\blacksquare$

Problem 17.15
(a) Compute the Wronskian of $y_1(t) = te^t$ and $y_2(t) = t^2e^t$.
(a) Are they linearly independent on [0,1]? Explain your answer.

Solution.
(a)

$$W(t) = \left| \begin{array}{cc} te^t & t^2e^t \\ e^t + te^t & 2te^t + t^2e^t \end{array} \right| = t^2e^{2t}$$

(b) Since y_2 is not a constant multiple of y_1, $\{y_1, y_2\}$ is linearly independent set $\blacksquare$

Problem 17.16
Determine if the following set of functions are linearly independent or linearly dependent,
(a) $y_1(t) = 9\cos 2t$ and $y_2(t) = 2\cos^2 t - 2\sin^2 t$.
(b) $y_1(t) = 2t^2$ and $y_2(t) = t^4$.

Solution.
(a) Since $y_1(t) = 9(\cos^2 t - \sin^2 t) = \frac{9}{2}y_2$, $\{y_1, y_2\}$ is linearly dependent.
(b) Suppose that $c_1(2t^2)+c_2t^4 = 0$ for all t. Letting $t = 1$ we find $2c_1+c_2 = 0$. Letting $t = 2$ we find $c_1+2c_2 = 0$. Solving we find $c_1 = c_2 = 0$ so that $\{y_1, y_2\}$ linearly independent ■

Problem 17.17
Without solving, determine the Wronskian of two solutions to the following differential equation.

$$t^4y'' - 2t^3y' - t^8y = 0.$$

Hint: Use Abel's Theorem

Solution.
We have $p(t) = -\frac{2}{t}$. Then W satisfies the differential equation $W' - \frac{2}{t}W = 0$. Solving for W using the method of integrating factor we find $W(t) = t^2$ ■

Problem 17.18
Without solving, determine the Wronskian of two solutions to the following differential equation.

$$y'' - 4ty' + \sin ty = 0.$$

Solution.
We have $p(t) = -4t$. Then $W' - 4tW = 0$. Solving for W we find $W(t) = e^{4t}$ ■

Problem 17.19
Let $y_1(t)$ and $y_2(t)$ be any two differentiable functions on a closed interval $a \le t \le b$.
(a) Show that if $W(y_1(t), y_2(t)) \neq 0$ for some $a \le t \le b$ then y_1 and y_2 are linearly independent.
(b) Show that the two functions $y_1(t) = t^2$ and $y_2(t) = t|t|$ are linearly independent with $W(t) = 0$ for all t. Thus, a set of functions could be linearly independent on some interval and yet have a vanishing Wronskian.

Solution.
(a) Suppose that $c_1y_1 + c_2y_2 = 0$ for all $a \le t \le b$. Then $c_1y_1' + c_2y_2' = 0$ for all $a \le t \le b$. Solving this system of linear equation in the unknowns c_1 and c_2 using elimination we find $c_1W(t) = 0$. Since $W(t) \neq 0$ then $c_1 = 0$. Similarly, $c_2 = 0$. Thus, $\{y_1, y_2\}$ is linearly independent.
(b) Suppose that $c_1t^2 + c_2t^3 = 0$ for all t. For $t = 1$ we get $c_1 + c_2 = 0$. for $t = -1$ we find $c_1 - c_2 = 0$. Thus, $c_1 = c_2 = 0$ so that $\{y_1, y_2\}$ are linearly independent. Moreover $W(0) = 0$.

Problem 17.20
Show that the two functions $y_1(t) = 1 - t$ and $y_2(t) = t^3$ cannot be both solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

if $p(t)$ and $q(t)$ are continuous in $-1 \le t \le 5$.

Solution.
Suppose that y_1 and y_2 are solutions. Since $W(1) = 1$, $\{y_1, y_2\}$ is a fundamental set and therefore $w(t) \neq 0$ for $-1 < t < 5$ according to Abel's Theorem. But $W(1.5) = 0$ a contradiction. Thus, y_1 and y_2 can't both be solutions to the differential equation ■

18 Second Order Linear Homogeneous Equations with Constant Coefficients

Problem 18.1
Solve the initial value problem

$$y'' + y' - 2y = 0,\ y(0) = 3,\ y'(0) = -3.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 + r - 2 = 0$ has roots $r = 1$ and $r = -2$ so that the general solution is given by

$$y(t) = c_1 e^t + c_2 e^{-2t}.$$

The initial conditions and $y'(t) = c_1 e^t - 2c_2 e^{-t}$ lead to the system $c_1 + c_2 = 3$ and $c_1 - 2c_2 = -3$. Solving this system we find $c_1 = 1$ and $c_2 = 2$. Hence, the unique solution to the initial value problem is

$$y(t) = e^t + 2e^{-t}.$$

$$\lim_{t\to-\infty} y(t) = \infty \text{ and } \lim_{t\to\infty} y(t) = \infty \ \blacksquare$$

Problem 18.2
Solve the initial value problem

$$y'' - 4y' + 3y = 0,\ y(0) = -1,\ y'(0) = 1.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 - 4r + 3 = 0$ has roots $r = 1$ and $r = 3$ so that the general solution is given by

$$y(t) = c_1 e^t + c_2 e^{3t}.$$

The initial conditions and $y'(t) = c_1 e^t + 3c_2 e^{3t}$ lead to the system $c_1 + c_2 = -1$ and $c_1 + 3c_2 = 1$. Solving this system we find $c_1 = 1$ and $c_2 = -2$. Hence, the unique solution to the initial value problem is

$$y(t) = e^t - 2e^{3t}.$$

$$\lim_{t\to-\infty} y(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{3t}\left(1 - \frac{2}{e^{2t}}\right) = \infty \blacksquare$$

Problem 18.3
Solve the initial value problem

$$y'' - y = 0,\ y(0) = 1,\ y'(0) = -1.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 - 1 = 0$ has roots $r = -1$ and $r = 1$ so that the general solution is given by

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

The initial conditions and $y'(t) = c_1 e^t - c_2 e^{-t}$ lead to the system $c_1 + c_2 = 1$ and $c_1 - c_2 = -1$. Solving this system we find $c_1 = 0$ and $c_2 = 1$. Hence, the unique solution to the initial value problem is

$$y(t) = e^{-t}.$$

$$\lim_{t\to-\infty} y(t) = \infty \text{ and } \lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 18.4
Solve the initial value problem

$$y'' + 5y' + 6y = 0,\ y(0) = 1,\ y'(0) = -1.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 + 5r + 6 = 0$ has roots $r = -2$ and $r = -3$ so that the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

The initial conditions and $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ lead to the system $c_1 + c_2 = 1$ and $2c_1 + 3c_2 = 1$. Solving this system we find $c_1 = 2$ and $c_2 = -1$. Hence, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} + e^{-3t}.$$

$$\lim_{t\to-\infty} y(t) = \lim_{t\to-\infty} e^{-3t}(2e^t - 1) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 18.5
Solve the initial value problem

$$y'' - 4y = 0,\ y(3) = 0,\ y'(3) = 0.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 - 4 = 0$ has roots $r = -2$ and $r = 2$ so that the general solution is given by

$$y(t) = c_1e^{2t} + c_2e^{-2t}.$$

The initial conditions and $y'(t) = 2c_1e^{2t} - 2c_2e^{-2t}$ lead to the system $c_1e^6 + c_2e^{-6} = 0$ and $2c_1e^6 - 2c_2e^{-6} = 0$. Solving this system we find $c_1 = 0$ and $c_2 = 0$. Hence, the unique solution to the initial value problem is $y(t) \equiv 0$.

$$\lim_{t\to-\infty} y(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 18.6
Solve the initial value problem

$$2y'' - 3y' = 0,\ y(-2) = 3,\ y'(-2) = 0.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $2r^2 - 3r = 0$ has roots $r = 0$ and $r = 1.5$ so that the general solution is given by

$$y(t) = c_1e^{1.5t} + c_2.$$

The initial conditions and $y'(t) = 1.5c_1e^{1.5t}$ lead to the system $c_1e^{-3} + c_2 = 3$ and $c_1 = 0$. Solving this system we find $c_2 = 3$. Hence, the unique solution to the initial value probem $y(t) \equiv 3$.

$$\lim_{t\to-\infty} y(t) = 3 \text{ and } \lim_{t\to\infty} y(t) = 3 \blacksquare$$

Problem 18.7
Solve the initial value problem

$$y'' + 4y' + 2y = 0, \ y(0) = 0, \ y'(0) = 4.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $r^2 + 4r + 2 = 0$ has roots $r = -2 - \sqrt{2}$ and $r = -2 + \sqrt{2}$ so that the general solution is given by

$$y(t) = c_1 e^{(-2-\sqrt{2})t} + c_2 e^{(-2+\sqrt{2})t}.$$

The initial conditions and $y'(t) = c_1(-2-\sqrt{2})e^{(-2-\sqrt{2})t} + c_2(-2+\sqrt{2})e^{(-2+\sqrt{2})t}$ lead to the system $c_1 + c_2 = 0$ and $(-2-\sqrt{2})c_1 + (-2+\sqrt{2})c_2 = 4$. Solving this system we find $c_1 = -2\sqrt{2}$ and $c_2 = 2\sqrt{2}$. Hence, the unique solution to the initial value problem is

$$y(t) = -2\sqrt{2}e^{(-2-\sqrt{2})t} + 2\sqrt{2}e^{(-2+\sqrt{2})t}.$$

$$\lim_{t\to-\infty} y(t) = \lim_{t\to-\infty} e^{(-2-\sqrt{2})t}[-2\sqrt{2} + 2\sqrt{2}e^{2\sqrt{2}t} = -\infty \text{ and}$$
$$\lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 18.8
Solve the initial value problem

$$2y'' - y = 0, \ y(0) = -2, \ y'(0) = \sqrt{2}.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic equation $2r^2 - 1 = 0$ has roots $r = -\frac{\sqrt{2}}{2}$ and $r = \frac{\sqrt{2}}{2}$ so that the general solution is given by

$$y(t) = c_1 e^{\frac{\sqrt{2}}{2}t} + c_2 e^{-\frac{\sqrt{2}}{2}t}.$$

The initial conditions and $y'(t) = \frac{\sqrt{2}}{2}c_1 e^{\frac{\sqrt{2}}{2}t} - \frac{\sqrt{2}}{2}c_2 e^{-\frac{\sqrt{2}}{2}t}$ lead to the system $c_1 + c_2 = -2$ and $c_1 - c_2 = 2$. Solving this system we find $c_1 = 0$ and $c_2 = -2$. Hence, the unique solution to the initial value problem is

$$y(t) = -2e^{-\frac{\sqrt{2}}{2}t}.$$

$$\lim_{t\to-\infty} y(t) = -\infty \text{ and } \lim_{t\to\infty} y(t) = 0 \blacksquare$$

Problem 18.9

Consider the initial value problem $y'' + \alpha y' + \beta y = 0,\ y(0) = 1,\ y'(0) = y'_0$, where α, β, and y'_0 are constants. It is known that one solution of the differential equation is $y_1(t) = e^{-3t}$ and that the solution of the initial value problem satisfies $\lim_{t\to\infty} y(t) = 2$. Determine the constants α, β, and y'_0.

Solution.

Since $r = -3$ is a solution to the characteristic equation, we obtain $(-3)^2 + \alpha(-3) + \beta = 0$ or $-3\alpha + \beta = -9$. Also, since $\lim_{t\to\infty} = 2$, the second root for the characteristic equation must be $r = 0$. In this case, $\beta = 0$ and solving for α we find $\alpha = 3$. Hence, $y(t) = c_1 e^{-3t} + c_2$. Since $\lim_{t\to-\infty} = 2$ we find $c_2 = 2$. Since $y(0) = 1$ we find $c_1 + 2 = 1$ so that $c_1 = -1$. Thus, $y(t) = -e^{-3t} + 2$ and $y'(t) = 3e^{-3t}$. Therefore, $y'_0 = y'(0) = 3$ $\blacksquare$

Problem 18.10

Consider the initial value problem $y'' + \alpha y' + \beta y = 0,\ y(0) = 3,\ y'(0) = 5$. The differential equation has a fundamental set of solutions $\{y_1, y_2\}$. It is known that $y_1(t) = e^{-t}$ and that the Wronskian formed by the two members of the fundamental set is $W(t) = 4e^{2t}$.

(a) Determine $y_2(t)$.

(b) Determine the constants α and β.

(c) Solve the initial value problem.

Solution.

(a) The second solution is of the form $y_2(t) = e^{rt}$. In this case,

$$W(t) = \begin{vmatrix} e^{-t} & e^{rt} \\ -e^{-t} & re^{rt} \end{vmatrix} = (r+1)e^{(r-1)t}$$

But $W(t) = 4e^{2t}$ and this leads to $r = 3$. Hence, $y_2(t) = e^{3t}$.

(b) Since $r = -1$ and $r = 3$ are the roots for the characteristic equation, we have $(r+1)(r-3) = 0$ or $r^2 - 2r - 3 = 0$. This implies that $y'' - 2y' - 3y = 0$ so that $\alpha = -2$ and $\beta = -3$

(c) The initial conditions and $y'(t) = -c_1 e^{-t} + 3c_2 e^{3t}$ lead to the system $c_1 + c_2 = 3$ and $-c_1 + 3c_2 = 5$. Solving this system we find $c_1 = 1$ and $c_2 = 2$. Thus,

$$y(t) = e^{-t} + 2e^{3t} \blacksquare$$

Problem 18.11
Obtain the general solution to the differential equation $y''' - 5y'' + 6y' = 0$.

Solution.
Let $u = y'$. Then $u' = y''$ and $u'' = y'''$ so that the given equation becomes

$$u'' - 5u' + 6u = 0.$$

The characteristic equation $r^2 - 5r + 6 = 0$ has roots $r = 2$ and $r = 3$ so that the general solution is given by

$$u(t) = c_1 e^{2t} + c_2 e^{3t}.$$

But $y'(t) = u(t)$ so that

$$y(t) = \frac{c_1}{2}e^{2t} + \frac{c_2}{3}e^{3t} + c_3 = c_1 e^{2t} + c_2 e^{3t} + c_3 \blacksquare$$

Problem 18.12
A particle of mass m moves along the x-axis and is acted upon by a drag force proportional to its velocity. The drag constant is denoted by k. If $x(t)$ represents the particle position at time t, Newton's law of motion leads to the differential equation $mx''(t) = -kx'(t)$.
(a) Obtain the general solution to this second order linear differential equation.
(b) Solve the initial value problem if $x(0) = x_0$ and $x'(0) = v_0$.
(c) What is $\lim_{t\to\infty} x(t)$?

Solution.
(a) The characteristic equation is $mr^2 + kr = 0$ with roots $r = 0$ and $r = -\frac{k}{m}$. Thus, the general solution is

$$x(t) = c_1 + c_2 e^{-\frac{k}{m}t}.$$

(b) The initial conditions and $x'(t) = -\frac{k}{m}c_2 e^{-\frac{k}{m}t}$ lead $c_1 = x_0 + \frac{m}{k}v_0$ and $c_2 = -\frac{m}{k}v_0$. Hence,

$$x(t) = \left(x_0 + \frac{m}{k}v_0\right) - \frac{m}{k}v_0 e^{-\frac{k}{m}t}.$$

(c) $\lim_{t\to\infty} x(t) = x_0 + \frac{m}{k}v_0 \blacksquare$

Problem 18.13
Solve the initial-value problem $4y'' - y = 0, y(0) = 2, y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

Solution.
The characteristic equation $4r^2 - 1 = 0$ has roots $r = -\frac{1}{2}$ and $r = \frac{1}{2}$. Thus,

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{t}{2}}.$$

The initial conditions and $y'(t) = -\frac{c_1}{2}e^{-\frac{t}{2}} + \frac{c_2}{2}e^{\frac{t}{2}}$ lead to the system $c_1 + c_2 = 2$ and $c_1 - c_2 = -2\beta$. Solving this system we find $c_1 = 1 - \beta$ and $c_2 = 1 + \beta$. Thus,

$$y(t) = (1-\beta)e^{-\frac{t}{2}} + (1+\beta)e^{\frac{t}{2}}.$$

Since $\lim_{t\to\infty} y(t) = 0$ we find $\beta = -1$ ■

Problem 18.14
Find a homogeneous second-order linear ordinary differential equation whose general solution is $y(t) = c_1 e^{2t} + c_2 e^{-t}$.

Solution.
The roots for the characteristic equation are $r = 2$ and $r = -1$ so that $(r-2)(r+1) = 0$ and hence $r^2 - r - 2 = 0$. The homogeneous equation is then $y'' - y' - 2y = 0$ ■

Problem 18.15
Find the general solution of the differential equation $y'' - 3y' - 4y = 0$.

Solution.
The characteristic equation $r^2 - 3r - 4 = 0$ has roots $r = -1$ and $r = 4$. Thus,

$$y(t) = c_1 e^{-t} + c_2 e^{4t} \blacksquare$$

Problem 18.16
Find the general solution of the differential equation $y'' + 4y' - 5y = 0$.

Solution.
The characteristic equation $r^2 + 4r - 5 = 0$ has roots $r = 1$ and $r = -5$. Thus,

$$y(t) = c_1 e^{t} + c_2 e^{-5t} \blacksquare$$

Problem 18.17
Find the general solution of the differential equation $-3y'' + 2y' + y = 0$.

Solution.
The characteristic equation $-3r^2 + 2r + 1 = 0$ has roots $r = 1$ and $r = -\frac{1}{3}$. Thus,

$$y(t) = c_1 e^t + c_2 e^{\frac{t}{3}} \blacksquare$$

Problem 18.18
Solve the initial-value problem: $y'' + 3y' - 4y = 0,\ y(0) = -1,\ y'(0) = 1$.

Solution.
The characteristic equation $r^2 + 3r - 4 = 0$ has roots $r = 1$ and $r = -4$. Thus,

$$y(t) = c_1 e^t + c_2 e^{-4t}.$$

The initial conditions and $y'(t) = c_1 e^t - 4c_2 e^{-4t}$ lead to the system $c_1 + c_2 = -1$ and $c_1 - 4c_2 = 1$. Solving this system we find $c_1 = -\frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus,

$$y(t) = -\frac{1}{2}(e^t + e^{-4t}) \blacksquare$$

Problem 18.19
Solve the initial-value problem: $2y'' + 5y' - 3y = 0,\ y(0) = 2,\ y'(0) = 1$.

Solution.
The characteristic equation $2r^2 + 5r - 3 = 0$ has roots $r = -3$ and $r = \frac{1}{2}$. Thus,

$$y(t) = c_1 e^{-3t} + c_2 e^{\frac{t}{2}}.$$

The initial conditions and $y'(t) = -3c_1 e^{-3t} + \frac{c_2}{2} e^{\frac{t}{2}}$ lead to the system $c_1 + c_2 = 2$ and $-3c_1 + \frac{c_2}{2} = 1$. Solving this system we find $c_1 = 0$ and $c_2 = 2$. Thus,

$$y(t) = 2e^{\frac{t}{2}} \blacksquare$$

Problem 18.20
Show that if λ is a root of $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, then $e^{\lambda t}$ is a solution of $ay''' + by'' + cy' + dy = 0$.

Solution.
We have

$$\begin{aligned} ay''' + by'' + cy' + dy =& a\lambda^3 e^{\lambda t} + b\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + de^{\lambda t} \\ =& (a\lambda^3 + b\lambda^2 + c\lambda + d)e^{\lambda t} = 0 \blacksquare \end{aligned}$$

19 Repeated Roots and the Method of Reduction of Order

In Problems 19.1 - 19.5 answer the following questions.

(a) Obtain the general solution of the differential equation.
(b) Impose the initial conditions to obtain the unique solution of the initial value problem.
(c) Describe the behavior of the solution as $t \to -\infty$ and $t \to \infty$.

Problem 19.1

$$9y'' - 6y' + y = 0,\ y(3) = -2,\ y'(3) = -\frac{5}{3}.$$

Solution.
(a) The characteristic equation $9r^2 - 6r + 1 = 0$ has the roots $r_1 = r_2 = \frac{1}{3}$. The general solution is then

$$y(t) = c_1 e^{\frac{t}{3}} + c_2 t e^{\frac{t}{3}}.$$

(b) The initial conditions and $y'(t) = \frac{c_1}{3} e^{\frac{t}{3}} + c_2 e^{\frac{t}{3}} + \frac{c_2}{3} e^{\frac{t}{3}}$ lead to the system $c_1 + 3c_2 = -2e^{-1}$ and $c_1 + 6c_2 = -5e^{-1}$. Solving this system we find $c_1 = e^{-1}$ and $c_2 = -e^{-1}$. Thus, the unique solution is

$$y(t) = e^{\frac{t}{3}-1}(1-t).$$

(c)

$$\lim_{t\to-\infty} y(t) = \lim_{t\to-\infty} \frac{1-t}{e^{1-\frac{t}{3}}} = \lim_{t\to-\infty} \frac{-1}{-1/3e^{1-\frac{t}{3}}} = 0.$$

Now, for large t we have $t-1 \geq 1$ so that $e^{\frac{t}{3}-1}(t-1) \geq e^{\frac{t}{3}-1}$. Since $e^{\frac{t}{3}-1} \to \infty$ as $t \to \infty$ we have $e^{\frac{t}{3}-1}(t-1) \to \infty$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} y(t) = -\lim_{t\to\infty} e^{\frac{t}{3}-1}(t-1) = -\infty \blacksquare$$

Problem 19.2

$$25y'' + 20y' + 4y = 0,\ y(5) = 4e^{-2},\ y'(5) = -\frac{3}{5}e^{-2}.$$

Solution.
(a) The characteristic equation $25r^2+20r+4=0$ has the roots $r_1=r_2=-\frac{2}{5}$. The general solution is then

$$y(t)=c_1e^{-\frac{2t}{5}}+c_2te^{-\frac{2t}{5}}.$$

(b) The initial conditions and $y'(t)=-\frac{2c_1}{5}e^{-\frac{2t}{5}}+c_2e^{-\frac{2t}{4}}-\frac{2c_2}{5}e^{-\frac{2t}{5}}$ lead to the system $c_1+5c_2=4$ and $2c_1+5c_2=3$. Solving this system we find $c_1=-1$ and $c_2=1$. Thus, the unique solution is

$$y(t)=e^{-\frac{2t}{5}}(t-1).$$

(c)

$$\lim_{t\to-\infty}y(t)=-\lim_{t\to-\infty}e^{\frac{-2t}{5}}(1-t)=-\infty$$

and

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}e^{\frac{-2t}{5}}(t-1)=\lim_{t\to\infty}\frac{1}{\frac{2}{5}e^{\frac{2t}{5}}}=0 \blacksquare$$

Problem 19.3

$$y''-4y'+4y=0,\ y(1)=-4,\ y'(1)=0.$$

Solution.
(a) The characteristic equation $r^2-4r+4=0$ has the roots $r_1=r_2=2$. The general solution is then

$$y(t)=c_1e^{2t}+c_2te^{2t}.$$

(b) The initial conditions and $y'(t)=2c_1e^{2t}+c_2e^{2t}+2c_2te^{2t}$ lead to the system $c_1-c_2=2e^2$ and $2c_1-c_2=e^2$. Solving this system we find $c_1=-e^2$ and $c_2=-3e^2$. Thus, the unique solution is

$$y(t)=-e^{2t+2}(1+3t)$$

(c)

$$\lim_{t\to-\infty}y(t)=-\lim_{t\to-\infty}-\frac{1+3t}{e^{-2t-2}}=-\lim_{t\to-\infty}\frac{3}{-2e^{-2t-2}}=0$$

and

$$\lim_{t\to\infty} y(t) = -\lim_{t\to\infty}(1+3t)e^{2t+2} = -\infty \blacksquare$$

Problem 19.4

$$y'' + 2\sqrt{2}y' + y = 0, \; y(0) = 1, \; y'(0) = 0.$$

Solution.
(a) The characteristic equation $r^2 + 2\sqrt{2}r + 1 = 0$ has the roots $r_1 = r_2 = -\sqrt{2}$. The general solution is then

$$y(t) = c_1 e^{-\sqrt{2}t} + c_2 t e^{-\sqrt{2}t}.$$

(b) The initial conditions and $y'(t) = -\sqrt{2}c_1 e^{-\sqrt{2}t} + c_2 e^{-\sqrt{2}t} - \sqrt{2}c_2 t e^{-\sqrt{2}t}$ lead to $c_1 = 0$ and $c_2 = \sqrt{2}$. Thus, the unique solution is

$$y(t) = e^{-\sqrt{2}}(1 + \sqrt{2}t).$$

(c)

$$\lim_{t\to-\infty} y(t) = -\lim_{t\to-\infty} e^{-\sqrt{2}t}(-1-3t) = -\infty$$

and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1+\sqrt{2}t}{e^{\sqrt{2}t}} = \lim_{t\to\infty} \frac{\sqrt{2}}{\sqrt{2}e^{\sqrt{2}t}} = 0 \blacksquare$$

Problem 19.5

$$3y'' + 2\sqrt{3}y' + y = 0, \; y(0) = 2\sqrt{3}, \; y'(0) = 3.$$

Solution.
(a) The characteristic equation $3r^2 + 2\sqrt{3}r + 1 = 0$ has the roots $r = r_1 = r_2 = -\frac{1}{\sqrt{3}}$. The general solution is then

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

(b) The initial conditions and $y'(t) = rc_1 e^{rt} + c_2 e^{rt} + rc_2 t e^{rt}$ lead to $c_1 = 2\sqrt{3}$ and $c_2 = 5$. Thus, the unique solution is

$$y(t) = e^{-\frac{t}{\sqrt{3}}}(5t + 2\sqrt{3}).$$

(c)

$$\lim_{t\to-\infty} y(t) = -\lim_{t\to-\infty} e^{-\frac{t}{\sqrt{3}}}(-5t - 2\sqrt{3}) = -\infty$$

and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{2\sqrt{3}+5t}{e^{\frac{t}{\sqrt{3}}}} = \lim_{t\to\infty} \frac{5}{(1/\sqrt{3})e^{\frac{t}{\sqrt{3}}}} = 0 \blacksquare$$

In Problems 19.6 - 19.9, one solution, $y_1(t)$, of the differential equation is given.
(a) Find a second solution of the form $y_2(t) = u(t)y_1(t)$.
(b) Compute the Wronskian formed by the solutions $y_1(t)$ and $y_2(t)$. On what intervals is the Wronskian continuous and nonzero?
(c) Rewrite the differential equation in the form $y'' + p(t)y' + q(t)y = 0$. On what interval(s) are both $p(t)$ and $q(t)$ continuous? How does this observation compare with the interval(s) determined in part (b)?

Problem 19.6

$$ty'' - (2t+1)y' + (t+1)y = 0, y_1(t) = e^t.$$

Solution.
(a) Let $y_2(t) = ue^t$. Then $y_2' = u'e^t + ue^t$ and $y_2'' = u''e^t + 2u'e^t + ue^t$. Substituting into the equation and simplifying we find

$$tu'' - u' = 0.$$

Let $w = u'$ so that $w' - \frac{1}{t}w = 0$. Solving this last equation by the method of integrating factor we find $w(t) = ct$. Now find $u(t) = \int ctdt = ct^2 + c'$. Choose $c = 1$ and $c' = 0$ we obtain $u(t) = t^2$. Thus, $y_2(t) = t^2e^t$.
(b)

$$W(t) = \begin{vmatrix} e^t & t^2e^t \\ e^t & (2te^t + t^2e^t) \end{vmatrix} = 2te^{2t}$$

W(t) is continuous and nonzero on $(-\infty, 0) \cup (0, \infty)$.
(c)

$$y'' - \left(2 + \frac{1}{t}\right)y' + \left(1 + \frac{1}{t}\right)y = 0, \quad p(t) = -\left(2 + \frac{1}{t}\right), \; q(t) = 1 + \frac{1}{t}.$$

The functions $p(t)$ and $q(t)$ are continuous on $(-\infty, 0) \cup (0, \infty)$ $\blacksquare$

Problem 19.7

$$y'' - (2\cot t)y' + (1 + 2\cot^2 t)y = 0, \ y_1(t) = \sin t.$$

Solution.
(a) Let $y_2(t) = u \sin t$. Then $y_2' = u\cos t + u'\sin t$ and $y_2'' = -u\sin t + 2\cos t u' + \sin t u''$. Substituting into the equation and simplifying we find

$$u'' = 0$$

Thus, $u(t) = ct + c'$. Choose $c = 1$ and $c' = 0$ to otbain $u(t) = t$. Thus, $y_2(t) = t\sin t$.
(b)

$$W(t) = \begin{vmatrix} \sin t & t\sin t \\ \cos t & (\sin t + t\cos t) \end{vmatrix} = \sin^2 t.$$

W(t) is continuous and nonzero on for all $t \neq n\pi$ where n is an integer.
(c)

$$y'' - 2\cot t y' + \left(1 + 2\cot^2 t\right) y = 0, \quad p(t) = -2\cot t, \ q(t) = 1 + 2\cot^2 t.$$

The functions $p(t)$ and $q(t)$ are continuous for all $t \neq n\pi$ where n is an integer ■

Problem 19.8

$$y'' + 4ty' + (2 + 4t^2)y = 0, \ y_1(t) = e^{-t^2}.$$

Solution.
(a) Let $y_2(t) = ue^{-t^2}$. Then $y_2' = u'e^{-t^2} - 2tue^{-t^2}$ and $y_2'' = u''e^{-t^2} - 4u'te^{-t^2} + 4t^2ue^{-t^2}$. Substituting into the equation and simplifying we find

$$u'' = 0.$$

Thus, $u(t) = ct + c'$. Choose $c = 1$ and $c' = 0$ to otbain $u(t) = t$. Thus, $y_2(t) = te^{-t^2}$.
(b)

$$W(t) = \begin{vmatrix} e^{-t^2} & te^{-t^2} \\ -2te^{-t^2} & (e^{-t^2} - 2t^2e^{-t^2}) \end{vmatrix} = e^{-2t^2}.$$

W(t) is continuous and nonzero on $(-\infty, \infty)$.
(c) $p(t) = 4t, \ q(t) = 2 + 4t^2$. The functions $p(t)$ and $q(t)$ are continuous in $(-\infty, \infty)$ ■

Problem 19.9

$$y'' - \left(2 + \frac{n-1}{t}\right)y' + \left(1 + \frac{n-1}{t}\right)y = 0,$$

where n is a positive integer, $y_1(t) = e^t$.

Solution.
(a) Let $y_2(t) = ue^t$. Then $y_2' = u'e^t + ue^t$ and $y_2'' = u''e^t + 2u'e^t + ue^t$. Substituting into the equation and simplifying we find

$$tu'' - (n-1)u' = 0.$$

Let $w = u'$ so that $w' - \frac{n-1}{t}w = 0$. Solving this last equation by the method of integrating factor we find $w(t) = ct^{n-1}$. Now find $u(t) = \int ct^{n-1}dt = ct^n + c'$. Choose $c = 1$ and $c' = 0$ we obtain $u(t) = t^n$. Thus, $y_2(t) = t^n e^t$.
(b)

$$W(t) = \begin{vmatrix} e^t & t^n e^t \\ e^t & (nt^{n-1}e^t + t^n e^t) \end{vmatrix} = nt^{n-1}e^{2t}.$$

W(t) is continuous and nonzero on $(-\infty, \infty)$ for $n = 1$ and on $(-\infty, 0) \cup (0, \infty)$ for $n \geq 2$.
(c) $p(t) = -\left(2 + \frac{n-1}{t}\right)$ and $q(t) = \left(1 + \frac{n-1}{t}\right)$. The functions $p(t)$ and $q(t)$ are continuous on $(-\infty, 0) \cup (0, \infty)$ ■

Problem 19.10
The graph of a solution $y(t)$ of the differential equation $4y'' + 4y' + y = 0$ passes through the points $(1, e^{-\frac{1}{2}})$ and $(2, 0)$. Determine $y(0)$ and $y'(0)$.

Solution.
The characteristic equation $4r^2 + 4r + 1 = 0$ has the roots $r_1 = r_2 = -\frac{1}{2}$ so that the general solution is

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 t e^{-\frac{t}{2}}.$$

Since $y(2) = 0$ we find $c_1 + 2c_2 = 0$. Since $y(1) = e^{-\frac{1}{2}}$ we find $c_1 + c_2 = 1$. Solving the system of two equations we find $c_1 = 2$ and $c_2 = -1$. Hence,

$$y(t) = 2e^{-\frac{t}{2}} - te^{-\frac{t}{2}}.$$

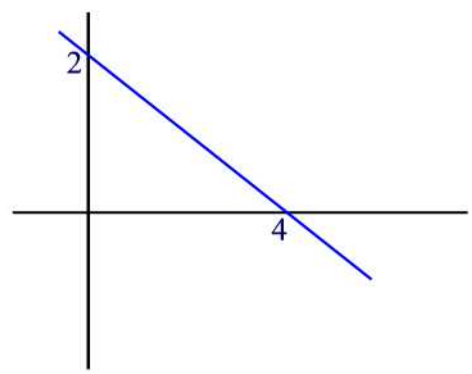
Now, $y(0) = 2$. Also, replacing $t = 0$ in $y'(t) = -2e^{-\frac{t}{2}} + \frac{t}{2}e^{-\frac{t}{2}}$ to obtain $y'(0) = -2$ ■

Problem 19.11
Find a homogeneous second order linear differential equation whose general solution is given by $y(t) = c_1e^{-3t} + c_2te^{-3t}$.

Solution.
The characteristic equation has the double roots $r_1 = r_2 = -3$ so that $r^2 + 6r + 9 = 0$. Hence, the differential equation is

$$y'' + 6y' + 9y = 0 \blacksquare$$

Problem 19.12
The graph shown below is the solution of $y'' - 2\alpha y' + \alpha^2 y = 0, y(0) = y_0,\ y'(0) = y_0$. Determine the constants α, y_0, and y_0' as well as the solution $y(t)$.



Solution.
Since the solution is a straight line, $y'' = 0$. Hence, $\alpha = 0$. On the other hand, the general solution has the form $y(t) = mt + b$. From the graph we see that $b = 2$ and $m = -\frac{1}{2}$. Thus, $y(t) = -\frac{t}{2} + 2$. Finally, $y(0) = y_0 = 2$ and $y_0' = y'(0) = -\frac{1}{2}$ $\blacksquare$

Problem 19.13
Show that if λ is a double root of $at^3 + bt^2 + ct + d = 0$, then $te^{\lambda t}$ is also a solution of $ay''' + by'' + cy' + dy = 0$.

Solution.
Since λ is a double root we find $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$ and $3a\lambda^2 + 2d\lambda + c = 0$. Let $y(t) = te^{\lambda t}$. Then $y' = e^{\lambda t} + \lambda te^{\lambda t},\ y'' = 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t},\ y''' = 3\lambda^2 e^{\lambda t} + \lambda^3 te^{\lambda t}$. Substituting into the equation we find

$$\begin{aligned} ay''' + by'' + cy' + dy &= [(a\lambda + b\lambda^2 + c\lambda + d)t + (3a\lambda^2 + 2b\lambda + c)]e^{\lambda t} \\ &= 0 \blacksquare \end{aligned}$$

Problem 19.14
Find the general solution of $y'' - 6y' + 9y = 0$.

Solution.
The characteristic equation $r^2 - 6r + 9 = 0$ has double roots $r_1 = r_2 = 3$ so the general solution is

$$y(t) = c_1 e^{3t} + c_2 te^{3t} \blacksquare$$

Problem 19.15
Find the general solution of $4y'' - 4y' + y = 0$.

Solution.
The characteristic equation $4r^2 - 4r + 1 = 0$ has double roots $r_1 = r_2 = \frac{1}{2}$ so the general solution is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 te^{\frac{t}{2}} \blacksquare$$

Problem 19.16
Solve the initial-value problem: $y'' + y' + \frac{y}{4} = 0,\ y(0) = 2,\ y'(0) = 0$.

Solution.
The characteristic equation $r^2 + r + \frac{1}{4} = 0$ has double roots $r_1 = r_2 = -\frac{1}{2}$ so the general solution is

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 te^{-\frac{t}{2}}.$$

Since $y(0) = 2$ we find $c_1 + c_2 = 2$. Since $y'(0) = 0$ we find $c_1 - 2c_2 = 0$. Solving this system we find $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Hence, the unique solution is

$$y(t)\frac{2}{3}e^{-\frac{t}{2}} + \frac{1}{3}te^{-\frac{t}{2}} \blacksquare$$

Problem 19.17
The method of reduction of order can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = uy_1$ and show that y is a solution of the nonhomogeneous if u is a solution of

$$y_1 u'' + [2y_1' + py_1]u' = g.$$

The latter equation is a first-order linear equation for u'.

Solution.
Inser ting$y(t) = uy_1$ and its first and second order derivatives into the given equation we obtain

$$\begin{aligned} g(t) =& y'' + p(t)y' + q(t)y \\ =& u''y_1 + 2u'y_1' + uy_1'' + p(t)u'y_1 + p(t)uy_1' + q(t)uy_1 \\ =& u(t)(y_1'' + p(t)y_1' + q(t)y_1) + u''y_1 + (2y_1' + p(t)y_1)u' \\ =& u''y_1 + (2y_1' + p(t)y_1)u'. \end{aligned}$$

It follows that if u is a solution to

$$y_1u'' + (p(t)y_1 + 2y_1')u' = g(t)$$

then $y = uy_1$ is a solution to the given differential equation ■

Problem 19.18
Given that $y_1(t) = t^2$ is a solution of

$$t^2y'' - 3ty' + 4y = 0, \ t > 0$$

find the general solution.

Solution.
We will use the method of reduction of order to find the second solution. Let $y_2(t) = ut^2$. Then $y_2' = u't^2 + 2tu$ and $y_2'' = u''t^2 + 4tu' + 2u$. Substituting into the differential equation and simplifying we find

$$u'' + \frac{1}{t}u' = 0.$$

Letting $w = u'$ we find $w' + \frac{1}{t}w = 0$. Solving this differential equation using the method of integrating factor we find $w(t) = \frac{c}{t}$. Now, find u by integration to obtain $u(t) = c\ln t + c'$. Let $c = 1$ and $c' = 0$ to obtain $u(t) = \ln t$. Finally, $y_2(t) = t^2 \ln t$ ■

Problem 19.19
Let $y_1(t)$ be a nonzero solution of the third-order homogeneous linear ODE

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0.$$

Use the substitution $y = uy_1$ to reduce the problem to a second-order linear equation.

Solution.
We have $y' = u'y_1 + uy_1'$, $y'' = 2u'y_1 + u''y_1 + uy_1''$, $y''' = 3u''y_1 + 2u'y_1' + u'y_1'' + u'''y_1 + uy_1'''$. Susbtituting these into the differential equation to obtain

$$\begin{aligned}y''' + p(t)y'' + q(t)y' + r(t)y =&y_1u''' + (y_1''' + p(t)y_1'' + q(t)y_1' + r(t)y_1)u\\&+(3y_1 + p(t)y_1')u'' + (2y_1' + y_1'' + 2py_1' + q(t)y_1)u'\\=&y_1u''' + (3y_1 + p(t)y_1')u'' + (2y_1' + y_1'' + 2py_1' + q(t)y_1)u'.\end{aligned}$$

Letting $z = u'$ we obtain the second order linear differential equation

$$y_1z'' + (3y_1 + p(t)y_1')z' + (2y_1' + y_1'' + 2py_1' + q(t)y_1)z = 0 \blacksquare$$

Problem 19.20
The following problem indicates a second way for finding the second root. It is known as the **method of reduction of order.** Consider the differential equation $y'' + p(t)y' + q(t)y = 0$ having one solution $y_1(t)$.
(a) If $y_2(t) = u(t)y_1(t)$ is a solution then show that the differential equation satisfied by $u(t)$ is given by

$$y_1u'' + (2y_1' + py_1)u' = 0.$$

(b) Use the substitution $v = u'$ to reduce the equation in part(a) into a first order linear differential equation in v.
(c) Solve the equation in part(b) for v.
(d) Find $u(t)$ and then $y_2(t)$.

Solution.
(a) Inserting y_2, y_2', and y_2'' into the equation we find

$$\begin{aligned}0 =&(u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + uy_1') + quy_1\\=&u(y_1'' + py_1' + qy_1) + y_1u'' + (2y_1' + py_1)u'\\=&y_1u'' + (2y_1' + py_1)u'.\end{aligned}$$

(b) Letting $v = u'$ then v satisfies the differential equation

$$v' + \left(\frac{2y_1'}{y_1} + p\right)v = 0.$$

(c) Solving the differential equation in part(b) using the method of integrating factor we find

$$v(t) = Ce^{-\int\left(\frac{2y_1'}{y_1} + p\right)dt} = C\frac{e^{-\int p(t)dt}}{y_1^2(t)}.$$

(d) Since $u' = v$ we have

$$u(t) = C \int \frac{e^{-\int p(t)dt}}{y_1^2(t)}.$$

Choose $C = 1$ so that

$$y_2(t) = \left(\int \frac{e^{-\int p(t)dt}}{y_1^2(t)} \right) y_1(t) \blacksquare$$

20 Characteristic Equations with Complex Roots

Problem 20.1
For any $z = \alpha + i\beta$ we define the conjugate of z to be the complex number $\overline{z} = \alpha - i\beta$. show that $\alpha = \frac{1}{2}(z + \overline{z})$ and $\beta = \frac{1}{2i}(z - \overline{z})$.

Solution.
Adding z and $\overline{z}$ we find $2\alpha = z + \overline{z}$. Hence, $\alpha = \frac{1}{2}(z + \overline{z})$. Next, subtracting $\overline{z}$ from z we find $2i\beta = z\overline{z}$. Therefore, $\beta = \frac{1}{2i}(z - \overline{z}$ $\blacksquare$

Problem 20.2
Write each of the complex numbers in the form $\alpha + i\beta$, where α and β are real numbers.

1. $2e^{i\frac{\pi}{3}}$.
2. $(2 - i)e^{i\frac{3\pi}{2}}$.
3. $(\sqrt{2}e^{i\frac{\pi}{6}})^4$.

Solution.
Recall Euler's function: $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$.

1. $2e^{i\frac{\pi}{3}} = 2\cos\left(\frac{\pi}{3}\right) + 2i\sin\left(\frac{\pi}{3}\right) = 1 + i\sqrt{3}$.
2. $(2 - i)e^{i\frac{3\pi}{2}} = 2ie^{i\frac{3\pi}{2}} + e^{i\frac{3\pi}{2}} = -2i + -1$.
3. $(\sqrt{2}e^{i\frac{\pi}{6}})^4 = (\sqrt{2})^4 e^{i\frac{2\pi}{3}} = 4(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -2 + 2i\sqrt{3}$ $\blacksquare$

Problem 20.3
Write each functions in the form $Ae^{\alpha t}\cos\beta t + iB\sin\beta t$, where α, β, A, and B are real numbers.

1. $2e^{i\sqrt{2}t}$.
2. $-\frac{1}{2}e^{2t + i(t+\pi)}$.
3. $(\sqrt{3}e^{(1+i)t})^3$.

Solution.
1. $2e^{i\sqrt{2}t} = 2\cos\sqrt{2}t + 2i\sin\sqrt{2}t$.
2. $-\frac{1}{2}e^{2t+i(t+\pi)} = -\frac{1}{2}e^{2t}\cos(t + \pi) - \frac{1}{2}e^{2t}\sin(t + \pi) = \frac{1}{2}e^{2t}\cos t + \frac{1}{2}ie^{2t}\sin t$.
3. $(\sqrt{3}e^{(1+i)t})^3 = 3\sqrt{3}e^{3(1+i)t} = e^{3t}\cos(3t) + ie^{3t}\sin(3t)$ $\blacksquare$

In Problems 20.4 - 20.8
(a) Determine the roots of the characteristic equation.
(b) Obtain the general solution as a linear combination of real-valued solutions.
(c) Impose the initial conditions and solve the initial value problem.

Problem 20.4

$$y'' + 2y' + 2y = 0,\ y(0) = 3,\ y'(0) = -1.$$

Solution.
(a) The characteristic equation $r^2 + 2r + 2 = 0$ has roots $r_1 = -1 - i$ and $r_2 = -1 + i$.
(b) $y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$.
(c) The initial conditions and $y'(t) = e^{-t} \cos t(c_2 - c_1) - e^{-t} \sin t(c_1 + c_2)$ lead to the equations $c_1 = 3$ and $-c_1 + c_2 = -1$. Solving we find $c_1 = 3$ and $c_2 = 2$. Hence, the unique solution to the initial value problem is

$$y(t) = 3e^{-t} \cos t + 2e^{-t} \sin t \ \blacksquare$$

Problem 20.5

$$2y'' - 2y' + y = 0,\ y(-\pi) = 1,\ y'(-\pi) = -1.$$

Solution.
(a) The characteristic equation $2r^2 - 2r + 1 = 0$ has roots $r_1 = \frac{1}{2}(1 - i)$ and $r_2 = \frac{1}{2}(1 + i)$.
(b) $y(t) = e^{\frac{t}{2}}(c_1 \cos \frac{t}{2} + c_2 \sin \frac{t}{2})$.
(c) The initial conditions and $y'(t) = \frac{1}{2}e^{\frac{t}{2}} \cos \frac{t}{2}(c_1 + c_2) + \frac{1}{2}e^{\frac{t}{2}} \sin \frac{t}{2}(-c_1 + c_2)$ lead to the equations $-e^{-\frac{\pi}{2}}c_2 = 1$ and $c_2 - c_1 = e^{\frac{\pi}{2}}$. Solving we find $c_1 = -e^{\frac{\pi}{2}}$ and $c_2 = 3e^{\frac{\pi}{2}}$. Hence, the unique solution to the initial value problem is

$$y(t) = -e^{\frac{1}{2}(t+\pi)}(3 \cos \frac{t}{2} + \sin \frac{t}{2}) \ \blacksquare$$

Problem 20.6

$$y'' + 4y' + 5y = 0,\ y(\frac{\pi}{2}) = \frac{1}{2},\ y'(\frac{\pi}{2}) = -2.$$

Solution.
(a) The characteristic equation $r^2+4r+5=0$ has roots $r_1=-2-i$ and $r_2=-2+i$.
(b) $y(t)=e^{-2t}(c_1\cos t+c_2\sin t)$.
(c) The initial conditions and $y'(t)=e^{-2t}\cos t(c_2-2c_1)-e^{-2t}\sin t(c_1+2c_2)$ lead to the equations $e^{-\pi}c_2=\frac{1}{2}$ and $c_1+2c_2=e^{\pi}$. Solving we find $c_1=e^{\pi}$ and $c_2=\frac{1}{2}e^{\pi}$. Hence, the unique solution to the initial value problem is

$$y(t)=e^{\pi-2t}(\cos t+\frac{1}{2}\sin t) \blacksquare$$

Problem 20.7

$$y''+4\pi^2y=0,\ y(1)=2,\ y'(1)=1.$$

Solution.
(a) The characteristic equation $r^2+4\pi^2=0$ has roots $r_1=-2\pi i$ and $r_2=2\pi i$.
(b) $y(t)=c_1\cos 2\pi t+c_2\sin 2\pi t$.
(c) The initial conditions and $y'(t)=2\pi c_2\cos 2\pi t-2\pi c_1\sin 2\pi t$ lead to the equations $c_1=2$ and $2\pi c_2=1$. Solving we find $c_1=2$ and $c_2=(2\pi)^{-1}$. Hence, the unique solution to the initial value problem is

$$y(t)=2\cos 2\pi t+(2\pi)^{-1}\sin 2\pi t \blacksquare$$

Problem 20.8

$$9y''+\pi^2y=0,\ y(3)=2,\ y'(3)=-\pi.$$

Solution.
(a) The characteristic equation $9r^2+\pi^2=0$ has roots $r_1=-\frac{\pi}{3}i$ and $r_2=\frac{\pi}{3}i$.
(b) $y(t)=c_1\cos\frac{\pi}{3}t+c_2\sin\frac{\pi}{3}t$.
(c) The initial conditions and $y'(t)=\frac{\pi}{3}c_2\cos\frac{\pi}{3}t-\frac{\pi}{3}c_1\sin\frac{\pi}{3}t$ lead to the equations $-c_1=2$ and $-\frac{\pi}{3}c_2=-\pi$. Solving we find $c_1=-2$ and $c_2=3$. Hence, the unique solution to the initial value problem is

$$y(t)=-2\cos\frac{\pi}{3}t+3\sin\frac{\pi}{3}t \blacksquare$$

In Problems 20.9 - 20.10, the function $y(t)$ is a solution of the initial value problem $y''+ay'+by=0,\ y(t_0)=y_0, y'(t_0)=y_0'$, where the point t_0 is specified. Determine the constants a, b, y_0, and y_0'.

Problem 20.9

$$y(t) = 2\sin 2t + \cos 2t,\ t_0 = \frac{\pi}{4}.$$

Solution.
The roots of the characteristic equation are $r_{1,2} = \pm 2i$ so that the characteristic equation is $r^2 + 4 = 0$. Hence, the corresponding differential equation is $y'' + 4y = 0$. From this we find $a = 0$ and $b = 4$. Now, $y_0 = y(\frac{\pi}{4}) = 2\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 2$. Finally, $y_0' = y'(\frac{\pi}{4}) = 4\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} = -2$ ■

Problem 20.10

$$y(t) = e^{t-\frac{\pi}{6}}\cos 2t - e^{t-\frac{\pi}{6}}\sin 2t,\ t_0 = \frac{\pi}{6}.$$

Solution.
The roots of the characteristic equation are $r_{1,2} = 1 \pm 2i$ so that the characteristic equation is $r^2 - 2r + 5 = 0$. Hence, the corresponding differential equation is $y'' - 2y' + 5y = 0$. From this we find $a = -2$ and $b = 5$. Now, $y_0 = y(\frac{\pi}{6}) = \cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}$. Finally, $y_0' = y'(\frac{\pi}{6}) = \cos\frac{\pi}{3} - \sin\frac{\pi}{3} - 2\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3} = -\frac{1}{2} - \frac{3\sqrt{3}}{2}$ ■

In Problems 20.11 - 20.13, rewrite the function $y(t)$ in the form $y(t) = Ke^{\alpha t}\cos\beta t - \delta)$, where $0 \le \delta < 2\pi$. Use this representation to sketch a graph of the given function, on a domain sufficiently large to display its main features.

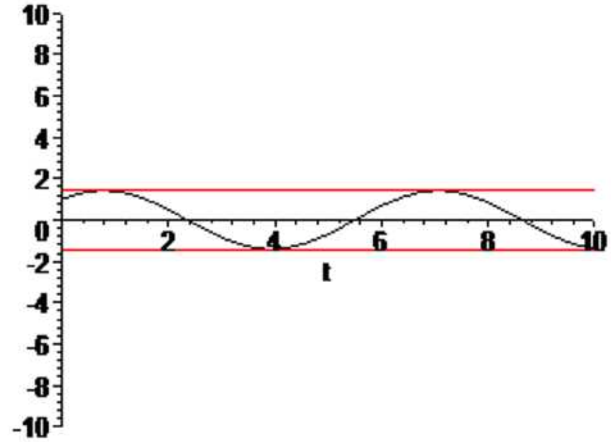
Problem 20.11

$$y(t) = \sin t + \cos t.$$

Solution.
We have $c_1 = 1$ and $c_2 = 1$ so that $K = \sqrt{1^2 + 1^2} = \sqrt{2}$. Moreover, $\cos\delta = \frac{c_1}{K} = \frac{\sqrt{2}}{2}$ and $\sin\delta = \frac{\sqrt{2}}{2}$. Thus, $\delta = \frac{\pi}{4}$ and

$$y(t) = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right).$$

The graph of $y(t)$ is given below ■



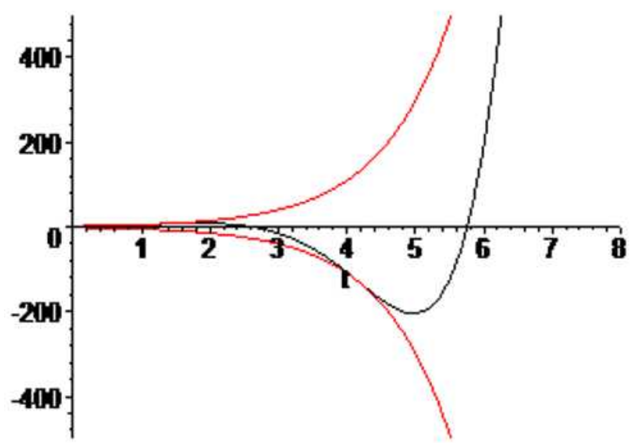
Problem 20.12

$$y(t) = e^t \cos t + \sqrt{3}e^t \sin t.$$

Solution.
We have $c_1 = 1$ and $c_2 = \sqrt{3}$ so that $K = \sqrt{1+3} = 2$. Moreover, $\cos\delta = \frac{c_1}{K} = \frac{1}{2}$ and $\sin\delta = \frac{\sqrt{3}}{2}$. Thus, $\delta = \frac{\pi}{3}$ and

$$y(t) = 2e^t\left(\cos\left(t - \frac{\pi}{3}\right)\right).$$

The graph of $y(t)$ is given below ■



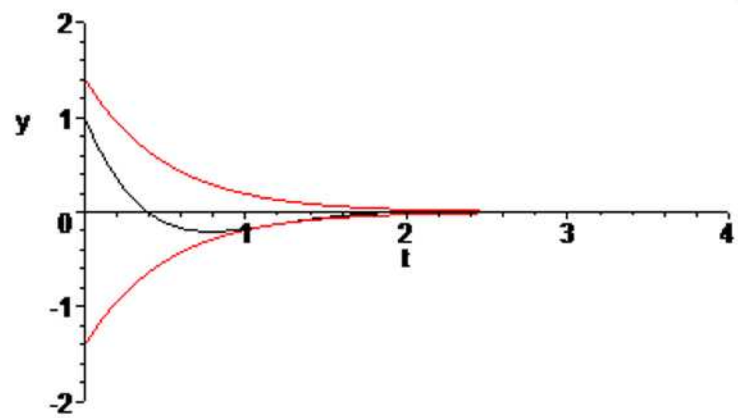
Problem 20.13

$$y(t) = e^{-2t}\cos 2t - e^{-2t}\sin 2t.$$

Solution.
We have $c_1 = 1$ and $c_2 = -$ so that $K = \sqrt{1+1} = \sqrt{2}$. Moreover, $\cos\delta = \frac{c_1}{K} = \frac{1}{\sqrt{2}}$ and $\sin\delta = -\frac{1}{\sqrt{2}}$. Thus, $\delta = \frac{7\pi}{4}$ and

$$y(t) = \sqrt{2}e^{-2t}\left(\cos\left(2t - \frac{7\pi}{4}\right)\right).$$

The graph of $y(t)$ is given below ■



Problem 20.14
Consider the differential equation $y'' + ay' + 9y = 0$, where a is a real number. Suppose that we know the Wronskian of a fundamental set of solutions of this differential equation is constant: $W(t) = 1$ for all real numbers t. Find the general solution of this differential equation.

Solution.
First we need to find a. Since $W'(t) = -aW(t)$ we obtain $a = 0$ so that $y'' + 9y = 0$. The characteristic equation is $r^2 + 9 = 0$ and has complex roots $r_{1,2} = \pm 3i$. Thus, the general solution is given by

$$y(t)c_1\cos 3t + c_2\sin 3t \blacksquare$$

Problem 20.15
Rewrite $2\cos 7t - 11\sin 7t$ in phase-angle form. Give the exact function (so your answer will involve the inverse tangent function).

Solution.
We have $c_1 = 2$ and $c_2 = -11$ so that $K = \sqrt{4+121} = \sqrt{125} = 5\sqrt{5}$. Furthermore, $\tan\delta = -\frac{11}{2}$ so that $\delta = -\arctan\left(\frac{11}{2}\right)$. Hence,

$$y(t) = 5\sqrt{5}\cos\left(t + \arctan\left(\frac{11}{2}\right)\right) \blacksquare$$

Problem 20.16
Find a homogeneous linear ordinary differential equation whose general solution is $y(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t)$.

Solution.
The roots to the characteristic equation are $r_{1,2} = 2 \pm 3i$ so that the characteristic equation is $r^2 - 4r + 13 = 0$ and the corresponding differential equation is

$$y'' - 4y' + 13 = 0 \blacksquare$$

Problem 20.17
Rewrite $y(t) = 5e^{(5-2i)t} - 3e^{(5+2i)t}$, without complex exponents, using sines and cosines. What ODE of the form $ay'' + by' + cy = 0$, has y as a solution?

Solution.
Using Euler's formula we have $e^{(5-2i)t} = e^{5t}(\cos 2t - i\sin 2t)$ and $e^{(5+2i)t} = e^{5t}(\cos 2t + i\sin 2t)$. Thus, $y(t) = 2e^{5t}\cos 2t - 8ie^{5t}\sin 2t$. The characteristic roots are $r_{1,2} = 5 \pm 2i$ so that the characteristic equation is $y'' - 10r + 29$ and the corresponding differential equation is

$$y'' - 10y' + 29y = 0 \blacksquare$$

Problem 20.18
Consider the function $y(t) = 3\cos 2t - 4\sin 2t$. Find a second order linear IVP that y satisfies.

Solution.
The roots to the characteristic equation are $r_{1,2} = \pm 2i$ so that the characteristic equation is $r^2 + 4 = 0$ and the corresponding differential equation is

$$y'' + 4y = 0 \blacksquare$$

Problem 20.19
An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0, \ t > 0$$

where α and β are real constants is called an **Euler equation.** Show that the substitution $u(t) = \ln t$ transforms Euler equation into an equation with constant coefficients.

Solution.
Since $x = \ln t$ we have $\frac{dx}{dt} = \frac{1}{t}$. But $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$. Moreover, $\frac{d^2y}{dt^2} = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2} = \frac{1}{t^2}\left((\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$. Hence,

$$\begin{aligned} 0 =& t^2y'' + \alpha ty' + \beta y \\ =& t^2\left(\frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\right) + \alpha t\left(\frac{1}{t}\frac{dy}{dx}\right) + \beta y \\ =& \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y \blacksquare \end{aligned}$$

Problem 20.20
Use the result of the previous problem to solve the differential equation $t^2y''+ ty' + y = 0$.

Solution.
Here we have $\alpha = \beta = 1$ so that

$$\frac{d^2y}{dx^2} + y = 0.$$

The characteristic equation is $r^2 + 1 = 0$ with complex roots $r_{1,2} = - \pm i$. The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

or

$$y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t) \blacksquare$$

21 Applications of Homogeneous Second Order Linear Differential Equations: Unforced Mechanical Vibrations

Problem 21.1
A 10-kg mass, when attached to the end of a spring hanging vertically, stretches the spring 30 mm. Assume the mass is then pulled down another 70 mm and released (with no initial velocity).
(a) Determine the spring constant k.
(b) State the initial value problem (giving numerical values for all the constants) for $y(t)$, where $y(t)$ denotes the displacement (in meters) of the mass from its equilibrium rest position. Assuming that y is measured positive in the downward direction.
(c) Solve the initial value problem formulated in part (b).

Solution.
(a) $k = \frac{mg}{Y} = \frac{10(9.8)}{0.03} = 3266.7$ N/m.
(b) $my'' + ky = 0,\ y(0) = 0.07,\ y'(0) = 0$ or $y'' + 326.67,\ y(0) = 0.07,\ y'(0) = 0$.
(c) The characteristic equation $r^2 + 326.67 = 0$ has the complex roots $r_{1,2} = \pm 18.074i$. Thus, the general solution is

$$y(t) = c_1 \cos 18.074t + c_2 \sin 18.074t.$$

Since $y(0) = 0.07$ we find $c_1 = 0.07$. Since $y'(0) = 0$ we find $c_2 = 0$. Thus,

$$y(t) = 0.07 \cos(18.074t) \blacksquare$$

Problem 21.2
A 20-kg mass was initially at rest, attached to the end of a vertically hanging spring. When given an initial velocity of 2 m/s from its equilibrium rest position, the mass was observed to attain a maximum displacement of 0.2 m from its equilibrium position. What is the value of the spring constant k?

Solution.
The initial-value problem is given by

$$20y'' + ky = 0,\ y(0) = 0,\ y'(0) = 2.$$

The general solution to the differential equation is given by

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\right).$$

Since $y(0) = 0$ we have $c_1 = 0$. Since $y'(0) = 0.2$ we find $c_2 = 2\sqrt{\frac{m}{k}}$. Thus,

$$y(t) = 2\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}\right).$$

Since y is maximum when $y = 2\sqrt{\frac{m}{k}}$ we obtain $2\sqrt{\frac{m}{k}} = 0.2$ or $k = \frac{10}{0.01} = 2000$ N/m ■

Problem 21.3
A spring-mass-dashpot system consists of a 10-kg mass attached to a spring with spring constant $k = 100\ N/m$; the dashpot has damping constant $\gamma = 7\ kg/s$. At time $t = 0$, the system is set into motion by pulling the mass down 0.5 m from its equilibrium rest position while simultaneously giving it an initial downward velocity of 1 m/s.
(a) State the initial value problem to be solved for $y(t)$, the displacement from equilibrium (in meters) measured positive in the downward direction. Give numerical values to all constants involved.
(b) Solve the initial value problem. What is $\lim_{t\to\infty} y(t)$? Explain why your answer for this limit makes sense from a physical perspective.

Solution.
(a) $y'' + 0.7y' + 10y = 0,\ y(0) = 0.5,\ y'(0) = 1.$
(b) The characteristic equation $r^2 + 0.7r + 10 = 0$ has complex roots $r_{1,2} = -0.35 \pm 3.143$. Thus, the general solution to the differential equation is

$$y(t) = e^{-0.35t}(c_1 \cos(3.143t) + c_2 \sin(3.143t).$$

Since $y(0) = 0.5$ we find $c_1 = 0.5$. Since $y'(0) = 1$ we find $c_2 = 0.374$. Hence,

$$y(t)e^{-0.35t}(0.5\cos(3.143t) + 0.374\sin(3.143t).$$

Clearly, $y(t) \to 0$ as $t \to \infty$. Physically, the damping force dissipates the energy of the system, causing the motion to decrease ■

Problem 21.4
A spring and dashpot system is to be designed for a 32-lb weight so that the overall system is critically damped.
(a) How must the damping constant γ and spring constant k be related?
(b) Assume the system is to be designed so that the mass, when given an initial velocity of 4 ft/sec from its rest position, will have a maximum displacement of 6 in. What values of damping constant γ and constant k are required?

Solution.
(a) For a critically damped system $\gamma^2 - 4mk = 0$ but $m = \frac{32}{32} = 1$ kg so that $\gamma^2 = 4k$.
(b) In a critically damped case the general solution has the form

$$y(t) = c_1 e^{-\frac{\gamma}{2}t} + c_2 t e^{-\frac{\gamma}{2}t}.$$

Since $y(0) = 0$ we find $c_1 = 0$. Also, since $y'(0) = 4$ we find $c_2 = 4$. Thus,

$$y(t) = 4te^{-\frac{\gamma}{2}t}.$$

The function $y(t)$ achieves its maximum height of 6 in $= (0.0254)(6) = 0.1524$ m at time t_{max} such that $y'(t_{max}) = 0$. That is, when $1 - \frac{\gamma}{2}t_{max} = 0$. Solving we find $t_{max} = \frac{2}{\gamma}$. But

$$y(\frac{2}{\gamma}) = \frac{8}{\gamma}e^{-1} = 0.1524.$$

Solving for γ we find $\gamma \approx 19.311$ kg/s. Finally, $k = \frac{\gamma^2}{2} \approx 93.2311$ N/m ■

Problem 21.5
A mass-spring-dashpot system can be modeled by the second order equation

$$my'' + ky' + \gamma y = 0$$

where m is the mass, k is the spring constant and γ is the damping coefficient. A certain system of this type with $m = 1$ can also be modeled by the first order system

$$\left[\begin{array}{c} y \\ y' \end{array}\right]' = \left[\begin{array}{cc} 0 & 1 \\ -5 & -4 \end{array}\right]\left[\begin{array}{c} y \\ y' \end{array}\right]$$

What is the spring constant in this system? What is the damping coefficient?

Solution.
Performing the matrix algebra on the system to find

$$y'' + 4y' + 5y = 0.$$

Thus $k = 4$ and $\gamma = 5$ ■

Problem 21.6
Consider the mass-spring-dashpot system satisfying the differential equation

$$y'' + 2y' + 5y = 0.$$

Is this system overdamped, critically damped, or underdamped?

Solution.
Since $\gamma^2 - 4mk = 25 - 4(1)(2) = 17 > 0$, the system is overdamped ■

Problem 21.7
Consider a mass-spring-dashpot system for which $m = 1, \gamma = 6$, and $k = 13$.
(a) Find the general solution of the corresponding second order differential equation that describes the displacement function.
(b) Is the system over-damped, under-damped, or critically damped?

Solution.
(a) The displacement function $y(t)$ satisfies the differential equation

$$y'' + 13y' + 6y = 0.$$

(b) Since $\gamma^2 - 4mk = 36 - 52 = -16 < 0$, the system is underdamped ■

Problem 21.8
A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from equilibrium with a downward velocity of 10 cm/sec and there is no air resistance, then when does the mass return to equilibrium position for the first time?

Solution.
The differential equation describing the motion is given by

$$my'' + ky = 0$$

where $m = 0.1$ kg and $k = \frac{0.1(9.8)}{0.05} = 19.6$. Thus, $y'' + 196y = 0$. The general solution to this equation is

$$y(t) = c_1 \cos 14t + c_2 \sin 14t.$$

Since $y(0) = 0$ we find $c_1 = 0$. Since $y'(0) = 0.1$ we find $c_2 = \frac{5}{7}$. Hence,

$$y(t) = \frac{5}{7} \sin 14t.$$

The mass first returns to equilibrium when $14t = \pi$ or $t = \frac{1}{\pi}$ seconds ■

Problem 21.9
A mass weighing 8 lb stretches a spring 1.5 in. The mass is attached to a damper with coefficient γ. Determine γ so the system is critically damped.

Solution.
This occurs when $\gamma^2 = 4km$. Now $k = \frac{8\ lb}{1.5\ in} = 64\frac{lb}{ft}$ and $m = \frac{8}{32}\frac{lb\,\text{sec}^2}{ft}$ Thus $\gamma^2 = 4 \cdot 64 \cdot \frac{1}{4}\frac{lb^2\,\text{sec}^2}{ft^2} \Rightarrow \gamma = 8$ lb-sec/ft ■

22 The Structure of the General Solution of Linear Nonhomogeneous Equations

In Problems 22.1- 22.7, answer the following three questions.
(a) Verify that the given function, $y_p(t)$, is a particular solution of the differential equations.
(b) Determine the general solution,y_h, of the homogeneous equation.
(c) Find the general solution to the differential equation and impose the initial conditions to obtain the unique solution of the initial value problem.

Problem 22.1

$$y'' - y' - 2y = 4e^{-t}, y(0) = 0,\ y'(0) = 0,\quad y_p(t) = -\frac{4}{3}te^{-t}.$$

Solutions
(a) $y_p' = -\frac{4}{3}e^{-t} + \frac{4}{3}te^{-t},\ y_p'' = \frac{8}{3}e^{-t} - \frac{4}{3}te^{-t}.$

$$\begin{aligned} y_p'' - y_p' - 2y_p =& \frac{8}{3}e^{-t} - \frac{4}{3}te^{-t} + \frac{4}{3}e^{-t} - \frac{4}{3}te^{-t} + \frac{8}{3}te^{-t} \\ =& 4e^{-t}. \end{aligned}$$

(b) The associated characteristic equation $r^2 - r - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1e^{-t} + c_2e^{2t}.$$

(c) The general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{2t} - \frac{4}{3}te^{-t}$. The derivative of this function is given by $y'(t) = -c_1e^{-t} + 2c_2e^{2t} - \frac{4}{3}e^{-t} + \frac{4}{3}te^{-t}$. The condition $y(0) = 0$ leads to $c_1 + c_2 = 0$. The condition $y'(0) = 0$ leads to $-c_1 + 2c_2 = \frac{4}{3}$. Solving for c_1 and c_2 we find $c_1 = -\frac{4}{9}$ and $c_2 = \frac{4}{9}$. The unique solution is given by

$$y(t) = \frac{4}{9}(e^{2t} - e^{-t} + 3te^{-t})\ \blacksquare$$

Problem 22.2

$$y'' - 2y' - 3y = e^{2t},\ y(0) = 1,\ y'(0) = 0,\ y_p(t) = -\frac{1}{3}e^{2t}.$$

Solutions
(a) $y_p' = -\frac{2}{3}e^{2t},\ y_p'' = -\frac{4}{3}e^{2t}.$

$$\begin{aligned} y_p'' - y_p' - 3y_p =& -\frac{4}{3}e^{2t} + \frac{4}{3}e^{2t} + e^{2t} \\ =& e^{2t}. \end{aligned}$$

(b) The associated characteristic equation $r^2 - 2r - 3 = 0$ has roots $r_1 = -1$ and $r_2 = 3$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1e^{-t} + c_2e^{3t}.$$

(c) The general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{3t} - \frac{1}{3}e^{2t}$. The derivative of this function is given by $y'(t) = -c_1e^{-t} + 3c_2e^{3t} - \frac{2}{3}e^{2t}$. The condition $y(0) = 1$ leads to $c_1 + c_2 = \frac{4}{3}$. The condition $y'(0) = 0$ leads to $-c_1 + 3c_2 = \frac{2}{3}$. Solving for c_1 and c_2 we find $c_1 = \frac{3}{2}$ and $c_2 = \frac{1}{2}$. The unique solution is given by

$$y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} - \frac{1}{3}e^{2t} \ \blacksquare$$

Problem 22.3

$$y'' - y' - 2y = 10,\ y(-1) = 0,\ y'(-1) = 1,\ y_p(t) = -5.$$

Solutions
(a) $y_p' = y_p'' = 0.$

$$y_p'' - y_p' - 2y_p = 0 - 0 - 2(-5) = 10.$$

(b) The associated characteristic equation $r^2 - r - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1e^{-t} + c_2e^{2t}.$$

(c) The general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{2t} - 5$. The derivative of this function is given by $y'(t) = -c_1e^{-t} + 2c_2e^{2t}$. The condition $y(-1) = 0$ leads to $c_1e + c_2e^{-2} = 5$. The condition $y'(-1) = 1$ leads to $-c_1e + 2c_2e^{-2} = 1$. Solving for c_1 and c_2 we find $c_1 = \frac{3}{e}$ and $c_2 = 2e^2$. The unique solution is given by

$$y(t) = \frac{3}{e}e^{-t} + 2e^{2t+2} - 5 \ \blacksquare$$

Problem 22.4

$$y'' + y' = 2e^{-t},\ y(0) = 2,\ y'(0) = 2,\ y_p(t) = -2te^{-t}.$$

Solutions

(a) $y_p' = -2te^{-t} + 2te^{-t},\ y_p'' = 4e^{-t} - 2te^{-t}$.

$$\begin{aligned} y_p'' + y_p' =& 4e^{-t} - 2te^{-t} - 2e^{-t} + 2te^{-t} \\ =& 2e^{-t}. \end{aligned}$$

(b) The associated characteristic equation $r^2 + r = 0$ has roots $r_1 = 0$ and $r_2 = -1$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1 + c_2 e^{-t}.$$

(c) The general solution to the differential equation is $y(t) = c_1 + c_2 e^{-t} - 2te^{-t}$. The derivative of this function is given by $y'(t) = -c_2 e^{-t} - 2e^{-t} + 2te^{-t}$. The condition $y(0) = 2$ leads to $c_1 + c_2 = 2$. The condition $y'(0) = 2$ leads to $-c_2 - 2 = 2$. Solving for c_1 and c_2 we find $c_1 = 6$ and $c_2 = -4$. The unique solution is given by

$$y(t) = 6 - 4e^{-t} - 2te^{-t} \ \blacksquare$$

Problem 22.5

$$y'' + 4y = 10e^{t-\pi},\ y(\pi) = 2,\ y'(\pi) = 0,\ y_p(t) = 2e^{t-\pi}.$$

Solutions

(a) $y_p' = y_p'' = 2e^{t-\pi}$.

$$\begin{aligned} y_p'' + 4y_p' =& 2e^{t-\pi} + 8e^{t-\pi} \\ =& 10e^{t-\pi}. \end{aligned}$$

(b) The associated characteristic equation $r^2 + 4 = 0$ has roots $r_1 = -2i$ and $r_2 = 2i$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1 \cos 2t + c_2 \sin 2t.$$

(c) The general solution to the differential equation is $y(t) = c_1 \cos 2t + c_2 \sin 2t + 2e^{t-\pi}$. The derivative of this function is given by $y'(t) = -2c_2 \sin 2t + 2\cos 2t + 2e^{t-\pi}$. The condition $y(\pi) = 2$ leads to $c_1 + 2 = 2$. The condition

$y'(\pi) = 0$ leads to $2c_2 + 2 = 0$. Solving for c_1 and c_2 we find $c_1 = 0$ and $c_2 = -1$. The unique solution is given by

$$y(t) = -\sin 2t + 2e^{t-\pi} \blacksquare$$

Problem 22.6

$$y'' - 2y' + 2y = 5\sin t,\ y(\frac{\pi}{2}) = 1,\ y'(\frac{\pi}{2}) = 0,\ y_p(t) = 2\cos t + \sin t.$$

Solutions

(a) $y_p' = -2\sin t + \cos t,\ y_p'' = -2\cos t - \sin t.$

$$\begin{aligned} y_p'' - 2y_p' + 2y_p =& -2\cos t - \sin t + 4\sin t - 2\cos t + 4\cos t + 2\sin t \\ =& 5\sin t. \end{aligned}$$

(b) The associated characteristic equation $r^2 - 2r + 2 = 0$ has roots $r_1 = 1 - i$ and $r_2 = 1 + i$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = e^t(c_1\cos t + c_2\sin t).$$

(c) The general solution to the differential equation is $y(t) = e^t(c_1\cos t + c_2\sin t) + 2\cos t + \sin t$. The derivative of this function is given by $y'(t) = e^t\cos t(c_1 + c_2) + e^t\sin t(-c_1 + c_2) - 2\sin t + \cos t$. The condition $y(\frac{\pi}{2}) = 1$ leads to $e^{\frac{\pi}{2}} + 1 = 1$. The condition $y'(\frac{\pi}{2}) = 0$ leads to $-e^{\frac{\pi}{2}}c_1 - 2 = 0$. Solving for c_1 and c_2 we find $c_1 = -2e^{-\frac{\pi}{2}}$ and $c_2 = 0$. The unique solution is given by

$$y(t) = -2e^{t-\frac{\pi}{2}}\cos t + 2\cos t + \sin t \blacksquare$$

Problem 22.7

$$y'' - 2y' + y = t^2 + 4 + 2\sin t,\ y(0) = 1,\ y'(0) = 3,\ y_p(t) = t^2 + 4t + 10 + \cos t.$$

Solutions

(a) $y_p' = 2t + 4 - \sin t,\ y_p'' = 2 - \cos t.$

$$\begin{aligned} y_p'' - 2y_p' + y_p =& 2 - \cos t - 4t - 8 + 2\sin t + t^2 + 4t + 10 + \cos t \\ =& t^2 + 4 + 2\sin t. \end{aligned}$$

(b) The associated characteristic equation $r^2 - 2r + 1 = 0$ has roots $r_1 = r_2 = 1$. Hence, the general solution to the homogeneous differential equation is

$$y_h(t) = c_1 e^t + c_2 t e^t.$$

(c) The general solution to the differential equation is $y(t) = c_1 e^t + c_2 t e^t + t^2 + 4t + 10 + \cos t$. The derivative of this function is given by $y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + 2t + 4 - \sin t$. The condition $y(0) = 1$ leads to $c_1 + 10 + 1 = 1$. The condition $y'(0) = 3$ leads to $c_1 + c_2 + 4 = 3$. Solving for c_1 and c_2 we find $c_1 = -10$ and $c_2 = 9$. The unique solution is given by

$$y(t) = -10e^t + 9te^t + t^2 + 4t + 10 + \cos t \blacksquare$$

The functions $u_1(t), u_2(t)$, and $u_3(t)$ are solutions to the following differential equations

$$\begin{aligned} u_1'' + p(t)u_1' + q(t)u_1 =& 2e^{-t} - t - 1 \\ u_2'' + p(t)u_2' + q(t)u_2 =& 3t \\ u_3'' + p(t)u_3' + q(t)u_3 =& 2e^t + 1. \end{aligned}$$

In Problems 22.8 - 22.9, use the functions $u_1, u_2(t)$ and u_3 to construct a particular solution of the differential equation.

Problem 22.8

$$u'' + p(t)u' + q(t)u = e^t + 2t + \frac{1}{2}.$$

Solution.
The left-hand side of the given equation can be written as $e^t + 2t + \frac{1}{2} = \frac{1}{2}(2e^t + 1) + \frac{2}{3}(3t)$ so that by Theorem 22.2, the function $u(t) = \frac{1}{2}u_1(t) + \frac{2}{3}u_2(t)$ is the required particular solution $\blacksquare$

Problem 22.9

$$u'' + p(t)u' + q(t)u = \frac{e^t + e^{-t}}{2}.$$

Solution.
The left-hand side of the given equation can be written as $\frac{e^t+e^{-t}}{2} = \frac{1}{4}(2e^t + 1) + \frac{1}{4}(2e^{-t} - t - 1) + \frac{1}{12}(3t)$ so that by Theorem 22.2, the function $u(t) =$

$\frac{1}{4}u_1(t) + \frac{1}{4}u_2(t)\frac{1}{12}u_2(t)$ is the required particular solution ■

In Problems 22.10 - 22.13, determine the function $g(t)$.

Problem 22.10

$$y'' - 2y' - 3y = g(t), \quad y_p(t) = 3e^{5t}.$$

Solution.
We have $y_p' = 15e^{5t}$ and $y_p'' = 45e^{5t}$. Thus,

$$\begin{aligned} g(t) =& y_p'' - 2y_p' - 3y_p \\ =& 45e^{5t} - 30e^{5t} - 9e^{5t} \\ =& 6e^{5t} \blacksquare \end{aligned}$$

Problem 22.11

$$y'' - 2y' = g(t), \quad y_p(t) = 3t + \sqrt{t}, \ t > 0.$$

Solution.
We have $y_p' = 3 + \frac{1}{2\sqrt{t}}$ and $y_p'' = -\frac{1}{4t^{\frac{3}{2}}}$. Thus,

$$\begin{aligned} g(t) =& y_p'' - 2y_p' \\ =& -\frac{1}{4}t^{-\frac{3}{2}} - 6 - t^{-\frac{1}{2}} \blacksquare \end{aligned}$$

Problem 22.12

$$y'' + y' = g(t), \quad y_p(t) = \ln(1+t), \ t > -1.$$

Solution.
We have $y_p' = \frac{1}{1+t}$ and $y_p'' = -(1+t)^{-2}$. Thus,

$$\begin{aligned} g(t) =& y_p'' + y_p' \\ =& -\frac{1}{(1+t)^2} + \ln(1+t) \blacksquare \end{aligned}$$

Problem 22.13

$$y'' + 2y' + y = g(t), \quad y_p(t) = t - 2.$$

Solution.
We have $y_p' = 1$ and $y_p'' = 0$. Thus,

$$\begin{aligned} g(t) =& y_p'' - 2y_p' + y_p \\ =& 0 - 2 + t - 2 \\ =& t \blacksquare \end{aligned}$$

In Problems 22.14 - 22.16, the general solution of the nonhomogeneous differential equation $y'' + \alpha y' + \beta y = g(t)$ is given, where c_1 and c_2 are arbitrary constants. Determine the constants α and β and the function $g(t)$.

Problem 22.14

$$y(t) = c_1 e^t + c_2 e^{2t} + 2t^{-2t}.$$

Solution.
From the given general solution we see that the roots of the characteristic equation are $r_1 = 1$ and $r_2 = 2$. Thus, the characteristic equation is $(r-1)(r-2) = r^2 - 3r + 2 = 0$. The associated differential equation is $y'' - 3y' + 2y = 0$. Hence, $\alpha = -3$ and $\beta = 2$. Now,

$$\begin{aligned} g(t) =& y_p'' - 3y_p' + 2y_p \\ =& 8e^{-2t} + 12e^{-2t} + 4e^{-2t} = 24e^{-2t} \blacksquare \end{aligned}$$

Problem 22.15

$$y(t) = c_1 e^t + c_2 t e^t + t^2 e^t.$$

Solution.
From the given general solution we see that the roots of the characteristic equation are $r_1 = r_2 = 1$. Thus, the characteristic equation is $(r-1)(r-1) = r^2 + 2r + 1 = 0$. The associated differential equation is $y'' + 2y' + y = 0$. Hence, $\alpha = -2$ and $\beta = 1$. Now,

$$\begin{aligned} g(t) =& y_p'' + 2y_p' + y_p \\ =& 2e^t + 4te^t + t^2 e^t - 4te^t - 2t^2 e^t + t^2 e^t \\ =& 2e^t \blacksquare \end{aligned}$$

Problem 22.16

$$y(t) = c_1 \sin 2t + c_2 \cos 2t - 1 + \sin t.$$

Solution.
From the given general solution we see that the roots of the characteristic equation are $r_1 = -2i$ and $r_2 = 2i$. Thus, the characteristic equation is $(r-2i)(r+2i) = r^2+4 = 0$. The associated differential equation is $y''+4y = 0$. Hence, $\alpha = 0$ and $\beta = 4$. Now,

$$\begin{aligned}g(t) =& y_p'' + 4y_p\\ =& -\sin t + 4\sin t - 4\\ =& 3\sin t - 4 \blacksquare\end{aligned}$$

Problem 22.17
Given that the function $\frac{e^t}{5}$ satisfies the differential equation $y'' + 4y = e^t$, write a general solution of the differential equation $y'' + 4y = e^t$.

Solution.
First, we find y_h. The characteristic equation $r^2 + 4 = 0$ has the roots $r_{1,2} = \pm 2i$. Thus, $y_h(t) = c_1 \cos 2t + c_2 \sin 2t$. The general solution to the nonhomogeneous equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{e^t}{5} \blacksquare$$

Problem 22.18
Find the general solution to the differential equation

$$y^{(4)} + 9y'' = 24 + 108t^2$$

given a particular solution $y_p(t) = \cos 3t + \sin 3t + t^4$.

Solution.
Let $u = y''$. Then the given equation reduces to a second order differential equation

$$z'' + 9z = 24 + 108t^2.$$

The characteristic equation is $r^2 + 9 = 0$ so that the roots are $r_{1,2} = \pm 3i$. Thus, $z_h(t) = c_1 \cos 3t + c_2 \sin 3t$. Integrating this function twice we find

$y_h(t) = c_1 \cos 3t + c_2 \sin 3t + c_3 t + c_4$. Hence, the general solution to the given differential equation is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + c_3 t + c_4 + t^4 \blacksquare$$

Problem 22.19
Show that the general solution of the third-order linear ODE $y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$ is of the form $y = y_p + y_h$, where y_p is a particular solution, and y_h is the general solution of the corresponding homogeneous equation.

Solution.
All we have to do is verify that y if is any solution of $y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$, then $y - y_p$ is a solution of the homogeneous equation. Indeed

$$\begin{aligned}(y - y_p)''' + p(t)(y - y_p)'' + q(t)(y - y_p)' + r(t)(y - y_p) =\\ y''' - y_p''' + p(t)y'' - p(t)y_p'' + q(t)y' - q(t)y_p' + r(t)y - r(t)y_p =\\ (y''' + p(t)y'' + q(t)y' + r(t)y) - (y_p''' + p(t)y_p'' + q(t)y_p' + r(t)y_p) =\\ g(t) - g(t) = 0 \blacksquare\end{aligned}$$

23 The Method of Undetermined Coefficients

Problem 23.1
List an appropriate form for a particular solution of
(a) $y'' + 4y = t^2 e^{3t}$.
(b) $y'' + 4y = te^{2t}\cos t$.
(c) $y'' + 4y = 2t^2 + 5\sin 2t + e^{3t}$.
(d) $y'' + 4y = t^2 \cos 2t$.

Solution.
The general solution to the homogeneous equation is $y_h(t) = c_1 \cos 2t + c_2 \sin 2t$.
(a) $y_p(t) = (A_2t^2 + A_1t + A_0)e^{3t}$.
(b) $y_p(t) = (A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t$
(c) $y_p(t) = A_2t^2 + A_1t + A_0 + B_0t\cos 2t + C_0t\sin 2t + D_0e^{3t}$
(d) $y_p(t) = t(A_2t^2 + A_1t + A_0)\cos 2t + t(B_2t^2 + B_1t + B_0)\sin 2t$ ■

For each of the differential equations in Problems 23.2 - 23.8
(a) Determine the general solution $y_h(t)$ to the homogeneous equation.
(b) Use the method of undetermined coefficients to find a particular solution $y_p(t)$.
(c) Form the general solution.

Problem 23.2

$$y'' - y' = 5e^t - \sin 2t.$$

Solution.
(a) The characteristic equation is $r^2 - r = 0$, with roots $r_1 = 0$ and $r_2 = 1$. Thus,

$$y_h(t) = c_1 + c_2e^t$$

(b) Since e^t is a particular solution to the homogeneous equation, we put the term Cte^t into y_p. Thus,

$$y_p(t) = A\cos(2t) + B\sin(2t) + Cte^t$$

Then $y_p'(t) = -2A\sin(2t) + 2B\cos(2t) + Ce^t + Cte^t$, and $y_p''(t) = -4A\cos(2t) - 4B\sin(2t) + 2Ce^t + Cte^t$. Putting these into the equation we get

$$-4A\cos(2t) - 4B\sin(2t) + 2Ce^t + Cte^t + 2A\sin(2t) - 2B\cos(2t) - Ce^t - Cte^t = 5e^t - \sin(2t).$$

We collect together the terms from both sides with $\cos(2t)$ and get $-4A - 2B = 0$. From the $\sin(2t)$ terms we get $-4B + 2A = -1$. The te^t terms cancel out and the e^t terms give $C = 5$. Solving we get $A = -\frac{1}{10}$ and $B = \frac{1}{5}$. Hence our particular solution is

$$y_p(t) = -\frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + 5te^t.$$

(c) The general solution to the nonhomogeneous is

$$y(t) = c_1 + c_2e^t - \frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + 5te^t \blacksquare$$

Problem 23.3

$$y'' + 6y' + 8y = -3e^{-t}.$$

Solution.
(a) The characteristic equation is $r^2 + 6r + 8 = 0$, with roots $r_1 = -2$ and $r_2 = -4$. Thus,

$$y_h(t) = c_1e^{-2t} + c_2e^{-4t}$$

(b) We look for a solution of the form $y_p(t) = Ae^{-t}$. After plugging in

$$y_p(t) = Ae^{-t},\ y_p'(t) = -Ae^{-t},\ y_p''(t) = Ae^{-t},$$

into the equation, we obtain

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t} \Longrightarrow 3Ae^{-t} = -3e^{-t} \Longrightarrow A = -1.$$

Thus, a particular solution of the ODE is

$$y_p(t) = -e^{-t}$$

(c) The general solution of the ODE is

$$y(t) = c_1e^{-2t} + c_2e^{-4t} - e^{-t} \blacksquare$$

Problem 23.4

$$y'' + 9y = \sin 2t.$$

Solution.
(a) The characteristic equation is $r^2+9=0$, with roots $r_1=-3i$ and $r_2=3i$. Thus,

$$y_h(t)=c_1\cos 3t+c_2\sin 3t$$

(b) Let $y_p(t)=a\cos 2t+b\sin 2t$. After plugging in

$$y_p(t)=a\cos 2t+b\sin 2t,\ y_p'(t)=-2a\sin 2t+2b\cos 2t,\ y_p''(t)=-4a\cos 2t-4b\sin 2t,$$

into the equation, we obtain

$$-4a\cos 2t-4b\sin 2t+9a\cos 2t+9b\sin 2t=\sin 2t\Longrightarrow 5a\cos 2t+5b\sin 2t=\sin 2t\Longrightarrow a=0,b=\tfrac{1}{5}$$

A particular solution is

$$y_p(t)=\frac{1}{5}\sin 2t$$

(c) The general solution is

$$y(t)=c_1\cos 3t+c_2\sin 3t+\frac{1}{5}\sin 2t\ \blacksquare$$

Problem 23.5

$$y''+5y'+6y=4-t^2.$$

Solution.
(a) The characteristic equation is $r^2+5r+6=0$, with roots $r_1=-2$ and $r_2=-3$. Thus,

$$y_h(t)=c_1e^{-2t}+c_2e^{-3t}$$

(b) The nonhomogeneous term is a quadratic polynomial, so we look for a particular solution of the form

$$y_p(t)=at^2+bt+c\Longrightarrow y_p'(t)=2at+b\Longrightarrow y_p''(t)=2a.$$

The equation becomes:

$$\begin{aligned}y_p''+5y_p'+6y_p=&4-t^2\Longrightarrow\\2a+5(2at+b)+6(at^2+bt+c)=&4-t^2\Longrightarrow\\6at^2+(10a+6b)t+(2a+5b+6c)=&-t^2+4.\end{aligned}$$

Thus, a, b, c must satisfy:

$$6a = -1,\ 10a + 6b = 0,\ 2a + 5b + 6c = 4 \Longrightarrow a = -\frac{1}{6}, b = \frac{5}{18}, c = \frac{53}{108}.$$

So, a particular solution is

$$y_p(t) = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}$$

(c) The general solution is

$$y(t) = c_1e^{-2t} + c_2e^{-3t} - \frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108} \blacksquare$$

Problem 23.6

$$y'' + 5y' + 4y = te^{-t}.$$

Solution.
(a) The characteristic equation is $r^2 + 5r + 4 = 0$, with roots $r_1 = -1$ and $r_2 = -4$. Thus,

$$y_h(t) = c_1e^{-t} + c_2e^{-4t}$$

(b) Note that e^{-t} is a solution to the homogeneous equation so our trial function will take the form $y_p(t) = t(at + b)e^{-t}$. In this case,

$$\begin{aligned} y_p(t) = t(at + b)e^{-t} &\Longrightarrow y_p'(t) = (-at^2 + (2a - b)t + b)e^{-t} \\ &\Longrightarrow y_p''(t) = (at^2 + (-4a + b)t + (2a - 2b))e^{-t} \end{aligned}$$

Substituting, we get:

$$te^{-t} = y_p'' + 5y_p' + 4y_p = (6at + (2a + 3b))e^{-t} \Longrightarrow 6a = 1,\ 2a + 3b = 0 \Longrightarrow a = \tfrac{1}{6},\ b = -\tfrac{1}{9}.$$

Thus, a particular solution is

$$y_p(t) = \frac{1}{6}t^2e^{-t} - \frac{1}{9}te^{-t}$$

(c) The general solution is

$$y(t) = c_1e^{-t} + c_2e^{-4t} + \frac{1}{6}t^2e^{-t} - \frac{1}{9}te^{-t} \blacksquare$$

Problem 23.7

$$y'' + y = t\cos t.$$

Solution.
(a) The characteristic equation is $r^2 + 1 = 0$, with roots $r_1 = -i$ and $r_2 = i$. Thus,

$$y_h(t) = c_1\cos t + c_2\sin t$$

(b) The right side $t\cos t$ has the form $P_n(t)e^{\alpha t}\cos\beta t$, with $n = 1, \alpha = 0, \beta = 1$. Since $\cos t$ is a solution to the characteristic equation we should try a particular solution of the form

$$y_p(t) = t[(A_0t+A_1)\cos t+(B_0t+B_1)\sin t] = (A_0t^2+A_1t)\cos t+(B_0t^2+B_1t)\sin t$$

where A_0, A_1, B_0, B_1 are constant coefficients to be determined.
Substituting y_p into the differential equation, we have the identity

$$\begin{aligned}
t\cos t =& y_p'' + y_p \\
=&[(A_0t^2 + A_1t)\cos t + (B_0t^2 + B_1t)\sin t]'' \\
&+[(A_0t^2 + A_1t)\cos t + (B_0t^2 + B_1t)\sin t] \\
=&\{[2A_0\cos t + 2(2A_0t + A_1)(-\sin t) + (A_0t^2 + A_1t)(-\cos t)] \\
&+[2B_0\sin t + 2(2B_0t + B_1)\cos t + (B_0t^2 + B_1t)(-\sin t)]\} \\
&+[(A_0t^2 + A_1t)\cos t + (B_0t^2 + B_1t)\sin t] \\
=&[4B_0t + (2A_0 + 2B_1)]\cos t + [-4A_0t + (-2A_1 + 2B_0)]\sin t
\end{aligned}$$

Comparing both sides we find

$$4B_0 = 1,\ 2A_0 + 2B_1 = 0,\ \ -4A_0 = 0,\ \ -2A_1 + 2B_0 = 0$$

which give

$$A_0 = 0,\ A_1 = \frac{1}{4},\ B_0 = \frac{1}{4},\ B_1 = 0$$

Thus, a particular solution is

$$y_p(t) = \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t$$

(c) The general solution is

$$y(t) = c_1\cos t + c_2\sin t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t \blacksquare$$

Problem 23.8

$$y'' + 4y' + 4y = e^{-2t}.$$

Solution.
(a) The characteristic equation is $(r+2)^2 = 0$, with roots $r_1 = r_2 = -2$. Thus,

$$y_h(t) = c_1e^{-2t} + c_2te^{-2t}$$

(b) Since e^{-2t} and te^{-2t} are solutions to the homogeneous equation, a trial function would be $y_p(t) = At^2e^{-2t}$. where A is a constant coefficient to be determined.
Substituting y_p into the differential equation, we have the identity

$$\begin{aligned}e^{-2t} =& y_p'' + 4y_p' + 4y_p\\ =&(At^2e^{-2t})'' + 4(At^2e^{-2t})' + 4At^2e^{-2t}\\ =&[2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t}]\\ &+4[2Ate^{-2t} - 2At^2e^{-2t}] + 4[At^2e^{-2t}]\\ =&2Ae^{-2t}\end{aligned}$$

Comparing the both sides, we have $A = \frac{1}{2}$. Thus, the particular solution is

$$y_p(t) = \frac{1}{2}t^2e^{-2t}$$

(c) The general solution is

$$y(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{-2t} \blacksquare$$

Problem 23.9
Find a second-order linear ordinary differential equation whose general solution is $y(t) = c_1e^{2t} + c_2e^{-t} + 7t$.

Solution.
From the general solution $y_h(t) = c_1e^{2t} + c_2e^{-t}$ to the homogeneous equation, we see that the corresponding homogeneous equation is associated with the characteristic equation whose roots are $r = 2, -1$. Hence, $(r-2)(r+1) = 0$, or $r^2 - r - 2 = 0$. Thus, the differential equation is given by $y'' - y' - 2y = g(t)$, where $g(t)$ is to be determined by using the particular solution $y_p(t) = 7t$.

Using $y_p(t) = 7t$ in $y_p'' - y_p' - 2y_p = g(t)$, we get $0 - 7 - 2(7t) = g(t)$, or equivalently, $g(t) = -14t - 7$. Thus, our differential equation is given by $y'' - y' - 2y = -14t - 7$. ■

Problem 23.10
For the equation $y'' + 6y' + 8y = 5t + 6t^2e^{-4t} + 7\sin(4t)$, determine the form of the simplest particular solution if the method of undetermined coefficients is to be used. You do not need to evaluate the coefficients.

Solution.
The homogeneous equation has constant coefficients. The characteristic equation is $r^2 + 6r + 8 = 0$, or $(r+2)(r+4) = 0$. Hence, $r = -2, -4$. Since e^{-4t} is a solution of the homogeneous equation, a trial guess for the particular solution is

$$y_p(t) = A_1 + A_2t + t(B_1t^2 + B_2t + B_3)e^{-4t} + E_1\cos(4t) + E_2\sin(4t),$$

where the coefficients are to be determined ■

Problem 23.11
Find a linear ordinary differential equation whose general solution is $y(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t) + 3e^{3t}$.

Solution.
From the general solution $y_h(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t)$ of the homogeneous equation, we see that the corresponding characteristic equation has roots are $r = 2 \pm 3i$. Hence, $(r-2)^2 + 9 = 0$, or $r^2 - 4r + 4 + 9 = 0$, which simplifies to $r^2 - 4r + 13 = 0$. Thus, the differential equation is given by $y'' - 4y' + 13y = g(t)$, where $g(t)$ is to be determined by using the particular solution $y_p(t) = 3e^{3t.}$ We have $y_p(t) = 3e^{3t}, y_p'(t) = 9e^{3t}, y_p''(t) = 27e^{3t.}$ Using these in $y_p'' - 4y_p' + 13y_p = g(t)$, we get $27e^{3t} - 4(9e^{3t}) + 13(3e^{3t}) = g(t)$, or equivalently, $g(t) = (27 - 36 + 39)e^{3t} = 30e^{3t.}$ Thus, our differential equation is given by $y'' - 4y' + 13y = 30e^{3t}$ ■

Problem 23.12
Write down the form of a particular solution of

$$y'' - 4y' + 4y = t^3 + 1 + 2te^{2t} - \sin 2t.$$

Solution.
By linear property, if y_{p_1}, y_{p_2} and y_{p_3} are respectively particular solutions of the equations

$$\begin{aligned}
y'' - 4y' + 4y =& t^3 + 1 \\
y'' - 4y' + 4y =& 2te^{2t} \\
y'' - 4y' + 4y =& -\sin 2t
\end{aligned}$$

then $y_p = y_{p_1} + y_{p_2} + y_{p_3}$ is a particular solution of the differential equation. The characteristic equation to the homogeneous equation is $r^2 - 4r + 4 = 0$ with repeated roots $r_{1,2} = 2$. Thus, $y_h = c_1e^{2t} + c_2te^{2t}$. By the method of undetermined coefficients, we have

$$\begin{aligned}
y_{p_1} =& A_0t^3 + A_1t^2 + A_2t + A_3 \\
y_{p_2} =& t^2(B_0t + B_1)e^{2t} \\
y_{p_3} =& C_0 \cos 2t + C_1 \sin 2t
\end{aligned}$$

Therefore we have the form

$$y_p = A_0t^3 + A_1t^2 + A_2t + A_3 + t^2(B_0t + B_1)e^{2t} + C_0 \cos 2t + C_1 \sin 2t \blacksquare$$

Problem 23.13
Set up the appropriate form of a particular solution (don't find the constants) for the equation

$$y'' - 4y' + 5y = 3e^{2t} \cos t + e^{-t} + (4t^5 + t^2)e^t \sin(2t).$$

Solution.
The characteristic equation is $r^2 - 4r + 5 = 0$ with complex roots $r_{1,2} = 2 \pm i$. Thus, $y_h(t) = c_1e^{2t} \cos t + c_2e^{2t} \sin t$. Hence the form of the particular solution is

$$\begin{aligned}
y_p(t) =& t(A \cos t + B \sin t)e^{2t} + Ce^{-t} \\
&+ (Dt^5 + Et^4 + Ft^3 + Gt^2 + Ht + I)e^t \sin(2t) \\
&+ (Kt^5 + Lt^4 + Mt^3 + Nt^2 + Ot + P)e^t \cos(2t) \blacksquare
\end{aligned}$$

Problem 23.14
Find the general solution to the differential equation

$$y'' - 4y' + 3y = e^{3t} + t^2.$$

Solution.
The associated homogeneous differential equation is $y'' - 4y' + 3y = 0$, and its characteristic equation is $r^2 - 4r + 3 = 0$ with roots $r_1 = 1$ and $r_2 = 3$. Therefore the general solution to the homogeneous equation is $y_h(t) = c_1e^t + c_2e^{3t}$.
We get a particular solution y_p to the given DE by using the method of undetermined coefficients. Because e^{3t} is a solution of the homogeneous DE, we know there exists a solution y_{p_1} with the form $y_{p_1}(t) = Ate^{3t}$, for some constant A. Plugging this into the DE, we find $A = 1/2$. We know when the right side is t^2 that there is a solution of the form $y_{p_2}(t) = Bt^2 + Ct + D$. Plugging this form into the DE and solving, we get $y_{p_2}(t) = \frac{1}{3}t^2 + \frac{8}{9}t + \frac{26}{27}$. Putting it all together, our general solution to the given DE is

$$y(t) = c_1e^t + c_2e^{3t} + \frac{1}{2}te^{3t} + \frac{1}{3}t^2 + \frac{8}{9}t + \frac{26}{27} \blacksquare$$

Problem 23.15
For each of the following nonhomogeneous 2nd order linear differential equations, propose a particular solution, with undetermined coefficients. Do not proceed to solve for the undetermined coefficients.
(a) $2y'' - 5y' - 3y = 7t + t^2e^{-\frac{t}{2}}$.
(b) $2y'' - 2y' + 5y = 7t + t^2e^{-\frac{t}{2}}$.

Solution.
The proposed particular solution for the method of undetermined coefficients depends almost exclusively on the RHS of the nonhomogeneous DE, but has to be modified slightly if the RHS involves a solution to the associated homogenous equation.
(a) The general solution to the homogeneous equation is $y_h(t) = c_1e^{3t} + c_2e^{-\frac{t}{2}}$. Since $e^{-\frac{t}{2}}$ is a solution to the homogenous equation, the proposed solution has the form:

$$y_p(t) = (A + Bt) + t(C + Dt + Et^2)e^{-\frac{t}{2}}.$$

(b) The general solution to the homogeneous equation is $y_h(t) = e^{\frac{t}{2}}\left(c_1\cos\left(\frac{3t}{2}\right) + c_2\sin\left(\frac{3t}{2}\right)\right)$. The RHS of the differential equation does not involve a solution to the associated homogenous equation, so the proposed solution has the form:

$$y_p(t) = (A + Bt) + (C + Dt + Et^2)e^{-\frac{t}{2}} \blacksquare$$

Problem 23.16
Solve the following initial value problem:

$$y'' - 5y' - 14y = -14t^2 - 10t - 26, \quad y(0) = 0, \ y'(0) = 0.$$

Solution.
The associated homogeneous differential equation

$$y'' - 5y' - 14y = 0$$

has characteristic equation $r^2 - 5r - 14 = 0$ with roots $r_1 = -2$ and $r_2 = 7$. Therefore, the general solution to the homogeneous equation is

$$y(t) = c_1 e^{-2t} + c_2 e^{7t}$$

Our trial function is

$$y_p(t) = At^2 + Bt + C$$

Plugging this expression into our equation, we obtain:

$$\begin{aligned}(2A) - 5(2At + B) - 14(At^2 + Bt + C) =& -14At^2 + (-10A - 14B)t + (2A - 5B - 14C)\\ =& -14t^2 - 10t - 26\end{aligned}$$

Equating coefficients of like powers of t we find

$$-14A = -14, \quad -10A - 14B = -10, \quad 2A - 5B - 14C = -26$$

Solving these equations we find $A = 1, \quad B = 0$, and $C = 2$. Hence,

$$y_p(t) = t^2 + 2$$

and

$$y(t) = c_1 e^{-2t} + c_2 e^{7t} + t^2 + 2$$

Now, c_1 and c_2 satisfy the equations

$$\begin{aligned}y(0) =& c_1 + c_2 + 2 = 0\\ y'(0) =& -2c_1 + 7c_2 = 0\end{aligned}$$

Solving these algebraic equations for c_1 and c_2, we find $c_1 = -\frac{14}{9}$ and $c_2 = -\frac{4}{9}$. Hence, the unique solution to the initial-value problem is

$$y(t) = -\frac{4}{9}e^{7t} - \frac{14}{9}e^{-2t} + t^2 + 2 \blacksquare$$

In Problems 17.17 - 17.18, we consider the differential equation $y''+\alpha y'+\beta y = g(t)$. The nonhomogeneous term $g(t)$ and the form of the particular solution prescribed by the method of undetermined coefficients are given. Determine α and β.

Problem 23.17

$$g(t) = t + e^{3t}, \quad y_p(t) = A_1t^2 + A_0t + B_0te^{3t}.$$

Solution.
Since $y_p(t) = t(A_0 + A_1t) + B_0te^{3t}$ we know that 0 and 3 are solutions to the characteristic equation. That is, $r^2-3r = 0$ so that the associated differential equation is $y'' - 3y' = 0$. Hence, $\alpha = -3$ and $\beta = 0$ ■

Problem 23.18

$$g(t) = -e^t+\sin 2t+e^t \sin 2t, \ \ y_p(t) = A_0e^t+B_0t\cos 2t+C_0t\sin 2t+D_0e^t \cos 2t+E_0e^t \sin 2t.$$

Solution.
From the expression of $y_p(t)$ we know that the roots of the characteristic equation are $r = \pm 2i$ so that the characteristic equation is $r^2 + 4 = 0$. The associated differential equation is then $y'' + 4y = 0$ and so $\alpha = 0$ and $\beta = 4$ ■

Problem 23.19
Solve using undetermined coefficients:

$$y'' + y' - 2y = t + \sin 2t, y(0) = 1, y'(0) = 0.$$

Solution.
The characteristic equation is $r^2+r-2 = 0$, which has roots $r_1 = -2$ and $r_2 = 1$. The homogeneous solution is thus $y_h(t) = c_1e^t + c_2e^{-2t}$. A trial function for the particular solution has the form $y_p(t) = At + B + C\sin 2t + D\cos 2t$. Plugging into the differential equation, we get:

$$-4C\sin 2t-4D\cos 2t+A+2C\cos 2t-2D\sin 2t-2At-2B-2C\sin 2t-2D\cos 2t = t+\sin 2t$$

Matching coefficients, we see

$$-6C - 2D = 1, \ \ -6D + 2C = 0, \ \ -2A = 1, \ A - 2B = 0,$$

whence we get the particular solution

$$y_p(t) = -\frac{1}{2}t - \frac{1}{4} - \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t,$$

and so the general solution is

$$y(t) = c_1 e^t + c_2 e^{-2t} - \frac{1}{2}t - \frac{1}{4} - \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t$$

Now, we must match initial conditions. Since

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} - \frac{1}{2} - \frac{3}{10}\cos 2t + \frac{1}{10}\sin 2t,$$

plugging in conditions at $t = 0$ gives:

$$\begin{aligned} y(0) =& c_1 + c_2 - \frac{1}{4} - \frac{1}{20} = 1 \\ y'(0) =& c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} = 0 \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 =& \frac{13}{10} \\ c_1 - 2c_2 =& \frac{4}{5} \end{aligned}$$

So $c_2 = \frac{1}{6}, c_1 = \frac{17}{15}$, and we have our solution, $y(t) = \frac{17}{15}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}t - \frac{1}{4} - \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t$ ■

24 The Method of Variation of Parameters

Problem 24.1
Solve $y'' + y = \sec t$ by variation of parameters.

Solution.
The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and

$$y_h(t) = c_1 \cos t + c_2 \sin t$$

Also, $y_1(t) = \cos t$ and $y_2(t) = \sin t$ so that $W(t) = \cos^2 t + \sin^2 t = 1$. Now,

$$u_1 = -\int \sin t \sec t dt = \int \frac{d(\cos t)}{\cos t} = \ln|\cos t|$$

and

$$u_2 = \int \cos t \sec t dt = \int dt = t$$

Hence, the particular solution is given by

$$y_p(t) = \ln|\cos t| \cos t + t \sin t$$

and the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + \ln|\cos t| \cos t + t \sin t \blacksquare$$

Problem 24.2
Solve $y'' - y = e^t$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution.
The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots $r = \pm 1$. The homogeneous solution is

$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

Undetermined Coefficients Summary. The basic trial solution method gives initial trial solution $y_p(t) = d_1 t e^t$ since 1 is a root of the characteristic equation. Substitution into $y'' - y = e^t$ gives $2d_1 e^t + d_1 t e^t - d_1 t e^t = e^t$. Cancel e^t and equate coefficients of like powers of t to find $d_1 = 1/2$. Then $y_p = \frac{te^t}{2}$.
Variation of Parameters Summary. The homogeneous solution $y_h(t) =$

$c_1e^t + c_2e^{-t}$ found above implies $y_1 = e^t,\ y_2 = e^{-t}$ is a suitable independent pair of solutions. Their Wronskian is $W = -2$. The variation of parameters formula applies:

$$y_p(t) = e^t \int \frac{e^{-t}}{2} e^t dt - e^{-t} \int \frac{e^t}{2} e^t dt$$

Integration, followed by setting all constants of integration to zero, gives $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$.

Differences. The two methods give respectively $y_p(t) = \frac{te^t}{2}$ and $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$. The solutions $y_p(t) = \frac{te^t}{2}$ and $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$ differ by the homogeneous solution $-\frac{e^t}{4}$. In both cases, the general solution is

$$y(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2 ■

Problem 24.3

Solve the following 2nd order equation using the variation of parameters method:

$$y'' + 4y = t^2 + 8\cos 2t.$$

Solution.

The characterisitc equation $r^2 + 4 = 0$ has roots $r = \pm 2i$ so that $y_h(t) = c_1 \cos 2t + c_2 \sin 2t$. Hence, $y_1(t) = \cos 2t, y_2(t) = \sin 2t$, and $W(t) = 2$. Thus,

$$\begin{aligned}
y_p =& -\cos 2t \int \frac{\sin 2t(t^2 + 8\cos 2t)}{2} dt + \sin 2t \int \frac{\cos 2t(t^2 + 8\cos 2t)}{2} dt \\
=& -\cos 2t(\frac{1}{4}t \sin 2t + \frac{1}{8}\cos 2t - \frac{1}{4}t^2 \cos 2t - \cos^2 2t) \\
&+ \sin 2t(\frac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t + \frac{1}{4}t^2 \sin 2t + 2t + \frac{1}{2}\sin 4t) \\
=& -\frac{1}{8} + \frac{1}{4}t^2 + \cos^2 2t \cos 2t + 2t \sin 2t + \frac{1}{2}\sin 4t \sin 2t
\end{aligned}$$

The general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + 2t \sin 2t \ ■$$

Problem 24.4
Find a particular solution by the variation of parameters to the equation

$$y'' + 2y' + y = e^{-t}\ln t.$$

Solution.
The characteristic equation

$$r^2 + 2r + 1 = 0$$

has roots $r_1 = r_2 = -1$, so the fundamental solutions of the reduced equation are

$$y_1(t) = e^{-t},\ y_2(t) = te^{-t}$$

Compute the Wronskian.

$$\begin{aligned}W(t) =& \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}\\ =& e^{-t}(e^{-t} - te^{-t}) + e^{-t}\cdot te^{-t}\\ =& e^{-2t} - te^{-2t} + te^{-2t}\\ =& e^{-2t}\end{aligned}$$

Compute $u_1(t)$.

$$\begin{aligned}u_1(t) =& -\int \frac{y_2(t)g(t)}{W(t)}dt\\ =& -\int \frac{te^{-t}\cdot e^{-t}\ln t}{e^{-2t}}dt\\ =& -\int t\ln t dt = -\frac{t^2}{2}\ln t + \int \frac{t^2}{2}\cdot\frac{1}{t}dt\\ =& -\frac{t^2}{2}\ln t + \frac{t^2}{4}\end{aligned}$$

Compute $u_2(t)$.

$$\begin{aligned}u_2(t) =& \int \frac{y_1(t)g(t)}{W(t)}dt\\ =& \int \frac{e^{-t}\cdot e^{-t}\ln t}{e^{-2t}}dt\\ =& \int \ln t dt = t\ln t - \int t\cdot\frac{1}{t}dt\\ =& t\ln t - t\end{aligned}$$

Note. We used integration by parts to compute the integrals $\int t\ln t dt$ and $\int \ln t dt$.
The particular solution to our complete equation is

$$\begin{aligned}
y_p(t) =& u_1(t)y_1(t) + u_2(t)y_2(t)\\
=&\left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right)e^{-t} + (t\ln t - t)te^{-t}\\
=&\frac{t^2}{2}\ln te^{-t} - \frac{3t^2}{4}e^{-t}\\
=&(\frac{t^2}{2}\ln t - \frac{3t^2}{4})e^{-t} \blacksquare
\end{aligned}$$

Problem 24.5
Solve the following initial value problem by using variation of parameters:

$$y'' + 2y' - 3y = te^t,\ y(0) = -\frac{1}{64},\ y'(0) = \frac{59}{64}.$$

Solution.
From the characteristic equation, we obtain $y_1(t) = e^t, y_2(t) = e^{-3t}$ and $W(t) = -4e^{-2t}$. Integration then yields

$$\begin{aligned}
u_1(t) =& -\int \frac{e^{-3t}te^t}{-4e^{-2t}}dt = \frac{t^2}{8}\\
u_2(t) =& \int \frac{e^t te^t}{-4e^{-2t}}dt = -\frac{1}{16}te^{4t} + \frac{e^{4t}}{64}
\end{aligned}$$

Thus. $y_p(t) = \frac{e^t}{64}(8t^2 - 4t + 1)$ and the general solution is

$$y(t) = c_1e^t + c_2e^{-3t} + \frac{t^2}{8}e^t - \frac{1}{16}te^t$$

Initial conditions:

$$\begin{aligned}
y(0) =& c_1 + c_2 = -\frac{1}{64}\\
y'(0) =& c_1 - 3c_2 - \frac{4}{64} = \frac{59}{64}
\end{aligned}$$

These are satisfied by $c_1 = \frac{15}{64}$ and $c_2 = -\frac{1}{4}$. Finally the solution to the initial value problem is

$$y = \frac{e^t}{64}(8t^2 - 4t + 15) - \frac{1}{4}e^{-3t} \blacksquare$$

Problem 24.6
(a) Verify that $\{e^{\sqrt{t}}, e^{-\sqrt{t}}\}$ is a fundamental set for the equation

$$4ty'' + 2y' - y = 0$$

on the interval $(0, \infty)$. You may assume that the given functions are solutions to the equation.
(b) Use the method of variation of parameters to find one solution to the equation

$$4ty'' + 2y' - y = 4\sqrt{t}e^{\sqrt{t}}.$$

Solution.
(a) Usually the first thing to do would be to check that $y_1(t) = e^{\sqrt{t}}$ and $y_2(t) = e^{-\sqrt{t}}$ really are solutions to the equation. However, the question says that this can be assumed and so we move on to the next step, which is to check that the Wronskian of the two solutions is non-zero on $(0, \infty)$. We have

$$y_1' = \frac{1}{2\sqrt{t}}e^{\sqrt{t}} \text{ and } y_2' = -\frac{1}{2\sqrt{t}}e^{-\sqrt{t}}$$

and so

$$W(t) = y_1y_2' - y_1'y_2 = -\frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{t}} = -\frac{1}{\sqrt{t}}$$

This is indeed non-zero and so $\{e^{\sqrt{t}}, e^{-\sqrt{t}}\}$ is a fundamental set for the homogeneous equation.
(b) The variation of parameters formula says that

$$y = -y_1 \int \frac{y_2 g}{W(t)} dt + y_2 \int \frac{y_1 g}{W(t)} dt$$

is a solution to the nonhomogeneous equation in the form $y'' + py' + qy = g$. To get the right g, we have to divide the equation through by $4t$ and so $g = \frac{1}{\sqrt{t}}e^{\sqrt{t}}$. Thus

$$\begin{aligned} y = &- e^{\sqrt{t}} \int \frac{e^{-\sqrt{t}}(\frac{1}{\sqrt{t}})e^{\sqrt{t}}}{-1/\sqrt{t}} dt + e^{-\sqrt{t}} \int \frac{e^{\sqrt{t}}(\frac{1}{\sqrt{t}})e^{\sqrt{t}}}{-1/\sqrt{t}} dt \\ =&e^{\sqrt{t}} \int dt - e^{-\sqrt{t}} \int e^{2\sqrt{t}} dt \\ =&te^{\sqrt{t}} - e^{-\sqrt{t}} \int e^{2\sqrt{t}} dt \end{aligned}$$

To evaluate the integral, we substitute $u = 2\sqrt{t}$ so that $dt = \frac{1}{2}udu$. We get

$$\int e^{2\sqrt{t}}dt = \frac{1}{2}\int ue^u du = \frac{1}{2}(u-1)e^u = (\sqrt{t}-1/2)e^{2\sqrt{t}}.$$

Thus

$$y = (t - \sqrt{t} + 1/2)e^{\sqrt{t}}$$

is one solution to the equation. You might notice that the 1/2 can be dropped (because $(1/2)e^{\sqrt{t}}$ is a solution to the homogeneous equation) so that

$$y = (t - \sqrt{t})e^{\sqrt{t}}$$

would also work ■

Problem 24.7
Use the method of variation of parameters to find the general solution to the equation

$$y'' + y = \sin t.$$

Solution.
The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ so that the solution to the homogeneous equation is $y_h(t) = c_1 \cos t + c_2 \sin t$. The Wronskian is $W(\cos t, \sin t) = 1$. Now $u_1'(t) = -\sin^2 t = \frac{\cos(2t)-1}{2}$. Hence $u_1(t) = \frac{1}{2}(\frac{1}{2}\sin(2t) - t)$. Similarly, $u_2'(t) = \sin t \cos t$. Hence $u_2(t) = \frac{1}{2}\sin^2 t$. So $y_p(t) = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$. The general solution is given by

$$y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2}t\cos t \;■$$

Problem 24.8
Consider the differential equation

$$t^2y'' + 3ty' - 3y = 0, \;\; t > 0.$$

(a) Determine r so that $y = t^r$ is a solution.
(b) Use (a) to find a fundamental set of solutions.
(c) Use the method of variation of parameters for finding a particular solution to

$$t^2y'' + 3ty' - 3y = \frac{1}{t^3}, \;\; t > 0.$$

Solution.
(a) Inserting y, y', and y'' into the equation we find $r^2+2r-3=0$. Solving for r to obtain $r_1=1$ and $r_2=-3$.
(b) Let $y_1(t)=t$ and $y_2(t)=t^{-3}$. Since

$$W(t)=\begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

$\{y_1, y_2\}$ is a fundamental set of solutions for $t>0$.
(c) Recall that the variation of parameters formula states that if y_1 and y_2 form a fundamental solution set for $y''+p(t)y'+q(t)y=0$, then $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ is a particular solution to the equation $y''+p(t)y'+q(t)y=g(t)$, where

$$\begin{aligned} u_1(t) &= -\int \frac{t^{-3}t^{-5}}{-4t^{-3}}dt = -\frac{1}{16}t^{-4} \\ u_2(t) &= \int \frac{t\cdot t^{-5}}{-4t^{-3}}dt = -\frac{1}{4}\ln t \end{aligned}$$

Thus,

$$y_p(t) = -\frac{1}{16}t^{-3} - \frac{1}{4}t^{-3}\ln t \ \blacksquare$$

Problem 24.9
Use the method of variation of parameters to find the general solution to the differential equations

$$y''+y=\sin^2 t.$$

Solution.
The characterisitc equation $r^2+1=0$ has roots $r=\pm i$ so that $y_1(t)=\cos t,\ y_2(t)=\sin t$, and $W(t)=1$. Hence,

$$\begin{aligned} u_1(t) &= -\int \sin t\sin^2 t dt = \int (1-\cos^2 t)d(\cos t) = \cos t - \frac{1}{3}\cos^3 t \\ u_2(t) &= \int \cos t\sin^2 t dt = \frac{1}{3}\sin^3 t \end{aligned}$$

Thus,

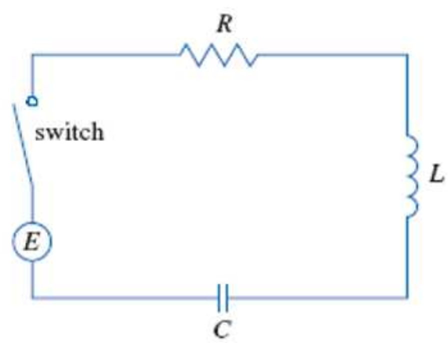
$$y_p(t) = \cos^2 t - \frac{1}{3}\cos^4 t + \frac{1}{3}\sin^4 t$$

and

$$y(t) = c_1\cos t + c_2\sin t + \cos^2 t - \frac{1}{3}\cos^4 t + \frac{1}{3}\sin^4 t \ \blacksquare$$

25 Applications of Nonhomogeneous Second Order Linear Differential Equations: Forced Mechanical Vibrations

Problem 25.1
Find the charge and current at time t in the circuit below if $R = 40\Omega, L = 1\ H, C = 16 \times 10^{-4}\ F$, and $E(t) = 100 \cos 10t$ and the initial charge and current are both zero.



Solution.
With the given values of L, R, C, and $E(t)$, the equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

becomes

$$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100 \cos(10t).$$

The characteristic equation is

$$r^2 + 40r + 625 = 0$$

with roots

$$r_{1,2} = \frac{-40 \pm \sqrt{40^2 - 4 \times 625}}{2} = -20 \pm 15i$$

so the general solution to the homogeneous equation is

$$Q_h(t) = e^{-20t}(c_1 \cos(15t) + c_2 \sin(15t)).$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A \cos(10t) + B \sin(10t).$$

Then

$$\begin{aligned}Q_p'(t) =& -10A\sin(10t)+10B\cos(10t)\\ Q_p''(t) =& -100A\cos(10t)-100B\sin(10t)\end{aligned}$$

Substituting into the nonhomogeneous equation and factoring we obtain

$$(525A+400B)\cos(10t)+(-400A+525B)\sin(10t)=100\cos(10t).$$

Equating coefficients we find

$$\begin{aligned}525A+400B =& 100\\ -400A+525B =& 0\end{aligned}$$

Solving this system by the method of elimination we find $A=\frac{84}{607}$ and $B=\frac{64}{697}$, so a particular solution is

$$Q_p(t)=\frac{1}{697}(84\cos(10t)+64\sin(10t))$$

and the general solution is

$$Q(t)=e^{-20t}(c_1\cos(15t)+c_2\sin(15t))+\frac{1}{697}(84\cos(10t)+64\sin(10t)).$$

Imposing the condition $Q(0)=0$ we get

$$Q(0)=c_1+\frac{84}{697}=0\Longrightarrow c_1=-\frac{84}{697}.$$

To impose the other initial condition we first differentiate to find the current:

$$\begin{aligned}I=\frac{dQ}{dt}=&e^{-20t}[(-20c_1+15c_2)\cos(15t)+(-15c_1-20c_2)\sin(10t)]\\ &+\frac{40}{697}(16\cos(10t)-21\sin(10t))\end{aligned}$$

Thus,

$$I(0)=-20c_1+15c_2+\frac{640}{697}=0\Longrightarrow c_2=-\frac{464}{2091}.$$

Thus, the formula for the charge is

$$Q(t)=\frac{4}{697}\left[\frac{e^{-20t}}{3}(-63\cos(15t)-116\sin(15t))+(21\cos(10t)+16\sin(10t))\right]$$

and the expression for the current is

$$I(t)=\frac{1}{2091}\left[e^{-20t}(-1920\cos(15t)+13060\sin(15t))+120(-21\sin(10t)+16\cos(10t))\right]\ \blacksquare$$

Problem 25.2
A series circuit consists of a resistor with $R = 20\Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.005\ F$, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

Solution.
Here the initial-value problem for the charge is

$$Q'' + 20Q' + 500Q = 12, Q(0) = Q'(0) = 0.$$

The characteristic equation is

$$r^2 + 20r + 500 = 0$$

with roots

$$r_{1,2} = -10 \pm 20i$$

so that the general solution to the homogeneous equation is

$$Q_h(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t).$$

Using the undetermined coefficients method we try the solution $Q_p(t) = A$ which by substitution we find $A = \frac{3}{125}$. Hence, the general solution is

$$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}.$$

Using the initial condition $Q(0) = 0$ we find $c_1 + \frac{3}{125} = 0$ which implies that $c_1 = -\frac{3}{125}$. Using the condition $Q'(0) = 0$ where

$$I(t) = Q'(t) = e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_1 - 20c_2)\sin 20t]$$

we find $-10c_1 + 20c_2 = 0$. Solving for c_2 we find $c_2 = \frac{3}{250}$. Thus, the formula for the charge is

$$Q(t) = -\frac{e^{-10t}}{250}((6\cos(20t) + 3\sin(20t)) + \frac{3}{125}$$

and the expression for the current is

$$I(t) = \frac{3}{5}e^{-10t}\sin(20t) \blacksquare$$

Problem 25.3
The battery in previous problem is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$. Find the charge at time t.

Solution.
As in the previous exercise, $Q_h(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$. Using the undetermined coefficients method we try the solution $Q_p(t) = A \cos 10t + B \sin 10t$ which by substitution we find

$$(-100A+200B+500A)\cos 10t+(-100B-200A+500B)\sin 10t = 12\sin 10t.$$

Hence, we obtain the system

$$\begin{aligned} -100A + 200B + 500A =& 0 \\ -100B - 200A + 500B =& 12 \end{aligned}$$

Solving this system by elimination we find $A = -\frac{3}{250}$ and $B = \frac{3}{125}$. Hence, the general solution is

$$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250}\cos 10t + frac3125 \sin 10t.$$

Using the initial condition $Q(0) = 0$ we find

$$c_1 - \frac{3}{250} = 0 \Longrightarrow c_1 = \frac{3}{250}.$$

Using the condition $Q'(0) = 0$ where

$$I(t) = Q'(t) = e^{-10t}[(-10c_1+20c_2)\cos 20t+(-10c_1-20c_2)\sin 20t]+\frac{3}{25}\sin 10t+\frac{6}{25}\cos 10t$$

we find $\frac{6}{25} - 10c_1 + 20c_2 = 0$. Solving for c_2 we find $c_2 = -\frac{3}{500}$. Thus, the formula for the charge is

$$Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right] - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t \ \blacksquare$$

Problem 25.4
A series circuit contains a resistor with $R = 24\ \Omega$, an inductor with $L = 2\ H$, a capacitor with $C = 0.005\ F$, and a 12-V battery. The initial charge is $Q = 0.001\ C$ and the initial current is 0.
(a) Find the charge and current at time t.
(b) Graph the charge and current functions.

Solution.
(a) With the given values of L, R, C, and $E(t)$, the equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

becomes

$$2\frac{d^2Q}{dt^2} + 24\frac{dQ}{dt} + 200Q = 12.$$

The characteristic equation is

$$r^2 + 12r + 100 = 0$$

with roots

$$r_{1,2} = \frac{-12 \pm \sqrt{12^2 - 4 \times 100}}{2} = -6 \pm 8i$$

so the general solution to the homogeneous equation is

$$Q_h(t) = e^{-6t}(c_1 \cos(8t) + c_2 \sin(8t)).$$

For the method of undetermined coefficients we try the particular solution $Q_p(t) = A$ which by substitution leads to $A = \frac{3}{50}$. Hence, the general solution is

$$Q(t) = e^{-6t}(c_1 \cos(8t) + c_2 \sin(8t)) + \frac{3}{50}.$$

Imposing the condition $Q(0) = 0.001$ we get

$$Q(0) = c_1 + \frac{3}{50} = 0 \Longrightarrow c_1 = -\frac{3}{50}.$$

To impose the other initial condition we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-6t}[(-6c_1 + 8c_2)\cos(8t) + (-8c_1 - 6c_2)\sin(8t).$$

Thus,

$$I(0) = -6c_1 + 8c_2 = 0 \Longrightarrow c_2 = -\frac{9}{200}.$$

Thus, the formula for the charge is

$$Q(t) = -\frac{3}{50}e^{-6t}(\cos(8t) + \frac{1}{4}\sin(8t)) + \frac{3}{50}$$

and the expression for the current is

$$I(t) = \frac{3}{4}e^{-6t}\sin 8t.$$

(b) Use a graphing calculator ■

Problem 25.5
A vibrating spring with damping is modeled by the differential equation

$$y'' + 2y' + 4y = 0.$$

(a) Find the general solution to the equation. Show each step of the process.
(b) Is the solution under damped, over damped or critically damped?
(c) Suppose that the damping were changed, keeping the mass and the spring the same, until the system became critically damped. Write the differential equation which models this critically damped system. Do not solve.
(d) What is the steady state (long time) solution to

$$y'' + 2y' + 4y = \cos(2t)?$$

Solution.
(a) The characteristic equation is $r^2 + 2r + 4 = 0$ with roots $r_{1,2} = -1 \pm i\sqrt{3}$. Thus, the general solution is $y_h(t) = e^{-t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t)$.
(b) Since the discriminant of the DE is negative, the solution is under damped. See Chapter 30.
(c) The differential equation is $y'' + 4y' + 4y = 0$.
(d) The general solution to the homogeneous equation tends to zero in the long time. Thus, the steady-state solution is the particular solution to the nonhomogeneous equation which can be found by the method of undetermined coefficients. We try $y_p = A\cos 2t + B\sin 2t$. In this case, $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$. Substituting into the differential equation we find

$$4B\cos 2t - 2A\sin 2t = \cos 2t.$$

Thus, $A = 0$ and $B = \frac{1}{4}$. In this case, the steady-state solution is $y(t) = \frac{1}{4}\sin 2t$ ■

Problem 25.6
A vertical spring with a spring constant equal to 108 lb/ft has a 96 lb weight attached to it. A dashpot (or a shock absorber) with a damping coefficient c = 36 lb-sec/ft is attached to the weight. Suppose that a downward force of $f(t) = 72\cos 6t$ is applied to the weight. If the weight is released from rest at the equilibrium position at time $t = 0$
(a) show that the differential equation governing the displacement $y(t)$ is

$$y'' + 12y' + 36y = 24\cos 6t$$

where g = 32 ft/sec is used .
(b) Find the solution satisfying the equation established in Part (a) and the given initial conditions.

Solution.
(a) We are given $32m = 96, k = 36, \gamma = 108$, and $E(t) = 72\cos 6t$. Thus, that the differential equation governing the displacement $y(t)$ is

$$y'' + 12y' + 36y = 24\cos 6t.$$

(b) The characteristic equation is $r^2 + 12r + 36 = 0$ with repeated roots $r_{1,2} = -6$. Thus, $y_h(t) = e^{-6t}(c_1 + c_2 t)$. For a particular solution we try $y_p = A\cos 6t + B\sin 6t$. We have $y_p' = -6A\sin 6t + 6B\cos 6t$ and $y_p'' = -36A\cos 6t - 36B\sin 6t$. Substituting into the differential equation we find

$$36B\cos 6t - 36A\sin 6t = 24\cos 6t.$$

Hence, $A = 0$ and $B = \frac{2}{3}$ and $y_p = \frac{2}{3}\sin 6t$. The general solution is $y = e^{-6t}(c_1 + c_2 t) + \frac{2}{3}\sin 6t$. Using the conditions $y(0) = y'(0) = 0$ we find $c_1 = 0$ andd $c_2 = -4$. Hence, $y(t) = -4te^{-6t} + \frac{2}{3}\sin 6t$ ■

Problem 25.7
A six Newton weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant of the spring being 27 Newtons per meter. The weight comes to rest in its equilibrium position, and beginning at t = 0 an external force given by $F(t) = 12\cos(20t)$ is applied to the system. Determine the resulting displacement as a function of time, assuming damping is negligible.

Solution.
We have the differential equation

$$6y'' + 27y = 12\cos 20t.$$

We also have $F = 12, \omega_0 = \frac{3}{\sqrt{2}}, \omega = 20$. Thus, the displacement function is

$$y(t) = \frac{24}{400 - \frac{9}{4}} \sin \frac{20 - 3/\sqrt{2}}{2} t \sin \frac{20 + 3/\sqrt{3}}{2} t \blacksquare$$

Problem 25.8
An inductor of 5 henries is connected in series with a capacitor of 1/180 farads, a resistor of 60 ohms and a voltage-supply given by $E(t) = 120\cos 6t$ in volts. Suppose that both the charge Q and the current I are zero initially.
(a) Show that the differential equation governing the charge $Q(t)$ is

$$Q'' + 12Q' + 36Q = 24\cos 6t$$

(b) Find the charge $Q(t)$ satisfying the equation of Part (a) and the given initial conditions.

Solution.
(a) We are given that $L = 5, C = \frac{1}{180}, R = 60$, and $E(t) = 12\cos 6t$. Substituting in the equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

we obtain

$$Q'' + 12Q' + 36Q = 24\cos 6t$$

(b) The characteristic equation is

$$r^2 + 12r + 36 = 0$$

with roots

$$r_{1,2} = \frac{-12 \pm \sqrt{12^2 - 4 \times 36}}{2} = -6$$

so the general solution to the homogeneous equation is

$$Q_h(t) = e^{-6t}(c_1 + c_2 t).$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A\cos(6t) + B\sin(6t).$$

Then

$$\begin{aligned} Q_p'(t) =& -6A\sin(6t) + 6B\cos(6t) \\ Q_p''(t) =& -36A\cos(6t) - 36B\sin(6t) \end{aligned}$$

Substituting into the nonhomogeneous equation and factoring we obtain

$$72B\cos(6t) - 72A\sin(6t) = 24\cos(6t).$$

Equating coefficients we find $A = 0$ and $B = \frac{1}{3}$ and so a particular solution is

$$Q_p(t) = \frac{1}{3}\sin(6t)$$

and the general solution is

$$Q(t) = e^{-6t}(c_1 + c_2 t) + \frac{1}{3}\sin(6t).$$

Imposing the condition $Q(0) = 0$ we get

$$Q(0) = c_1 = 0.$$

To impose the other initial condition we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-6t}c_2(1-t) + 2\cos 6t.$$

Thus,

$$I(0) = c_2 + 2 = 0 \Longrightarrow c_2 = -2.$$

Thus, the formula for the charge is

$$Q(t) = -2te^{-6t} + \frac{1}{3}\sin(6t) \blacksquare$$

Problem 25.9
An inductor of 4 H is connected in series with a capacitor of 0.25 F and a resistor of 10 , without supplied voltage. Suppose that at $t = 0$, there is a charge of 1/3 coulomb on the capacitor but no current.
(a) Write down the differential equation for the charge, $Q(t)$, and the initial conditions.
(b) Find the charge as a function of time t.

Solution.
(a) We are given that $L = 4, C = 0.25, R = 10$, and $E(t) = 0$. Substituting in the equation
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$
we obtain
$$4Q'' + 10Q' + 4Q = 0.$$
(b) The characteristic equation is
$$2r^2 + 5r + 2 = 0$$
with roots
$$r_{1,2} = \frac{-5 \pm \sqrt{25 - 16}}{4} = -\frac{5}{4} \pm \frac{3}{4}.$$
Thus, the charge is given by
$$Q(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-2t} \blacksquare$$

Problem 25.10
Consider the IVP, $y'' + by' + 9y = \sin \omega t,\ y(0) = 0,\ y'(0) = 0$. For what values of b and ω is the solution periodic? For what values are there frequency beats? Solve the system in the resonant case and sketch the solution.

Solution.
The solution to the IVP is periodic when $b = 0$ and $\omega \neq \omega_0$ where $\omega_0 = 3$. For frequency beats, we must have $b = 0$ and $\omega \neq 3$. In the resonant case, we have the IVP
$$y'' + 9y = \sin 3t, y(0) = y'(0) = 0.$$
The characteristic equation is $r^2 + 9 = 0$ with roots $r_{1,2} = \pm 3i$. Thus, $y_h(t) = c_1 \cos 3t + c_2 \sin 3t$. For a particular solution, we try $y_p = t(A \cos 3t + B \sin 3t)$. In this case, $y_p' = A \cos 3t + B \sin 3t + t(-3A \sin 3t + 3B \cos 3t)$ and $y_p'' = -6A \sin 3t + 6B \cos 3t + t(-9A \cos 3t - 9B \sin 3t)$. Substituting into the differential equation we find
$$-6A \sin 3t + 6B \cos 3t = \sin 3t.$$
Thus, $A = -\frac{1}{6}$ and $B = 0$ so that $y_p = -\frac{1}{6}t \cos 3t$. The general solution to the differential equation is then $y(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{6}t \cos 3t$. Now using

the initial conditions $y(0) = y'(0) = 0$ we find $c_1 = 0$ and $c_2 = \frac{1}{9}$. Finally, the solution to the IVP is

$$y(t) = \frac{1}{9}\sin 3t - \frac{1}{6}t\cos 3t \blacksquare$$