

A First Course in Quasi-Linear Partial  
Differential Equations  
for Physical Sciences and Engineering  
Solution Manual

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## Preface

This manuscript provides the complete and detailed solutions to *A First Course in Partial Differential Equations for Physical Sciences and Engineering*. Distribution of this book in any form is prohibited.

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## Solutions to Section 1

### Problem 1.1

Classify the following equations as either ODE or PDE.

(a)  $(y''')^4 + \frac{t^2}{(y')^2+4} = 0.$

(b)  $\frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{y-x}{y+x}.$

(c)  $y'' - 4y = 0.$

### Solution.

(a) ODE with dependent variable  $y$  and independent variable  $x$ .

(b) PDE with dependent variable  $u$  and independent variables  $x$  and  $y$ .

(c) ODE with dependent variable  $y$  and independent variable  $x$  ■

### Problem 1.2

Write the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

in the coordinates  $s = x$ ,  $t = x - y$ .

### Solution.

We have

$$u_x = u_s s_x + u_t t_x = u_s + u_t$$

$$u_{xx} = u_{ss} s_x + u_{st} t_x + u_{st} s_x + u_{tt} t_x = u_{ss} + 2u_{st} + u_{tt}$$

$$u_{xy} = u_{ss} s_y + u_{st} t_y + u_{st} s_y + u_{tt} t_y = -u_{st} - u_{tt}$$

$$u_y = u_s s_y + u_t t_y = -u_t$$

$$u_{yy} = -u_{st} s_y - u_{tt} t_y = u_{tt}.$$

Substituting these expressions into the given equation we find

$$u_{ss} = 0 \quad \blacksquare$$

### Problem 1.3

Write the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

in the coordinates  $s = x + y$ ,  $t = 2x$ .

**Solution.**

We have

$$\begin{aligned}u_x &= u_s s_x + u_t t_x = u_s + 2u_t \\u_{xx} &= u_{ss} s_x + u_{st} t_x + 2u_{st} s_x + 2u_{tt} t_x = u_{ss} + 4u_{st} + 4u_{tt} \\u_{xy} &= u_{ss} s_y + u_{st} t_y + 2u_{st} s_y + 2u_{tt} t_y = u_{ss} + 2u_{st} \\u_y &= u_s s_y + u_t t_y = u_s \\u_{yy} &= u_{ss} s_y + u_{st} t_y = u_{ss}.\end{aligned}$$

Substituting these expressions into the given equation we find

$$u_{ss} + u_{tt} = 0 \blacksquare$$

**Problem 1.4**

For each of the following PDEs, state its order and whether it is linear or non-linear. If it is linear, also state whether it is homogeneous or nonhomogeneous:

- (a)  $uu_x + x^2 u_{yyy} + \sin x = 0$ .
- (b)  $u_x + e^{x^2} u_y = 0$ .
- (c)  $u_{tt} + (\sin y) u_{yy} - e^t \cos y = 0$ .

**Solution.**

- (a) Order 3, non-linear.
- (b) Order 1, linear, homogeneous.
- (c) Order 2, linear, non-homogeneous  $\blacksquare$

**Problem 1.5**

For each of the following PDEs, determine its order and whether it is linear or not. For linear PDEs, state also whether the equation is homogeneous or not; For nonlinear PDEs, circle all term(s) that are not linear.

- (a)  $x^2 u_{xx} + e^x u = x u_{xyy}$ .
- (b)  $e^y u_{xxx} + e^x u = -\sin y + 10x u_y$ .
- (c)  $y^2 u_{xx} + e^x u u_x = 2x u_y + u$ .
- (d)  $u_x u_{xy} + e^x u u_y = 5x^2 u_x$ .
- (e)  $u_t = k^2(u_{xx} + u_{yy}) + f(x, y, t)$ .

**Solution.**

- (a) Linear, homogeneous, order 3.

- (b) Linear, non-homogeneous, order 3. The inhomogeneity is  $-\sin y$ .
- (c) Non-linear, order 2. The non-linear term is  $e^x uu_x$ .
- (d) Non-linear, order 3. The non-linear terms are  $u_x u_{xxy}$  and  $e^x uu_y$ .
- (e) Linear, non-homogeneous, order 2. The inhomogeneity is  $f(x, y, t)$  ■

**Problem 1.6**

Which of the following PDEs are linear?

- (a) **Laplace's equation:**  $u_{xx} + u_{yy} = 0$ .
- (b) **Convection (transport) equation:**  $u_t + cu_x = 0$ .
- (c) **Minimal surface equation:**  $(1 + Z_y^2)Z_{xx} - 2Z_x Z_y Z_{xy} + (1 + Z_x^2)Z_{yy} = 0$ .
- (d) **Korteweg-Vries equation:**  $u_t + 6uu_x = u_{xxx}$ .

**Solution.**

- (a) Linear.
- (b) Linear.
- (c) Non-linear where all the terms are non-linear.
- (d) Non-linear with non-linear term  $6uu_x$  ■

**Problem 1.7**

Classify the following differential equations as ODEs or PDEs, linear or non-linear, and determine their order. For the linear equations, determine whether or not they are homogeneous.

- (a) The **diffusion equation** for  $u(x, t)$  :

$$u_t = ku_{xx}.$$

- (b) The **wave equation** for  $w(x, t)$  :

$$w_{tt} = c^2 w_{xx}.$$

- (c) The **thin film equation** for  $h(x, t)$  :

$$h_t = -(hh_{xxx})_x.$$

- (d) The **forced harmonic oscillator** for  $y(t)$  :

$$y_{tt} + \omega^2 y = F \cos(\omega t).$$

- (e) The **Poisson Equation** for the electric potential  $\Phi(x, y, z)$  :

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x, y, z).$$

where  $\rho(x, y, z)$  is a known charge density.

(f) **Burger's equation** for  $h(x, t)$  :

$$h_t + hh_x = \nu h_{xx}.$$

**Solution.**

- (a) PDE, linear, second order, homogeneous.
- (b) PDE, linear, second order, homogeneous.
- (c) PDE, quasi-linear (non-linear), fourth order.
- (d) ODE, linear, second order, non-homogeneous.
- (e) PDE, linear, second order, non-homogeneous.
- (f) PDE, quasilinear (non-linear), second order ■

**Problem 1.8**

Write down the general form of a linear second order differential equation of a function in three variables.

**Solution.**

$$A(x, y, z)u_{xx} + B(x, y, z)u_{xy} + C(x, y, z)u_{yy} + E(x, y, z)u_{xz} + F(x, y, z)u_{yz} + G(x, y, z)u_{zz} + H(x, y, z)u_x + I(x, y, z)u_y + J(x, y, z)u_z + K(x, y, z)u = L(x, y, z) \blacksquare$$

**Problem 1.9**

Give the orders of the following PDEs, and classify them as linear or non-linear. If the PDE is linear, specify whether it is homogeneous or non-homogeneous.

- (a)  $x^2u_{xxy} + y^2u_{yy} - \log(1 + y^2)u = 0$
- (b)  $u_x + u^3 = 1$
- (c)  $u_{xxyy} + e^x u_x = y$
- (d)  $uu_{xx} + u_{yy} - u = 0$
- (e)  $u_{xx} + u_t = 3u$ .

**Solution.**

- (a) Order 3, linear, homogeneous.
- (b) Order 1, non-linear.
- (c) Order 4, linear, non-homogeneous
- (d) Order 2, non-linear.
- (e) Order 2, linear, homogeneous ■

**Problem 1.10**

Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

Use the change of variables  $v(x, y) = y - 2x$  and  $w(x, y) = x$  to show that  $u_{ww} = 0$ .

**Solution.**

Using the chain rule we find

$$\begin{aligned} u_x &= -2u_v + u_w \\ u_{xx} &= 4u_{vv} - 4u_{vw} + u_{ww} \\ u_y &= u_v \\ u_{yy} &= u_{vv} \\ u_{xy} &= -2u_{vv} + u_{vw}. \end{aligned}$$

Substituting these into the given PDE we find  $u_{ww} = 0$  ■

**Problem 1.11**

Write the one dimensional wave equation  $u_{tt} = c^2 u_{xx}$  in the coordinates  $v = x + ct$  and  $w = x - ct$ .

**Solution.**

We have

$$\begin{aligned} u_t &= cu_v - cu_w \\ u_{tt} &= c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww} \\ u_x &= u_v + u_w \\ u_{xx} &= u_{vv} + 2u_{vw} + u_{ww}. \end{aligned}$$

Substituting we find  $u_{vw} = 0$  ■

**Problem 1.12**

Write the PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

in the coordinates  $v(x, y) = y - 3x$  and  $w(x, y) = x + y$ .



**Solution.**

We have

$$\begin{aligned}u_x &= -3u_v + u_w \\u_{xx} &= -3(-3u_v + u_w)_v + (-3u_v + u_w)_w = 9u_{vv} - 6u_{vw} + u_{ww} \\u_{xy} &= -3u_{vv} + u_{vw} - 3u_{vw} + u_{ww} = -3u_{vv} - 2u_{vw} + u_{ww} \\u_y &= u_v + u_w \\u_{yy} &= (u_v + u_w)_v + (u_v + u_w)_w = u_{vv} + 2u_{vw} + u_{ww}.\end{aligned}$$

Substituting into the PDE we find  $u_{vw} = 0$  ■

**Problem 1.13**

Write the PDE

$$au_x + bu_y = 0$$

in the coordinates  $s(x, y) = ax + by$  and  $t(x, y) = bx - ay$ . Assume  $a^2 + b^2 > 0$ .

**Solution.**

According to the chain rule for the derivative of a composite function, we have

$$\begin{aligned}u_x &= u_s s_x + u_t t_x = au_s + bu_t \\u_y &= u_s s_y + u_t t_y = bu_s - au_t.\end{aligned}$$

Substituting these into the given equation to obtain

$$a^2 u_s + abu_t + b^2 u_s - abu_t = 0$$

or

$$(a^2 + b^2)u_s = 0$$

and since  $a^2 + b^2 > 0$  we obtain

$$u_s = 0 \quad \blacksquare$$

**Problem 1.14**

Write the PDE

$$u_x + u_y = 1$$

in the coordinates  $s = x + y$  and  $t = x - y$ .

**Solution.**

Using the chain rule we find

$$u_x = u_s s_x + u_t t_x = u_s + u_t$$

$$u_y = u_s s_y + u_t t_y = u_s - u_t.$$

Substituting these into the PDE to obtain  $u_s = \frac{1}{2}$  ■

**Problem 1.15**

Write the PDE

$$au_t + bu_x = u, \quad a, b \neq 0$$

in the coordinates  $v = ax - bt$  and  $w = \frac{1}{a}t$ .

**Solution.**

We have  $u_t = -bu_v + \frac{1}{a}u_w$  and  $u_x = au_v$ . Substituting we find  $u_w = u$  ■

## Solutions to Section 2

### Problem 2.1

Determine  $a$  and  $b$  so that  $u(x, y) = e^{ax+by}$  is a solution to the equation

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$$

### Solution.

We have  $u_{xxxx} = a^4 e^{ax+by}$ ,  $u_{yyyy} = b^4 e^{ax+by}$ , and  $u_{xxyy} = a^2 b^2 e^{ax+by}$ . Thus, substituting these into the equation we find

$$(a^4 + 2a^2 b^2 + b^4) e^{ax+by} = 0.$$

Since  $e^{ax+by} \neq 0$ , we must have  $a^4 + 2a^2 b^2 + b^4 = 0$  or  $(a^2 + b^2) = 0$ . This is true only when  $a = b = 0$ . Thus,  $u(x, y) = 1$  ■

### Problem 2.2

Consider the following differential equation

$$t u_{xx} - u_t = 0.$$

Suppose  $u(t, x) = X(x)T(t)$ . Show that there is a constant  $\lambda$  such that  $X'' = \lambda X$  and  $T' = \lambda t T$ .

### Solution.

Substituting into the differential equation we find

$$t X'' T - X T' = 0$$

or

$$\frac{X''}{X} = \frac{T'}{tT}.$$

The LHS is a function of  $x$  only whereas the RHS is a function of  $t$  only. This is true only when both sides are constant. That is, there is  $\lambda$  such that

$$\frac{X''}{X} = \frac{T'}{tT} = \lambda$$

and this leads to the two ODEs  $X'' = \lambda X$  and  $T' = \lambda t T$  ■

**Problem 2.3**

Consider the initial value problem

$$xu_x + (x + 1)yu_y = 0, \quad x, y > 1$$

$$u(1, 1) = e.$$

Show that  $u(x, y) = \frac{xe^x}{y}$  is the solution to this problem.

**Solution.**

We have  $xu_x + (x+1)yu_y = \frac{x}{y}(e^x + xe^x) + (x+1)y\left(-\frac{xe^x}{y^2}\right) = 0$  and  $u(1, 1) = e$  ■

**Problem 2.4**

Show that  $u(x, y) = e^{-2y} \sin(x - y)$  is the solution to the initial value problem

$$u_x + u_y + 2u = 0, \quad x, y > 1$$

$$u(x, 0) = \sin x.$$

**Solution.**

We have  $u_x + u_y + 2u = e^{-2y} \cos(x - y) - 2e^{-2y} \sin(x - y) - e^{-2y} \cos(x - y) + 2e^{-2y} \sin(x - y) = 0$  and  $u(x, 0) = \sin x$  ■

**Problem 2.5**

Solve each of the following differential equations:

- (a)  $\frac{du}{dx} = 0$  where  $u = u(x)$ .
- (b)  $\frac{\partial u}{\partial x} = 0$  where  $u = u(x, y)$ .

**Solution.**

(a) The general solution to this equation is  $u(x) = C$  where  $C$  is an arbitrary constant.

(b) The general solution is  $u(x, y) = f(y)$  where  $f$  is an arbitrary function of  $y$  ■

**Problem 2.6**

Solve each of the following differential equations:

- (a)  $\frac{d^2u}{dx^2} = 0$  where  $u = u(x)$ .
- (b)  $\frac{\partial^2 u}{\partial x \partial y} = 0$  where  $u = u(x, y)$ .

**Solution.**

(a) The general solution to this equation is  $u(x) = C_1x + C_2$  where  $C_1$  and  $C_2$  are arbitrary constants.

(b) We have  $u_y = f(y)$  where  $f$  is an arbitrary differentiable function of  $y$ . Hence,  $u(x, y) = \int f(y)dy + g(x)$  ■

**Problem 2.7**

Show that  $u(x, y) = f(y + 2x) + xg(y + 2x)$ , where  $f$  and  $g$  are two arbitrary twice differentiable functions, satisfy the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

**Solution.**

Let  $v(x, y) = y + 2x$ . Then

$$\begin{aligned} u_x &= 2f_v(v) + g(v) + 2xg_v(v) \\ u_{xx} &= 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ u_y &= f_v(v) + xg_v(v) \\ u_{yy} &= f_{vv}(v) + xg_{vv}(v) \\ u_{xy} &= 2f_{vv}(v) + g_v(v) + 2xg_{vv}(v). \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} - 4u_{xy} + 4u_{yy} &= 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ &\quad - 8f_{vv}(v) - 4g_v(v) - 8xg_{vv}(v) \\ &\quad + 4f_{vv}(v) + 4xg_{vv}(v) = 0 \quad \blacksquare \end{aligned}$$

**Problem 2.8**

Find the differential equation whose general solution is given by  $u(x, t) = f(x - ct) + g(x + ct)$ , where  $f$  and  $g$  are arbitrary twice differentiable functions in one variable.

**Solution.**

Let  $v = x - ct$  and  $w = x + ct$ . We have

$$\begin{aligned} u_x &= f_v v_x + g_w w_x = f_v + g_w \\ u_{xx} &= f_{vv} v_x + g_{ww} w_x = f_{vv} + g_{ww} \\ u_t &= f_v v_t + g_w w_t = -cf_v + cg_w \\ u_{tt} &= -cf_{vv} v_t + cg_{ww} w_t = c^2 f_{vv} + c^2 g_{ww} \end{aligned}$$

Hence,  $u$  satisfies the wave equation  $u_{tt} = c^2 u_{xx}$  ■

**Problem 2.9**

Let  $p : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function in one variable. Prove that

$$u_t = p(u)u_x$$

has a solution satisfying  $u(x, t) = f(x + p(u)t)$ , where  $f$  is an arbitrary differentiable function. Then find the general solution to  $u_t = (\sin u)u_x$ .

**Solution.**

Let  $v = x + p(u)t$ . Using the chain rule we find

$$u_t = f_v \cdot v_t = f_v \cdot (p(u) + p_u u_t t).$$

Thus

$$(1 - t f_v p_u) u_t = f_v p.$$

If  $1 - t f_v p_u \equiv 0$  on any  $t$ -interval  $I$  then  $f_v p \equiv 0$  on  $I$  which implies that  $f_v \equiv 0$  or  $p \equiv 0$  on  $I$ . But either condition will imply that  $t f_v p_u = 0$  and this will imply that  $1 = 1 - t f_v p_u = 0$ , a contradiction. Hence, we must have  $1 - t f_v p_u \neq 0$ . In this case,

$$u_t = \frac{f_v p}{1 - t f_v p_u}.$$

Likewise,

$$u_x = f_v \cdot (1 + p_u u_x t)$$

or

$$u_x = \frac{f_v}{1 - t f_v p_u}.$$

It follows that  $u_t = p(u)u_x$ .

If  $u_t = (\sin u)u_x$  then  $p(u) = \sin u$  so that the general solution is given by

$$u(x, t) = f(x + t \sin u)$$

where  $f$  is an arbitrary differentiable function in one variable ■

**Problem 2.10**

Find the general solution to the pde

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Hint: See Problem 1.2.

**Solution.**

Using Problem 1.2, we found  $u_{ss} = 0$ . Hence,  $u(s, t) = sf(t) + g(t)$  where  $f$  and  $g$  are arbitrary differentiable functions. In terms of  $x$  and  $y$  we find  $u(x, y) = xf(x - y) + g(x - y)$  ■

**Problem 2.11**

Let  $u(x, t)$  be a function such that  $u_{xx}$  exists and  $u(0, t) = u(L, t) = 0$  for all  $t \in \mathbb{R}$ . Prove that

$$\int_0^L u_{xx}(x, t)u(x, t)dx \leq 0.$$

**Solution.**

Using integration by parts, we compute

$$\begin{aligned} \int_0^L u_{xx}(x, t)u(x, t)dx &= u_x(x, t)u(x, t)|_{x=0}^L - \int_0^L u_x^2(x, t)dx \\ &= u_x(L, t)u(L, t) - u_x(0, t)u(0, t) - \int_0^L u_x^2(x, t)dx \\ &= - \int_0^L u_x^2(x, t)dx \leq 0 \end{aligned}$$

Note that we have used the boundary conditions  $u(0, t) = u(L, t) = 0$  and the fact that  $u_x^2(x, t) \geq 0$  for all  $x \in [0, L]$  ■

**Problem 2.12**

Consider the initial value problem

$$u_t + u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = 1.$$

(a) Show that  $u(x, t) \equiv 1$  is a solution to this problem.

(b) Show that  $u_n(x, t) = 1 + \frac{e^{n^2 t}}{n} \sin nx$  is a solution to the initial value problem

$$u_t + u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = 1 + \frac{\sin nx}{n}.$$

(c) Find  $\sup\{|u_n(x, 0) - 1| : x \in \mathbb{R}\}$ .

(d) Find  $\sup\{|u_n(x, t) - 1| : x \in \mathbb{R}, t > 0\}$ .

(e) Show that the problem is ill-posed.

**Solution.**

- (a) This can be done by plugging in the equations.  
 (b) Plug in.  
 (c) We have  $\sup\{|u_n(x, 0) - 1| : x \in \mathbb{R}\} = \frac{1}{n} \sup\{|\sin nx| : x \in \mathbb{R}\} = \frac{1}{n}$ .  
 (d) We have  $\sup\{|u_n(x, t) - 1| : x \in \mathbb{R}\} = \frac{e^{n^2 t}}{n}$ .  
 (e) We have  $\lim_{t \rightarrow \infty} \sup\{|u_n(x, t) - 1| : x \in \mathbb{R}, t > 0\} = \lim_{t \rightarrow \infty} \frac{e^{n^2 t}}{n} = \infty$ .  
 Hence, the solution is unstable and thus the problem is ill-posed ■

**Problem 2.13**

Find the general solution of each of the following PDEs by means of direct integration.

- (a)  $u_x = 3x^2 + y^2$ ,  $u = u(x, y)$ .  
 (b)  $u_{xy} = x^2 y$ ,  $u = u(x, y)$ .  
 (c)  $u_{xtt} = e^{2x+3t}$ ,  $u = u(x, t)$ .

**Solution.**

- (a)  $u(x, y) = x^3 + xy^2 + f(y)$ , where  $f$  is an arbitrary differentiable function.  
 (b)  $u(x, y) = \frac{x^3 y^2}{6} + F(x) + g(y)$ , where  $F(x) = \int f(x) dx$  and  $g(y)$  is an arbitrary differentiable function.  
 (c)  $u(x, t) = \frac{1}{18} e^{2x+3t} + t \int f_1(x) dx + \int f_2(x) dx + g(t)$  ■

**Problem 2.14**

Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

- (a) Use the change of variables  $v(x, y) = y - 2x$  and  $w(x, y) = x$  to show that  $u_{ww} = 0$ .  
 (b) Find the general solution to the given PDE.

**Solution.**

- (a) Using the chain rule we find

$$\begin{aligned} u_x &= -2u_v + u_w \\ u_{xx} &= 4u_{vv} - 4u_{vw} + u_{ww} \\ u_y &= u_v \\ u_{yy} &= u_{vv} \\ u_{xy} &= -2u_{vv} + u_{vw}. \end{aligned}$$



Substituting these into the given PDE we find  $u_{ww} = 0$ .

(b) Solving the equation  $u_{ww} = 0$  we find  $u_w = f(v)$  and  $u(v, w) = wf(v) + g(v)$ . In terms of  $x$  and  $y$  the general solution is  $u(x, y) = xf(y - 2x) + g(y - 2x)$  ■

### Problem 2.15

Derive the general solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

by using the change of variables  $v = x + ct$  and  $w = x - ct$ .

### Solution.

We have

$$\begin{aligned} u_t &= cu_v - cu_w \\ u_{tt} &= c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww} \\ u_x &= u_v + u_w \\ u_{xx} &= u_{vv} + 2u_{vw} + u_{ww} \end{aligned}$$

Substituting we find  $u_{vw} = 0$  and solving this equation we find  $u_v = f(v)$  and  $u(v, w) = F(v) + G(w)$  where  $F(v) = \int f(v)dv$ .

Finally, using the fact that  $v = x + ct$  and  $w = x - ct$ ; we get d'Alembert's solution to the one-dimensional wave equation:

$$u(x, t) = F(x + ct) + G(x - ct)$$

where  $F$  and  $G$  are arbitrary differentiable functions ■

## Solutions to Section 3

### Problem 3.1

Solve the IVP:  $y' + 2ty = t$ ,  $y(0) = 0$

#### Solution.

Since  $p(t) = 2t$ , we find  $\mu(t) = e^{\int 2tdt} = e^{t^2}$ . Multiplying the given equation by  $e^{t^2}$  to obtain

$$(e^{t^2} y)' = te^{t^2}$$

Integrating both sides with respect to  $t$  and using substitution on the right-hand integral to obtain

$$e^{t^2} y = \frac{1}{2}e^{t^2} + C$$

Dividing the last equation by  $e^{t^2}$  to obtain

$$y(t) = Ce^{-t^2} + \frac{1}{2}$$

Since  $y(0) = 0$ , we find  $C = -\frac{1}{2}$ . Thus, the unique solution to the IVP is given by

$$y = \frac{1}{2}(1 - e^{-t^2}) \blacksquare$$

### Problem 3.2

Find the general solution:  $y' + 3y = t + e^{-2t}$

#### Solution.

Since  $p(t) = 3$ , the integrating factor is  $\mu(t) = e^{3t}$ . Thus, the general solution is

$$\begin{aligned} y(t) &= e^{-3t} \int e^{3t}(t + e^{-2t})dt + Ce^{-3t} \\ &= e^{-3t} \int (te^{3t} + e^t)dt + Ce^{-3t} \\ &= e^{-3t} \left( \frac{e^{3t}}{9}(3t - 1) + e^t \right) + Ce^{-3t} \\ &= \frac{3t-1}{9} + e^{-2t} + Ce^{-3t} \blacksquare \end{aligned}$$

### Problem 3.3

Find the general solution:  $y' + \frac{1}{t}y = 3 \cos t$ ,  $t > 0$

**Solution.**

Since  $p(t) = \frac{1}{t}$ , the integrating factor is  $\mu(t) = e^{\int \frac{dt}{t}} = e^{\ln t} = t$ . Using the method of integrating factor we find

$$\begin{aligned} y(t) &= \frac{1}{t} \int 3t \cos t dt + \frac{C}{t} \\ &= \frac{3}{t} (t \sin t + \cos t) + \frac{C}{t} \\ &= 3 \sin t + \frac{3 \cos t}{t} + \frac{C}{t} \blacksquare \end{aligned}$$

**Problem 3.4**

Find the general solution:  $y' + 2y = \cos(3t)$ .

**Solution.**

We have  $p(t) = 2$  so that  $\mu(t) = e^{2t}$ . Thus,

$$y(t) = e^{-2t} \int e^{2t} \cos(3t) dt + C e^{-2t}$$

But

$$\begin{aligned} \int e^{2t} \cos(3t) dt &= \frac{e^{2t}}{3} \sin(3t) - \frac{2}{3} \int e^{2t} \sin(3t) dt \\ &= \frac{e^{2t}}{3} \sin(3t) - \frac{2}{3} \left( -\frac{e^{2t}}{3} \cos(3t) + \frac{2}{3} \int e^{2t} \cos(3t) dt \right) \\ \frac{13}{9} \int e^{2t} \cos(3t) dt &= \frac{e^{2t}}{9} (3 \sin(3t) + 2 \cos(3t)) \\ \int e^{2t} \cos(3t) dt &= \frac{e^{2t}}{13} (3 \sin(3t) + 2 \cos(3t)) \end{aligned}$$

Hence,

$$y(t) = \frac{1}{13} (3 \sin(3t) + 2 \cos(3t)) + C e^{-2t} \blacksquare$$

**Problem 3.5**

Find the general solution:  $y' + (\cos t)y = -3 \cos t$ .

**Solution.**

Since  $p(t) = \cos t$  we have  $\mu(t) = e^{\sin t}$ . Thus,

$$\begin{aligned} y(t) &= e^{-\sin t} \int e^{\sin t} (-3 \cos t) dt + C e^{-\sin t} \\ &= -3 e^{-\sin t} e^{\sin t} + C e^{-\sin t} \\ &= C e^{-\sin t} - 3 \blacksquare \end{aligned}$$

**Problem 3.6**

Given that the solution to the IVP  $ty' + 4y = \alpha t^2$ ,  $y(1) = -\frac{1}{3}$  exists on the interval  $-\infty < t < \infty$ . What is the value of the constant  $\alpha$ ?

**Solution.**

Solving this equation with the integrating factor method with  $p(t) = \frac{4}{t}$  we find  $\mu(t) = t^4$ . Thus,

$$\begin{aligned} y &= \frac{1}{t^4} \int t^4(\alpha t)dt + \frac{C}{t^4} \\ &= \frac{\alpha}{6}t^2 + \frac{C}{t^4} \end{aligned}$$

Since the solution is assumed to be defined for all  $t$ , we must have  $C = 0$ . On the other hand, since  $y(1) = -\frac{1}{3}$  we find  $\alpha = -2$  ■

**Problem 3.7**

Suppose that  $y(t) = Ce^{-2t} + t + 1$  is the general solution to the equation  $y' + p(t)y = g(t)$ . Determine the functions  $p(t)$  and  $g(t)$ .

**Solution.**

The integrating factor is  $\mu(t) = e^{2t}$ . Thus,  $\int p(t)dt = 2t$  and this implies that  $p(t) = 2$ . On the other hand, the function  $t + 1$  is the particular solution to the nonhomogeneous equation so that  $(t + 1)' + 2(t + 1) = g(t)$ . Hence,  $g(t) = 2t + 3$  ■

**Problem 3.8**

Suppose that  $y(t) = -2e^{-t} + e^t + \sin t$  is the unique solution to the IVP  $y' + y = g(t)$ ,  $y(0) = y_0$ . Determine the constant  $y_0$  and the function  $g(t)$ .

**Solution.**

First, we find  $y_0 : y_0 = y(0) = -2 + 1 + 0 = -1$ . Next, we find  $g(t) : g(t) = y' + y = (-2e^{-t} + e^t + \sin t)' + (-2e^{-t} + e^t + \sin t) = 2e^{-t} + e^t + \cos t - 2e^{-t} + e^t + \sin t = 2e^t + \cos t + \sin t$  ■

**Problem 3.9**

Find the value (if any) of the unique solution to the IVP  $y' + (1 + \cos t)y = 1 + \cos t$ ,  $y(0) = 3$  in the long run?

**Solution.**

The integrating factor is  $\mu(t) = e^{\int(1+\cos t)dt} = e^{t+\sin t}$ . Thus, the general solution is

$$\begin{aligned} y(t) &= e^{-(t+\sin t)} \int e^{t+\sin t}(1 + \cos t)dt + Ce^{-(t+\sin t)} \\ &= 1 + Ce^{-(t+\sin t)} \end{aligned}$$

Since  $y(0) = 3$ , we find  $C = 2$  and therefore  $y(t) = 1 + 2e^{-(t+\sin t)}$ . Finally,

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} (1 + 2e^{-\sin t} e^{-t}) = 1 \blacksquare$$

**Problem 3.10**

Solve the initial value problem  $ty' = y + t$ ,  $y(1) = 7$

**Solution.**

Rewriting the equation in the form

$$y' - \frac{1}{t}y = 1$$

we find  $p(t) = -\frac{1}{t}$  and  $\mu(t) = \frac{1}{t}$ . Thus, the general solution is given by

$$y(t) = t \ln |t| + Ct$$

But  $y(1) = 7$  so that  $C = 7$ . Hence,

$$y(t) = t \ln |t| + 7t \blacksquare$$

**Problem 3.11**

Show that if  $a$  and  $\lambda$  are positive constants, and  $b$  is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that  $y \rightarrow 0$  as  $t \rightarrow \infty$ . Hint: Consider the cases  $a = \lambda$  and  $a \neq \lambda$  separately.

**Solution.**

Since  $p(t) = a$  we find  $\mu(t) = e^{at}$ . Suppose first that  $a = \lambda$ . Then

$$y' + ay = be^{-at}$$

and the corresponding general solution is

$$y(t) = bte^{-at} + Ce^{-at}$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{t \rightarrow \infty} \left( \frac{bt}{e^{at}} + \frac{C}{e^{at}} \right) \\ &= \lim_{t \rightarrow \infty} \frac{b}{ae^{at}} = 0 \end{aligned}$$

Now, suppose that  $a \neq \lambda$  then

$$y(t) = \frac{b}{a - \lambda} e^{-\lambda t} + Ce^{-at}$$

Thus,

$$\lim_{t \rightarrow \infty} y(t) = 0 \blacksquare$$

### Problem 3.12

Solve the initial-value problem  $y' + y = e^t y^2$ ,  $y(0) = 1$  using the substitution  $u(t) = \frac{1}{y(t)}$

#### Solution.

Substituting into the equation we find

$$u' - u = -e^t, \quad u(0) = 1$$

Solving this equation by the method of integrating factor with  $\mu(t) = e^{-t}$  we find

$$u(t) = -te^t + Ce^t$$

Since  $u(0) = 1$ ,  $C = 1$  and therefore  $u(t) = -te^t + e^t$ . Finally, we have

$$y(t) = (-te^t + e^t)^{-1} \blacksquare$$

### Problem 3.13

Solve the initial-value problem  $ty' + 2y = t^2 - t + 1$ ,  $y(1) = \frac{1}{2}$

#### Solution.

Rewriting the equation in the form

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Since  $p(t) = \frac{2}{t}$  we find  $\mu(t) = t^2$ . The general solution is then given by

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

Since  $y(1) = \frac{1}{2}$  we find  $C = \frac{1}{12}$ . Hence,

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2} \blacksquare$$

**Problem 3.14**

Solve  $y' - \frac{1}{t}y = \sin t$ ,  $y(1) = 3$ . Express your answer in terms of the **sine integral**,  $Si(t) = \int_0^t \frac{\sin s}{s} ds$ .

**Solution.**

Since  $p(t) = -\frac{1}{t}$  we find  $\mu(t) = \frac{1}{t}$ . Thus,

$$\begin{aligned} \left(\frac{1}{t}y\right)' &= \left(\int_0^t \frac{\sin s}{s}\right)' \\ \frac{1}{t}y(t) &= Si(t) + C \\ y(t) &= tSi(t) + Ct \end{aligned}$$

Since  $y(1) = 3$ ,  $C = 3 - Si(1)$ . Hence,  $y(t) = tSi(t) + (3 - Si(1))t \blacksquare$

## Solutions to Section 4

### Problem 4.1

Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

### Solution.

At first, this equation may not appear separable, so we must simplify the right hand side until it is clear what to do.

$$\begin{aligned} y' &= te^{t^2 - \ln y^2} \\ &= te^{t^2} \cdot e^{\ln\left(\frac{1}{y^2}\right)} \\ &= te^{t^2} \cdot \frac{1}{y^2} \\ &= \frac{t}{y^2} e^{t^2}. \end{aligned}$$

Separating the variables and solving the equation we find

$$\begin{aligned} y^2 y' &= te^{t^2} \\ \frac{1}{3} \int (y^3)' dt &= \int te^{t^2} dt \\ \frac{1}{3} y^3 &= \frac{1}{2} e^{t^2} + C \\ y^3 &= \frac{3}{2} e^{t^2} + C \blacksquare \end{aligned}$$

### Problem 4.2

Solve the (separable) differential equation

$$y' = \frac{t^2 y - 4y}{t + 2}.$$



**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\frac{y'}{y} &= \frac{t^2 - 4}{t + 2} = t - 2 \\ \int (\ln |y|)' dt &= \int (t - 2) dt \\ \ln |y| &= \frac{t^2}{2} - 2t + C \\ y(t) &= Ce^{\frac{t^2}{2} - 2t} \blacksquare\end{aligned}$$

**Problem 4.3**

Solve the (separable) differential equation

$$ty' = 2(y - 4).$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\frac{y'}{y - 4} &= \frac{2}{t} \\ \int (\ln |y - 4|)' dt &= \int \frac{2}{t} dt \\ \ln |y - 4| &= \ln t^2 + C \\ \ln \left| \frac{y - 4}{t^2} \right| &= C \\ y(t) &= Ct^2 + 4 \blacksquare\end{aligned}$$

**Problem 4.4**

Solve the (separable) differential equation

$$y' = 2y(2 - y).$$

**Solution.**

Separating the variables and solving (using partial fractions in the process)

we find

$$\begin{aligned}\frac{y'}{y(2-y)} &= 2 \\ \frac{y'}{2y} + \frac{y'}{2(2-y)} &= 2 \\ \frac{1}{2} \int (\ln |y|)' dt - \frac{1}{2} \int (\ln |2-y|)' dt &= \int 2 dt \\ \ln \left| \frac{y}{2-y} \right| &= 4t + C \\ \left| \frac{y}{2-y} \right| &= Ce^{4t} \\ y(t) &= \frac{2Ce^{4t}}{1 + Ce^{4t}} \blacksquare\end{aligned}$$

**Problem 4.5**

Solve the IVP

$$y' = \frac{4 \sin(2t)}{y}, \quad y(0) = 1.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}yy' &= 4 \sin(2t) \\ (y^2)' &= 8 \sin(2t) \\ \int (y^2)' dt &= \int 8 \sin(2t) dt \\ y^2 &= -4 \cos(2t) + C \\ y(t) &= \pm \sqrt{C - 4 \cos(2t)}.\end{aligned}$$

Since  $y(0) = 1$ , we find  $C = 5$  and hence

$$y(t) = \sqrt{5 - 4 \cos(2t)} \blacksquare$$

**Problem 4.6**

Solve the IVP:

$$yy' = \sin t, \quad y\left(\frac{\pi}{2}\right) = -2.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\int \left(\frac{y^2}{2}\right)' dt &= \int \sin t dt \\ \frac{y^2}{2} &= -\cos t + C \\ y^2 &= -2\cos t + C.\end{aligned}$$

Since  $y(\frac{\pi}{2}) = -2$ , we find  $C = 4$ . Thus,  $y(t) = \pm\sqrt{(-2\cos t + 4)}$ . From  $y(\frac{\pi}{2}) = -2$ , we have

$$y(t) = -\sqrt{(-2\cos t + 4)} \blacksquare$$

**Problem 4.7**

Solve the IVP:

$$y' + y + 1 = 0, \quad y(1) = 0.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}(\ln(y+1))' &= -1 \\ \ln(y+1) &= -t + C \\ y+1 &= Ce^{-t} \\ y(t) &= Ce^{-t} - 1.\end{aligned}$$

Since  $y(1) = 0$ , we find  $C = e$ . Thus,  $y(t) = e^{1-t} - 1 \blacksquare$

**Problem 4.8**

Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\int y' y^{-3} dt &= \int t dt \\ \int \left( \frac{y^{-2}}{-2} \right)' dt &= \frac{t^2}{2} + C \\ -\frac{1}{2y^2} &= \frac{t^2}{2} + C \\ y^2 &= \frac{1}{-t^2 + C}.\end{aligned}$$

Since  $y(0) = 2$ , we find  $C = \frac{1}{4}$ . Thus,  $y(t) = \pm \sqrt{\frac{4}{-4t^2+1}}$ . Since  $y(0) = 2$ , we have  $y(t) = \frac{2}{\sqrt{-4t^2+1}}$  ■

**Problem 4.9**

Solve the IVP:

$$y' = 1 + y^2, \quad y\left(\frac{\pi}{4}\right) = -1.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\frac{y'}{1+y^2} &= 1 \\ \arctan y &= t + C \\ y(t) &= \tan(t + C).\end{aligned}$$

Since  $y\left(\frac{\pi}{4}\right) = -1$ , we find  $C = \frac{\pi}{2}$ . Hence,  $y(t) = \tan\left(t + \frac{\pi}{2}\right) = -\cot t$  ■

**Problem 4.10**

Solve the IVP:

$$y' = t - ty^2, \quad y(0) = \frac{1}{2}.$$

**Solution.**

Separating the variables and solving we find

$$\begin{aligned}\frac{y'}{y^2 - 1} &= -t \\ \frac{y'}{y - 1} - \frac{y'}{y + 1} &= -2t \\ \ln \left| \frac{y - 1}{y + 1} \right| &= -t^2 + C \\ \frac{y - 1}{y + 1} &= Ce^{-t^2} \\ y(t) &= \frac{1 + Ce^{-t^2}}{1 - Ce^{-t^2}}.\end{aligned}$$

Since  $y(0) = \frac{1}{2}$ , we find  $C = -\frac{1}{3}$ . Thus,

$$y(t) = \frac{3 - e^{-t^2}}{3 + e^{-t^2}} \blacksquare$$

**Problem 4.11**

Solve the equation  $3u_y + u_{xy} = 0$  by using the substitution  $v = u_y$ .

**Solution.**

Letting  $v = u_y$  we obtain  $3v + v_x = 0$ . Solving this ODE by the method of separation of variables we find

$$\begin{aligned}\frac{v_x}{v} &= -3 \\ \ln |v(x, y)| &= -3x + f(y) \\ v(x, y) &= f(y)e^{-3x}.\end{aligned}$$

Hence,  $u(x, y) = \int f(y)e^{-3x}dy = F(y)e^{-3x} + G(x)$  where  $F(y) = \int f(y)dy$  ■

**Problem 4.12**

Solve the IVP

$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0.$$

**Solution.**

Separating the variables and solving we find

$$f(y)e^{-3x} \int (2y - \sin y)y' dt = \int (\sin t - t) dt \quad (4.1)$$

$$f(y)e^{-3x}y^2 + \cos y = -\cos t - \frac{t^2}{2} + C. \quad (4.2)$$

Since  $y(0) = 0$ , we find  $C = 2$ . Thus,

$$y^2 + \cos y + \cos t + \frac{t^2}{2} = 2 \blacksquare$$

**Problem 4.13**

State an initial value problem, with initial condition imposed at  $t_0 = 2$ , having implicit solution  $y^3 + t^2 + \sin y = 4$ .

**Solution.**

Differentiating both sides of the given equation we find

$$3y^2y' + \cos y + 2t = 0, \quad y(2) = 0 \blacksquare$$

**Problem 4.14**

Can the differential equation

$$\frac{dy}{dx} = x^2 - xy$$

be solved by the method of separation of variables? Explain.

**Solution.**

If we try to factor the right side of the ODE, we get

$$\frac{dy}{dx} = x(x - y).$$

The second factor is a function of both  $x$  and  $y$ . The ODE is not separable  $\blacksquare$

## Solutions to Section 5

### Problem 5.1

Classify each of the following PDE as linear, quasi-linear, semi-linear, or non-linear.

- (a)  $xu_x + yu_y = \sin(xy)$ .  
 (b)  $u_t + uu_x = 0$   
 (c)  $u_x^2 + u^3u_y^4 = 0$ .  
 (d)  $(x+3)u_x + xy^2u_y = u^3$

### Solution.

(a) Linear (b) Quasi-linear, non-linear (c) Non-linear (d) Semi-linear, non-linear ■

### Problem 5.2

Show that  $u(x, y) = e^x f(2x - y)$ , where  $f$  is a differentiable function of one variable, is a solution to the equation

$$u_x + 2u_y - u = 0.$$

### Solution.

Let  $w = 2x - y$ . Then  $u_x + 2u_y - u = e^x f(w) + 2e^x f_w(w) - 2e^x f_w(w) - e^x f(w) = 0$  ■

### Problem 5.3

Show that  $u(x, y) = x\sqrt{xy}$  satisfies the equation

$$xu_x - yu_y = u$$

subject to the constraint

$$u(y, y) = y^2, \quad y \geq 0.$$

### Solution.

We have  $xu_x - yu_y = x \left( \frac{3}{2}x^{\frac{1}{2}}y^{\frac{1}{2}} \right) - y \left( \frac{1}{2}x^{\frac{3}{2}}y^{-\frac{1}{2}} \right) = x\sqrt{xy} = u$ . Also,  $u(y, y) = y^2$  ■

### Problem 5.4

Show that  $u(x, y) = \cos(x^2 + y^2)$  satisfies the equation

$$-yu_x + xu_y = 0$$

subject to the constraint

$$u(0, y) = \cos y^2.$$

**Solution.**

We have  $-yu_x + xu_y = -2xy \sin(x^2 + y^2) + 2xy \sin(x^2 + y^2) = 0$ . Moreover,  $u(0, y) = \cos y^2$  ■

**Problem 5.5**

Show that  $u(x, y) = y - \frac{1}{2}(x^2 - y^2)$  satisfies the equation

$$\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{y}$$

subject to  $u(x, 1) = \frac{1}{2}(3 - x^2)$ .

**Solution.**

We have  $\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{x}(-x) + \frac{1}{y}(1 + y) = \frac{1}{y}$ . Moreover,  $u(x, 1) = \frac{1}{2}(3 - x^2)$  ■

**Problem 5.6**

Find a relationship between  $a$  and  $b$  if  $u(x, y) = f(ax + by)$  is a solution to the equation  $3u_x - 7u_y = 0$  for any differentiable function  $f$  such that  $f'(x) \neq 0$  for all  $x$ .

**Solution.**

Let  $v = ax + by$ . We have

$$u_x = f_v(v) \frac{d(ax + by)}{dx} = af_v(v)$$

$$u_y = f_v(v) \frac{d(ax + by)}{dy} = bf_v(v).$$

Hence, by substitution we find  $3a - 7b = 0$  ■

**Problem 5.7**

Reduce the partial differential equation

$$au_x + bu_y + cu = 0$$

to a first order ODE by introducing the change of variables  $s = ax + by$  and  $t = bx - ay$ .



**Solution.**

By the chain rule we find

$$u_x = u_s s_x + u_t t_x = a u_s + b u_t$$

$$u_y = u_s s_y + u_t t_y = b u_s - a u_t.$$

Thus,

$$0 = a u_x + b u_y + c u = (a^2 + b^2) u_s + c u$$

or

$$u_s + \frac{c u}{a^2 + b^2} = 0.$$

This is a first order linear ODE that can be solved using the method of separation of variables ■

**Problem 5.8**

Solve the partial differential equation

$$u_x + u_y = 1$$

by introducing the change of variables  $s = x + y$  and  $t = x - y$ .

**Solution.**

Using the chain rule we find

$$u_x = u_s s_x + u_t t_x = u_s + u_t$$

$$u_y = u_s s_y + u_t t_y = u_s - u_t$$

Substituting these into the PDE to obtain  $u_s = \frac{1}{2}$ . Solving this ODE we find  $u(s, t) = \frac{1}{2}s + f(t)$  where  $f$  is an arbitrary differentiable function in one variable. Now substituting for  $s$  and  $t$  we find  $u(x, y) = \frac{1}{2}(x + y) + f(x - y)$  ■

**Problem 5.9**

Show that  $u(x, y) = e^{-4x} f(2x - 3y)$  is a solution to the first-order PDE

$$3u_x + 2u_y + 12u = 0.$$

**Solution.**

We have

$$\begin{aligned}u_x &= -4e^{-4x}f(2x-3y) + 2e^{-4x}f'(2x-3y) \\u_y &= -3e^{-4x}f'(2x-3y)\end{aligned}$$

Thus,

$$\begin{aligned}3u_x + 2u_y + 12u &= -12e^{-4x}f(2x-3y) + 6e^{-4x}f'(2x-3y) \\&\quad -6e^{-4x}f'(2x-3y) + 12e^{-4x}f(2x-3y) = 0 \blacksquare\end{aligned}$$

**Problem 5.10**

Derive the general solution of the PDE

$$au_t + bu_x = u, \quad a, b \neq 0$$

by using the change of variables  $v = ax - bt$  and  $w = \frac{1}{a}t$ .

**Solution.**

We have  $u_t = -bu_v + \frac{1}{a}u_w$  and  $u_x = au_v$ . Substituting we find  $u_w = u$  and solving this equation we find  $u(v, w) = f(v)e^w$  where  $f$  is an arbitrary differentiable function in one variable. Thus,  $u(x, t) = f(ax - bt)e^{\frac{t}{a}}$  ■

**Problem 5.11**

Derive the general solution of the PDE

$$au_x + bu_y = 0, \quad a, b \neq 0$$

by using the change of variables  $s(x, y) = ax + by$  and  $t(x, y) = bx - ay$ . Assume  $a^2 + b^2 > 0$ .

**Solution.**

According to the chain rule for the derivative of a composite function, we have

$$\begin{aligned}u_x &= u_s s_x + u_t t_x = au_s + bu_t \\u_y &= u_s s_y + u_t t_y = bu_s - au_t\end{aligned}$$

Substituting these into the given equation to obtain

$$a^2 u_s + abu_t + b^2 u_s - abu_t = 0$$

or

$$(a^2 + b^2)u_s = 0$$

and since  $a^2 + b^2 > 0$  we obtain

$$u_s = 0.$$

Solving this equation, we find

$$u(s, t) = f(t)$$

where  $f$  is an arbitrary differentiable function of one variable. Now, in terms of  $x$  and  $y$  we find

$$u(x, y) = f(bx - ay) \blacksquare$$

**Problem 5.12**

Write the equation

$$u_t + cu_x + \lambda u = f(x, t)$$

in the coordinates  $v = x - ct$ ,  $w = t$ .

**Solution.**

Using the chain rule, we find

$$u_t = u_v v_t + u_w w_t = -cu_v + u_w$$

$$u_x = u_v v_x + u_w w_x = u_v$$

Substituting these into the original equation we obtain the equation

$$u_w + \lambda u = f(v + cw, w) \blacksquare$$

**Problem 5.13**

Suppose that  $u(x, t) = w(x - ct)$  is a solution to the PDE

$$xu_x + tu_t = Au$$

where  $A$  and  $c$  are constants. Let  $v = x - ct$ . Write the differential equation with unknown function  $w(v)$ .

**Solution.**

Using the chain rule we find

$$u_t = -cw_v$$

and

$$u_x = w_v.$$

Substitution into the original PDE gives

$$vw_v(v) = Aw(v) \blacksquare$$

## Solutions to Section 7

### Problem 7.1

Solve  $u_x + yu_y = y^2$  with the initial condition  $u(0, y) = \sin y$ .

#### Solution.

We have  $a = 1$ ,  $b = y$ , and  $f = y^2$ . Solving  $\frac{dy}{dx} = y$  we find  $y = k_1 e^x$ . Solving  $\frac{du}{dx} = y^2 = k_1^2 e^{2x}$  we find  $u = \frac{k_1^2}{2} e^{2x} + f(k_1) = \frac{1}{2} y^2 + f(k_1) = \frac{1}{2} y^2 + f(ye^{-x})$ . Using the initial condition  $u(0, y) = \sin y$  we find  $\sin y - \frac{1}{2} y^2 = f(y)$ . Hence,  $u(x, y) = \frac{1}{2} y^2 - \frac{1}{2} y^2 e^{-2x} e^{2x} + \sin(ye^{-x})$  ■

### Problem 7.2

Solve  $u_x + yu_y = u^2$  with the initial condition  $u(0, y) = \sin y$ .

#### Solution.

We have  $a = 1$ ,  $b = y$ , and  $f = u^2$ . Solving  $\frac{dy}{dx} = y$  we find  $y = k_1 e^x$ . Solving  $\frac{du}{dx} = u^2$  we find  $x + \frac{1}{u} = k_2$ . Thus,  $u(x, y) = \frac{1}{f(ye^{-x}) - x}$ . Using the initial condition  $u(0, y) = \sin y$  we find  $f(y) = \csc y$ . Hence,  $u(x, y) = \frac{1}{\csc(ye^{-x}) - x}$  ■

### Problem 7.3

Find the general solution of  $yu_x - xu_y = 2xyu$ .

#### Solution.

The system of ODEs is

$$\frac{dy}{dx} = -\frac{x}{y}, \quad \frac{du}{dx} = 2xu.$$

Solving the first equation, we find  $x^2 + y^2 = k_1$ . Solving the second equation, we find  $u = k_2 e^{x^2}$ . Hence,  $u(x, y) = e^{x^2} f(x^2 + y^2)$  where  $f$  is an arbitrary differentiable function in one variable ■

### Problem 7.4

Find the integral surface of the IVP:  $xu_x + yu_y = u$ ,  $u(x, 1) = 2 + e^{-|x|}$ .

#### Solution.

The system of ODEs is

$$\frac{dy}{dx} = \frac{y}{x}, \quad \frac{du}{dx} = \frac{u}{x}.$$

Solving the first equation, we find  $y = k_1x$ . Solving the second equation, we find  $u = k_2x$ . Hence,  $u(x, y) = xf\left(\frac{y}{x}\right)$  where  $f$  is an arbitrary differentiable function in one variable. From the initial condition  $u(x, 1) = 2 + e^{-|x|}$  we find  $f(x) = x(2 + e^{-\frac{1}{|x|}})$ . Hence, the integral surface is

$$u(x, y) = y(2 + e^{-|\frac{x}{y}|}) \blacksquare$$

### Problem 7.5

Find the unique solution to  $4u_x + u_y = u^2$ ,  $u(x, 0) = \frac{1}{1+x^2}$ .

#### Solution.

The system of ODEs can be written as

$$\frac{dx}{4} = \frac{dy}{1} = \frac{du}{u^2}.$$

Solving the equation  $\frac{dx}{4} = \frac{dy}{1}$  we find  $x - 4y = k_1$ . Solving the equation  $\frac{dy}{1} = \frac{du}{u^2}$  we find  $u(x, y) = \frac{1}{f(x-4y)-y}$ . Using the initial condition  $u(x, 0) = \frac{1}{1+x^2}$  we find  $f(x) = 1 + x^2$ . Hence,  $u(x, y) = \frac{1}{(x-4y)^2+1-y}$  ■

### Problem 7.6

Find the unique solution to  $e^{2y}u_x + xu_y = xu^2$ ,  $u(x, 0) = e^{x^2}$ .

#### Solution.

The system of ODEs can be written as

$$\frac{dx}{e^{2y}} = \frac{dy}{x} = \frac{du}{xu^2}.$$

Thus,  $x dx = e^{2y} dy$  which implies  $x^2 - e^{2y} = k_1$ . Solving the equation  $\frac{du}{u^2} = \frac{dy}{x}$  we find  $y + \frac{1}{u} = k_2 = f(x^2 - e^{2y})$ . Hence,

$$u(x, y) = \frac{1}{f(x^2 - e^{2y}) - y}.$$

Using the initial condition  $u(x, 0) = e^{x^2}$  we find  $f(x) = e^{-(x+1)}$ . Hence,

$$u(x, y) = \frac{1}{e^{-x^2+e^{2y}-1} - y} \blacksquare$$

**Problem 7.7**

Find the unique solution to  $xu_x + u_y = 3x - u$ ,  $u(x, 0) = \tan^{-1} x$ .

**Solution.**

The system of ODEs can be written as

$$\frac{dx}{x} = \frac{dy}{1} = \frac{du}{3x - u}.$$

Solving the equation  $\frac{dx}{x} = \frac{dy}{1}$  we find  $xe^{-y} = k_1$ . On the other hand, we have

$$\frac{dx}{x} = \frac{du}{3x - u} = \frac{d(3x - u)}{u} \implies (3x - u)d(3x - u) = udu.$$

Thus,  $(3x - u)^2 - u^2 = k_2 = f(xe^{-y})$  which leads to

$$u(x, y) = \frac{3}{2}x - \frac{1}{6x}f(xe^{-y}).$$

Using the initial condition,  $u(x, 0) = \tan^{-1} x$  we find  $f(x) = 9x^2 - 6x \tan^{-1} x$ . Hence,

$$u(x, y) = \frac{3}{2}x - \frac{9x^2e^{-2y} - 6xe^{-y} \tan^{-1}(xe^{-y})}{6x} = \frac{3}{2}x - \frac{3}{2}xe^{-2y} + e^{-y} \tan^{-1}(xe^{-y}) \blacksquare$$

**Problem 7.8**

Solve:  $xu_x - yu_y = 0$ ,  $u(x, x) = x^4$ .

**Solution.**

Solving the equation  $\frac{dy}{dx} = -\frac{y}{x}$  we find  $xy = k_1$ . Since the right-hand side is 0,  $u(x, y) = k_2 = f(k_1) = f(xy)$ . But  $u(x, x) = x^4 = f(x^2)$ . Hence,  $f(x) = x^2$ ,  $x \geq 0$ . Hence,

$$u(x, y) = x^2y^2, \quad xy \geq 0 \blacksquare$$

**Problem 7.9**

Find the general solution of  $yu_x - 3x^2yu_y = 3x^2u$ .

**Solution.**

Solving the equation  $\frac{dy}{dx} = -3x^2$  we find  $y + x^3 = k_1$ . Solving the equation  $\frac{du}{u} = -\frac{dy}{y}$  we find  $uy = k_2 = f(k_1) = f(y + x^3)$  where  $f$  is a differentiable function in one variable  $\blacksquare$

**Problem 7.10**

Find  $u(x, y)$  that satisfies  $yu_x + xu_y = 4xy^3$  subject to the boundary conditions  $u(x, 0) = -x^4$  and  $u(0, y) = 0$ .

**Solution.**

Solving the equation  $\frac{dy}{dx} = \frac{x}{y}$  we find  $y^2 - x^2 = k_1$ . On the other hand,  $du = 4y^3 dy$  so that  $u(x, y) = y^4 + f(y^2 - x^2)$ . Since  $u(x, 0) = -x^4$ , we have  $f(-x^2) = -x^4$  or  $f(x) = -x^2$  for  $x \leq 0$ . Since  $u(0, y) = 0$  we find  $f(y^2) = -y^4$  so that  $f(y) = -y^2$  for  $y \geq 0$ . Hence,  $f(x) = -x^2$  for all  $x$ . Finally,

$$u(x, y) = y^4 - (y^2 - x^2)^2 = 2x^2y^2 - x^4 \blacksquare$$



## Solutions to Section 8

### Problem 8.1

Find the solution to  $u_t + 3u_x = 0$ ,  $u(x, 0) = \sin x$ .

#### Solution.

Solving  $\frac{dt}{dx} = \frac{1}{3}$  we find  $x - 3t = k_1$ . Solving the equation  $\frac{du}{dx} = 0$  we find  $u(x, t) = k_2 = f(x - 3t)$  where  $f$  is a differentiable function in one variable. Since  $u(x, 0) = \sin x$ , we find  $\sin x = f(x)$ . Hence,  $u(x, t) = \sin(x - 3t)$  ■

### Problem 8.2

Solve the equation  $au_x + bu_y + cu = 0$ .

#### Solution.

Solving the equation  $\frac{dy}{dx} = \frac{b}{a}$  we find  $bx - ay = k_1$ . Solving the equation  $\frac{du}{dx} = -\frac{c}{a}u$  we find  $u(x, y) = k_2 e^{-\frac{c}{a}x} = f(bx - ay)e^{-\frac{c}{a}x}$  where  $f$  is a differentiable function in one variable ■

### Problem 8.3

Solve the equation  $u_x + 2u_y = \cos(y - 2x)$  with the initial condition  $u(0, y) = f(y)$  where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a given function.

#### Solution.

Solving the equation  $\frac{dy}{dx} = 2$  we find  $2x - y = k_1$ . Solving the equation  $\frac{du}{dx} = \cos(y - 2x)$  we find

$$u(x, y) = x \cos(y - 2x) + k_2 = x \cos(y - 2x) + g(2x - y)$$

where  $g$  is a differentiable function in one variable.

Since  $u(0, y) = f(y)$ , we obtain  $f(y) = g(-y)$  or  $g(y) = f(-y)$ . Thus,

$$u(x, y) = x \cos(y - 2x) + f(y - 2x) \blacksquare$$

### Problem 8.4

Show that the initial value problem  $u_t + u_x = x$ ,  $u(x, x) = 1$  has no solution.

#### Solution.

Solving the equation  $\frac{dy}{dx} = 1$  we find  $x - y = k_1$ . Solving the equation  $\frac{du}{dx} = x$  we find  $u(x, y) = \frac{1}{2}x^2 + f(x - y)$  where  $f$  is a differentiable function of one variable. Since  $u(x, x) = 1$  we find  $1 = \frac{1}{2}x^2 + f(0)$  or  $f(0) = 1 - \frac{x^2}{2}$  which is impossible since  $f(0)$  is a constant. Hence, the given initial value problem has no solution ■

**Problem 8.5**

Solve the transport equation  $u_t + 2u_x = -3u$  with initial condition  $u(x, 0) = \frac{1}{1+x^2}$ .

**Solution.**

Solving the equation  $\frac{dt}{dx} = \frac{1}{2}$  we find  $x - 2t = k_1$ . Solving the equation  $\frac{du}{dx} = -\frac{3}{2}u$  we find  $u(x, t) = f(x - 2t)e^{-\frac{3}{2}x}$ . Since  $u(x, 0) = \frac{1}{1+x^2}$  we find  $f(x) = \frac{e^{\frac{3}{2}x}}{1+x^2}$ . Hence,

$$u(x, t) = \frac{e^{-3t}}{1 + (x - 2t)^2} \blacksquare$$

**Problem 8.6**

Solve  $u_t + u_x - 3u = t$  with initial condition  $u(x, 0) = x^2$ .

**Solution.**

Solving the equation  $\frac{dt}{dx} = 1$  we find  $x - t = k_1$ . Solving the equation  $\frac{du}{dx} = 3u + t = 3u + x + k_1$  by the method of integrating factor, we find

$$u(x, t) = -\frac{1}{3}t - \frac{1}{9} + f(x - t)e^{3x}.$$

But  $u(x, 0) = x^2$  which leads to  $f(x) = e^{-3x} \left(x^2 + \frac{1}{9}\right)$ . Hence,

$$u(x, t) = e^{3t} \left[ (x - t)^2 + \frac{1}{9} \right] - \frac{1}{3}t - \frac{1}{9} \blacksquare$$

**Problem 8.7**

Show that the decay term  $\lambda u$  in the transport equation with decay

$$u_t + cu_x + \lambda u = 0$$

can be eliminated by the substitution  $w = ue^{\lambda t}$ .

**Solution.**

Using the chain rule we find  $w_t = u_t e^{\lambda t} + \lambda u e^{\lambda t}$  and  $w_x = u_x e^{\lambda t}$ . Substituting these equations into the original equation we find

$$w_t e^{-\lambda t} - \lambda u + cw_x e^{-\lambda t} + \lambda u = 0$$

or

$$w_t + cw_x = 0 \blacksquare$$

**Problem 8.8** (*Well-Posed*)

Let  $u$  be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x)$$

and  $v$  be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = g(x)$$

where  $f$  and  $g$  are continuously differentiable functions.

(a) Show that  $w(x, t) = u(x, t) - v(x, t)$  is the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x) - g(x)$$

(b) Write an explicit formula for  $w$  in terms of  $f$  and  $g$ .

(c) Use (b) to conclude that the transport problem is well-posed. That is, a small change in the initial data leads to a small change in the solution.

**Solution.**

(a)  $w(x, t)$  is a solution to the equation follows from the principle of superposition. Moreover,  $w(x, 0) = u(x, 0) - v(x, 0) = f(x) - g(x)$ .

(b)  $w(x, t) = f(x - ct) - g(x - ct)$ .

(c) From (b) we see that

$$\sup_{x,t} \{|u(x, t) - v(x, t)|\} = \sup_x \{|f(x) - g(x)|\}.$$

Thus, small changes in the initial data produces small changes in the solution.

Hence, the problem is a well-posed problem ■

**Problem 8.9**

Solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \quad x > 0, \quad t > 0$$

$$u(x, 0) = 0, \quad u(0, t) = g(t), \quad t > 0.$$

**Solution.**

Solving  $\frac{dt}{dx} = \frac{1}{c}$  we find  $x - ct = k_1$ . Solving the equation  $\frac{du}{dx} = -\frac{\lambda}{c}u$  we find  $u(x, t) = f(x - ct)e^{-\frac{\lambda}{c}x}$ . From the condition  $u(0, t) = g(t)$  we find  $f(-ct) = g(t)$  or  $f(t) = g\left(-\frac{t}{c}\right)$ . Thus,

$$u(x, t) = g\left(t - \frac{x}{c}\right) e^{-\frac{\lambda}{c}x}.$$

This is valid only for  $x < ct$  since  $g$  is defined on  $(0, \infty)$ . Also, this expression will not satisfy  $u(x, 0) = 0$ . So we define  $u(x, t) = 0$  for  $x \geq ct$ . That is, the solution to the initial boundary value problem is

$$u(x, t) = \begin{cases} g\left(t - \frac{x}{c}\right) e^{-\frac{\lambda}{c}x} & \text{if } x < ct \\ 0 & \text{if } x \geq ct \blacksquare \end{cases}$$

**Problem 8.10**

Solve the first-order equation  $2u_t + 3u_x = 0$  with the initial condition  $u(x, 0) = \sin x$ .

**Solution.**

Solving the equation  $\frac{dt}{dx} = \frac{2}{3}$  we find  $2x - 3t = k_1$ . Solving the equation  $\frac{du}{dx} = 0$  we find  $u(x, t) = k_2 = f(2x - 3t)$  where  $f$  is an arbitrary differentiable function. Using the initial condition we find  $f(2x) = \sin x$  or  $f(x) = \sin\left(\frac{x}{2}\right)$ . The final answer is  $u(x, t) = \sin\left(\frac{2x-3t}{2}\right)$  ■

**Problem 8.11**

Solve the PDE  $u_x + u_y = 1$ .

**Solution.**

Solving the equation  $\frac{dy}{dx} = 1$  we find  $x - y = k_1$ . Solving the equation  $\frac{du}{dx} = 1$  we find  $u(x, y) = x + f(x - y)$  where  $f$  is an arbitrary differentiable function in one variable ■

## Solutions to Section 9

### Problem 9.1

Find the general solution of the PDE  $\ln(y+u)u_x + u_y = -1$ .

#### Solution.

The characteristic equations are  $\frac{dx}{\ln(y+u)} = \frac{dy}{1} = \frac{du}{-1}$ . Using the second and third fractions we find that  $y+u = k_1$ . Now, from the first and second fractions we have  $\frac{dx}{\ln k_1} = \frac{dy}{1}$  so that  $x+k_2 = y \ln k_1$ . Hence,  $y \ln(y+u) - x = k_2$ . Hence, the general solution is given by  $u = -y + f(y \ln(y+u) - x)$  where  $f$  is an arbitrary differentiable function ■

### Problem 9.2

Find the general solution of the PDE  $x(y-u)u_x + y(u-x)u_y = u(x-y)$ .

#### Solution.

The characteristic equations are given by  $\frac{dx}{x(y-u)} = \frac{dy}{y(u-x)} = \frac{du}{u(x-y)}$ . We have

$$\frac{dx + dy + du}{x(y-u) + y(u-x) + u(x-y)} = \frac{du}{u(x-y)}$$

or

$$\frac{d(x+y+z)}{0} = \frac{du}{u(x-y)}.$$

Hence,  $x+y+z = k_1$ . On the other hand we have

$$\frac{\frac{dx}{x} + \frac{dy}{y}}{-(x-y)} = \frac{\frac{du}{u}}{x-y}.$$

This implies that

$$\frac{dx}{x} + \frac{dy}{y} = -\frac{du}{u}$$

or

$$\ln xyu = k$$

that is  $xyu = k_2$ . Hence, the general solution is given by  $u = \frac{f(x+y+z)}{xy}$  where  $f$  is an arbitrary differentiable function ■

### Problem 9.3

Find the general solution of the PDE  $u(u^2 + xy)(xu_x - yu_y) = x^4$ .

**Solution.**

The characteristic equations are  $\frac{dx}{ux(u^2+xy)} = -\frac{dy}{yu(u^2+xy)} = \frac{du}{x^4}$ . From the first and second fractions we get  $\frac{dx}{x} = -\frac{dy}{y}$ . Upon integration we find  $xy = k_1$ . From first and third fractions we get  $x^3 dx = (u^3 + uxy)du$  or  $x^3 dx = (u^3 + k_1 u)du$ . Integration leads to  $\frac{x^4}{4} = \frac{u^4}{4} + \frac{k_1}{2}u^2 + k_2$  or  $x^4 - u^4 - 2k_1 u^2 = k_2$ . Substituting for  $k_1$  we find  $x^4 - u^4 - 2xyu^2 = k_2$ . Hence, the general solution is given by  $x^4 - u^4 - 2xyu^2 = f(xy)$  where  $f$  is an arbitrary differentiable function ■

**Problem 9.4**

Find the general solution of the PDE  $(y + xu)u_x - (x + yu)u_y = x^2 - y^2$ .

**Solution.**

The characteristic equations are  $\frac{dx}{y+xu} = -\frac{dy}{x+yu} = \frac{du}{x^2-y^2}$ . We have

$$\frac{xdx + ydy - udu}{xy + x^2u - xy - y^2u - ux^2 + uy^2} = \frac{du}{x^2 - y^2}.$$

Thus,  $xdx + ydy - udu = 0$  or  $x^2 + y^2 - u^2 = k_1$ . On the other hand,

$$\frac{ydx + xdy + udu}{y^2 + xyu - x^2 - xyu + x^2 - y^2} = \frac{du}{x^2 - y^2}.$$

That is,  $ydx + xdy + udu = 0$ . Hence,  $2xy + u^2 = k_2$ . The general solution is  $2xy + u^2 = f(x^2 + y^2 - u^2)$  where  $f$  is an arbitrary differentiable function ■

**Problem 9.5**

Find the general solution of the PDE  $(y^2 + u^2)u_x - xyu_y + xu = 0$ .

**Solution.**

The characteristic equations are  $\frac{dx}{y^2+u^2} = \frac{dy}{-xy} = \frac{du}{-xu}$ . Using the last two fractions we find  $\frac{dy}{y} = \frac{du}{u}$  which leads to  $\frac{y}{u} = k_1$ . On the other hand, we have

$$\frac{xdx + ydy + udu}{xy^2 + xu^2 - xy^2 - xu^2} = \frac{du}{-xu}.$$

Thus,  $xdx + ydy + udu = 0$  or  $x^2 + y^2 + u^2 = k_2$ . The general solution is  $x^2 + y^2 + u^2 = f\left(\frac{y}{u}\right)$  where  $f$  is an arbitrary differentiable function ■

**Problem 9.6**

Find the general solution of the PDE  $u_t + uu_x = x$ .

**Solution.**

The characteristic equations are

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{x}.$$

Solving  $\frac{dx}{u} = \frac{du}{x}$  we find  $u^2 - x^2 = k_1$ . Solving  $\frac{dt}{1} = \frac{d(x+u)}{x+u}$  we find  $x + u = k_2 e^t = e^t f(u^2 - x^2)$  where  $f$  is an arbitrary differentiable function ■

**Problem 9.7**

Find the general solution of the PDE  $(y - u)u_x + (u - x)u_y = x - y$ .

**Solution.**

The characteristic equations are

$$\frac{dx}{y - u} = \frac{dy}{u - x} = \frac{du}{x - y}.$$

We have

$$\frac{dx + dy + du}{y - u + u - x + x - y} = \frac{dx + dy + du}{0}$$

so that  $dx + dy + du = 0$ . Hence,  $x + y + u = k_1$ . Likewise,

$$\frac{xdx + ydy + udu}{x(y - u) + y(u - x) + u(x - y)} = \frac{xdx + ydy + udu}{0}$$

so that  $xdx + ydy + udu = 0$ . Hence,  $x^2 + y^2 + u^2 = k_2$ . The general solution is given by

$$x^2 + y^2 + u^2 = f(x + y + u)$$

where  $f$  is an arbitrary differentiable function ■

**Problem 9.8**

Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u.$$

**Solution.**

The characteristic equations are

$$\frac{dx}{x(y^2 + u)} = \frac{dy}{-y(x^2 + u)} = \frac{du}{(x^2 - y^2)u}.$$

We first note that

$$\frac{\frac{dx}{x}}{y^2 + u} = \frac{\frac{dy}{y}}{-(x^2 + u)} = \frac{\frac{du}{u}}{x^2 - y^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{du}{u}}{0}.$$

Thus,

$$\frac{dx}{x} + \frac{dy}{y} + \frac{du}{u} = 0$$

which gives  $xyu = k_1$ . Likewise, we have

$$\frac{xdx + ydy - du}{x^2(y^2 + u) - y^2(x^2 + u) - (x^2 - y^2)u} = \frac{xdx + ydy - du}{0}.$$

Thus,  $xdx + ydy - du = 0$  and this implies that  $x^2 + y^2 - 2u = k_2$ . The general solution is given by

$$x^2 + y^2 - 2u = f(xyu)$$

where  $f$  is an arbitrary differentiable function ■

**Problem 9.9**

Solve

$$\sqrt{1 - x^2}u_x + u_y = 0.$$

**Solution.**

The characteristic equations are

$$\frac{dx}{\sqrt{1 - x^2}} = \frac{dy}{1} = \frac{du}{0}.$$

From the last fraction, we have  $u(x, y) = k_1$ . From the first two fractions, we have  $y = \sin^{-1} x + k_2 = \sin^{-1} x + f(u)$  where  $f$  is a differentiable function ■

**Problem 9.10**

Solve

$$u(x + y)u_x + u(x - y)u_y = x^2 + y^2.$$



**Solution.**

The characteristic equations are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2}.$$

Each of these ratio is equivalent to

$$\frac{ydx + xdy - udu}{0} = \frac{xdx - ydy - udu}{0}$$

or

$$\frac{d(xy - \frac{u^2}{2})}{0} = \frac{\frac{1}{2}(x^2 - y^2 - u^2)}{0}.$$

Hence,

$$\frac{1}{2}(x^2 - y^2 - u^2) = f(xy - \frac{u^2}{2})$$

where  $f$  is a differentiable function ■

## Solutions to Section 10

### Problem 10.1

Solve

$$(y - u)u_x + (u - x)u_y = x - y$$

with the condition  $u(x, \frac{1}{x}) = 0$ .

### Solution.

The characteristic equations are

$$\frac{dx}{y - u} = \frac{dy}{u - x} = \frac{du}{x - y}.$$

We have

$$\frac{dx + dy + du}{y - u + u - x + x - y} = \frac{dx + dy + du}{0}$$

so that  $dx + dy + du = 0$ . Hence,  $x + y + u = c_1$ . Likewise,

$$\frac{xdx + ydy + udu}{x(y - u) + y(u - x) + u(x - y)} = \frac{xdx + ydy + udu}{0}$$

so that  $xdx + ydy + udu = 0$ . Hence,  $x^2 + y^2 + u^2 = c_2$ . The general solution is given by

$$f(x + y + u, x^2 + y^2 + u^2) = 0$$

where  $f$  is an arbitrary differentiable function. Now, using the Cauchy data  $u = 0$  when  $xy = 1$  we find  $c_1^2 = (x + y)^2 = x^2 + y^2 + 2xy = c_2 + 2$ . Hence, the integral surface is described by

$$(x + y + u)^2 = x^2 + y^2 + u^2 + 2$$

and the unique solution is given by

$$u(x, y) = \frac{1 - xy}{x + y}, \quad x + y \neq 0 \blacksquare$$

### Problem 10.2

Solve the linear equation

$$yu_x + xu_y = u,$$

with the Cauchy data  $u(x, 0) = x^3$ .

**Solution.**

The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}.$$

Using the first two fractions we find  $x^2 - y^2 = c_1$ . Now, since

$$\frac{du}{u} = \frac{dx + dy}{x + y}$$

we can write  $u = c_2(x + y)$ . Hence, the general solution is given by

$$f(x^2 - y^2, \frac{u}{x + y}) = 0$$

or

$$u = (x + y)g(x^2 - y^2)$$

where  $f$  and  $g$  are arbitrary differentiable functions. Using the Cauchy data we find  $g(x^2) = x^2$ , that is  $g(x) = x$ . Consequently, the unique solution is given by

$$u(x, y) = (x + y)(x^2 - y^2) \blacksquare$$

**Problem 10.3**

Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the Cauchy data  $u(x, -x) = 1$ .

**Solution.**

The characteristic equations are

$$\frac{dx}{x(y^2 + u)} = \frac{dy}{-y(x^2 + u)} = \frac{du}{(x^2 - y^2)u}.$$

We first note that

$$\frac{\frac{dx}{x}}{y^2 + u} = \frac{\frac{dy}{y}}{-(x^2 + u)} = \frac{\frac{du}{u}}{x^2 - y^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{du}{u}}{0}.$$

Thus,

$$\frac{dx}{x} + \frac{dy}{y} + \frac{du}{u} = 0$$

which gives  $xyu = c_1$ . Likewise, we have

$$\frac{xdx + ydy - du}{x^2(y^2 + u) - y^2(x^2 + u) - (x^2 - y^2)u} = \frac{xdx + ydy - du}{0}.$$

Thus,  $xdx + ydy - du = 0$  and this implies that  $x^2 + y^2 - 2u = c_2$ . The general solution is given by

$$f(xyu, x^2 + y^2 - 2u) = 0$$

where  $f$  is an arbitrary differentiable function. Using the Cauchy data we see that  $f(-x^2, 2x^2 - 2) = 0$  which implies that  $f(x, y) = 2x + y + 2$ . Hence, the unique solution is given by

$$2xyu + x^2 + y^2 - 2u + 2 = 0 \blacksquare$$

#### Problem 10.4

Solve

$$xu_x + yu_y = xe^{-u}$$

with the Cauchy data  $u(x, x^2) = 0$ .

#### Solution.

The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}.$$

Using the first two fractions we find  $\frac{y}{x} = c_1$ . Using the first and the last fractions we find  $dx = e^u du$  or  $x - e^u = c_2$ . Hence, the general solution is given by

$$f\left(\frac{y}{x}, x - e^u\right) = 0$$

where  $f$  is an arbitrary differentiable function. Using the Cauchy data we find  $f(x, x - 1) = 0$  so that  $f(x, y) = -x + y + 1$ . Hence, the unique integral surface is described by

$$-\frac{y}{x} + x - e^u + 1 = 0$$

or

$$u(x, y) = \ln\left(x + 1 - \frac{y}{x}\right) \blacksquare$$

**Problem 10.5**

Solve the initial value problem

$$xu_x + u_y = 0, \quad u(x, 0) = f(x)$$

using the characteristic equations in parametric form.

**Solution.**

The initial curve in parametric form is

$$\Gamma : x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = f(t).$$

Since

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

the initial value problem has a unique solution. The characteristic equations in parametric form are

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = 0.$$

Solving we find

$$x(s, t) = \alpha(t)e^s, \quad y(s, t) = s + \beta(t), \quad u(s, t) = \gamma(t).$$

But

$$x(0, t) = t, \quad y(0, t) = 0, \quad u(0, t) = f(t).$$

Hence,

$$x(s, t) = te^s, \quad y(s, t) = s, \quad u(s, t) = f(t).$$

Now,  $s = y$  and  $t = xe^{-y}$ . Hence,  $u(x, y) = f(xe^{-y})$  ■

**Problem 10.6**

Solve the initial value problem

$$u_t + au_x = 0, \quad u(x, 0) = f(x).$$

**Solution.**

The initial curve parametrization is given by

$$\Gamma : x_0(w) = w, t_0(w) = 0, u_0(w) = f(w).$$

Since

$$a(x_0(w), t_0(w), u_0(w)) \frac{dt_0}{dw}(w) - b(x_0(w), t_0(w), u_0(w)) \frac{dx_0}{dw}(w) = -1 \neq 0$$

the initial value problem has a unique solution. The characteristic curves are solutions to the system

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = a, \quad \frac{du}{ds} = 0.$$

Solving this system we find

$$t(s, w) = s + \alpha(w), \quad x(s, w) = as + \beta(w), \quad u(s, w) = \gamma(w).$$

But  $x(0, w) = w$ ,  $t(0, w) = 0$ , and  $u(0, w) = f(w)$  so that we find

$$x(s, w) = as + w, \quad t(s, w) = s, \quad u(s, w) = f(w).$$

Using the first two equations we find  $s = t$ ,  $w = x - at$ . Hence, the unique solution is given by  $u(t, x) = f(x - at)$  ■

**Problem 10.7**

Solve the initial value problem

$$au_x + u_y = u^2, \quad u(x, 0) = \cos x$$

**Solution.**

The initial curve parametrization is given by

$$\Gamma : x_0(t) = t, y_0(t) = 0, u_0(t) = \cos t.$$

Since

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

the initial value problem has a unique solution. The characteristic curves are solutions to the system

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = u^2.$$

Solving this system we find

$$x(s, t) = as + \alpha(t), \quad y(s, t) = s + \beta(t), \quad u(s, t) = -\frac{1}{s + \gamma(t)}.$$

But  $x(0, t) = t$ ,  $y(0, t) = 0$ , and  $u(0, t) = \cos t$  so that we find

$$x(s, t) = as + t, \quad y(s, t) = s, \quad u(s, t) = \frac{1}{\sec t - s}.$$

The first two equations lead to  $s = y$  and  $t = x - ay$ . Substituting into the third equation we find

$$u(x, y) = \frac{1}{\sec(x - ay) - y} \blacksquare$$

### Problem 10.8

Solve the initial value problem

$$u_x + xu_y = u, \quad u(1, y) = h(y).$$

### Solution.

The initial curve parametrization is given by

$$\Gamma : \quad x_0(t) = 1, \quad y_0(t) = t, \quad u_0(t) = h(t).$$

Since

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = 1 \neq 0$$

the initial value problem has a unique solution. The characteristic curves are solutions to the system

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = x, \quad \frac{du}{ds} = u.$$

Solving the first equation we find  $x(s, t) = s + \alpha(t)$ . Since  $x(0, t) = 1$  we find  $x(s, t) = s + 1$ . Hence, the second equation above becomes  $\frac{dy}{ds} = s + 1$ . Solving we obtain  $y(s, t) = \frac{s^2}{2} + s + \beta(t)$ . Since  $y(0, t) = t$  we find  $y(s, t) = \frac{s^2}{2} + s + t$ . Next, we have  $\frac{du}{ds} = u$  so that  $u(s, t) = \gamma(t)e^s$ . Since  $u(0, t) = h(t)$  we find  $u(s, t) = h(t)e^s$ .

Now we need to solve for  $s$  and  $t$  in terms of  $x$  and  $y$ . In particular,  $x = s + 1$  implies that  $s = x - 1$ . Therefore,  $y = \frac{(x-1)^2}{2} + x - 1 + t$  which implies that  $t = y - \frac{(x-1)^2}{2} - (x - 1)$ . And as a result, we have found the solution

$$u(x, y) = h \left( y - \frac{(x-1)^2}{2} - (x-1) \right) e^{x-1} \blacksquare$$

### Problem 10.9

Solve the initial value problem

$$uu_x + u_y = 0, \quad u(x, 0) = f(x).$$

### Solution.

The initial curve parametrization is given by

$$\Gamma : \quad x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = f(t).$$

Since

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

the initial value problem has a unique solution. The characteristic curves are solutions to the system

$$\frac{dx}{ds} = u, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = 0.$$

Solving the second equation we find  $y(s, t) = s + \beta(t)$ . Since  $y(0, t) = 0$  we find  $y(s, t) = s$ . Solving the last equation we find  $u(s, t) = \gamma(t)$ . But  $u(0, t) = f(t)$  so that  $u(s, t) = f(t)$ . Now,  $\frac{dx}{ds} = f(t)$  so that  $x(s, t) = f(t)s + \alpha(t)$ . Since  $x(0, t) = t$  we conclude that  $x(s, t) = f(t)s + t$ . Solving  $s$  and  $t$  in terms of  $x$  and  $y$  we find  $s = y$  and  $t = x - f(t)s = x - f(t)y = x - uy$ . Thus,  $u(x, y) = f(x - uy)$  so that  $u$  is defined implicitly  $\blacksquare$



**Problem 10.10**

Solve the initial value problem

$$\sqrt{1-x^2}u_x + u_y = 0, \quad u(0, y) = y.$$

**Solution.**

The initial curve parametrization is given by

$$\Gamma : \quad x_0(t) = 0, \quad y_0(t) = t, \quad u_0(t) = t.$$

Since

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = 1 \neq 0$$

the initial value problem has a unique solution. The characteristic curves are solutions to the system

$$\frac{dx}{ds} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = 0.$$

From the last equation, we have  $u(s, t) = \gamma(t)$ . Since  $u(0, t) = t$ , we find  $u(s, t) = t$ . From the second equation, we have  $y(s, t) = s + \beta(t)$ . Since  $y(0, t) = t$ , we find  $y(s, t) = s + t$ . From the first equation, we find  $x(s, t) = \sin(s + \alpha(t))$ . Since  $x(0, t) = 0$ , we find  $x(s, t) = \sin s$ . Solving  $s$  and  $t$  in terms of  $x$  and  $y$ , we find  $s = \arcsin x$  and  $t = y - \arcsin x$ . Hence, the solution to the problem is  $u(x, y) = y - \arcsin x$  ■

**Problem 10.11**

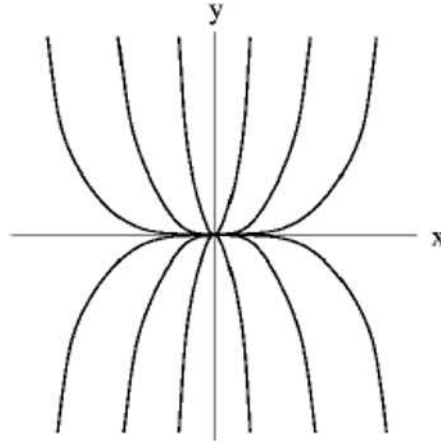
Consider

$$xu_x + 2yu_y = 0.$$

- (i) Find and sketch the characteristics.
- (ii) Find the solution with  $u(1, y) = e^y$ .
- (iii) What happens if you try to find the solution satisfying either  $u(0, y) = g(y)$  or  $u(x, 0) = h(x)$  for given functions  $g$  and  $h$ ?
- (iv) Explain, using your picture of the characteristics, what goes wrong at  $(x, y) = (0, 0)$ .

**Solution.**

(i) The characteristics are solutions to the ODE  $\frac{dy}{dx} = \frac{2y}{x}$ . Solving we find  $y = Cx^2$ . Thus, the characteristics are parabolas in the plane centered at the origin. See figure below.



(ii) The general solution is  $u(x, y) = f(yx^{-2})$ , and so the solution satisfying the condition at  $u(1, y) = e^y$  is

$$u(x, y) = e^{yx^{-2}}.$$

(iii) In the first case, we cannot substitute  $x = 0$  into  $yx^{-2}$  (the argument of the function  $f$ , above) because  $x^{-2}$  is not defined at 0. Similarly, in the second case, we'd need to find a function  $f$  so that  $f(0) = h(x)$ . If  $h$  is not constant, it is not possible to satisfy this condition for all  $x \in \mathbb{R}$ .

(iv) All characteristics intersect at  $(0, 0)$ . Since the solution is constant along any characteristic, if the solution is not exactly constant for all  $(x, y)$ , then the limit of  $u(x, y)$  as  $(x, y) \rightarrow (0, 0)$  is different if we approach  $(0, 0)$  along different characteristics. Therefore, the method doesn't work at that point ■

**Problem 10.12**

Solve the equation  $u_x + u_y = u$  subject to the condition  $u(x, 0) = \cos x$ .

**Solution.**

The initial curve in  $\mathbb{R}^3$  can be given parametrically as

$$\Gamma : x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = \cos t.$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

so by Theorem 11.1 the given Cauchy problem has a unique solution. To find this solution, we solve the system of ODEs

$$\begin{aligned} \frac{dx}{ds} &= 1 \\ \frac{dy}{ds} &= 1 \\ \frac{du}{ds} &= u. \end{aligned}$$

Solving this system we find

$$x(s, t) = s + \alpha(t), \quad y(s, t) = s + \beta(t), \quad u(s, t) = \gamma(t)e^s.$$

But  $x(0, t) = t$  so that  $\alpha(t) = t$ . Similarly,  $y(0, t) = 0$  so that  $\beta(t) = 0$  and  $u(0, t) = \cos t$  implies  $\gamma(t) = \cos t$ . Hence, the unique solution is given parametrically by the equations

$$x(s, t) = t + s, \quad y(s, t) = s, \quad u(s, t) = e^s \cos t.$$

Solving the first two equations for  $s$  and  $t$  we find

$$s = y, \quad t = x - y$$

and substituting these into the third equation we find

$$u(x, y) = e^y \cos(x - y) \blacksquare$$

### Problem 10.13

(a) Find the general solution of the equation

$$u_x + yu_y = u.$$

(b) Find the solution satisfying the Cauchy data  $u(x, 3e^x) = 2$ .

(c) Find the solution satisfying the Cauchy data  $u(x, e^x) = e^x$ .

**Solution.**

(a) The characteristic equations in non-parametric form are

$$\frac{dx}{1} = \frac{dy}{y} = \frac{du}{u}.$$

Using the first two fractions we find  $y = C_1 e^x$ . Using the first and the last fractions we find  $u = C_2 e^x$ . Thus, the general solution is given by

$$f(ye^{-x}, ue^{-x}) = 0$$

or

$$u = e^x f(ye^{-x})$$

where  $f$  is an arbitrary differentiable function.

(b) We want  $2 = u(x, 3x) = e^x f(3e^x e^{-x}) = e^x f(3)$ . This equation is impossible so this Cauchy problem has no solutions.

(c) We want  $e^x = e^x f(e^x e^{-x}) \implies f(1) = 1$ . In this case, there are infinitely many solutions to this Cauchy problem, namely,  $u(x, y) = e^x f(ye^{-x})$  where  $f$  is an arbitrary function satisfying  $f(1) = 1$  ■

**Problem 10.14**

Solve the Cauchy problem

$$u_x + 4u_y = x(u + 1)$$

$$u(x, 5x) = 1.$$

**Solution.**

The characteristic equations are  $\frac{dx}{1} = \frac{dy}{4} = \frac{du}{x(u+1)}$ . Solving we find  $4x - y = C_1$  and  $u + 1 = C_2 e^{\frac{x^2}{2}}$ . Thus, the general solution is given by  $f(4x - y, (u + 1)e^{-\frac{x^2}{2}}) = 0$  or  $u = -1 + e^{\frac{x^2}{2}} f(4x - y)$ .

Using the condition  $u(x, 5x) = 1$  we obtain  $e^{\frac{x^2}{2}} f(-x) = 2$  or  $f(x) = 2e^{-\frac{x^2}{2}}$ . Thus,

$$u(x, y) = -1 + 2e^{\frac{x^2}{2}} e^{-\frac{(4x-y)^2}{2}} \blacksquare$$

**Problem 10.15**

Solve the Cauchy problem

$$u_x - u_y = u$$

$$u(x, -x) = \sin x.$$

**Solution.**

The characteristic equations are  $\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}$ . Solving we find  $x+y = C_1$  and  $u = C_2 e^x$ . Thus, the general solution is  $f(x+y, ue^{-x}) = 0$  or  $u = e^x f(x+y)$ . Using the condition  $u(x, -x) = \sin x$  we find  $f(0) = e^{-x} \sin x$  which is an impossible equation. Hence, the Cauchy problem has no solutions ■

**Problem 10.16**

(a) Find the characteristics of the equation

$$yu_x + xu_y = 0.$$

(b) Sketch some of the characteristics.

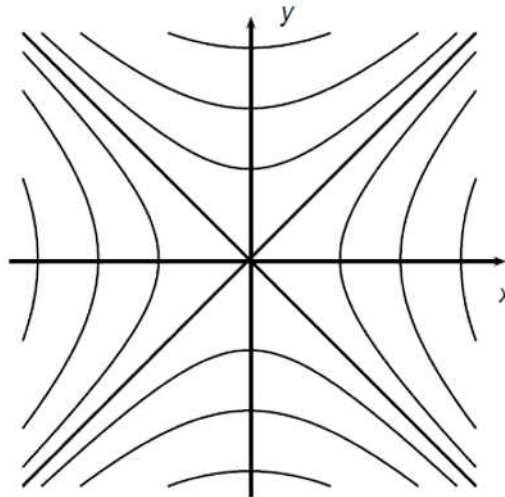
(c) Find the solution satisfying the boundary condition  $u(0, y) = e^{-y^2}$ .

(d) In which region of the plane is the solution uniquely determined?

**Solution.**

(a) The characteristics satisfy the ODE  $\frac{dy}{dx} = \frac{x}{y}$ . Solving this equation we find  $x^2 - y^2 = C$ . Thus, the characteristics are hyperbolas.

(b)



(c) The general solution to the PDE is  $u(x, y) = f(x^2 - y^2)$  where  $f$  is an arbitrary differentiable function. Since  $u(0, y) = e^{-y^2}$  we find  $f(y^2) = e^{-y^2}$ . Hence,  $u(x, y) = e^{x^2 - y^2}$ .

(d) This solution is only defined in the region covered by characteristics that

cross the  $y$  axis:  $y^2 - x^2 > 0$ . The solution in the region  $y^2 - x^2 < 0$  can be any function of the form  $u(x, y) = f(x^2 - y^2)$  ■

**Problem 10.17**

Consider the equation  $u_x + yu_y = 0$ . Is there a solution satisfying the extra condition

- (a)  $u(x, 0) = 1$
- (b)  $u(x, 0) = x$ ?

If yes, give a formula; if no, explain why.

**Solution.**

(a) Solving the ODE  $\frac{dy}{dx} = y$  we find the characteristics  $ye^{-x} = C$ . Thus,  $u(x, y) = f(ye^{-x})$ . If  $u(x, 0) = 1$  then we choose  $f$  to be any arbitrary differentiable function satisfying  $f(0) = 1$ .

(b) The line  $y = 0$  is a characteristic so that  $u$  has to be constant there. Hence, there is no solution satisfying the condition  $u(x, 0) = x$  ■

## Solutions to Section 11

### Problem 11.1

Classify each of the following equation as hyperbolic, parabolic, or elliptic:

- (a) Wave propagation:  $u_{tt} = c^2 u_{xx}$ ,  $c > 0$ .
- (b) Heat conduction:  $u_t = c u_{xx}$ ,  $c > 0$ .
- (c) Laplace's equation:  $\Delta u = u_{xx} + u_{yy} = 0$ .

### Solution.

- (a) We have  $A = 1$ ,  $B = 0$  and  $C = -c^2$  so that  $B^2 - 4AC = 4c^2 > 0$ . Thus, the given equation is of hyperbolic type.
- (b) We have  $A = 0$ ,  $B = 0$  and  $C = c$  so that  $B^2 - 4AC = 0$ . Thus, the given equation is of parabolic type.
- (c) We have  $A = 1$ ,  $B = 0$  and  $C = 1$  so that  $B^2 - 4AC = -4 < 0$ . Thus, the given equation is of elliptic type ■

### Problem 11.2

Classify the following linear scalar PDE with constant coefficients as hyperbolic, parabolic or elliptic.

- (a)  $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$ .
- (b)  $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u = 0$ .
- (c)  $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0$ .

### Solution.

- (a) We have  $A = 1$ ,  $B = 4$  and  $C = 5$  so that  $B^2 - 4AC = 16 - 20 = -4 < 0$ . Thus, the given equation is of elliptic type.
- (b) We have  $A = 1$ ,  $B = -4$  and  $C = 4$  so that  $B^2 - 4AC = 16 - 16 = 0$ . Thus, the given equation is of parabolic type.
- (c) We have  $A = 1$ ,  $B = 2$  and  $C = -3$  so that  $B^2 - 4AC = 4 + 12 = 16 > 0$ . Thus, the given equation is of hyperbolic type ■

### Problem 11.3

Find the region(s) in the  $xy$ -plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch these regions.

**Solution.**

We have  $A = 1 + x$ ,  $B = 2xy$ , and  $C = -y^2$  so that  $B^2 - 4AC = 4x^2y^2 + 4y^2(1 + x) = 4y^2(x^2 + x + 1)$ .

- The PDE is of hyperbolic type if  $4y^2(x^2 + x + 1) > 0$ . This is true for all  $y \neq 0$ . Graphically, this is the  $xy$ -plane with the  $x$ -axis removed.
- The PDE is of parabolic type if  $4y^2(x^2 + x + 1) = 0$ . Since  $x^2 + x + 1 > 0$  for all  $x \in \mathbb{R}$ , we must have  $y = 0$ . Graphically, this is  $x$ -axis.
- The PDE is of elliptic type if  $4y^2(x^2 + x + 1) < 0$  which can not happen ■

**Problem 11.4**

Show that  $u(x, t) = \cos x \sin t$  is a solution to the problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x, 0) &= 0 \\ u_t(x, 0) &= \cos x \\ u_x(0, t) &= 0 \end{aligned}$$

for all  $x, t > 0$ .

**Solution.**

We have

$$\begin{aligned} u_x(x, t) &= -\sin x \sin t, \\ u_{xx}(x, t) &= -\cos x \sin t, \\ u_t(x, t) &= \cos x \cos t, \\ u_{tt}(x, t) &= -\cos x \sin t. \end{aligned}$$

Thus,

$$\begin{aligned} u_{xx}(x, t) &= -\cos x \sin t = u_{tt}(x, t), \\ u(x, 0) &= \cos x \sin 0 = 0, \\ u_t(x, 0) &= \cos x \cos 0 = \cos x, \\ u_x(0, t) &= -\sin 0 \sin t = 0 \blacksquare \end{aligned}$$

**Problem 11.5**

Classify each of the following PDE as linear, quasilinear, semi-linear, or non-linear.



- (a)  $u_t + uu_x = uu_{xx}$   
 (b)  $xu_{tt} + tu_{yy} + u^3u_x^2 = t + 1$   
 (c)  $u_{tt} = c^2u_{xx}$   
 (d)  $u_{tt}^2 + u_x = 0$ .

**Solution.**

(a) Quasi-linear (b) Semi-linear (c) Linear (d) Nonlinear ■

**Problem 11.6**

Show that, for all  $(x, y) \neq (0, 0)$ ,  $u(x, y) = \ln(x^2 + y^2)$  is a solution of

$$u_{xx} + u_{yy} = 0,$$

and that, for all  $(x, y, z) \neq (0, 0, 0)$ ,  $u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$  is a solution of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

**Solution.**

We have

$$\begin{aligned} u_x &= \frac{2x}{x^2 + y^2} \\ u_{xx} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} \\ u_y &= \frac{2y}{x^2 + y^2} \\ u_{yy} &= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \end{aligned}$$

Plugging these expressions into the equation we find  $u_{xx} + u_{yy} = 0$ . Similar argument holds for the second part of the problem ■

**Problem 11.7**

Consider the eigenvalue problem

$$\begin{aligned} u_{xx} &= \lambda u, \quad 0 < x < L \\ u_x(0) &= k_0 u(0) \\ u_x(L) &= -k_L u(L) \end{aligned}$$

with Robin boundary conditions, where  $k_0$  and  $k_L$  are given positive numbers and  $u = u(x)$ . Can this system have a nontrivial solution  $u \neq 0$  for  $\lambda > 0$ ?  
 Hint: Multiply the first equation by  $u$  and integrate over  $x \in [0, L]$ .

**Solution.**

Multiplying the equation by  $u$  and integrating, we obtain

$$\begin{aligned}\lambda \int_0^L u^2(x) dx &= \int_0^L uu_{xx}(x) dx \\ &= [u(L)u_x(L) - u(0)u_x(0)] - \int_0^L u_x^2(x) dx \\ &= - \left[ k_L u(L)^2 + k_0 u(0)^2 + \int_0^L u_x^2(x) dx \right]\end{aligned}$$

For  $\lambda > 0$ , because  $k_0, k_L > 0$ , the right-hand side is nonpositive and the left-hand side is nonnegative. Therefore, both sides must be zero, and there can be no solution other than  $u \equiv 0$ , which is the trivial solution ■

**Problem 11.8**

Show that  $u(x, y) = f(x)g(y)$ , where  $f$  and  $g$  are arbitrary differentiable functions, is a solution to the PDE

$$uu_{xy} = u_x u_y.$$

**Solution.**

Substitute  $u(x, y) = f(x)g(y)$  into the left side of the equation to obtain  $f(x)g(y)(f(x)g(y))_{xy} = f(x)g(y)f'(x)g'(y)$ . Now, substitute the same thing into the right side to obtain  $(f(x)g(y))_x(f(x)g(y))_y = f'(x)g(y)f(x)g'(y) = f(x)g(y)f'(x)g'(y)$ . So the sides are equal, which means  $f(x)g(y)$  is a solution ■

**Problem 11.9**

Show that for any  $n \in \mathbb{N}$ , the function  $u_n(x, y) = \sin nx \sinh ny$  is a solution to the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

**Solution.**

We have

$$(u_n)_{xx} = -n^2 \sin nx \sinh ny \text{ and } (u_n)_{yy} = n^2 \sin nx \sinh ny$$

Hence,  $\Delta u_n = 0$  ■

**Problem 11.10**

Solve

$$u_{xy} = xy.$$

**Solution.**

Integrate both sides with respect to  $y$  to obtain  $u_x(x, y) = \frac{xy^2}{2} + f(x)$ . Next, integrate both sides w.r.t.  $x$  to obtain  $u(x, y) = \frac{x^2y^2}{4} + F(x) + G(y)$ , where  $F(x) = \int f(x)dx$  can be any function of  $x$ , since  $f(x)$  itself is an arbitrary function of  $x$  ■

**Problem 11.11**

Classify each of the following second-order PDEs according to whether they are hyperbolic, parabolic, or elliptic:

- (a)  $2u_{xx} - 4u_{xy} + 7u_{yy} - u = 0$ .
- (b)  $u_{xx} - 2\cos x u_{xy} - \sin^2 x u_{yy} = 0$ .
- (c)  $yu_{xx} + 2(x-1)u_{xy} - (y+2)u_{yy} = 0$ .

**Solution.**

(a) We have  $A = 2$ ,  $B = -4$ ,  $C = 7$  so  $B^2 - 4AC = 16 - 56 = -40 < 0$ . So this equation is elliptic everywhere in  $\mathbb{R}^2$ .

(b) We have  $A = 1$ ,  $B = -2\cos x$ ,  $C = -\sin^2 x$  so  $B^2 - 4AC = 4\cos^2 x + 4\sin^2 x = 4 > 0$ . So this equation is hyperbolic everywhere in  $\mathbb{R}^2$ .

(c) We have  $A = y$ ,  $B = 2(x-1)$ ,  $C = -(y+2)$  so  $B^2 - 4AC = 4(x-1)^2 + 4y(y+2) = 4[(x-1)^2 + (y+1)^2 - 4]$ . The equation is parabolic if  $(x-1)^2 + (y+1)^2 = 4$ . It is hyperbolic if  $(x-1)^2 + (y+1)^2 > 4$  and elliptic if  $(x-1)^2 + (y+1)^2 < 4$  ■

**Problem 11.12**

Let  $c > 0$ . By computing  $u_x, u_{xx}, u_t$ , and  $u_{tt}$  show that

$$u(x, t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds$$

is a solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

where  $f$  is twice differentiable function and  $g$  is a differentiable function. Then compute and simplify  $u(x, 0)$  and  $u_t(x, 0)$ .

**Solution.**

Using the chain rule we find

$$\begin{aligned} u_t(x, t) &= \frac{1}{2}(cf'(x+ct) - cf'(x-ct)) + \frac{1}{2c}[g(x+ct)(c) - g(x-ct)(-c)] \\ &= \frac{c}{2}(f'(x+ct) - f'(x-ct)) + \frac{1}{2}(g(x+ct) + g(x-ct)) \\ u_{tt} &= \frac{c^2}{2}(f''(x+ct) + f''(x-ct)) + \frac{c}{2}(g'(x+ct) - g'(x-ct)) \\ u_x(x, t) &= \frac{1}{2}(f'(x+ct) + f'(x-ct)) + \frac{1}{2c}[g(x+ct) - g(x-ct)] \\ u_{xx}(x, t) &= \frac{1}{2}(f''(x+ct) + f''(x-ct)) + \frac{1}{2c}[g'(x+ct) - g'(x-ct)]. \end{aligned}$$

By substitution we see that  $c^2u_{xx} = u_{tt}$ . Moreover,

$$u(x, 0) = \frac{1}{2}(f(x) + f(x)) + \frac{1}{2c} \int_x^x g(s)ds = f(x)$$

and

$$u_t(x, 0) = g(x) \blacksquare$$

**Problem 11.13**

Consider the second-order PDE

$$yu_{xx} + u_{xy} - x^2u_{yy} - u_x - u = 0.$$

Determine the region  $D$  in  $\mathbb{R}^2$ , if such a region exists, that makes this PDE: (a) hyperbolic, (b) parabolic, (c) elliptic.

**Solution.**

We have  $A = y$ ,  $B = 1$ , and  $C = -x^2$ . Thus,  $B^2 - 4AC = 1 + 4yx^2$ . We have then (a)  $1 + 4x^2y > 0$ , (b)  $1 + 4x^2y = 0$ , (c)  $1 + 4x^2y < 0$  ■

**Problem 11.14**

Consider the second-order hyperbolic PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

Use the change of variables  $v(x, y) = y - 3x$  and  $w(x, y) = x + y$  to solve the given equation.

**Solution.**

We have

$$\begin{aligned}u_x &= -3u_v + u_w \\u_{xx} &= 9u_{vv} - 6u_{vw} + u_{ww} \\u_{xy} &= -3u_{vv} - 2u_{vw} + u_{ww} \\u_y &= u_v + u_w \\u_{yy} &= (u_v + u_w)_v + (u_v + u_w)_w = u_{vv} + 2u_{vw} + u_{ww}\end{aligned}$$

Substituting into the PDE we find  $u_{vw} = 0$ . Solving this equation we find  $u(v, w) = f(v) + g(w)$ . In terms of  $x$  and  $y$  we have

$$u(x, y) = f(y - 3x) + g(x + y) \blacksquare$$

**Problem 11.15**

Solve the Cauchy problem

$$\begin{aligned}u_{xx} + 2u_{xy} - 3u_{yy} &= 0. \\u(x, 2x) &= 1, \quad u_x(x, 2x) = x.\end{aligned}$$

**Solution.**

From the previous exercise we have

$$u(x, y) = f(y - 3x) + g(x + y).$$

From the Cauchy data  $u(x, 2x) = 1$  we find

$$1 = f(-x) + g(3x). \tag{11.3}$$

Now from the Cauchy data  $u_x(x, 2x) = x$  we find

$$x = -3f'(-x) + g'(3x). \tag{11.4}$$

Differentiate (11.3) with respect to  $x$  we find

$$-f'(-x) + 3g'(3x) = 0 \tag{11.5}$$

Multiply (11.5) by  $-3$  to obtain

$$3f'(-x) - 9g'(3x) = 0 \tag{11.6}$$

Add (11.4) and (11.6) to obtain  $x = -8g'(3x)$  or  $g'(x) = -\frac{x}{24}$ . Integrating, we find  $g(x) = -\frac{x^2}{48} + C$ . Now, from (11.3) we have  $1 = f(-x) + g(3x)$  or  $f(-x) = 1 + \frac{9}{48}x^2 - C = f(x)$ . Thus,

$$u(x, y) = f(y - 3x) + g(x + y) = \frac{10x^2 + y^2 - 7xy + 6}{6} \blacksquare$$

## Solutions to Section 12

### Problem 12.1

Show that if  $v(x, t)$  and  $w(x, t)$  satisfy equation 12.1 then  $\alpha v + \beta w$  is also a solution to 12.1, where  $\alpha$  and  $\beta$  are constants.

### Solution.

Let  $z(x, t) = \alpha v(x, t) + \beta w(x, t)$ . Then we have

$$\begin{aligned} c^2 z_{xx} &= c^2 \alpha v_{xx} + c^2 \beta w_{xx} \\ &= \alpha v_{tt} + \beta w_{tt} \\ &= z_{tt} \blacksquare \end{aligned}$$

### Problem 12.2

Show that any linear time independent function  $u(x, t) = ax + b$  is a solution to equation 12.1.

### Solution.

Indeed we have  $c^2 u_{xx}(x, t) = 0 = u_{tt}(x, t)$  ■

### Problem 12.3

Find a solution to 12.1 that satisfies the homogeneous conditions  $u(x, 0) = u(0, t) = u(L, t) = 0$ .

### Solution.

Clearly the trivial solution  $u(x, t) = 0$  for all  $x$  and  $t$  is an answer to the question ■

### Problem 12.4

Solve the initial value problem

$$\begin{aligned} u_{tt} &= 9u_{xx} \\ u(x, 0) &= \cos x \\ u_t(x, 0) &= 0. \end{aligned}$$

### Solution.

According to Example 12.1, the unique solution is given by

$$u(x, t) = \frac{1}{2}(\cos(x - 3t) + \cos(x + 3t)) \blacksquare$$

**Problem 12.5**

Solve the initial value problem

$$\begin{aligned}u_{tt} &= u_{xx} \\ u(x, 0) &= \frac{1}{1+x^2} \\ u_t(x, 0) &= 0.\end{aligned}$$

**Solution.**

According to Example 12.1 with  $w(x) = 0$ , the unique solution is given by

$$u(x, t) = \frac{1}{2} \left[ \frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2} \right] \blacksquare$$

**Problem 12.6**

Solve the initial value problem

$$\begin{aligned}u_{tt} &= 4u_{xx} \\ u(x, 0) &= 1 \\ u_t(x, 0) &= \cos(2\pi x).\end{aligned}$$

**Solution.**

We have  $v(x) = 1$  and  $w(x) = \cos(2\pi x)$ . The unique solution is given by

$$\begin{aligned}u(x, t) &= \frac{1}{2} \left[ 2 + \frac{1}{2} \int_{x-2t}^{x+2t} \cos(2\pi s) ds \right] \\ &= 1 + \frac{1}{4} \left[ \frac{1}{2\pi} \sin(2\pi s) \right]_{x-2t}^{x+2t} \\ &= 1 + \frac{1}{8\pi} [\sin(2\pi x + 4\pi t) - \sin(2\pi x - 4\pi t)] \blacksquare\end{aligned}$$

**Problem 12.7**

Solve the initial value problem

$$\begin{aligned}u_{tt} &= 25u_{xx} \\ u(x, 0) &= v(x) \\ u_t(x, 0) &= 0\end{aligned}$$

where

$$v(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$



**Solution.**

The general solution is given by

$$u(x, t) = \frac{1}{2}(v(x + 5t) + v(x - 5t)).$$

Thus,

$$u(x, t) = \begin{cases} 1 & \text{if } x - 5t < 0 \text{ and } x + 5t < 0 \\ \frac{1}{2} & \text{if } x - 5t < 0 \text{ and } x + 5t > 0 \\ \frac{1}{2} & \text{if } x - 5t > 0 \text{ and } x + 5t < 0 \\ 0 & \text{if } x - 5t > 0 \text{ and } x + 5t > 0 \blacksquare \end{cases}$$

**Problem 12.8**

Solve the initial value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x, 0) &= e^{-x^2} \\ u_t(x, 0) &= \cos^2 x. \end{aligned}$$

**Solution.**

We have

$$\begin{aligned} u(x, t) &= \frac{1}{2}[e^{-(x+ct)^2} + e^{-(x-ct)^2}] + \frac{1}{2c} \int_{x-ct}^{x+ct} \cos^2 s ds \\ &= \frac{1}{2}[e^{-(x+ct)^2} + e^{-(x-ct)^2}] + \frac{t}{2} + \frac{1}{4c} \cos(2x) \sin(2ct) \blacksquare \end{aligned}$$

**Problem 12.9**

Prove that the wave equation,  $u_{tt} = c^2 u_{xx}$  satisfies the following properties, which are known as invariance properties. If  $u(x, t)$  is a solution, then

- (i) Any translate,  $u(x - y, t)$  where  $y$  is a fixed constant, is also a solution.
- (ii) Any derivative, say  $u_x(x, t)$ , is also a solution.
- (iii) Any dilation,  $u(ax, at)$ , is a solution, for any fixed constant  $a$ .

**Solution.**

Just plug the translated/differentiated/dilated solution into the wave equation and check that it is a solution  $\blacksquare$

**Problem 12.10**

Find  $v(r)$  if  $u(r, t) = \frac{v(r)}{r} \cos nt$  is a solution to the PDE

$$u_{rr} + \frac{2}{r} u_r = u_{tt}.$$

**Solution.**

We have

$$\begin{aligned}u_r &= \frac{v'(r)}{r} \cos nt - \frac{v(r)}{r^2} \cos nt \\u_{rr} &= \frac{v''(r)}{r} \cos nt - 2\frac{v'(r)}{r^2} \cos nt + 2\frac{v(r)}{r^3} \cos nt \\u_t &= -n\frac{v(r)}{r} \sin nt \\u_{tt} &= -n^2\frac{v(r)}{r} \cos nt\end{aligned}$$

Avoiding the trivial solution  $u = 0$ , we cancel  $\cos nt$  and find from  $u_{rr} + \frac{2}{r}u_r = u_{tt}$  the ODE

$$\frac{v''(r)}{r} = -n^2\frac{v(r)}{r}$$

or

$$v''(r) + n^2v(r) = 0.$$

Solving this equation we find

$$v(r) = A \cos(nr) + B \sin(nr) \blacksquare$$

**Problem 12.11**

Find the solution of the wave equation on the real line ( $-\infty < x < +\infty$ ) with the initial conditions

$$u(x, 0) = e^x, \quad u_t(x, 0) = \sin x.$$

**Solution.**

The general solution is given by

$$u(x, t) = \frac{1}{2}[e^{x-ct} + e^{x+ct}] + \frac{1}{c} \int_{x-ct}^{x+ct} \sin s ds$$

Thus,

$$u(x, t) = \frac{1}{2}[e^{x-ct} + e^{x+ct}] + \frac{1}{c}(\cos(x-ct) - \cos(x+ct)) \blacksquare$$

**Problem 12.12**

The total energy of the string (the sum of the kinetic and potential energies) is defined as

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2) dx.$$

(a) Using the wave equation derive the equation of conservation of energy

$$\frac{dE(t)}{dt} = c^2(u_t(L, t)u_x(L, t) - u_t(0, t)u_x(0, t)).$$

(b) Assuming fixed ends boundary conditions, that is the ends of the string are fixed so that  $u(0, t) = u(L, t) = 0$ , for all  $t > 0$ , show that the energy is constant.

(c) Assuming free ends boundary conditions for both  $x = 0$  and  $x = L$ , that is both  $u(0, t)$  and  $u(L, t)$  vary with  $t$ , show that the energy is constant.

**Solution.**

(a) We have

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_0^L u_t u_{tt} dx + \int_0^L c^2 u_x u_{xt} dx \\ &= \int_0^L u_t u_{tt} dx + c^2 u_t(L, t) u_x(L, t) - c^2 u_t(0, t) u_x(0, t) - c^2 \int_0^L u_t u_{xx} dx \\ &= c^2 u_t(L, t) u_x(L, t) - c^2 u_t(0, t) u_x(0, t) + \int_0^L u_t (u_{tt} - c^2 u_{xx}) dx \\ &= c^2 (u_t(L, t) u_x(L, t) - u_t(0, t) u_x(0, t)) \end{aligned}$$

since  $u_{tt} - c^2 u_{xx} = 0$ .

(b) Since the ends are fixed, we have  $u_t(0, t) = u_t(L, t) = 0$ . From (a) we have

$$\frac{dE}{dt}(t) = c^2 (u_t(L, t) u_x(L, t) - u_t(0, t) u_x(0, t)) = 0.$$

(c) Assuming free ends boundary conditions, that is  $u_x(0, t) = u_x(L, t) = 0$ , we find  $\frac{dE}{dt}(t) = 0$  ■

**Problem 12.13**

For a wave equation with damping

$$u_{tt} - c^2 u_{xx} + du_t = 0, \quad d > 0, \quad 0 < x < L$$

with the fixed ends boundary conditions show that the total energy decreases.

**Solution.**

Using the previous exercise, we find

$$\frac{dE}{dt}(t) = -d \int_0^L (u_t)^2 dx.$$

The right-hand side is nonpositive, so the energy either decreases or is constant. The latter case can occur only if  $u_t(x, t)$  is identically zero, which means that the string is at rest ■

**Problem 12.14**

(a) Verify that for any twice differentiable  $R(x)$  the function

$$u(x, t) = R(x - ct)$$

is a solution of the wave equation  $u_{tt} = c^2 u_{xx}$ . Such solutions are called traveling waves.

(b) Show that the potential and kinetic energies (see Exercise 12.12) are equal for the traveling wave solution in (a).

**Solution.**

(a) By the chain rule we have  $u_t(x, t) = -cR'(x - ct)$  and  $u_{tt}(x, t) = c^2 R''(x - ct)$ . Likewise,  $u_x(x, t) = R'(x - ct)$  and  $u_{xx} = R''(x - ct)$ . Thus,  $u_{tt} = c^2 u_{xx}$ .

(b) We have

$$\frac{1}{2} \int_0^L (u_t)^2 dx = \int_0^L \frac{c^2}{2} [R'(x - ct)]^2 dx = \int_0^L \frac{c^2}{2} (u_x)^2 dx \quad \blacksquare$$

**Problem 12.15**

Find the solution of the Cauchy wave equation

$$u_{tt} = 4u_{xx}$$

$$u(x, 0) = x^2, \quad u_t(x, 0) = \sin 2x.$$

Simplify your answer as much as possible.

**Solution.**

The solution is

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{c} \int_{x-ct}^{x+ct} g(s) ds.$$

Here,  $f(x) = x^2$ ,  $g(x) = \sin 2x$ , and  $c = 2$ . Thus,

$$\begin{aligned} u(x, t) &= \frac{1}{2}[(x - 2t)^2 + (x + 2t)^2] + \frac{1}{4} \int_{x-2t}^{x+2t} \sin 2s ds \\ &= x^2 + 4t^2 - \frac{1}{8} \cos 2s \Big|_{x-2t}^{x+2t} \\ &= x^2 + 4t^2 - \frac{1}{8} \cos(2x + 4t) + \frac{1}{8} \cos(2x - 4t) \\ &= x^2 + 4t^2 + \frac{1}{4} \sin 2x \sin 4t \blacksquare \end{aligned}$$

## Solutions to Section 13

### Problem 13.1

Show that if  $u(x, t)$  and  $v(x, t)$  satisfy equation (13.1) then  $\alpha u + \beta v$  is also a solution to (13.1), where  $\alpha$  and  $\beta$  are constants.

#### Solution.

Let  $z(x, t) = \alpha u(x, t) + \beta v(x, t)$ . Then we have

$$\begin{aligned} kz_{xx} &= k\alpha u_{xx} + k\beta v_{xx} \\ &= \alpha u_t + \beta v_t \\ &= z_t \blacksquare \end{aligned}$$

### Problem 13.2

Show that any linear time independent function  $u(x, t) = ax + b$  is a solution to equation (13.1).

#### Solution.

Indeed we have  $ku_{xx}(x, t) = 0 = u_t(x, t)$  ■

### Problem 13.3

Find a linear time independent solution  $u$  to (13.1) that satisfies  $u(0, t) = T_0$  and  $u(L, T) = T_L$ .

#### Solution.

Let  $u(x, t) = ax + b$ . From the assumptions of the problem we must have  $b = T_0$  and  $a = \frac{T_L - T_0}{L}$ . Thus,  $u(x, t) = T_0 + \frac{T_L - T_0}{L}x$  ■

### Problem 13.4

Show that to solve (13.1) with the boundary conditions  $u(0, t) = T_0$  and  $u(L, t) = T_L$  it suffices to solve (13.1) with the **homogeneous boundary conditions**  $u(0, t) = u(L, t) = 0$ .

#### Solution.

Let  $\bar{u}$  be the solution to (13.1) that satisfies  $\bar{u}(0, t) = \bar{u}(L, t) = 0$ . Let  $w(x, t)$  be the time independent solution to (13.1) that satisfies  $w(0, t) = T_0$  and  $w(L, t) = T_L$ . That is,  $w(x, t) = T_0 + \frac{T_L - T_0}{L}x$ . From Exercise 13.1, the function  $u(x, t) = \bar{u}(x, t) + w(x, t)$  is a solution to (13.1) that satisfies  $u(0, t) = T_0$  and  $u(L, t) = T_L$  ■

**Problem 13.5**

Find a solution to (13.1) that satisfies the conditions  $u(x, 0) = u(0, t) = u(L, t) = 0$ .

**Solution.**

Clearly the trivial solution  $u(x, t) = 0$  for all  $x$  and  $t$  is an answer to the question ■

**Problem 13.6**

Let (I) denote equation (13.1) together with initial condition  $u(x, 0) = f(x)$ , where  $f$  is not the zero function, and the homogeneous boundary conditions  $u(0, t) = u(L, t) = 0$ . Suppose a nontrivial solution to (I) can be written in the form  $u(x, t) = X(x)T(t)$ . Show that  $X$  and  $T$  satisfy the ODE

$$X'' - \frac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0$$

for some constant  $\lambda$ .

**Solution.**

Substituting  $u(x, t) = X(x)T(t)$  into (13.1) we obtain

$$k \frac{X''}{X} = \frac{T'}{T}.$$

Since  $X$  only depends on  $x$  and  $T$  only depends on  $t$ , we must have that there is a constant  $\lambda$  such that

$$k \frac{X''}{X} = \lambda \text{ and } \frac{T'}{T} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \frac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0 \quad \blacksquare$$

**Problem 13.7**

Consider again the solution  $u(x, t) = X(x)T(t)$ . Clearly,  $T(t) = T(0)e^{-\lambda t}$ . Suppose that  $\lambda > 0$ .

(a) Show that  $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$ , where  $\alpha = \frac{\lambda}{k}$  and  $A$  and  $B$  are arbitrary constants.

(b) Show that  $A$  and  $B$  satisfy the two equations  $A + B = 0$  and  $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$ .

(c) Show that  $A = 0$  leads to a contradiction.

(d) Using (b) and (c) show that  $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$ . Show that this equality leads to a contradiction. We conclude that  $\lambda < 0$ .

**Solution.**

(a) Letting  $\alpha = \frac{\lambda}{k} > 0$  we obtain the ODE  $X'' - \alpha X = 0$  whose general solution is given by  $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$  for some constants  $A$  and  $B$ .

(b) The condition  $u(0, t) = 0$  implies that  $X(0) = 0$  which in turn implies  $A + B = 0$ . Likewise, the condition  $u(L, t) = 0$  implies  $Ae^{\sqrt{L\alpha}} + Be^{-L\sqrt{\alpha}} = 0$ . Hence,  $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$ .

(c) If  $A = 0$  then  $B = 0$  and  $u(x, t)$  is the trivial solution which contradicts the assumption that  $u$  is non-trivial. Hence, we must have  $A \neq 0$ .

(d) Using (b) and (c) we obtain  $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$  or  $e^{2L\sqrt{\alpha}} = 1$ . This equation is impossible since  $2L\sqrt{\alpha} > 0$ . Hence, we must have  $\lambda < 0$  so that  $X(x) = A \cos(x\sqrt{-\alpha}) + B \sin(x\sqrt{-\alpha})$  ■

**Problem 13.8**

Consider the results of the previous exercise.

(a) Show that  $X(x) = c_1 \cos \beta x + c_2 \sin \beta x$  where  $\beta = \sqrt{\frac{-\lambda}{k}}$ .

(b) Show that  $\lambda = \lambda_n = -\frac{kn^2\pi^2}{L^2}$ , where  $n$  is an integer.

**Solution.**

(a) Now, write  $\beta = \sqrt{\frac{-\lambda}{k}}$ . Then we obtain the equation  $X'' + \beta^2 X = 0$  whose general solution is given by

$$X(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

(b) Using  $X(0) = 0$  we obtain  $c_1 = 0$ . Since  $c_2 \neq 0$  we must have  $\sin \beta L = 0$ . Thus,  $\lambda = -\frac{kn^2\pi^2}{L^2}$ , where  $n$  is an integer ■

**Problem 13.9**

Show that  $u(x, t) = \sum_{k=1}^n u_k(x, t)$ , where  $u_n(x, t) = c_n e^{\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$  satisfies (13.1) and the homogeneous boundary conditions.

**Solution.**

For each integer  $n \geq 0$  we have  $u_n(x, t) = \frac{c_n}{T(0)} T(0) e^{\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$  is a solution to (13.1). By superposition,  $u(x, t)$  is also a solution to (13.1). Moreover,  $u(0, t) = u(L, t) = 0$  since  $u_n(0, t) = u_n(L, t) = 0$  ■

**Problem 13.10**

Suppose that a wire is stretched between 0 and  $a$ . Describe the boundary



conditions for the temperature  $u(x, t)$  when

- (i) the left end is kept at 0 degrees and the right end is kept at 100 degrees;  
and  
(ii) when both ends are insulated.

**Solution.**

(i)  $u(0, t) = 0$  and  $u(a, t) = 100$  for  $t > 0$ .

(ii)  $u_x(0, t) = u_x(a, t) = 0$  for  $t > 0$  ■

**Problem 13.11**

Let  $u_t = u_{xx}$  for  $0 < x < \pi$  and  $t > 0$  with boundary conditions  $u(0, t) = 0 = u(\pi, t)$  and initial condition  $u(x, 0) = f(x)$ . Let  $E(t) = \int_0^\pi (u_t^2 + u_x^2) dx$ . Show that  $E'(t) < 0$ .

**Solution.**

Solving this problem we find  $u(x, t) = e^{-t} \sin x$ . We have

$$E(t) = \int_0^\pi [e^{-2t} \sin^2 x + e^{-2t} \cos^2 x] dx = \int_0^\pi e^{-2t} dx = \pi e^{-2t}.$$

Thus,  $E'(t) = -2\pi e^{-2t} < 0$  for all  $t > 0$  ■

**Problem 13.12**

Suppose

$$u_t = u_{xx} + 4, \quad u_x(0, t) = 5, \quad u_x(L, t) = 6, \quad u(x, 0) = f(x).$$

Calculate the total thermal energy of the one-dimensional rod (as a function of time).

**Solution.**

We have

$$\frac{d}{dt} \int_0^L u(x, t) dx = u_x|_0^L + 4L = 1 + 4L.$$

Thus,

$$E(t) = \int_0^L f(x) dx + (1 + 4L)t \quad \blacksquare$$

**Problem 13.13**

Consider the heat equation

$$u_t = ku_{xx}$$

for  $x \in (0, 1)$  and  $t > 0$ , with boundary conditions  $u(0, t) = 2$  and  $u(1, t) = 3$  for  $t > 0$  and initial conditions  $u(x, 0) = x$  for  $x \in (0, 1)$ . A function  $v(x)$  that satisfies the equation  $v''(x) = 0$ , with conditions  $v(0) = 2$  and  $v(1) = 3$  is called a **steady-state solution**. Find  $v(x)$ .

**Solution.**

Solving the equation  $v''(x) = 0$  we find  $v(x) = ax + b$ . Using the conditions  $v(0) = 2$  and  $v(1) = 3$  we find  $v(x) = x + 2$  ■

**Problem 13.14**

Consider the equation for the one-dimensional rod of length  $L$  with given heat energy source:

$$u_t = u_{xx} + q(x).$$

Assume that the initial temperature distribution is given by  $u(x, 0) = f(x)$ . Find the equilibrium (steady state) temperature distribution in the following cases.

- (a)  $q(x) = 0, u(0, t) = 0, u(L, t) = T$ .
- (b)  $q(x) = 0, u_x(0, t) = 0, u(L, t) = T$ .
- (c)  $q(x) = 0, u(0, t) = T, u_x(L, t) = \alpha$ .

**Solution.**

Recall that a steady-state solution is a solution that does not depend on time (i.e.  $u_t = 0$ ).

(a) We have  $v''(x) = 0 \implies v(x) = c_1x + c_2$ . But  $v(0) = 0$  and  $v(L) = T$  so that  $c_1 = \frac{T}{L}$  and  $c_2 = 0$ . Thus, the steady-state solution is  $v(x) = \frac{T}{L}x$ .

(b) We have  $v(x) = c_1x + c_2$  with  $v'(0) = 0$  and  $v(L) = T$ . Thus,  $c_1 = 0$  and  $c_2 = T$  so that  $v(x) = T$ .

(c) We have  $v(x) = c_1x + c_2$  with  $v(0) = T$  and  $v'(L) = \alpha$ . Thus,  $c_1 = \alpha$  and  $c_2 = T$  so that  $v(x) = \alpha x + T$  ■

**Problem 13.15**

Consider the equation for the one-dimensional rod of length  $L$  with insulated ends:

$$c\rho u_t = Ku_{xx}, \quad u_x(0, t) = u_x(L, t) = 0.$$

- (a) Give the expression for the total thermal energy of the rod.  
 (b) Show using the equation and the boundary conditions that the total thermal energy is constant.

**Solution.**

(a)  $E(t) = \int_0^L c\rho u(x, t)dx.$

(b) We integrate the equation in  $x$  from 0 to  $L$  :

$$\int_0^L c\rho u_t(x, t)dx = \int_0^L Ku_{xx}dx = Ku_x(x, t)|_0^L = 0,$$

since  $u_x(0, t) = u_x(L, t) = 0$ . The left-hand side can also be written as

$$\frac{d}{dt} \int_0^L c\rho u(x, t)dx = E'(t).$$

Thus, we have shown that  $E'(t) = 0$  so that  $E(t)$  is constant ■

**Problem 13.16**

Suppose

$$u_t = u_{xx} + x, \quad u(x, 0) = f(x), \quad u_x(0, t) = \beta, \quad u_x(L, t) = 7.$$

- (a) Calculate the total thermal energy of the one-dimensional rod (as a function of time).  
 (b) From part (a) find the value of  $\beta$  for which a steady-state solution exist.  
 (c) For the above value of  $\beta$  find the steady state solution.

**Solution.**

(a) The total thermal energy is

$$E(t) = \int_0^L u(x, t)dx.$$

We have

$$\frac{dE}{dt} = \int_0^L u_t(x, t)dx = u_x|_0^L + \int_0^L xdx = (7 - \beta) + \frac{L^2}{2}.$$

(b) The steady solution (equilibrium) is possible if the right-hand side vanishes:

$$(7 - \beta) + \frac{L^2}{2} = 0$$

Solving this equation for  $\beta$  we find  $\beta = 7 + \frac{L^2}{2}$ .

(c) By integrating the equation  $u_{xx} + x = 0$  we find the steady solution

$$u(x) = -\frac{x^3}{6} + C_1x + C_2$$

From the condition  $u_x(0) = \beta$  we find  $C_1 = \beta$ . The steady solution should also have the same value of the total energy as the initial condition. This means

$$\int_0^L \left( -\frac{x^3}{6} + \beta x + C_2 \right) dx = \int_0^L f(x) dx = E(0).$$

Performing the integration and then solving for  $C_2$  we find

$$C_2 = \frac{1}{L} \int_0^L f(x) dx + \frac{L^3}{24} - \beta \frac{L}{2}.$$

Therefore, the steady-state solution is

$$u(x) = \frac{1}{L} \int_0^L f(x) dx + \frac{L^3}{24} - \beta \frac{L}{2} + \beta x - \frac{x^3}{6} \blacksquare$$

## Solutions to Section 14

### Problem 14.1

Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$ . Define  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

- (a) Show that the sequence  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to  $f$ .  
 (b) Show that the sequence  $\{f_n\}_{n=1}^{\infty}$  does not converge uniformly to  $f$ . Hint: Suppose otherwise. Let  $\epsilon = 0.5$  and get a contradiction by using a point  $(0.5)^{\frac{1}{N}} < x < 1$ .

### Solution.

- (a) For all  $0 \leq x < 1$  we have  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = 0$ . Also,  $\lim_{n \rightarrow \infty} f_n(1) = 1$ . Hence, the sequence  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to  $f$ .  
 (b) Suppose the contrary. Let  $\epsilon = \frac{1}{2}$ . Then there exists a positive integer  $N$  such that for all  $n \geq N$  we have

$$|f_n(x) - f(x)| < \frac{1}{2}$$

for all  $x \in [0, 1]$ . In particular, we have

$$|f_N(x) - f(x)| < \frac{1}{2}$$

for all  $x \in [0, 1]$ . Choose  $(0.5)^{\frac{1}{N}} < x < 1$ . Then  $|f_N(x) - f(x)| = x^N > 0.5 = \epsilon$  which is a contradiction. Hence, the given sequence does not converge uniformly ■

### Problem 14.2

Consider the sequence of functions

$$f_n(x) = \frac{nx + x^2}{n^2}$$

defined for all  $x$  in  $\mathbb{R}$ . Show that this sequence converges pointwise to a function  $f$  to be determined.

**Solution.**

For every real number  $x$ , we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx + x^2}{n^2} = \lim_{n \rightarrow \infty} \frac{x}{n} + \lim_{n \rightarrow \infty} \frac{x^2}{n^2} = 0$$

Thus,  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to the zero function on  $\mathbb{R}$  ■

**Problem 14.3**

Consider the sequence of functions

$$f_n(x) = \frac{\sin(nx + 3)}{\sqrt{n+1}}$$

defined for all  $x$  in  $\mathbb{R}$ . Show that this sequence converges pointwise to a function  $f$  to be determined.

**Solution.**

For every real number  $x$ , we have

$$-\frac{1}{\sqrt{n+1}} \leq f_n(x) \leq \frac{1}{\sqrt{n+1}}.$$

Moreover,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the squeeze rule for sequences, we obtain

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

for all  $x$  in  $\mathbb{R}$ . Thus,  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to the zero function on  $\mathbb{R}$  ■

**Problem 14.4**

Consider the sequence of functions defined by  $f_n(x) = n^2 x^n$  for all  $0 \leq x \leq 1$ . Show that this sequence does not converge pointwise to any function.

**Solution.**

First of all, observe that  $f_n(0) = 0$  for every  $n$  in  $\mathbb{N}$ . So the sequence  $\{f_n(0)\}_{n=1}^{\infty}$  is constant and converges to zero. Now suppose  $0 < x < 1$  then  $n^2 x^n = n^2 e^{n \ln x}$ . But  $\ln x < 0$  when  $0 < x < 1$ , it follows that

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \text{ for } 0 < x < 1$$

Finally,  $f_n(1) = n^2$  for all  $n$ . So,

$$\lim_{n \rightarrow \infty} f_n(1) = \infty.$$

Therefore,  $\{f_n\}_{n=1}^{\infty}$  is not pointwise convergent on  $[0, 1]$  ■

### Problem 14.5

Consider the sequence of functions defined by  $f_n(x) = (\cos x)^n$  for all  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . Show that this sequence converges pointwise to a noncontinuous function to be determined.

### Solution.

For  $-\frac{\pi}{2} \leq x < 0$  and  $0 < x \leq \frac{\pi}{2}$  we have

$$\lim_{n \rightarrow \infty} (\cos x)^n = 0.$$

For  $x = 0$  we have  $f_n(0) = 1$  for all  $n$  in  $\mathbb{N}$ . Therefore,  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \leq x < 0 \text{ and } 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if } x = 0 \end{cases} \blacksquare$$

### Problem 14.6

Consider the sequence of functions  $f_n(x) = x - \frac{x^n}{n}$  defined on  $[0, 1)$ .

(a) Does  $\{f_n\}_{n=1}^{\infty}$  converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

(b) Does  $\{f'_n\}_{n=1}^{\infty}$  converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

### Solution.

(a) Let  $\epsilon > 0$  be given. Let  $N$  be a positive integer such that  $N > \frac{1}{\epsilon}$ . Then for  $n \geq N$

$$\left| x - \frac{x^n}{n} - x \right| = \frac{|x|^n}{n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus, the given sequence converges uniformly (and pointwise) to the function  $f(x) = x$ .

(b) Since  $\lim_{n \rightarrow \infty} f'_n(x) = 1$  for all  $x \in [0, 1)$ , the sequence  $\{f'_n\}_{n=1}^{\infty}$  converges

pointwise to  $f'(x) = 1$ . However, the convergence is not uniform. To see this, let  $\epsilon = \frac{1}{2}$  and suppose that the convergence is uniform. Then there is a positive integer  $N$  such that for  $n \geq N$  we have

$$|1 - x^{n-1} - 1| = |x|^{n-1} < \frac{1}{2}.$$

In particular, if we let  $n = N + 1$  we must have  $x^N < \frac{1}{2}$  for all  $x \in [0, 1)$ . But  $x = (\frac{1}{2})^{\frac{1}{N}} \in [0, 1)$  and  $x^N = \frac{1}{2}$  which contradicts  $x^N < \frac{1}{2}$ . Hence, the convergence is not uniform ■

**Problem 14.7**

Let  $f_n(x) = \frac{x^n}{1+x^n}$  for  $x \in [0, 2]$ .

- (a) Find the pointwise limit  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  on  $[0, 2]$ .  
 (b) Does  $f_n \rightarrow f$  uniformly on  $[0, 2]$ ?

**Solution.**

- (a) The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

- (b) The convergence cannot be uniform because if it were  $f$  would have to be continuous ■

**Problem 14.8**

For each  $n \in \mathbb{N}$  define  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  by  $f_n(x) = \frac{n + \cos x}{2n + \sin^2 x}$ .

- (a) Show that  $f_n \rightarrow \frac{1}{2}$  uniformly.  
 (b) Find  $\lim_{n \rightarrow \infty} \int_2^7 f_n(x) dx$ .

**Solution.**

- (a) Let  $\epsilon > 0$  be given. Note that

$$|f_n(x) - \frac{1}{2}| = \left| \frac{2 \cos x - \sin^2 x}{2(2n + \sin^2 x)} \right| \leq \frac{3}{4n}.$$

Since  $\lim_{n \rightarrow \infty} \frac{3}{4n} = 0$  we can find a positive integer  $N$  such that if  $n \geq N$  then  $\frac{3}{4n} < \epsilon$ . Thus, for  $n \geq N$  and all  $x \in \mathbb{R}$  we have

$$|f_n(x) - \frac{1}{2}| \leq \frac{3}{4n} < \epsilon.$$



This shows that  $f_n \rightarrow \frac{1}{2}$  uniformly on  $\mathbb{R}$  and also on  $[2, 7]$ .

(b) We have

$$\lim_{n \rightarrow \infty} \int_2^7 f_n x dx = \int_2^7 \lim_{n \rightarrow \infty} f_n x dx = \int_2^7 \frac{1}{2} dx = \frac{5}{2} \blacksquare$$

**Problem 14.9**

Show that the sequence defined by  $f_n(x) = (\cos x)^n$  does not converge uniformly on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

**Solution.**

We have proved earlier that this sequence converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \leq x < 0 \text{ and } 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if } x = 0 \end{cases}$$

Therefore, uniform convergence cannot occur for this given sequence  $\blacksquare$

**Problem 14.10**

Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of functions such that

$$\sup\{|f_n(x)| : 2 \leq x \leq 5\} \leq \frac{2^n}{1 + 4^n}.$$

(a) Show that this sequence converges uniformly to a function  $f$  to be found.

(b) What is the value of the limit  $\lim_{n \rightarrow \infty} \int_2^5 f_n(x) dx$ ?

**Solution.**

(a) Using the squeeze rule we find

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x)| : 2 \leq x \leq 5\} = 0.$$

Thus,  $\{f_n\}_{n=1}^{\infty}$  converges uniformly to the zero function.

(b) We have

$$\lim_{n \rightarrow \infty} \int_2^5 f_n(x) dx = \int_2^5 0 dx = 0 \blacksquare$$

## Solutions to Section 15

### Problem 15.1

Let  $f$  and  $g$  be two functions with common domain  $D$  and common period  $T$ . Show that

- (a)  $fg$  is periodic of period  $T$ .
- (b)  $c_1f + c_2g$  is periodic of period  $T$ , where  $c_1$  and  $c_2$  are real numbers.

### Solution.

- (a) We have  $(fg)(x + T) = f(x + T)g(x + T) = f(x)g(x) = (fg)(x)$ .
- (b) We have  $(c_1f + c_2g)(x + T) = c_1f(x + T) + c_2g(x + T) = c_1f(x) + c_2g(x) = (c_1f + c_2g)(x)$  ■

### Problem 15.2

Show that for  $m \neq n$  we have

- (a)  $\int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$  and
- (b)  $\int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$ .

### Solution.

- (a) For  $n \neq m$  we have

$$\begin{aligned} \int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx &= -\frac{1}{2} \int_{-L}^L \left[ \cos\left(\frac{(m+n)\pi}{L}x\right) - \cos\left(\frac{(m-n)\pi}{L}x\right) \right] dx \\ &= -\frac{1}{2} \left[ \frac{L}{(m+n)\pi} \sin\left(\frac{(m+n)\pi}{L}x\right) \right. \\ &\quad \left. - \frac{L}{(m-n)\pi} \sin\left(\frac{(m-n)\pi}{L}x\right) \right]_{-L}^L \\ &= 0 \end{aligned}$$

where we used the trigonometric identity

$$\sin a \sin b = \frac{1}{2}[-\cos(a+b) + \cos(a-b)].$$

(b) For  $n \neq m$  we have

$$\begin{aligned} \int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx &= \frac{1}{2} \int_{-L}^L \left[ \sin\left(\frac{(m+n)\pi}{L}x\right) - \sin\left(\frac{(m-n)\pi}{L}x\right) \right] dx \\ &= \frac{1}{2} \left[ -\frac{L}{(m+n)\pi} \cos\left(\frac{(m+n)\pi}{L}x\right) \right. \\ &\quad \left. + \frac{L}{(m-n)\pi} \cos\left(\frac{(m-n)\pi}{L}x\right) \right]_{-L}^L \\ &= 0 \end{aligned}$$

where we used the trigonometric identity

$$\cos a \sin b = \frac{1}{2} [\sin(a+b) - \sin(a-b)] \blacksquare$$

### Problem 15.3

Compute the following integrals:

- (a)  $\int_{-L}^L \cos^2\left(\frac{n\pi}{L}x\right) dx$ .  
 (b)  $\int_{-L}^L \sin^2\left(\frac{n\pi}{L}x\right) dx$ .  
 (c)  $\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx$ .

### Solution.

(a) Using the trigonometric identity  $\cos^2 a = \frac{1+\cos 2a}{2}$  we can write

$$\begin{aligned} \int_{-L}^L \cos^2\left(\frac{n\pi}{L}x\right) dx &= \frac{1}{2} \int_{-L}^L \left( 1 + \cos\left(\frac{2n\pi}{L}x\right) \right) dx \\ &= \frac{1}{2} \left[ x + \left(\frac{L}{2n\pi}\right) \sin\left(\frac{2n\pi}{L}x\right) \right]_{-L}^L = L. \end{aligned}$$

(b) Using the trigonometric identity  $\sin^2 a = \frac{1-\cos 2a}{2}$  we can write

$$\begin{aligned} \int_{-L}^L \sin^2\left(\frac{n\pi}{L}x\right) dx &= \frac{1}{2} \int_{-L}^L \left( 1 - \cos\left(\frac{2n\pi}{L}x\right) \right) dx \\ &= \frac{1}{2} \left[ x - \left(\frac{L}{2n\pi}\right) \sin\left(\frac{2n\pi}{L}x\right) \right]_{-L}^L = L \end{aligned}$$

(c) Using the trigonometric identity  $\cos a \sin a = \frac{\cos 2a}{2}$  we can write

$$\begin{aligned} \int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx &= \frac{1}{2} \int_{-L}^L \cos\left(\frac{2n\pi}{L}x\right) dx \\ &= \left(\frac{L}{4n\pi}\right) \left[\sin\left(\frac{2n\pi}{L}x\right)\right]_{-L}^L = 0 \blacksquare \end{aligned}$$

**Problem 15.4**

Find the Fourier coefficients of

$$f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ \pi, & 0 < x < \pi \\ 0, & x = 0, \pi \end{cases}$$

on the interval  $[-\pi, \pi]$ .

**Solution.**

We have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0 \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= - \int_{-\pi}^0 \cos nx dx + \int_0^{\pi} \cos nx dx = 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= - \int_{-\pi}^0 \sin nx dx + \int_0^{\pi} \sin nx dx \\ &= \frac{2}{n} [1 - (-1)^n] \blacksquare \end{aligned}$$

**Problem 15.5**

Find the Fourier series of  $f(x) = x^2 - \frac{1}{2}$  on the interval  $[-1, 1]$ .

**Solution.**

We have

$$\begin{aligned}
 a_0 &= \int_{-1}^1 \left(x^2 - \frac{1}{2}\right) dx = -\frac{1}{3} \\
 a_n &= \int_{-1}^1 \left(x^2 - \frac{1}{2}\right) \cos n\pi x dx \\
 &= \int_{-1}^1 x^2 \cos n\pi x dx - \frac{1}{2} \int_{-1}^1 \cos n\pi x dx \\
 &= x^2 \left[ \frac{\sin(n\pi x)}{n\pi} \right]_{-1}^1 - \int_{-1}^1 2x \frac{\sin(n\pi x)}{n\pi} dx - \frac{1}{2} \left[ \frac{\sin(n\pi x)}{n\pi} \right]_{-1}^1 \\
 &= 2x \left[ \frac{\cos(n\pi x)}{(n\pi)^2} \right]_{-1}^1 - \int_{-1}^1 2 \frac{\cos(n\pi x)}{(n\pi)^2} dx \\
 &= \frac{4}{(n\pi)^2} (-1)^n. \\
 b_n &= \int_{-1}^1 \left(x^2 - \frac{1}{2}\right) \sin n\pi x dx = 0.
 \end{aligned}$$

Note that  $b_n = 0$  because the integrand is odd. Hence,

$$f(x) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} (-1)^n \cos(n\pi x) \blacksquare$$

**Problem 15.6**

Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -2\pi < x < -\pi \\ 0, & -\pi < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$$

**Solution.**

From the graph of  $f(x)$  we see that  $f$  is an odd function on  $(-2\pi, 2\pi)$ . Thus,

$f(x) \cos\left(\frac{nx}{2}\right)$  is odd so that  $a_n = 0$  for all  $n \in \mathbb{N}$ . Now,

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) dx = 0 \\ b_n &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) \sin\left(\frac{nx}{2}\right) dx \\ &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin\left(\frac{nx}{2}\right) dx \\ &= \frac{1}{\pi} \int_{\pi}^{2\pi} \sin\left(\frac{nx}{2}\right) dx \\ &= -\frac{2}{n\pi} \cos\left(\frac{nx}{2}\right) \Big|_{\pi}^{2\pi} \\ &= \frac{2}{n\pi} \left[ \cos\left(\frac{n\pi}{2}\right) - (-1)^n \right] \end{aligned}$$

Hence,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[ \cos\left(\frac{n\pi}{2}\right) - (-1)^n \right] \sin\left(\frac{nx}{2}\right) \blacksquare$$

### Problem 15.7

Find the Fourier series of the function

$$f(x) = \begin{cases} 1+x, & -2 \leq x \leq 0 \\ 1-x, & 0 < x \leq 2. \end{cases}$$

### Solution.

From the graph of  $f(x)$  we see that  $f$  is an even function on  $[-2, 2]$ . Thus,

$f(x) \sin\left(\frac{n\pi}{2}x\right)$  is odd so that  $b_n = 0$  for all  $n \in \mathbb{N}$ . Now,

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \left[ \int_{-2}^0 (1+x) dx + \int_0^2 (1-x) dx \right] = 0 \\ a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \int_0^2 (1-x) \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \int_0^2 \cos\left(\frac{n\pi}{2}x\right) dx - \int_0^2 x \cos\left(\frac{n\pi}{2}x\right) dx \\ &= -\frac{4}{(n\pi)^2} \cos\left(\frac{n\pi}{2}x\right) \Big|_0^2 \\ &= \frac{4}{(n\pi)^2} [1 - (-1)^n] \end{aligned}$$

Hence,

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} [1 - (-1)^n] \cos\left(\frac{n\pi}{2}x\right) \blacksquare$$

**Problem 15.8**

Show that  $f(x) = \frac{1}{x}$  is not piecewise continuous on  $[-1, 1]$ .

**Solution.**

Since the sided limits at the point of discontinuity  $x = 0$  do not exist, the function is not piecewise continuous in  $[-1, 1]$  ■

**Problem 15.9**

Assume that  $f(x)$  is continuous and has period  $2L$ . Prove that

$$\int_{-L}^L f(x) dx = \int_{-L+a}^{L+a} f(x) dx$$

is independent of  $a \in \mathbb{R}$ . In particular, it does not matter over which interval the Fourier coefficients are computed as long as the interval length is  $2L$ . [Remark: This result is also true for piecewise continuous functions].

**Solution.**

Define the function

$$g(a) = \int_{-L+a}^{L+a} f(x) dx.$$

Using the fundamental theorem of calculus, we have

$$\begin{aligned}\frac{dg}{da} &= \frac{d}{da} \int_{-L+a}^{L+a} f(x) dx \\ &= f(L+a) - f(-L+a) = f(-L+a+2L) - f(-L+a) \\ &= f(-L+a) - f(-L+a) = 0\end{aligned}$$

Hence,  $g$  is a constant function, and in particular we can write  $g(a) = g(0)$  for all  $a \in \mathbb{R}$  which gives the desired result ■

**Problem 15.10**

Consider the function  $f(x)$  defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 3 \end{cases}$$

and extended periodically with period 3 to  $\mathbb{R}$  so that  $f(x+3) = f(x)$  for all  $x$ .

- (i) Find the Fourier series of  $f(x)$ .
- (ii) Discuss its limit: In particular, does the Fourier series converge pointwise or uniformly to its limit, and what is this limit?
- (iii) Plot the graph of  $f(x)$  and the limit of the Fourier series.

**Solution.**

- (i) The Fourier series is computed for functions of period  $2L$ . Since this function has period 3,  $L = 3/2$ . By the previous problem, we can compute the coefficients over any interval of length 3, so we might as well use  $[0, 3]$ .



Using the formulas for the coefficients, we obtain:

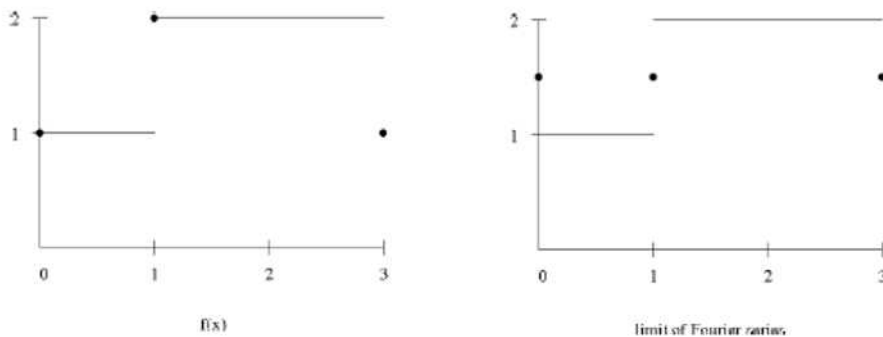
$$\begin{aligned}
 a_0 &= \frac{1}{L} \int_{-L}^L f(x) dx = \frac{2}{3} \left[ \int_0^1 dx + \int_1^3 2 dx \right] = \frac{10}{3} \\
 a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{3} \left[ \int_0^1 \cos\left(\frac{2n\pi x}{3}\right) dx + \int_1^3 2 \cos\left(\frac{2n\pi x}{3}\right) dx \right] \\
 &= \frac{2}{3} \frac{3}{2n\pi} \left[ \left( \sin\left(\frac{2n\pi}{3}\right) - 0 \right) + 2 \left( \sin 2n\pi - \sin\left(\frac{2n\pi}{3}\right) \right) \right] \\
 &= -\frac{1}{n\pi} \sin\left(\frac{2n\pi}{3}\right) \\
 b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{3} \left[ \int_0^1 \sin\left(\frac{2n\pi x}{3}\right) dx + \int_1^3 2 \sin\left(\frac{2n\pi x}{3}\right) dx \right] \\
 &= -\frac{2}{3} \frac{3}{2n\pi} \left[ \left( \cos\left(\frac{2n\pi}{3}\right) - 1 \right) + 2 \left( \cos 2n\pi - \cos\left(\frac{2n\pi}{3}\right) \right) \right] \\
 &= -\frac{1}{n\pi} \left( -\cos\left(\frac{2n\pi}{3}\right) + 1 \right)
 \end{aligned}$$

Thus, the Fourier series is

$$f(x) = \frac{10}{3} + \sum_{n=1}^{\infty} \left[ -\frac{1}{n\pi} \sin\left(\frac{2n\pi}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) - \frac{1}{n\pi} \left( -\cos\left(\frac{2n\pi}{3}\right) + 1 \right) \sin\left(\frac{2n\pi x}{3}\right) \right].$$

(ii) Using the theorem discussed in class, because this function and its derivative are piecewise continuous, the Fourier series will converge to the function at each point of continuity. At any point of discontinuity, the Fourier series will converge to the average of the left and right limits.

(iii)

**Problem 15.11**

For the following functions  $f(x)$  on the interval  $-L < x < L$ , determine the coefficients  $a_n$ ,  $n = 0, 1, 2, \dots$  and  $b_n$ ,  $n \in \mathbb{N}$  of the Fourier series expansion.

- (a)  $f(x) = 1$ .  
 (b)  $f(x) = 2 + \sin\left(\frac{\pi x}{L}\right)$ .  
 (c)  $f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$   
 (d)  $f(x) = x$ .

**Solution.**

- (a)  $a_0 = 2$ ,  $a_n = b_n = 0$  for  $n \in \mathbb{N}$ .  
 (b)  $a_0 = 4$ ,  $a_n = 0$ ,  $b_1 = 1$ , and  $b_n = 0$ .  
 (c)  $a_0 = 1$ ,  $a_n = 0$ ,  $b_n = \frac{1}{\pi n}[1 - (-1)^n]$ ,  $n \in \mathbb{N}$ .  
 (d)  $a_0 = a_n = 0$ ,  $b_n = \frac{2L}{\pi n}(-1)^{n+1}$ ,  $n \in \mathbb{N}$  ■

**Problem 15.12**

Let  $f(t)$  be the function with period  $2\pi$  defined as

$$f(t) = \begin{cases} 2 & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq 2\pi \end{cases}$$

$f(t)$  has a Fourier series and that series is equal to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Find  $\frac{a_3}{b_3}$ .

**Solution.**

We have

$$a_3 = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 2 \cos 3x dx = -\frac{2}{3\pi}$$

and

$$b_3 = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 2 \sin 3x dx = \frac{2}{3\pi}.$$

Thus,  $\frac{a_3}{b_3} = -1$  ■

**Problem 15.13**

Let  $f(x) = x^3$  on  $[-\pi, \pi]$ , extended periodically to all of  $\mathbb{R}$ . Find the Fourier coefficients  $a_n$ ,  $n = 1, 2, 3, \dots$ .

**Solution.**

Since the extension is an odd function, we must have  $a_n = 0$  for all  $n \in \mathbb{N}$  ■

**Problem 15.14**

Let  $f(x)$  be the square wave function

$$f(x) = \begin{cases} -\pi & -\pi \leq x < 0 \\ \pi & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of  $\mathbb{R}$ . To what value does the Fourier series of  $f(x)$  converge when  $x = 0$ ?

**Solution.**

$f(x)$  is piecewise smooth function with discontinuity at  $x = 0$ . Thus, the Fourier series of  $f(x)$  at  $x = 0$  converges to

$$\frac{f(0^-) + f(0^+)}{2} = \frac{-\pi + \pi}{2} = 0 \quad \blacksquare$$

**Problem 15.15**

(a) Find the Fourier series of

$$f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 2 & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of  $\mathbb{R}$ . Simplify your coefficients as much as possible.

(b) Use (a) to evaluate the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$ . Hint: Evaluate the Fourier series at  $x = \frac{\pi}{2}$ .

**Solution.**

(a) We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 dx + \int_0^{\pi} dx \right] = 3$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \left[ \frac{1}{n} - \frac{(-1)^n}{n} \right]$$

Thus,

$$f(x) = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

(b) By the convergence theorem we have

$$\frac{1}{2} \left[ f\left(\frac{\pi^-}{2}\right) + f\left(\frac{\pi^-}{2}\right) \right] = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi}{2}}{2n-1}.$$

This implies

$$2 = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

and this reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \blacksquare$$

## Solutions to Section 16

### Problem 16.1

Give an example of a function that is both even and odd.

#### Solution.

Let  $f(x)$  be such a function. Since  $f$  is both even and odd, we must have  $f(x) = -f(x)$ . This implies that  $2f(x) = 0$  and therefore  $f(x) = 0$  for all  $x$  in the domain of  $f$  ■

### Problem 16.2

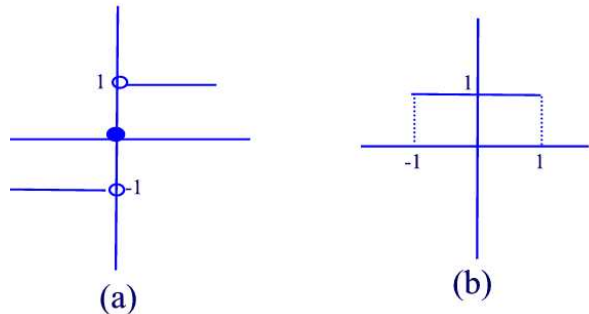
Graph the odd and even extensions of the function  $f(x) = 1$ ,  $0 \leq x \leq 1$ .

#### Solution.

We have

$$f_{\text{odd}}(x) = \begin{cases} 1 & 0 < x \leq 1 \\ -1 & -1 \leq x < 0 \\ 0 & x = 0 \end{cases}$$

and  $f_{\text{even}}(x) = 1$  for  $-1 \leq x \leq 1$ . The odd extension of  $f$  is shown in (a) while the even extension is shown in (b) ■



### Problem 16.3

Graph the odd and even extensions of the function  $f(x) = L - x$  for  $0 \leq x \leq L$ .

#### Solution.

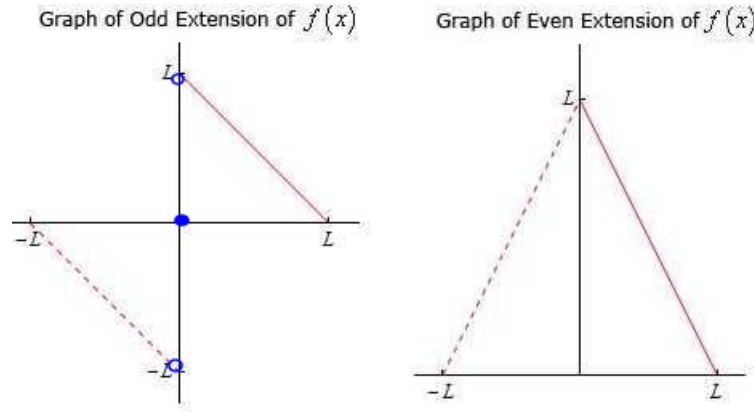
We have

$$f_{\text{odd}}(x) = \begin{cases} L - x & 0 < x \leq L \\ -L - x & -L \leq x < 0 \\ 0 & x = 0 \end{cases}$$

and

$$f_{\text{even}}(x) = \begin{cases} L - x & 0 \leq x \leq L \\ L + x & -L \leq x \leq 0 \end{cases}$$

The odd extension is shown in (a) while the even extension is shown in (b) ■



**Problem 16.4**

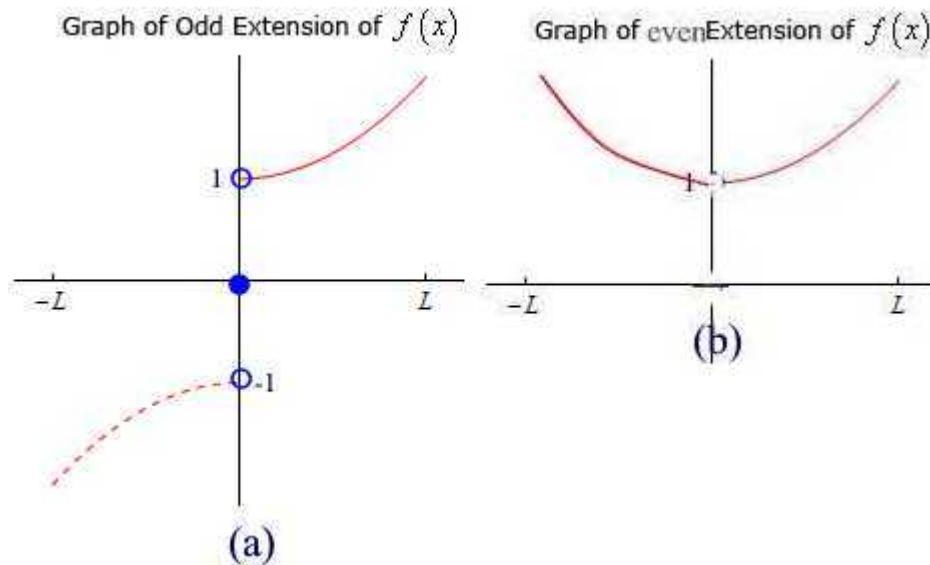
Graph the odd and even extensions of the function  $f(x) = 1 + x^2$  for  $0 \leq x \leq L$ .

**Solution.**

We have  $f_{\text{even}}(x) = 1 + x^2$  for  $-L \leq x \leq L$  while

$$f_{\text{odd}}(x) = \begin{cases} 1 + x^2 & 0 < x \leq L \\ -1 - x^2 & -L \leq x < 0 \\ 0, & x = 0 \end{cases}$$

The odd extension is shown in (a) while the even extension is shown in (b) ■



**Problem 16.5**

Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

**Solution.**

We have

$$a_0 = \frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) dx \right] = \frac{\pi}{2}$$

and for  $n \in \mathbb{N}$

$$a_n = \frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \cos nx dx \right].$$

Using integration by parts we find

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \cos nx dx &= \left[ \frac{x}{n} \sin nx \right]_0^{\frac{\pi}{2}} - \frac{1}{n} \int_0^{\frac{\pi}{2}} \sin nx dx \\ &= \frac{\pi \sin(n\pi/2)}{2n} + \frac{1}{n^2} [\cos nx]_0^{\frac{\pi}{2}} \\ &= \frac{\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2} \end{aligned}$$

while

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \cos nx dx &= \left[ \frac{(\pi - x)}{n} \sin nx \right]_{\frac{\pi}{2}}^{\pi} + \frac{1}{n} \int_{\frac{\pi}{2}}^{\pi} \sin nx dx \\ &= -\frac{\pi \sin(n\pi/2)}{2n} - \frac{1}{n^2} [\cos nx]_{\frac{\pi}{2}}^{\pi} \\ &= -\frac{\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2} \cos(n\pi). \end{aligned}$$

Thus, when  $n \in \mathbb{N}$

$$a_n = \frac{2}{\pi n^2} [2 \cos(n\pi/2) - 1 - (-1)^n],$$

and the Fourier cosine series of  $f(x)$  is

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [2 \cos(n\pi/2) - 1 - (-1)^n] \cos nx \blacksquare$$

**Problem 16.6**

Find the Fourier cosine series of  $f(x) = x$  on the interval  $[0, \pi]$ .

**Solution.**

We have

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \\ &= \frac{2}{\pi} \left[ \left[ \frac{x}{n} \sin nx \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx \right] \\ &= \frac{2}{n^2 \pi} [\cos nx]_0^{\pi} = \frac{2}{n^2 \pi} [(-1)^n - 1] \end{aligned}$$

Hence, the Fourier cosine of  $f$  is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [(-1)^n - 1] \cos nx \blacksquare$$

**Problem 16.7**

Find the Fourier sine series of  $f(x) = 1$  on the interval  $[0, \pi]$ .

**Solution.**

We have

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx dx = \frac{2}{n\pi} [1 - (-1)^n].$$

Hence, the Fourier sine series of  $f$  is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin nx \blacksquare$$

**Problem 16.8**

Find the Fourier sine series of  $f(x) = \cos x$  on the interval  $[0, \pi]$ .



**Solution.**

We have

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi \cos x \sin nx dx \\
 &= \frac{1}{\pi} \int_0^\pi [\sin(n+1)x - \sin(1-n)x] dx \\
 &= \frac{1}{\pi} \left[ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(1-n)x}{1-n} \right]_0^\pi \\
 &= \frac{2n}{\pi} \left( \frac{1 - (-1)^n}{n^2 - 1} \right)
 \end{aligned}$$

Hence, the Fourier sine series is

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} n \left( \frac{1 - (-1)^n}{n^2 - 1} \right) \sin nx \blacksquare$$

**Problem 16.9**

Find the Fourier cosine series of  $f(x) = e^{2x}$  on the interval  $[0, 1]$ .

**Solution.**

We have

$$a_0 = 2 \int_0^1 e^{2x} dx = e^2 - 1$$

and using integration by parts twice one finds

$$a_n = 2 \int_0^1 e^{2x} \cos n\pi x dx = \frac{4[(-1)^n e^2 - 1]}{4 + n^2 \pi^2}.$$

Hence, the Fourier cosine series is given by

$$f(x) = \frac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty} \frac{4[(-1)^n e^2 - 1]}{4 + n^2 \pi^2} \cos(n\pi x) \blacksquare$$

**Problem 16.10**

For the following functions on the interval  $[0, L]$ , find the coefficients  $b_n$  of the Fourier sine expansion.

- (a)  $f(x) = \sin\left(\frac{2\pi}{L}x\right)$ .
- (b)  $f(x) = 1$
- (c)  $f(x) = \cos\left(\frac{\pi}{L}x\right)$ .

**Solution.**

The coefficients  $b_n$  are given by the formula

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}.$$

- (a) If  $f(x) = \sin\left(\frac{2\pi}{L}x\right)$  then  $b_n = 0$  if  $n \neq 2$  and  $b_2 = 1$ .  
 (b) If  $f(x) = 1$  then

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{n\pi} [1 - (-1)^n].$$

- (c) If  $f(x) = \cos\left(\frac{\pi}{L}x\right)$  then

$$b_1 = \frac{2}{L} \int_0^L \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{L}x\right) dx = 0$$

and for  $n \neq 1$  we have

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{1}{2} \frac{2}{L} \int_0^L \left[ \sin\left(\frac{\pi x}{L}\right) (1+n) - \sin\left(\frac{\pi x}{L}\right) (1-n) \right] dx \\ &= \frac{1}{L} \left[ -\frac{L}{(1+n)\pi} \cos\left(\frac{\pi x}{L}\right) (1+n) + \frac{L}{(1-n)\pi} \cos\left(\frac{\pi x}{L}\right) (1-n) \right]_0^L \\ &= \frac{2n}{(n^2-1)\pi} [1 - (-1)^n] \blacksquare \end{aligned}$$

**Problem 16.11**

For the following functions on the interval  $[0, L]$ , find the coefficients  $a_n$  of the Fourier cosine expansion.

- (a)  $f(x) = 5 + \cos\left(\frac{\pi}{L}x\right)$ .  
 (b)  $f(x) = x$   
 (c)

$$f(x) = \begin{cases} 1 & 0 < x \leq \frac{L}{2} \\ 0 & \frac{L}{2} < x \leq L \end{cases}$$

**Solution.**

- (a)  $a_0 = 10$  and  $a_1 = 1$ , and  $a_n = 0$  for  $n \neq 1$ .  
 (b)  $a_0 = L$  and  $a_n = \frac{2L}{(\pi n)^2} [(-1)^n - 1]$ ,  $n \in \mathbb{N}$ .  
 (c)  $a_0 = 1$  and  $a_n = \frac{2}{\pi n} \sin\left(\frac{\pi n}{2}\right)$ ,  $n \in \mathbb{N}$  ■

**Problem 16.12**

Consider a function  $f(x)$ , defined on  $0 \leq x \leq L$ , which is even (symmetric) around  $x = \frac{L}{2}$ . Show that the even coefficients ( $n$  even) of the Fourier sine series are zero.

**Solution.**

By definition of Fourier sine coefficients,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

The symmetry around  $x = \frac{L}{2}$  can be written as

$$f\left(\frac{L}{2} + x\right) = f\left(\frac{L}{2} - x\right)$$

for all  $x \in \mathbb{R}$ . To use this symmetry it is convenient to make the change of variable  $x - \frac{L}{2} = u$  in the above integral to obtain

$$b_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2} + u\right) \sin\left[\frac{n\pi}{L}\left(\frac{L}{2} + u\right)\right] du.$$

Since  $f\left(\frac{L}{2} + u\right)$  is even in  $u$  and for  $n$  even  $\sin\left[\frac{n\pi}{L}\left(\frac{L}{2} + u\right)\right] = \sin\left(\frac{n\pi u}{L}\right)$  is odd in  $u$ , the integrand of the above integral is odd in  $u$  for  $n$  even. Since the integral is from  $-\frac{L}{2}$  to  $\frac{L}{2}$  we must have  $b_{2n} = 0$  for  $n = 0, 1, 2, \dots$  ■

**Problem 16.13**

Consider a function  $f(x)$ , defined on  $0 \leq x \leq L$ , which is odd around  $x = \frac{L}{2}$ . Show that the even coefficients ( $n$  even) of the Fourier cosine series are zero.

**Solution.**

By definition of Fourier cosine coefficients,

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

The anti-symmetry around  $x = \frac{L}{2}$  can be written as

$$f\left(\frac{L}{2} - y\right) = -f\left(\frac{L}{2} + y\right)$$

for all  $y \in \mathbb{R}$ . To use this symmetry it is convenient to make the change of variable  $x = \frac{L}{2} + y$  in the above integral to obtain

$$a_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2} + y\right) \cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] dy.$$

Since  $f\left(\frac{L}{2} + y\right)$  is odd in  $y$  and for  $n$  even  $\cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] = \pm \cos\left(\frac{n\pi y}{L}\right)$  is even in  $y$ , the integrand of the above integral is odd in  $y$  for  $n$  even. Since the integral is from  $-\frac{L}{2}$  to  $\frac{L}{2}$  we must have  $a_{2n} = 0$  for all  $n = 0, 1, 2, \dots$  ■

**Problem 16.14**

The Fourier sine series of  $f(x) = \cos\left(\frac{\pi x}{L}\right)$  for  $0 \leq x \leq L$  is given by

$$\cos\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}$$

where

$$b_1 = 0, \quad b_n = \frac{2n}{(n^2 - 1)\pi} [1 - (-1)^n].$$

Using term-by-term integration, find the Fourier cosine series of  $\sin\left(\frac{n\pi x}{L}\right)$ .

**Solution.**

Integrate both sides from 0 to  $x$  we find

$$\frac{L}{\pi} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{Lb_n}{\pi n} \left(1 - \cos\left(\frac{n\pi x}{L}\right)\right).$$

Thus,

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{a_0}{2} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos\left(\frac{n\pi x}{L}\right)$$

where

$$\frac{a_0}{2} = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

It is more convenient to calculate  $a_0$  using the definition rather than trying to sum the series;

$$a_0 = \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) dx = \frac{4}{\pi}.$$

Hence,

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1 - (-1)^n}{n^2 - 1} \cos\left(\frac{n\pi x}{L}\right) \quad \blacksquare$$

**Problem 16.15**

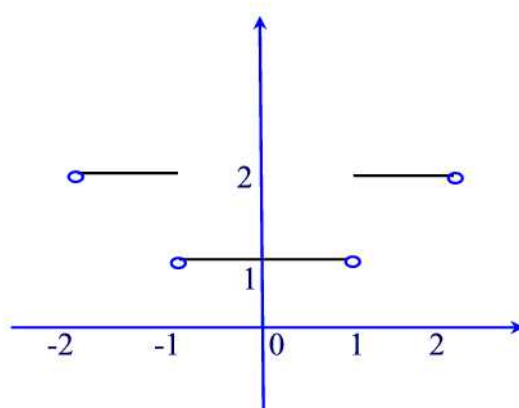
Consider the function

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 2 \end{cases}$$

- (a) Sketch the even extension of  $f$ .  
 (b) Find  $a_0$  in the Fourier series for the even extension of  $f$ .  
 (c) Find  $a_n$  ( $n = 1, 2, \dots$ ) in the Fourier series for the even extension of  $f$ .  
 (d) Find  $b_n$  in the Fourier series for the even extension of  $f$ .  
 (e) Write the Fourier series for the even extension of  $f$ .

**Solution.**

(a)



(b)  $a_0 = \frac{2}{2} \int_0^2 f(x) dx = 3.$

(c) We have

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 2 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_0^1 + 2 \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_1^2 \\ &= -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

(d)  $b_n = 0$  since  $f(x) \sin\left(\frac{n\pi x}{2}\right)$  is odd in  $-2 \leq x \leq 2$ .

(e)

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left( -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{2}\right) \blacksquare$$

## Solutions to Section 17

### Problem 17.1

Solve using the separation of variables method

$$\Delta u + \lambda u = 0.$$

### Solution.

We look for a solution of the form  $u(x, y) = X(x)Y(y)$ . Substituting in the given equation, we obtain

$$X''Y + XY'' + \lambda XY = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtract both sides for  $\frac{X''(x)}{X(x)}$ , we find:

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda = \delta.$$

where  $\delta$  is a constant. This results in the following two ODEs

$$X'' + \delta X = 0 \text{ and } Y'' + (\lambda - \delta)Y = 0.$$

- If  $\delta > 0$  and  $\lambda - \delta > 0$  then

$$\begin{aligned} X(x) &= A \cos \delta x + B \sin \delta x \\ Y(y) &= C \cos(\lambda - \delta)y + D \sin(\lambda - \delta)y \end{aligned}$$

- If  $\delta > 0$  and  $\lambda - \delta < 0$  then

$$\begin{aligned} X(x) &= A \cos \delta x + B \sin \delta x \\ Y(y) &= C e^{-\sqrt{-(\lambda-\delta)}y} + D e^{\sqrt{-(\lambda-\delta)}y} \end{aligned}$$

- If  $\delta = \lambda > 0$  then

$$\begin{aligned} X(x) &= A \cos \delta x + B \sin \delta x \\ Y(y) &= Cy + D \end{aligned}$$

- If  $\delta = \lambda < 0$  then

$$\begin{aligned} X(x) &= Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x} \\ Y(y) &= Cy + D \end{aligned}$$

- If  $\delta < 0$  and  $\lambda - \delta > 0$  then

$$\begin{aligned} X(x) &= Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x} \\ Y(y) &= C \cos(\lambda - \delta)y + D \sin(\lambda - \delta)y \end{aligned}$$

- If  $\delta < 0$  and  $\lambda - \delta < 0$  then

$$\begin{aligned} X(x) &= Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x} \\ Y(y) &= Ce^{-\sqrt{(\lambda-\delta)}y} + De^{\sqrt{(\lambda-\delta)}y} \blacksquare \end{aligned}$$

### Problem 17.2

Solve using the separation of variables method

$$u_t = ku_{xx}.$$

#### Solution.

Let's assume that the solution can be written in the form  $u(x, t) = X(x)T(t)$ . Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since  $X$  only depends on  $x$  and  $T$  only depends on  $t$ , we must have that there is a constant  $\lambda$  such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

Next, we consider the three cases of the sign of  $\lambda$ .

#### Case 1: $\lambda = 0$

In this case,  $X'' = 0$  and  $T' = 0$ . Solving these equations we find  $X(x) = ax + b$  and  $T(t) = c$ .

#### Case 2: $\lambda > 0$

In this case,  $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$  and  $T(t) = Ce^{k\lambda t}$ .

#### Case 3: $\lambda < 0$

In this case,  $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$  and  $T(t) = Ce^{k\lambda t}$  ■

**Problem 17.3**

Derive the system of ordinary differential equations for  $R(r)$  and  $\Theta(\theta)$  that is satisfied by solutions to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

**Solution.**

Suppose that a solution  $u(r, \theta)$  of the given equation can be written in the form  $u(r, \theta) = R(r)\Theta(\theta)$ . Substituting in the given equation we obtain

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Dividing by  $R\Theta$  (under the assumption that  $R\Theta \neq 0$ ) we obtain

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -r^2 \frac{R''(r)}{R(r)} - r \frac{R'(r)}{R(r)}.$$

The left-hand side is independent of  $r$  whereas the right-hand side is independent of  $\theta$  so that there is a constant  $\lambda$  such that

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} = \lambda.$$

This results in the following ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

and

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0.$$

The second equation is known as **Euler's equation** ■

**Problem 17.4**

Derive the system of ordinary differential equations and boundary conditions for  $X(x)$  and  $T(t)$  that is satisfied by solutions to

$$u_{tt} = u_{xx} - 2u, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0 = u_x(1, t) \quad t > 0$$

of the form  $u(x, t) = X(x)T(t)$ . (Note: you do not need to solve for  $X$  and  $T$ .)

**Solution.**

First, plug  $u(x, t) = X(x)T(t)$  into the equation for the boundary conditions to obtain



$$X(0)T(t) = 0 \text{ and } X'(1)T(t) = 0.$$

Since this must hold for all  $t > 0$ , we either have  $T(t) = 0$  for all  $t > 0$ , which leads to the trivial solution, so we throw this possibility out, or

$$X(0) = 0 \text{ and } X(1) = 0$$

which we keep. Plug  $u(x, t) = X(x)T(t)$  into the equation and rearrange terms to obtain

$$\frac{T''}{T} = \frac{X'' - 2X}{X}.$$

Since one side depends only on  $t$  and the other only on  $x$ , they must both be constant:

$$\frac{T''}{T} = \frac{X'' - 2X}{X} = \lambda.$$

Writing this as two separate equations, we obtain

$$\begin{aligned} X'' &= (2 + \lambda)X \\ T'' &= \lambda T. \end{aligned}$$

Thus, the final set of ODEs and boundary conditions is:

$$X'' = (2 + \lambda)X, \quad T'' = \lambda T, \quad X(0) = 0, \quad X(1) = 0 \quad \blacksquare$$

### Problem 17.5

Derive the system of ordinary differential equations and boundary conditions for  $X(x)$  and  $T(t)$  that is satisfied by solutions to

$$u_t = k u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u(x, 0) = f(x), \quad u(0, t) = 0 = u_x(L, t) \quad t > 0$$

of the form  $u(x, t) = X(x)T(t)$ . (Note: you do not need to solve for  $X$  and  $T$ .)

### Solution.

Plug  $u(x, t) = X(x)T(t)$  into the equation and rearrange terms to obtain

$$\frac{T'}{kT} = \frac{X''}{X}.$$

Since one side depends only on  $t$  and the other only on  $x$ , they must both be constant:

$$\frac{T'}{kT} = \frac{X''}{X} = \lambda.$$

Writing this as two separate equations, we obtain

$$X'' - \lambda X = 0$$

$$T' = k\lambda T.$$

Next, plug  $u(x, t) = X(x)T(t)$  into the equation for the boundary conditions to obtain

$$X'(0)T(t) = 0 \text{ and } X'(L)T(t) = 0.$$

Since this must hold for all  $t > 0$ , we either have  $T(t) = 0$  for all  $t > 0$ , which leads to the trivial solution, so we throw this possibility out, or

$$X'(0) = 0 = X'(L)$$

which we keep. Using the initial value condition  $u(x, 0) = f(x)$  we find  $X(x)T(0) = f(x)$ .

Thus, the final set of ODEs and boundary conditions is:

$$X'' - \lambda X = 0, \quad T' = k\lambda T, \quad X'(0) = 0 = X'(L) \quad \blacksquare$$

### Problem 17.6

Find all product solutions of the PDE  $u_x + u_t = 0$ .

#### Solution.

Substitute  $u(x, t) = X(x)T(t)$  into the given equation we find

$$X'(x)T(t) + X(x)T'(t) = 0.$$

Divide through by  $X(x)T(t)$  we obtain

$$\frac{X'}{X} = -\frac{T'}{T}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $t$ . This says that they must equal to a constant. That is,

$$\frac{X'}{X(x)} = -\frac{T'}{T} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X' = \lambda X \text{ and } T' = -\lambda T.$$

Solving this system of ODEs we find  $X(x) = C_1 e^{\lambda x}$  and  $T(t) = C_2 e^{-\lambda t}$ . The product solutions are given by

$$u(x, t) = C e^{\lambda(x-t)} \blacksquare$$

**Problem 17.7**

Derive the system of ordinary differential equations for  $X(x)$  and  $Y(y)$  that is satisfied by solutions to

$$3u_{yy} - 5u_{xxx} + 7u_{xy} = 0.$$

of the form  $u(x, y) = X(x)Y(y)$ .

**Solution.**

Substitute  $u(x, t) = X(x)Y(y)$  into the given equation we find

$$3XY'' - 5X'''Y' + 7X''Y' = 0.$$

Divide through by  $XY'$  we obtain

$$3\frac{Y''}{Y'} = \frac{5X''' - 7X''}{X}.$$

The left hand side is a function of  $y$  while the right hand side is a function of  $x$ . This says that they must equal to a constant. That is,

$$3\frac{Y''}{Y'} = \frac{5X''' - 7X''}{X} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$5X''' - 7X'' - \lambda X = 0 \text{ and } 3Y'' - \lambda Y' = 0 \blacksquare$$

**Problem 17.8**

Find the general solution by the method of separation of variables.

$$u_{xy} + u = 0.$$

**Solution.**

Substitute  $u(x, t) = X(x)Y(y)$  into the given equation we find

$$X'Y' + XY = 0$$

which can be separated as

$$\frac{X'}{X} = -\frac{Y}{Y'}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X'}{X} = -\frac{Y'}{Y} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X' - \lambda X = 0 \text{ and } Y' + \frac{1}{\lambda}Y = 0.$$

Solving these equations using the method of separation of variable for ODEs we find  $X(x) = Ae^{\lambda x}$  and  $Y(y) = Be^{-\frac{y}{\lambda}}$ . Thus, the general solution is given by

$$u(x, y) = Ce^{\lambda x - \frac{y}{\lambda}} \blacksquare$$

**Problem 17.9**

Find the general solution by the method of separation of variables.

$$u_x - yu_y = 0.$$

**Solution.**

Substitute  $u(x, t) = X(x)Y(y)$  into the given equation we find

$$X'Y - yXY' = 0$$

which can be separated as

$$\frac{X'}{X} = -\frac{yY'}{Y}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X'}{X} = \frac{yY'}{Y} = \lambda.$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X' - \lambda X = 0 \text{ and } yY' - \lambda Y = 0$$

Solving these equations using the method of separation of variable for ODEs we find  $X(x) = Ae^{\lambda x}$  and  $Y(y) = By^\lambda$ . Thus, the general solution is given by

$$u(x, y) = Ce^{\lambda x}y^\lambda \blacksquare$$

**Problem 17.10**

Find the general solution by the method of separation of variables.

$$u_{tt} - u_{xx} = 0.$$

**Solution.**

We look for a solution of the form  $u(x, y) = X(x)T(t)$ . Substituting in the wave equation, we obtain

$$X''(x)T(t) - X(x)T''(t) = 0.$$

Assuming  $X(x)T(t)$  is nonzero, dividing for  $X(x)T(t)$  we find:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $t$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } T'' - \lambda T = 0.$$

The solutions of these equations depend on the sign of  $\lambda$ .

- If  $\lambda > 0$  then the solutions are given

$$\begin{aligned} X(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ T(t) &= Ce^{\sqrt{\lambda}t} + De^{-\sqrt{\lambda}t} \end{aligned}$$

where  $A, B, C,$  and  $D$  are constants. In this case,

$$u(x, t) = k_1 e^{\sqrt{\lambda}(x+t)} + k_2 e^{\sqrt{\lambda}(x-t)} + k_3 e^{-\sqrt{\lambda}(x+t)} + k_4 e^{-\sqrt{\lambda}(x-t)}.$$

- If  $\lambda = 0$  then

$$\begin{aligned} X(x) &= Ax + B \\ T(t) &= Ct + D \end{aligned}$$

where  $A, B,$  and  $C$  are arbitrary constants. In this case,

$$u(x, t) = k_1 xt + k_2 x + k_3 t + k_4.$$

- If  $\lambda < 0$  then

$$\begin{aligned} X(x) &= A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x \\ T(t) &= A \cos \sqrt{-\lambda}t + B \sin \sqrt{-\lambda}t \end{aligned}$$

where  $A, B, C,$  and  $D$  are arbitrary constants. In this case,

$$u(x, t) = k_1 \cos \sqrt{-\lambda}x \cos \sqrt{-\lambda}t + k_2 \cos \sqrt{-\lambda}x \sin \sqrt{-\lambda}t + k_3 \sin \sqrt{-\lambda}x \cos \sqrt{-\lambda}t + k_4 \sin \sqrt{-\lambda}x \sin \sqrt{-\lambda}t \blacksquare$$

**Problem 17.11**

For the following PDEs find the ODEs implied by the method of separation of variables.

- (a)  $u_t = kr(ru_r)_r$
- (b)  $u_t = ku_{xx} - \alpha u$
- (c)  $u_t = ku_{xx} - au_x$
- (d)  $u_{xx} + u_{yy} = 0$
- (e)  $u_t = ku_{xxxx}$ .

**Solution.**

Details can be verified easily and therefore are omitted.

- (a)  $u(r, t) = R(r)T(t)$ ,  $T'(t) = k\lambda T$ ,  $r(rR')' = \lambda R$ .
- (b)  $u(x, t) = X(x)T(t)$ ,  $T' = \lambda T$ ,  $kX'' - (\alpha + \lambda)X = 0$ .
- (c)  $u(x, t) = X(x)T(t)$ ,  $T' = \lambda T$ ,  $kX'' - aX' = \lambda X$ .
- (d)  $u(x, t) = X(x)Y(y)$ ,  $X'' = \lambda X$ ,  $Y'' = -\lambda Y$ .
- (e)  $u(x, t) = X(x)T(t)$ ,  $T' = k\lambda T$ ,  $X'''' = \lambda X$  ■

**Problem 17.12**

Find all solutions to the following partial differential equation that can be obtained via the separation of variables.

$$u_x - u_y = 0.$$

**Solution.**

Assume  $u(x, y) = X(x)Y(y)$ . Then by substitution into the given PDE we find  $X'Y - XY' = 0$  or

$$\frac{X'}{X} = \frac{Y'}{Y}.$$

Since the left-hand side is independent of  $y$  and the right-hand side is independent from  $x$ , there must be a constant  $\lambda$  such that

$$\frac{X'}{X} = \frac{Y'}{Y} = \lambda.$$

This leads to the system of ODEs

$$X' = \lambda X, \quad Y' = \lambda Y$$

whose solution is  $X(x) = Ae^{\lambda x}$  and  $Y(y) = Be^{\lambda y}$ . Thus,  $u(x, y) = Ce^{\lambda(x+y)}$  ■

**Problem 17.13**

Separate the PDE  $u_{xx} - u_y + u_{yy} = u$  into two ODEs with a parameter. You do not need to solve the ODEs.

**Solution.**

Assume  $u(x, y) = X(x)Y(y)$ . Then by substitution into the given PDE we find  $X''Y - XY' + XY'' = XY$  or

$$\frac{X''}{X} = \frac{Y'}{Y} - \frac{Y''}{Y} + 1.$$

Since the left-hand side is independent of  $y$  and the right-hand side is independent from  $x$ , there must be a constant  $\lambda$  such that

$$\frac{X''}{X} = \frac{Y'}{Y} - \frac{Y''}{Y} + 1 = \lambda.$$

This leads to the system of ODEs

$$X'' = \lambda X, \quad Y' - Y'' + Y = \lambda Y \quad \blacksquare$$

## Solutions to Section 18

### Problem 18.1

Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is  $f(x) = \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right)$ .

#### Solution.

Let  $u(x, t)$  be the temperature of the bar. The boundary conditions are  $u(0, t) = u(2, t) = 0$  for any  $t > 0$ . The initial condition is  $u(x, 0) = \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right)$ . The solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2}x\right) e^{-\frac{n^2\pi^2}{4}t}$$

where

$$C_n = \int_0^2 \left( \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right) \right) \sin\left(\frac{n\pi}{2}x\right) dx.$$

Simple algebra shows that  $C_1 = 1$ ,  $C_5 = 3$ , and  $C_n = 0$  otherwise. Hence,

$$u(x, t) = \sin\left(\frac{\pi}{2}x\right) e^{-\frac{\pi^2}{4}t} + 3\sin\left(\frac{5\pi}{2}x\right) e^{-\frac{25\pi^2}{4}t} \blacksquare$$

### Problem 18.2

Find the temperature in a homogeneous bar of heat conducting material of length  $L$  with its end points kept at zero and initial temperature distribution given by  $f(x) = \frac{dx}{L^2}(L - x)$ ,  $0 \leq x \leq L$ .

#### Solution.

The solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{kn^2\pi^2}{L^2}t}$$



where

$$\begin{aligned}
 C_n &= \frac{2}{L} \int_0^L \frac{dx}{L^2} (L-x) \sin\left(\frac{n\pi}{L}x\right) dx \\
 &= \frac{2d}{L^3} \left[ x(L-x) \cdot \left(-\frac{L}{n\pi}\right) \cos\left(\frac{n\pi}{L}x\right) \right. \\
 &\quad \left. - (L-2x) \cdot \left(-\frac{L^2}{n^2\pi^2}\right) \cdot \sin\left(\frac{n\pi}{L}x\right) \right. \\
 &\quad \left. + (-2) \left(\frac{-L^3}{n^3\pi^3}\right) \cdot \left(-\cos\left(\frac{n\pi}{L}x\right)\right) \right]_0^L \\
 &= \frac{2d}{L^3} \left[ 0 + 0 - \frac{2L^3}{n^3\pi^3} [(-1)^n - 1] \right] \\
 &= \frac{8d}{n^3\pi^3}
 \end{aligned}$$

if  $n$  is odd and 0 otherwise. Therefore the temperature distribution in the bar is

$$u(x, t) = \frac{8d}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\left(\frac{(2n-1)\pi}{L}x\right) e^{-\frac{k(2n-1)^2\pi^2}{L^2}t} \blacksquare$$

### Problem 18.3

Find the temperature in a thin metal rod of length  $L$ , with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod  $f(x) = \sin\left(\frac{\pi}{L}x\right)$ .

### Solution.

The solution is given by

$$u(x, t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\frac{4n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned}
 C_0 &= \frac{2}{L} \int_0^L \sin\left(\frac{\pi}{L}x\right) dx \\
 &= \frac{2}{\pi} \left[ -\cos\left(\frac{\pi}{L}x\right) \right]_0^L = \frac{4}{\pi} \\
 C_1 &= \frac{2}{L} \int_0^L \sin\left(\frac{\pi}{L}x\right) \cos\left(\frac{\pi}{L}x\right) dx = 0 \\
 C_n &= \frac{2}{L} \int_0^L \sin\left(\frac{\pi}{L}x\right) \cos\left(n\frac{\pi}{L}x\right) dx \\
 &= \frac{1}{L} \int_0^L \left[ \sin\left(\frac{(n+1)\pi}{L}x\right) + \sin\left(\frac{(1-n)\pi}{L}x\right) \right] dx \\
 &= \frac{1}{L} \left[ -\frac{L}{(n+1)\pi} \cos\left(\frac{(n+1)\pi}{L}x\right) \right. \\
 &\quad \left. - \frac{L}{(1-n)\pi} \cos\left(\frac{(1-n)\pi}{L}x\right) \right]_0^L \\
 &= -\frac{2}{\pi(n^2-1)} [(-1)^{n+1} + 1] \\
 &= \frac{4}{\pi(n^2-1)}
 \end{aligned}$$

if  $n \geq 2$  is even and 0 otherwise. Thus, the temperature  $u(x, t)$  in the rod is given by

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos\left(\frac{2n\pi}{L}x\right) e^{-k\frac{4n^2\pi^2}{L^2}t} \blacksquare$$

#### Problem 18.4

Solve the following heat equation with Dirichlet boundary conditions

$$u_t = ku_{xx}$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = \begin{cases} 1 & 0 \leq x < \frac{L}{2} \\ 2 & \frac{L}{2} \leq x \leq L. \end{cases}$$

#### Solution.

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned} C_n &= \frac{2}{L} \int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) dx + \frac{4}{L} \int_{\frac{L}{2}}^L \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2}{L} \left[ -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \right]_0^{\frac{L}{2}} + \frac{4}{L} \left[ -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \right]_{\frac{L}{2}}^L \\ &= \frac{2}{n\pi} + \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \cos(n\pi) \end{aligned}$$

Thus,

$$C_n = \begin{cases} -\frac{4}{n\pi} & n = 2, 6, 10, \dots \\ 0 & n = 4, 8, 12, \dots \\ \frac{6}{n\pi} & n \text{ is odd} \blacksquare \end{cases}$$

### Problem 18.5

Solve

$$u_t = ku_{xx}$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = 6 \sin\left(\frac{9\pi}{L}x\right).$$

### Solution.

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned} C_n &= \frac{2}{L} \int_0^L 6 \sin\left(\frac{9\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= 6 \end{aligned}$$

if  $n = 9$  and 0 otherwise. Hence, the solution is given by

$$u(x, t) = 6 \sin\left(\frac{9\pi}{L}x\right) e^{-\frac{81\pi^2}{L^2}t} \blacksquare$$

**Problem 18.6**

Solve

$$u_t = ku_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= \begin{cases} 0 & 0 \leq x < \frac{L}{2} \\ 1 & \frac{L}{2} \leq x \leq L \end{cases} \end{aligned}$$

**Solution.**

The solution is given by

$$u(x, t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned} C_0 &= \frac{2}{L} \int_{\frac{L}{2}}^L dx = 1 \\ C_n &= \frac{2}{L} \int_{\frac{L}{2}}^L \cos\left(\frac{n\pi}{L}x\right) dx \\ &= -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Thus, for  $n \in \mathbb{N}$  we have

$$C_n = \begin{cases} -\frac{2}{n\pi} & n = 1, 5, 9, \dots \\ \frac{2}{n\pi} & n = 3, 7, 11, \dots \\ 0 & n \text{ is even} \end{cases}$$

So the solution is given by

$$u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

with the  $C_n$  defined as above ■**Problem 18.7**

Solve

$$u_t = ku_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= 6 + 4 \cos\left(\frac{3\pi}{L}x\right). \end{aligned}$$

**Solution.**

The solution is given by

$$u(x, t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned} C_0 &= \frac{2}{L} \int_0^L \left(6 + 4 \cos\left(\frac{3\pi}{L}x\right)\right) dx = 12 \\ C_n &= \frac{2}{L} \int_0^L \left(6 + 4 \cos\left(\frac{3\pi}{L}x\right)\right) \cos\left(\frac{n\pi}{L}x\right) dx \\ &= 4 \end{aligned}$$

if  $n = 3$  and 0 otherwise. Thus, the solution is given by

$$u(x, t) = 6 + 4 \cos\left(\frac{3\pi}{L}x\right) e^{-\frac{9\pi^2}{L^2}t} \blacksquare$$

**Problem 18.8**

Solve

$$u_t = k u_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= -3 \cos\left(\frac{8\pi}{L}x\right). \end{aligned}$$

**Solution.**

The solution is given by

$$u(x, t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

where

$$\begin{aligned} C_0 &= \frac{2}{L} \int_0^L \left[-3 \cos\left(\frac{8\pi}{L}x\right)\right] dx = 0 \\ C_n &= \frac{2}{L} \int_0^L \left(-3 \cos\left(\frac{8\pi}{L}x\right)\right) \cos\left(\frac{n\pi}{L}x\right) dx \\ &= -3 \end{aligned}$$

if  $n = 8$  and 0 otherwise. Thus, the solution is given by

$$u(x, t) = -3 \cos\left(\frac{8\pi}{L}x\right) e^{-\frac{64\pi^2}{L^2}t} \blacksquare$$

**Problem 18.9**

Find the general solution  $u(x, t)$  of

$$u_t = u_{xx} - u, \quad 0 < x < L, \quad t > 0$$

$$u_x(0, t) = 0 = u_x(L, t), \quad t > 0.$$

Briefly describe its behavior as  $t \rightarrow \infty$ .

**Solution.**

Using separation of variables and taking care to note of the boundary conditions, we see that the general solution is

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t}.$$

As  $t \rightarrow \infty$ ,  $e^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t} \rightarrow 0$  for each  $n \in \mathbb{N}$ . Hence,  $u(x, t) \rightarrow 0$  ■

**Problem 18.10** (*Energy method*)

Let  $u_1$  and  $u_2$  be two solutions to the Robin boundary value problem

$$u_t = u_{xx} - u, \quad 0 < x < 1, \quad t > 0$$

$$u_x(0, t) = u_x(1, t) = 0, \quad t > 0$$

$$u(x, 0) = g(x), \quad 0 < x < 1$$

Define  $w(x, t) = u_1(x, t) - u_2(x, t)$ .

(a) Show that  $w$  satisfies the initial value problem

$$w_t = w_{xx} - w, \quad 0 < x < 1, \quad t > 0$$

$$w(x, 0) = 0, \quad 0 < x < 1$$

(b) Define  $E(t) = \int_0^1 w^2(x, t) dx \geq 0$  for all  $t \geq 0$ . Show that  $E'(t) \leq 0$ . Hence,  $0 \leq E(t) \leq E(0)$  for all  $t > 0$ .

(c) Show that  $E(t) = 0$ ,  $w(x, t) = 0$ . Hence, conclude that  $u_1 = u_2$ .

**Solution.**

(a) Easy calculation.

(b) We have

$$\begin{aligned}
 E'(t) &= 2 \int_0^1 w(x, t) w_t(x, t) dx \\
 &= 2 \int_0^1 w(x, t) [w_{xx}(x, t) - w(x, t)] dx \\
 &= 2w(x, t)w_x(x, t)|_0^1 - 2 \left[ \int_0^1 w_x^2(x, t) dx + \int_0^1 w^2(x, t) dx \right] \\
 &= -2 \left[ \int_0^1 w_x^2(x, t) dx + \int_0^1 w^2(x, t) dx \right] \leq 0
 \end{aligned}$$

Hence,  $E$  is decreasing, and  $0 \leq E(t) \leq E(0)$  for all  $t > 0$ .

(c) Since  $w(x, 0) = 0$ , we must have  $E(0) = 0$ . Hence,  $E(t) = 0$  for all  $t \geq 0$ . This implies that  $w(x, t) = 0$  for all  $t > 0$  and all  $0 < x < 1$ . Therefore  $u_1(x, t) = u_2(x, t)$ . This means that the given problem has a unique solution ■

### Problem 18.11

Consider the heat induction in a bar where the left end temperature is maintained at 0, and the right end is perfectly insulated. We assume  $k = 1$  and  $L = 1$ .

- (a) Derive the boundary conditions of the temperature at the endpoints.  
 (b) Following the separation of variables approach, derive the ODEs for  $X$  and  $T$ .  
 (c) Consider the equation in  $X(x)$ . What are the values of  $X(0)$  and  $X(1)$ ? Show that solutions of the form  $X(x) = \sin \sqrt{-\lambda}x$ ,  $\lambda < 0$  satisfy the ODE and one of the boundary conditions. Can you choose a value of  $\lambda$  so that the other boundary condition is also satisfied?

### Solution.

(a)  $u(0, t) = 0$  and  $u_x(1, t) = 0$ .

(b) Let's assume that the solution can be written in the form  $u(x, t) = X(x)T(t)$ . Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since  $X$  only depends on  $x$  and  $T$  only depends on  $t$ , we must have that there is a constant  $\lambda$  such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0, t) = 0 = X(0)T(t) \implies X(0) = 0$$

and

$$u_x(1, t) = 0 = X'(1)T(t) \implies X'(1) = 0.$$

Note that  $T$  is not the zero function for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

(c) We have  $X' = \sqrt{-\lambda} \cos \sqrt{-\lambda}x$  and  $X'' = \lambda \sin \sqrt{-\lambda}x$ . Thus,  $X'' - \lambda X = 0$ . Moreover  $X(0) = 0$ . Now,  $X'(1) = 0$  implies  $\cos \sqrt{-\lambda} = 0$  or  $\sqrt{-\lambda} = (n - \frac{1}{2})\pi$ ,  $n \in \mathbb{N}$ . Hence,  $\lambda = - (n - \frac{1}{2})^2 \pi^2$  ■

### Problem 18.12

Using the method of separation of variables find the solution of the heat equation

$$u_t = k u_{xx}$$

satisfying the following boundary and initial conditions:

- (a)  $u(0, t) = u(L, t) = 0$ ,  $u(x, 0) = 6 \sin \left( \frac{9\pi x}{L} \right)$   
 (b)  $u(0, t) = u(L, t) = 0$ ,  $u(x, 0) = 3 \sin \left( \frac{\pi x}{L} \right) - \sin \left( \frac{3\pi x}{L} \right)$

### Solution.

(a) Let's assume that the solution can be written in the form  $u(x, t) = X(x)T(t)$ . Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on  $x$  and the RHS only depends on  $t$ , there must be a constant  $\lambda$  such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0, t) = 0 = X(0)T(t) \implies X(0) = 0$$

and

$$u(L, t) = 0 = X(L)T(t) \implies X(L) = 0.$$



Note that  $T$  is not the zero function for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Next, we consider the three cases of the sign of  $\lambda$ .

**Case 1:**  $\lambda = 0$

In this case,  $X'' = 0$ . Solving this equation we find  $X(x) = ax + b$ . Since  $X(0) = 0$  we find  $b = 0$ . Since  $X(L) = 0$  we find  $a = 0$ . Hence,  $X \equiv 0$  and  $u(x, t) \equiv 0$ . That is,  $u$  is the trivial solution.

**Case 2:**  $\lambda > 0$

In this case,  $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ . Again, the conditions  $X(0) = X(L) = 0$  imply  $A = B = 0$  and hence the solution is the trivial solution.

**Case 3:**  $\lambda < 0$

In this case,  $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$ . The condition  $X(0) = 0$  implies  $A = 0$ . The condition  $X(L) = 0$  implies  $B \sin \sqrt{-\lambda}L = 0$ . We must have  $B \neq 0$  otherwise  $X(x) = 0$  and this leads to the trivial solution. Since  $B \neq 0$ , we obtain  $\sin \sqrt{-\lambda}L = 0$  or  $\sqrt{-\lambda}L = n\pi$  where  $n \in \mathbb{N}$ . Solving for  $\lambda$  we find  $\lambda = -\frac{n^2\pi^2}{L^2}$ . Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin \frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$T' - \lambda kT = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n \in \mathbb{N}.$$

Hence, the functions

$$u_n(x, t) = C_n \sin \left( \frac{n\pi}{L}x \right) e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n \in \mathbb{N}$$

satisfy  $u_t = ku_{xx}$  and the boundary conditions  $u(0, t) = u(L, t) = 0$ .

Now, in order for these solutions to satisfy the initial value condition  $u(x, 0) = 6 \sin \left( \frac{9\pi x}{L} \right)$ , we invoke the superposition principle of linear PDE to write

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi}{L}x \right) e^{-\frac{n^2\pi^2}{L^2}kt}. \quad (18.7)$$

To determine the unknown constants  $C_n$  we use the initial condition  $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$  in (18.7) to obtain

$$6 \sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right).$$

By equating coefficients we find  $C_9 = 6$  and  $C_n = 0$  if  $n \neq 9$ . Hence, the solution to the problem is given by

$$u(x, t) = 6 \sin\left(\frac{9\pi x}{L}\right) e^{-\frac{81\pi^2}{L^2}kt}.$$

(b) Similar to (a), we find

$$u(x, t) = 3 \sin\left(\frac{\pi}{L}x\right) e^{-\frac{\pi^2}{L^2}kt} - \sin\left(\frac{3\pi}{L}x\right) e^{-\frac{9\pi^2}{L^2}kt} \blacksquare$$

### Problem 18.13

Using the method of separation of variables find the solution of the heat equation

$$u_t = k u_{xx}$$

satisfying the following boundary and initial conditions:

(a)  $u_x(0, t) = u_x(L, t) = 0$ ,  $u(x, 0) = \cos\left(\frac{\pi x}{L}\right) + 4 \cos\left(\frac{5\pi x}{L}\right)$ .

(b)  $u_x(0, t) = u_x(L, t) = 0$ ,  $u(x, 0) = 5$ .

### Solution.

(a) See the Neumann boundary case of Section 18. The answer is

$$u(x, t) = \cos\left(\frac{\pi x}{L}\right) e^{-\frac{\pi^2}{L^2}kt} + 4 \cos\left(\frac{5\pi x}{L}\right) e^{-\frac{25\pi^2}{L^2}kt}.$$

(b) The answer is

$$u(x, t) = 5 \blacksquare$$

### Problem 18.14

Find the solution of the following heat conduction partial differential equation

$$u_t = 8u_{xx}, \quad 0 < x < 4\pi, \quad t > 0$$

$$u(0, t) = u(4\pi, t) = 0, \quad t > 0$$

$$u(x, 0) = 6 \sin x, \quad 0 < x < 4\pi.$$

**Solution.**

The solution is given by the Fourier sine series

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx}{4}\right) e^{-\frac{n^2}{2}t}.$$

Using the condition  $u(x, 0) = 6 \sin x$  we find

$$6 \sin x = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx}{4}\right).$$

Thus,  $c_4 = 6$  and  $c_n = 0$  for  $n \neq 4$ . Finally,

$$u(x, t) = 6 \sin x e^{-8t} \blacksquare$$

## Solutions to Section 19

### Problem 19.1

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(a, y) = f_2(y), \\ u(0, y) = u(x, 0) = u(x, b) = 0. \end{cases}$$

### Solution.

Assume that the solution can be written in the form  $u(x, y) = X(x)Y(y)$ . Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtracting  $\frac{Y''(y)}{Y(y)}$  from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all  $0 \leq x \leq a$  and  $0 \leq y \leq b$

$$u(0, y) = 0 = X(0)Y(y) \implies X(0) = 0$$

$$u(a, y) = f_2(y) = X(a)Y(y)$$

$$u(x, 0) = 0 = X(x)Y(0) \implies Y(0) = 0$$

$$u(x, b) = 0 = X(x)Y(b) \implies Y(b) = 0$$

Note that  $X$  and  $Y$  are not the zero functions for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Consider the second equation: since  $Y'' + \lambda Y = 0$  the solution depends on the sign of  $\lambda$ . If  $\lambda = 0$  then  $Y(y) = Ay + B$ . Now, the conditions  $Y(0) = Y(b) = 0$  imply  $A = B = 0$  and so  $u \equiv 0$ . So assume that  $\lambda \neq 0$ . If  $\lambda < 0$  then  $Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}$ . Now, the condition  $Y(0) = Y(b) = 0$  imply  $A = B = 0$  and

hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have  $\lambda > 0$ . In this case,

$$Y(y) = A \cos \sqrt{\lambda}y + B \sin \sqrt{\lambda}y.$$

The condition  $Y(0) = 0$  implies  $A = 0$ . The condition  $Y(b) = 0$  implies  $B \sin \sqrt{\lambda}b = 0$ . We must have  $B \neq 0$  otherwise  $Y(y) = 0$  and this leads to the trivial solution. Since  $B \neq 0$ , we obtain  $\sin \sqrt{\lambda}b = 0$  or  $\sqrt{\lambda}b = n\pi$  where  $n \in \mathbb{N}$ . Solving for  $\lambda$  we find  $\lambda_n = \frac{n^2\pi^2}{b^2}$ . Thus, we obtain infinitely many solutions given by

$$Y_n(y) = \sin \left( \frac{n\pi}{b}y \right), \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \quad \lambda > 0$$

we obtain

$$X_n(x) = a_n e^{\sqrt{\lambda_n}x} + b_n e^{-\sqrt{\lambda_n}x} = A_n \cosh \left( \frac{n\pi}{b}x \right) + B_n \sinh \left( \frac{n\pi}{b}x \right), \quad n \in \mathbb{N}.$$

The boundary condition  $X(0) = 0$  implies  $A_n = 0$ . Hence, the functions

$$u_n(x, y) = B_n \sin \left( \frac{n\pi}{b}y \right) \sinh \left( \frac{n\pi}{b}x \right), \quad n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions  $u(0, y) = u(x, 0) = u(x, b) = 0$ .

Now, in order for these solutions to satisfy the boundary value condition  $u(a, y) = f_2(y)$ , we invoke the superposition principle of linear PDE to write

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi}{b}y \right) \sinh \left( \frac{n\pi}{b}x \right). \quad (19.8)$$

To determine the unknown constants  $B_n$  we use the boundary condition  $u(a, y) = f_2(y)$  in (19.8) to obtain

$$f_2(y) = \sum_{n=1}^{\infty} \left( B_n \sinh \left( \frac{n\pi}{b}a \right) \right) \sin \left( \frac{n\pi}{b}y \right).$$

Since the right-hand side is the Fourier sine series of  $f_2$  on the interval  $[0, b]$ , the coefficients  $B_n$  are given by

$$B_n = \left[ \frac{2}{b} \int_0^b f_2(y) \sin \left( \frac{n\pi}{b}y \right) dy \right] \left[ \sinh \left( \frac{n\pi}{b}a \right) \right]^{-1}. \quad (19.9)$$

Thus, the solution to the Laplace's equation is given by (19.8) with the  $B_n$ 's calculated from (19.9) ■

**Problem 19.2**

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = g_1(x), \\ u(0, y) = u(a, y) = u(x, b) = 0. \end{cases}$$

**Solution.**

Assume that the solution can be written in the form  $u(x, y) = X(x)Y(y)$ . Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtracting  $\frac{Y''(y)}{Y(y)}$  from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all  $0 \leq x \leq a$  and  $0 \leq y \leq b$

$$u(0, y) = 0 = X(0)Y(y) \implies X(0) = 0$$

$$u(a, y) = 0 = X(a)Y(y) \implies X(a) = 0$$

$$u(x, 0) = g_1(x) = X(x)Y(0)$$

$$u(x, b) = 0 = X(x)Y(b) \implies Y(b) = 0$$

Note that  $X$  and  $Y$  are not the zero functions for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Consider the first equation: since  $X'' - \lambda X = 0$  the solution depends on the sign of  $\lambda$ . If  $\lambda = 0$  then  $X(x) = Ax + B$ . Now, the conditions  $X(0) = X(a) = 0$  imply  $A = B = 0$  and so  $u \equiv 0$ . So assume that  $\lambda \neq 0$ . If  $\lambda > 0$  then  $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ . Now, the conditions  $X(0) = X(a) = 0$ ,  $\lambda \neq 0$  imply  $A = B = 0$  and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have  $\lambda < 0$ . In this case,

$$X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x.$$

The condition  $X(0) = 0$  implies  $A = 0$ . The condition  $X(a) = 0$  implies  $B \sin \sqrt{-\lambda}a = 0$ . We must have  $B \neq 0$  otherwise  $X(x) = 0$  and this leads to the trivial solution. Since  $B \neq 0$ , we obtain  $\sin \sqrt{-\lambda}a = 0$  or  $\sqrt{-\lambda}a = n\pi$  where  $n \in \mathbb{Z}$ . Solving for  $\lambda$  we find  $\lambda_n = -\frac{n^2\pi^2}{a^2}$ . Thus, we obtain infinitely many solutions given by

$$X_n(x) = \sin \frac{n\pi}{a}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$Y'' + \lambda Y = 0$$

we obtain

$$Y_n(y) = a_n e^{\sqrt{-\lambda_n}y} + b_n e^{-\sqrt{-\lambda_n}y} = A_n \cosh \sqrt{-\lambda_n}y + B_n \sinh \sqrt{-\lambda_n}y, \quad n \in \mathbb{N}.$$

However, this is not really suited for dealing with the boundary condition  $Y(b) = 0$ . So, let's also notice that the following is also a solution.

$$Y_n(y) = A_n \cosh \left( \frac{n\pi}{a}(y-b) \right) + B_n \sinh \left( \frac{n\pi}{a}(y-b) \right), \quad n \in \mathbb{N}.$$

Using the boundary condition  $Y(b) = 0$  we obtain  $A_n = 0$  for all  $n \in \mathbb{N}$ . Hence, the functions

$$u_n(x, y) = B_n \sin \frac{n\pi}{a}x \sinh \left( \frac{n\pi}{a}(y-b) \right), \quad n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions  $u(0, y) = u(a, y) = u(x, b) = 0$ .

Now, in order for these solutions to satisfy the boundary value condition  $u(x, 0) = g_1(x)$ , we invoke the superposition principle of linear PDE to write

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a}x \sinh \left( \frac{n\pi}{a}(y-b) \right). \quad (19.10)$$

To determine the unknown constants  $B_n$  we use the boundary condition  $u(x, 0) = g_1(x)$  in (19.10) to obtain

$$g_1(x) = \sum_{n=1}^{\infty} \left( B_n \sinh \left( -\frac{n\pi}{a}b \right) \right) \sin \frac{n\pi}{a}x.$$

Since the right-hand side is the Fourier sine series of  $f$  on the interval  $[0, a]$ , the coefficients  $B_n$  are given by

$$B_n = \left[ \frac{2}{a} \int_0^a g_1(x) \sin \left( \frac{n\pi}{a}x \right) dx \right] \left[ \sinh \left( -\frac{n\pi}{a}b \right) \right]^{-1}. \quad (19.11)$$

Thus, the solution to the Laplace's equation is given by (19.10) with the  $B_n$ 's calculated from (19.11) ■

**Problem 19.3**

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = u(0, y) = 0, \\ u(1, y) = 2y, u(x, 1) = 3 \sin \pi x + 2x \end{cases}$$

where  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Hint: Define  $U(x, y) = u(x, y) - 2xy$ .

**Solution.**

With the suggested hint we are supposed to solve the problem

$$\begin{cases} U_{xx} + U_{yy} = 0 \\ U(0, y) = U(1, y) = 0, \\ U(x, 0) = 0, U(x, 1) = 3 \sin \pi x \end{cases}$$

The solution is given by

$$U(x, y) = \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

where

$$B_n = \left[ 2 \int_0^1 3 \sin \pi x \sin n\pi x dx \right] [\sinh n\pi]^{-1}.$$

Simple integration shows that  $A_1 = \frac{3}{\sinh \pi}$  and  $A_n = 0$  otherwise. Hence,

$$U(x, y) = \frac{3}{\sinh \pi} \sin \pi x \sinh \pi y$$

and finally

$$u(x, y) = 2xy + \frac{3}{\sinh \pi} \sin \pi x \sinh \pi y \blacksquare$$

**Problem 19.4**

Show that  $u(x, y) = x^2 - y^2$  and  $u(x, y) = 2xy$  are harmonic functions.

**Solution.**

If  $u(x, y) = x^2 - y^2$  then  $u_{xx} = 2$  and  $u_{yy} = -2$  so that  $\Delta u = 0$ . If  $u(x, y) = 2xy$  then  $u_{xx} = u_{yy} = 0$  so that  $\Delta u = 0$  ■

**Problem 19.5**

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad -\frac{H}{2} \leq y \leq \frac{H}{2}$$



subject to

$$\begin{aligned} u(0, y) = u(L, y) = 0, \quad -\frac{H}{2} < y < \frac{H}{2} \\ u(x, -\frac{H}{2}) = f_1(x), \quad u(x, \frac{H}{2}) = f_2(x), \quad 0 \leq x \leq L. \end{aligned}$$

**Solution.**

Assume that the solution can be written in the form  $u(x, y) = X(x)Y(y)$ . Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtracting  $\frac{Y''(y)}{Y(y)}$  from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all  $0 \leq x \leq a$  and  $0 \leq y \leq b$

$$u(0, y) = 0 = X(0)Y(y) \implies X(0) = 0$$

$$u(L, y) = 0 = X(L)Y(y) \implies X(L) = 0$$

$$u(x, -\frac{H}{2}) = f_1(x) = X(x)Y(0)$$

$$u(x, \frac{H}{2}) = f_2(x) = X(x)Y(b)$$

Note that  $X$  and  $Y$  are not the zero functions for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Consider the first equation: since  $X'' - \lambda X = 0$  the solution depends on the sign of  $\lambda$ . If  $\lambda = 0$  then  $X(x) = Ax + B$ . Now, the conditions  $X(0) = X(L) = 0$  imply  $A = B = 0$  and so  $u \equiv 0$ . So assume that  $\lambda \neq 0$ . If  $\lambda > 0$  then  $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ . Now, the conditions  $X(0) = X(L) = 0$ ,  $\lambda \neq 0$  imply  $A = B = 0$  and hence the

solution is the trivial solution. Hence, in order to have a nontrivial solution we must have  $\lambda < 0$ . In this case,

$$X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x.$$

The condition  $X(0) = 0$  implies  $A = 0$ . The condition  $X(L) = 0$  implies  $B \sin \sqrt{-\lambda}L = 0$ . We must have  $B \neq 0$  otherwise  $X(x) = 0$  and this leads to the trivial solution. Since  $B \neq 0$ , we obtain  $\sin \sqrt{-\lambda}L = 0$  or  $\sqrt{-\lambda}L = n\pi$  where  $n \in \mathbb{Z}$ . Solving for  $\lambda$  we find  $\lambda_n = -\frac{n^2\pi^2}{L^2}$ . Thus, we obtain infinitely many solutions given by

$$X_n(x) = \sin \frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$Y'' + \lambda Y = 0$$

we obtain

$$Y_n(y) = a_n e^{\sqrt{-\lambda_n}y} + b_n e^{-\sqrt{-\lambda_n}y} = A_n \cosh \sqrt{-\lambda_n}y + B_n \sinh \sqrt{-\lambda_n}y, \quad n \in \mathbb{N}.$$

Thus, the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} [A_n \cosh \left( \frac{n\pi}{L}y \right) + B_n \sinh \left( \frac{n\pi}{L}y \right)] \sin \frac{n\pi}{L}x.$$

Now using the boundary condition  $u(x, -\frac{H}{2}) = f_1(x)$  we find

$$f_1(x) = \sum_{n=1}^{\infty} [A_n \cosh \left( -\frac{n\pi H}{2L} \right) + B_n \sinh \left( -\frac{n\pi H}{2L} \right)] \sin \frac{n\pi}{L}x$$

where

$$A_n \cosh \left( \frac{n\pi H}{2L} \right) - B_n \sinh \left( -\frac{n\pi H}{2L} \right) = \frac{2}{L} \int_0^L f_1(x) \sin \frac{n\pi}{L}x dx.$$

Likewise, using the boundary condition  $u(x, \frac{H}{2}) = f_2(x)$  we find

$$f_2(x) = \sum_{n=1}^{\infty} [A_n \cosh \left( \frac{n\pi H}{2L} \right) + B_n \sinh \left( 2\frac{n\pi H}{2L} \right)] \sin \frac{n\pi}{L}x$$

where

$$A_n \cosh \left( \frac{n\pi H}{2L} \right) + B_n \sinh \left( \frac{n\pi H}{2L} \right) = \frac{2}{L} \int_0^L f_2(x) \sin \frac{n\pi}{L}x dx.$$

Solving the above two equations in  $A_n$  and  $B_n$  we find

$$A_n = \left[ \frac{2}{L} \int_0^L (f_1(x) + f_2(x)) \sin \frac{n\pi}{L} x dx \right] \left[ \cosh \left( \frac{n\pi H}{2L} \right) \right]^{-1}$$

and

$$B_n = \left[ \frac{2}{L} \int_0^L (f_2(x) - f_1(x)) \sin \frac{n\pi}{L} x dx \right] \left[ \sinh \left( \frac{n\pi H}{2L} \right) \right]^{-1}$$

which completes the solution ■

### Problem 19.6

Consider a complex valued function  $f(z) = u(x, y) + iv(x, y)$  where  $i = \sqrt{-1}$ . We say that  $f$  is **holomorphic** or **analytic** if and only if  $f$  can be expressed as a power series in  $z$ , i.e.

$$u(x, y) + iv(x, y) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

(a) By differentiating with respect to  $x$  and  $y$  show that

$$u_x = v_y \text{ and } u_y = -v_x$$

These are known as the Cauchy-Riemann equations.

(b) Show that  $\Delta u = 0$  and  $\Delta v = 0$ .

### Solution.

(a) Differentiating term by term with respect to  $x$  we find

$$u_x + iv_x = \sum_{n=0}^{\infty} n a_n (x + iy)^{n-1}.$$

Likewise, differentiating term by term with respect to  $y$  we find

$$u_y + iv_y = \sum_{n=0}^{\infty} n a_n i (x + iy)^{n-1}.$$

Multiply this equation by  $i$  we find

$$-iu_y + v_y = \sum_{n=0}^{\infty} n a_n (x + iy)^{n-1}.$$

Hence,  $u_x + iv_x = v_y - iu_y$  which implies  $u_x = v_y$  and  $v_x = -u_y$ .

(b) We have  $u_{xx} = (v_y)_x = (v_x)_y = -u_{yy}$  so that  $\Delta u = 0$ . Similar argument for  $\Delta v = 0$  ■

**Problem 19.7**

Show that Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

**Solution.**

Polar and Cartesian coordinates are related by the expressions  $x = r \cos \theta$  and  $y = r \sin \theta$  where  $r = (x^2 + y^2)^{\frac{1}{2}}$  and  $\tan \theta = \frac{y}{x}$ . Using the chain rule we obtain

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta \\ u_{xx} &= u_{xr} r_x + u_{x\theta} \theta_x \\ &= \left( \cos \theta u_{rr} + \frac{\sin \theta}{r^2} u_\theta - \frac{\sin \theta}{r} u_{r\theta} \right) \cos \theta \\ &\quad + \left( -\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta} \right) \left( -\frac{\sin \theta}{r} \right) \\ u_y &= u_r r_y + u_\theta \theta_y = \sin \theta u_r + \frac{\cos \theta}{r} u_\theta \\ u_{yy} &= u_{yr} r_y + u_{y\theta} \theta_y \\ &= \left( \sin \theta u_{rr} - \frac{\cos \theta}{r^2} u_\theta + \frac{\cos \theta}{r} u_{r\theta} \right) \sin \theta \\ &\quad + \left( \cos \theta u_r + \sin \theta u_{r\theta} - \frac{\sin \theta}{r} u_\theta + \frac{\cos \theta}{r} u_{\theta\theta} \right) \left( \frac{\cos \theta}{r} \right) \end{aligned}$$

Substituting these equations into (19.1) we obtain the desired equation ■

**Problem 19.8**

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$u(x, 0) = 0, \quad u(x, 3) = \frac{x}{2}$$

$$u(0, y) = \sin\left(\frac{4\pi}{3}y\right), \quad u(2, y) = 7.$$

**Solution.**

We have

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y).$$

The solution  $u_1$  to the problem

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$u(x, 0) = u(x, 3) = u(0, y) = u(2, y) = 0$$

is the trivial solution, i.e.  $u_1 \equiv 0$ . The solution  $u_2$  to the problem

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$\begin{aligned} u(x, 0) &= 0, \quad u(x, 3) = \frac{x}{2} \\ u(0, y) &= u(2, y) = 0 \end{aligned}$$

is given by

$$u_2(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{2} x \sinh \left( \frac{n\pi}{2} y \right)$$

where

$$\begin{aligned} a_n &= \left[ \int_0^2 \frac{x}{2} \sin \left( \frac{n\pi}{2} x \right) dx \right] \left[ \sinh \left( \frac{3n\pi}{2} \right) \right]^{-1} \\ &= -\frac{2}{n\pi} \cdot \frac{(-1)^n}{\sinh \left( \frac{3n\pi}{2} \right)} \end{aligned}$$

Thus,

$$u_2(x, y) = \sum_{n=1}^{\infty} \left[ -\frac{2}{n\pi} \cdot \frac{(-1)^n}{\sinh \left( \frac{3n\pi}{2} \right)} \right] \sin \frac{n\pi}{2} x \sinh \left( \frac{n\pi}{2} y \right)$$

The solution  $u_3$  to the problem

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$\begin{aligned} u(x, 0) &= u(x, 3) = 0 \\ u(0, y) &= \sin \left( \frac{4\pi}{3} y \right), \quad u(2, y) = 0 \end{aligned}$$

is given by

$$u_3(x, y) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi}{3} y \right) \sinh \left( \frac{n\pi}{3} (x - 2) \right)$$

where

$$a_n = \frac{2}{3 \sinh\left(\frac{2n\pi}{3}\right)} \int_0^3 \sin\left(\frac{4\pi}{3}y\right) \sin\left(\frac{n\pi}{3}y\right) dy$$

Simple calculation shows that  $a_n = 0$  if  $n \neq 4$  and

$$a_4 = -\frac{1}{\sinh\left(\frac{8\pi}{3}\right)}.$$

Thus,

$$u_3(x, y) = \frac{1}{\sinh\left(\frac{8\pi}{3}\right)} \sinh\left(\frac{4\pi(x-2)}{3}\right) \sin\left(\frac{4\pi}{3}y\right).$$

Now, the solution  $u_3$  to the problem

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$\begin{aligned} u(x, 0) &= u(x, 3) = 0 \\ u(0, y) &= 0, \quad u(2, y) = 2 \end{aligned}$$

is given by

$$u_4(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{3}y\right) \sinh\left(\frac{n\pi}{3}x\right).$$

where

$$\begin{aligned} a_n &= \frac{2}{3 \sinh\left(\frac{2n\pi}{3}\right)} \int_0^3 7 \sin\left(\frac{n\pi}{3}y\right) dy \\ &= \frac{14(1 - (-1)^n)}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \end{aligned}$$

Hence,

$$u_4(x, y) = \sum_{n=1}^{\infty} \frac{14(1 - (-1)^n)}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \sin\left(\frac{n\pi}{3}y\right) \sinh\left(\frac{n\pi}{3}x\right) \blacksquare$$

### Problem 19.9

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad 0 \leq y \leq H$$

subject to

$$\begin{aligned} u_y(x, 0) &= 0, \quad u(x, H) = 0 \\ u(0, y) &= u(L, y) = 4 \cos\left(\frac{\pi y}{2H}\right). \end{aligned}$$

**Solution.**

Let's assume that the solution can be written in the form  $u(x, y) = X(x)Y(y)$ . Substituting in (21.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtracting  $\frac{Y''(y)}{Y(y)}$  from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all  $0 \leq x \leq L$  and  $0 \leq y \leq H$

$$u(x, H) = 0 = X(x)Y(H) \implies Y(H) = 0$$

$$u_y(x, 0) = 0 = X(x)Y'(0) \implies Y'(0) = 0$$

$$u(0, y) = 4 \cos\left(\frac{\pi y}{2H}\right) = X(0)Y(y)$$

$$u(L, y) = 4 \cos\left(\frac{\pi y}{2H}\right) = X(L)Y(y).$$

Note that  $X$  and  $Y$  are not the zero functions for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Consider the second equation: since  $Y'' + \lambda Y = 0$  the solution depends on the sign of  $\lambda$ . If  $\lambda = 0$  then  $Y(y) = Ay + B$ . Now, the conditions  $Y(H) = Y'(0) = 0$  imply  $A = B = 0$  and so  $u \equiv 0$ . So assume that  $\lambda \neq 0$ . If  $\lambda < 0$  then  $Y(y) = A \cosh \sqrt{\lambda}y + B \sinh \sqrt{\lambda}y$ . Now, the condition  $Y'(0) = 0$ ,  $\lambda \neq 0$  imply  $B = 0$ . The condition  $Y(H) = 0$  implies  $A \cosh \sqrt{\lambda}y = 0$ . Since  $\cosh x > 0$  for all  $x$  then we must have  $A = 0$  and therefore  $u \equiv 0$ .

Hence, in order to have a nontrivial solution we must have  $\lambda > 0$ . In this case,

$$Y(y) = A \cos \sqrt{\lambda}y + B \sin \sqrt{\lambda}y.$$

The condition  $Y'(0) = 0$  implies  $B = 0$ . The condition  $Y(H) = 0$  implies  $A \cos \sqrt{\lambda}H = 0$ . We must have  $A \neq 0$  otherwise  $Y(y) = 0$  and this leads to the trivial solution.

Since  $A \neq 0$ , we obtain  $\cos \sqrt{\lambda}H = 0$  or  $\sqrt{\lambda}H = (n - \frac{1}{2})\pi$  where  $n \in \mathbb{N}$ . Solving for  $\lambda$  we find  $\lambda_n = (n - \frac{1}{2})^2 \frac{\pi^2}{H^2}$ . Thus, we obtain infinitely many solutions given by

$$Y_n(x) = A \cos \left[ \left( n - \frac{1}{2} \right) \frac{\pi}{H} y \right], \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \quad \lambda > 0$$

we obtain

$$X_n(x) = a_n \sinh \sqrt{\lambda_n}x + b_n \sinh \sqrt{\lambda_n}(x - L), \quad n \in \mathbb{N}.$$

Hence, the general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} [A_n \sinh \sqrt{\lambda_n}x + B_n \sinh \sqrt{\lambda_n}(x - L)] \cos \sqrt{\lambda_n}y.$$

Using the boundary conditions  $u(0, y) = u(L, y) = 4 \cos \left( \frac{\pi y}{2H} \right)$  we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} B_n \sinh \sqrt{\lambda_n}(-L) \cos \sqrt{\lambda_n}y &= 4 \cos \left( \frac{\pi y}{2H} \right) \\ \sum_{n=1}^{\infty} A_n \sinh \sqrt{\lambda_n}L \cos \sqrt{\lambda_n}y &= 4 \cos \left( \frac{\pi y}{2H} \right) \end{aligned}$$

Comparing coefficients we find

$$-B_1 \sinh \frac{\pi L}{2H} = 4 \quad \text{and} \quad A_1 \sinh \frac{\pi L}{2H} = 4$$

and zero for  $n \neq 1$ . Hence,

$$u(x, y) = \frac{4}{\sinh \left( \frac{\pi L}{2H} \right)} \left\{ \sinh \left( \frac{\pi x}{2H} \right) - \sinh \left( \frac{\pi(x-L)}{2H} \right) \right\} \cos \frac{\pi y}{2H} \quad \blacksquare$$

### Problem 19.10

Solve

$$u_{xx} + u_{yy} = 0, \quad x > 0, \quad 0 \leq y \leq H$$

subject to

$$u(0, y) = f(y), \quad |u(x, 0)| < \infty$$

$$u_y(x, 0) = u_y(x, H) = 0.$$



**Solution.**

Let's assume that the solution can be written in the form  $u(x, y) = X(x)Y(y)$ . Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming  $X(x)Y(y)$  is nonzero, dividing for  $X(x)Y(y)$  and subtracting  $\frac{Y''(y)}{Y(y)}$  from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of  $x$  while the right hand side is a function of  $y$ . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where  $\lambda$  is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all  $x > 0$  and  $0 \leq y \leq H$

$$u(0, y) = f(y) = X(0)Y(y)$$

$$u_y(x, 0) = 0 = X(x)Y'(0) \implies Y'(0) = 0$$

$$u_y(x, H) = 0 = X(x)Y'(H) \implies Y'(H) = 0.$$

Note that  $X$  and  $Y$  are not the zero functions for otherwise  $u \equiv 0$  and this contradicts our assumption that  $u$  is the non-trivial solution.

Consider the second equation: since  $Y'' + \lambda Y = 0$  the solution depends on the sign of  $\lambda$ . If  $\lambda = 0$  then  $Y(y) = Ay + B$ . Now, the condition  $Y'(H) = 0$  implies  $A = 0$ . Hence,  $u \equiv C$ . But clearly we are looking for a non-constant solution. So assume that  $\lambda \neq 0$ . If  $\lambda < 0$  then  $Y(y) = A \cosh \sqrt{\lambda}y + B \sinh \sqrt{\lambda}y$ . Now, the condition  $Y'(0) = 0$ ,  $\lambda \neq 0$  imply  $B = 0$ . The condition  $Y'(H) = 0$  implies  $A \sinh \sqrt{\lambda}H = 0$  which implies that  $\lambda = 0$ .

Hence, in order to have a nontrivial solution we must have  $\lambda > 0$ . In this case,

$$Y(y) = A \cos \sqrt{\lambda}y + B \sin \sqrt{\lambda}y.$$

The condition  $Y'(0) = 0$  implies  $B = 0$ . The condition  $Y'(H) = 0$  implies  $A \sin \sqrt{\lambda}H = 0$ . We must have  $A \neq 0$  otherwise  $Y(y) = 0$  and this leads to the trivial solution. Since  $A \neq 0$ , we obtain  $\sin \sqrt{\lambda}H = 0$  or  $\sqrt{\lambda}H = n\pi$  where

$n \in \mathbb{Z}$ . Solving for  $\lambda$  we find  $\lambda_n = \frac{n^2\pi^2}{H^2}$ . Thus, we obtain infinitely many solutions given by

$$Y_n(x) = A \cos \frac{n\pi}{H}y, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \quad \lambda > 0$$

we obtain

$$X_n(x) = a_n e^{\sqrt{\lambda_n}x} + b_n e^{-\sqrt{\lambda_n}x}, \quad n \in \mathbb{N}.$$

Since the solution must be bounded, we must have  $a_n = 0$ . Hence,  $X_n(x) = b_n e^{-\sqrt{\lambda_n}x}$ .

Hence, the general solution is given by

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\sqrt{\lambda_n}x} \cos \sqrt{\lambda_n}y.$$

Using the boundary conditions  $u(0, y) = f(y)$  we obtain

$$\sum_{n=0}^{\infty} A_n \cos \sqrt{\lambda_n}y = f(y)$$

This is the Fourier cosine series of  $f$ . Hence,

$$A_0 = \frac{1}{H} \int_0^H f(y) dy$$

$$A_n = \frac{2}{H} \int_0^H f(y) \cos \frac{n\pi}{H}y dy \quad \blacksquare$$

### Problem 19.11

Consider Laplace's equation inside a rectangle

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad 0 \leq y \leq H$$

subject to the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) - u_y(x, 0) = 0, \quad u(x, H) = 20 \sin\left(\frac{\pi x}{L}\right) - 5 \sin\left(\frac{3\pi x}{L}\right).$$

Find the solution  $u(x, y)$ .

**Solution.**

Look for solutions of the form  $u(x, y) = X(x)Y(y)$ . Separation of variables gives

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

and

$$Y'' - \lambda Y = 0, \quad Y(0) - Y'(0) = 0.$$

From the first set of equations find eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}.$$

Solving the problem for  $Y(y)$  we find

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right), \quad n \in \mathbb{N}.$$

Using the condition  $Y(0) - Y'(0) = 0$  we find  $A_n = B_n \left(\frac{n\pi}{L}\right)$  and

$$Y_n(y) = B_n \left(\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right), \quad n \in \mathbb{N}.$$

Using the superposition principle we find

$$u(x, t) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right) \sin\left(\frac{n\pi}{L}x\right)$$

Using the boundary condition

$$u(x, H) = 20 \sin\left(\frac{\pi x}{L}\right) - 5 \sin\left(\frac{3\pi x}{L}\right)$$

we find

$$u(x, y) = \frac{20}{Y_1(H)} Y_1(y) \sin\left(\frac{\pi x}{L}\right) - \frac{5}{Y_3(H)} \sin\left(\frac{3\pi x}{L}\right) \blacksquare$$

**Problem 19.12**

Solve Laplace's equation  $u_{xx} + u_{yy} = 0$  in the rectangle  $0 < x, y < 1$  subject to the conditions

$$\begin{aligned} u(0, y) &= u(1, y) = 0 \\ u(x, 0) &= \sin(2\pi x), \quad u_x(x, 0) = -2\pi \sin(2\pi x). \end{aligned}$$

**Solution.**

The answer is  $u(x, y) = \sin(2\pi x)e^{-2\pi y}$  (detail left to the reader)  $\blacksquare$

**Problem 19.13**

Find the solution to Laplace's equation on the rectangle  $0 < x < 1, 0 < y < 1$  with boundary conditions

$$\begin{aligned}u(x, 0) &= 0, & u(x, 1) &= 1 \\u_x(0, y) &= u_x(1, y) = 0.\end{aligned}$$

**Solution.**

The answer is  $u(x, y) = y$  (detail left to the reader) ■

**Problem 19.14**

Solve Laplace's equation on the rectangle  $0 < x < a, 0 < y < b$  with the boundary conditions

$$\begin{aligned}u_x(0, y) &= -a, & u_x(a, y) &= 0 \\u_y(x, 0) &= b, & u_y(x, b) &= 0.\end{aligned}$$

**Solution.**

The answer is  $u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - axby + C$  where  $C$  is an arbitrary constant (detail left to the reader) ■

**Problem 19.15**

Solve Laplace's equation on the rectangle  $0 < x < \pi, 0 < y < 2$  with the boundary conditions

$$\begin{aligned}u(0, y) &= u(\pi, y) = 0 \\u_y(x, 0) &= 0, & u_y(x, 2) &= 2 \sin 3x - 5 \sin 10x.\end{aligned}$$

**Solution.**

The answer is

$$u(x, y) = \frac{2 \cosh 3y \sin 3x}{\cosh 6} - \frac{5 \cosh 10y \sin 10x}{\cosh 20}.$$

The details are left to the reader ■

## Solutions to Section 20

### Problem 20.1

Solve the Laplace's equation in the unit disk with  $u(1, \theta) = 3 \sin 5\theta$ .

#### Solution.

We have

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

so that

$$u(1, \theta) = C_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = 3 \sin 5\theta.$$

Comparing coefficients we find  $C_0 = A_n = 0$  for all  $n \in \mathbb{N}$  and  $B_n = 0$  for all  $n \neq 5$  and  $B_5 = 3$ . Thus, the solution to the problem is

$$u(r, \theta) = 3r^5 \sin 5\theta \blacksquare$$

### Problem 20.2

Solve the Laplace's equation in the upper half of the unit disk with  $u(1, \theta) = \pi - \theta$ .

#### Solution.

We have

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

where

$$\begin{aligned} C_0 &= \frac{1}{2\pi} \int_0^\pi (\pi - \theta) d\theta = \frac{\pi}{4} \\ A_n &= \frac{1}{\pi} \int_0^\pi (\pi - \theta) \cos n\theta d\theta = \frac{1 - (-1)^n}{n^2\pi} \\ B_n &= \frac{1}{\pi} \int_0^\pi (\pi - \theta) \sin n\theta d\theta = \frac{1}{n} \end{aligned}$$

Thus, the solution to the problem is

$$u(r, \theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} r^n \left[ \frac{1 - (-1)^n}{n^2\pi} \cos n\theta + \frac{\sin n\theta}{n} \right] \blacksquare$$

### Problem 20.3

Solve the Laplace's equation in the unit disk with  $u_r(1, \theta) = 2 \cos 2\theta$ .

**Solution.**

We have

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with

$$u_r(1, \theta) = \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta) = 2 \cos 2\theta.$$

Expanding this series and equating coefficients of like terms in both sides we find  $A_n = 0$  for  $n \neq 2$  and  $A_2 = 2$ . Moreover,  $B_n = 0$  for all  $n \in \mathbb{N}$ . Hence, the solution to the problem is

$$u(r, \theta) = C_0 + r^2 \cos 2\theta \blacksquare$$

**Problem 20.4**

Consider

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with

$$\begin{aligned} C_0 &= \frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi \\ A_n &= \frac{a_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \cos n\phi d\phi, \quad n = 1, 2, \dots \\ B_n &= \frac{b_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \sin n\phi d\phi, \quad n = 1, 2, \dots \end{aligned}$$

Using the trigonometric identity

$$\cos a \cos b + \sin a \sin b = \cos(a - b)$$

show that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[ 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi.$$

**Solution.**

Substituting  $C_0$ ,  $A_n$ , and  $B_n$  into the right-hand side of  $u(r, \theta)$  we find

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} f(\phi) [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[ 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi \blacksquare \end{aligned}$$

**Problem 20.5**

(a) Using Euler's formula from complex analysis  $e^{it} = \cos t + i \sin t$  show that

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}),$$

where  $i = \sqrt{-1}$ .

(b) Show that

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = 1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta - \phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta - \phi)}.$$

(c) Let  $q_1 = \frac{r}{a}e^{i(\theta - \phi)} = \frac{r}{a}[\cos(\theta - \phi) + i \sin(\theta - \phi)]$  and  $q_2 = \frac{r}{a}e^{-i(\theta - \phi)} = \frac{r}{a}[\cos(\theta - \phi) - i \sin(\theta - \phi)]$ . It is defined in complex analysis that the absolute value of a complex number  $z = x + iy$  is given by  $|z| = (x^2 + y^2)^{\frac{1}{2}}$ . Using these concepts, show that  $|q_1| < 1$  and  $|q_2| < 1$ .

**Solution.**

(a) We have  $e^{it} = \cos t + i \sin t$  and  $e^{-it} = \cos t - i \sin t$ . The result follows by adding these two equalities and dividing by 2.

(b) This follows from the fact that

$$\cos n(\theta - \phi) = \frac{1}{2}(e^{in(\theta - \phi)} + e^{-in(\theta - \phi)}).$$

(c) We have  $|q_1| = \frac{r}{a} \sqrt{\cos^2(\theta - \phi) + \sin^2(\theta - \phi)} = \frac{r}{a} < 1$  since  $0 < r < a$ . A similar argument shows that  $|q_2| < 1$  ■

**Problem 20.6**

(a) Show that

$$\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta - \phi)} = \frac{re^{i(\theta - \phi)}}{a - re^{i(\theta - \phi)}}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta - \phi)} = \frac{re^{-i(\theta - \phi)}}{a - re^{-i(\theta - \phi)}}$$

Hint: Each sum is a geometric series with a ratio less than 1 in absolute value so that these series converges.

(b) Show that

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2}.$$

**Solution.**

(a) The first sum is a convergent geometric series with ratio  $q_1$  and sum

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} &= \frac{\frac{r}{a} e^{i(\theta-\phi)}}{1 - q_1} \\ &= \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \end{aligned}$$

Similar argument for the second sum.

(b) We have

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) &= 1 + \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \\ &\quad + \frac{r e^{-i(\theta-\phi)}}{a - r e^{-i(\theta-\phi)}} \\ &= 1 + \frac{r}{a e^{-i(\theta-\phi)} - r} + \frac{r}{a e^{-i(\theta-\phi)} - r} \\ &= 1 + \frac{r}{a \cos(\theta - \phi) - r - ai \sin(\theta - \phi)} \\ &\quad + \frac{r}{a \cos(\theta - \phi) - r + ai \sin(\theta - \phi)} \\ &= 1 + \frac{r[a \cos(\theta - \phi) - r + ai \sin(\theta - \phi)]}{a^2 + 2ar \cos(\theta - \phi) + r^2} \\ &\quad + \frac{r[a \cos(\theta - \phi) - r - ai \sin(\theta - \phi)]}{a^2 - 2ar \cos(\theta - \phi) + r^2} \\ &= \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2} \blacksquare \end{aligned}$$

**Problem 20.7**

Show that

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi.$$

This is known as the **Poisson formula** in polar coordinates.



**Solution.**

We have

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[ 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi \\ &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi \blacksquare \end{aligned}$$

**Problem 20.8**

Solve

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

subject to

$$u(1, \theta) = \theta, \quad -\pi \leq \theta \leq \pi.$$

**Solution.**

We have

$$\begin{aligned} C_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta d\theta = 0 \\ A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos n\theta d\theta = 0 \\ B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin n\theta d\theta = 2(-1)^{n+1} \end{aligned}$$

Hence,

$$u(r, \theta) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\sin n\theta}{n} \blacksquare$$

**Problem 20.9**

The vibrations of a symmetric circular membrane where the displacement  $u(r, t)$  depends on  $r$  and  $t$  only can be describe by the one-dimensional wave equation in polar coordinates

$$u_{tt} = c^2 \left( u_{rr} + \frac{1}{r} u_r \right), \quad 0 < r < a, \quad t > 0$$

with initial condition

$$u(a, t) = 0, \quad t > 0$$

and boundary conditions

$$u(r, 0) = f(r), \quad u_t(r, 0) = g(r), \quad 0 < r < a.$$

(a) Show that the assumption  $u(r, t) = R(r)T(t)$  leads to the equation

$$\frac{1}{c^2} \frac{T''}{T} = \frac{1}{R} R'' + \frac{1}{r} \frac{R'}{R} = \lambda.$$

(b) Show that  $\lambda < 0$ .

**Solution.**

(a) Differentiating  $u(r, t) = R(r)T(t)$  with respect to  $r$  and  $t$  we find

$$u_{tt} = RT'' \quad \text{and} \quad u_r = R'T \quad \text{and} \quad u_{rr} = R''T.$$

Substituting these into the given PDE we find

$$RT'' = c^2 \left( R''T + \frac{1}{r} R'T \right)$$

Dividing both sides by  $c^2 RT$  we find

$$\frac{1}{c^2} \frac{T''}{T} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R}.$$

Since the RHS of the above equation depends on  $r$  only, and the LHS depends on  $t$  only, they must equal to a constant  $\lambda$ .

(b) The given boundary conditions imply

$$u(a, t) = 0 = R(a)T(t) \implies R(a) = 0$$

$$u(r, 0) = f(r) = R(r)T(0)$$

$$u_t(r, 0) = g(r) = R(r)T'(0).$$

If  $\lambda = 0$  then  $R'' + \frac{1}{r}R' = 0$  and this implies  $R(r) = C \ln r$ . Using the condition  $R(a) = 0$  we find  $C = 0$  so that  $R(r) = 0$  and hence  $u \equiv 0$ . If  $\lambda > 0$  then  $T'' - \lambda c^2 T = 0$ . This equation has the solution

$$T(t) = A \cos(c\sqrt{\lambda}t) + B \sin(c\sqrt{\lambda}t).$$

The condition  $u(r, 0) = f(r)$  implies that  $A = f(r)$  which is not possible. Hence,  $\lambda < 0$  ■

**Problem 20.10**

Cartesian coordinates and cylindrical coordinates are shown in Figure 22.1 below.

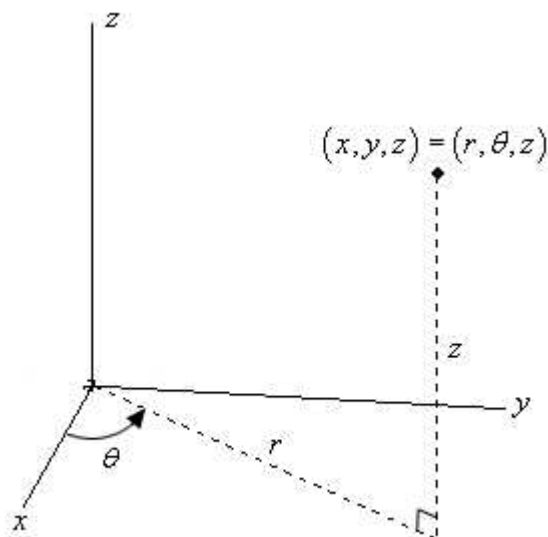


Figure 20.1

(a) Show that  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = z$ .

(b) Show that

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

**Solution.**

(a) Follows from the figure and the definitions of trigonometric functions in a right triangle.

(b) The result follows from Equation (20.1) ■

**Problem 20.11**

An important result about harmonic functions is the so-called the **maximum principle** which states: Any harmonic function  $u(x, y)$  defined in a domain  $\Omega$  satisfies the inequality

$$\min_{(x,y) \in \partial\Omega} u(x, y) \leq u(x, y) \leq \max_{(x,y) \in \partial\Omega} u(x, y), \quad \forall (x, y) \in \Omega$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$ .

Let  $u$  be harmonic in  $\Omega = \{(x, y) : x^2 + y^2 < 1\}$  and satisfies  $u(x, y) = 2 - x$  for all  $(x, y) \in \partial\Omega$ . Show that  $u(x, y) > 0$  for all  $(x, y) \in \Omega$ .

**Solution.**

By the maximum principle we have

$$\min_{(x,y) \in \partial\Omega} u(x,y) \leq u(x,y) \leq \max_{(x,y) \in \partial\Omega} u(x,y), \quad \forall (x,y) \in \Omega$$

But  $\min_{(x,y) \in \partial\Omega} u(x,y) = u(1,0) = 1$  and  $\max_{(x,y) \in \partial\Omega} u(x,y) = u(-1,0) = 3$ . Hence,

$$1 \leq u(x,y) \leq 3$$

and this implies that  $u(x,y) > 0$  for all  $(x,y) \in \Omega$  ■

**Problem 20.12**

Let  $u$  be harmonic in  $\Omega = \{(x,y) : x^2 + y^2 < 1\}$  and satisfies  $u(x,y) = 1 + 3x$  for all  $(x,y) \in \partial\Omega$ . Determine

(i)  $\max_{(x,y) \in \Omega} u(x,y)$

(ii)  $\min_{(x,y) \in \Omega} u(x,y)$

without solving  $\Delta u = 0$ .

**Solution.**

(i) The solution is not constant because it is not constant on the boundary. Therefore, the maximum is achieved on the boundary. The maximum value of the boundary data is  $u(1,0) = 4$ , which is therefore also the maximum value of the solution.

(ii) Similar to above, the minimum is achieved on the boundary, and is  $u(-1,0) = -2$  ■

**Problem 20.13**

Let  $u_1(x,y)$  and  $u_2(x,y)$  be harmonic functions on a smooth domain  $\Omega$  such that

$$u_1|_{\partial\Omega} = g_1(x,y) \text{ and } u_2|_{\partial\Omega} = g_2(x,y)$$

where  $g_1$  and  $g_2$  are continuous functions satisfying

$$\max_{(x,y) \in \partial\Omega} g_1(x,y) < \min_{(x,y) \in \partial\Omega} g_2(x,y).$$

Prove that  $u_1(x,y) < u_2(x,y)$  for all  $(x,y) \in \Omega \cup \partial\Omega$ .

**Solution.**

Using the maximum principle and the hypothesis on  $g_1$  and  $g_2$ , for all  $(x,y) \in$

$\Omega \cup \partial\Omega$  we have

$$\begin{aligned}
 \min_{(x,y) \in \partial\Omega} u_1(x,y) &= \min_{(x,y) \in \partial\Omega} g_1(x,y) \\
 &\leq u_1(x,y) \leq \max_{(x,y) \in \partial\Omega} u_1(x,y) \\
 &= \max_{(x,y) \in \partial\Omega} g_1(x,y) < \max_{(x,y) \in \partial\Omega} g_2(x,y) \\
 &\leq \min_{(x,y) \in \partial\Omega} g_1(x,y) = \min_{(x,y) \in \partial\Omega} u_2(x,y) \\
 &\leq u_2(x,y) \leq \max_{(x,y) \in \partial\Omega} u_2(x,y) = \max_{(x,y) \in \partial\Omega} g_2(x,y) \blacksquare
 \end{aligned}$$

**Problem 20.14**

Show that  $r^n \cos(n\theta)$  and  $r^n \sin(n\theta)$  satisfy Laplace's equation in polar coordinates.

**Solution.**

We have

$$\begin{aligned}
 \Delta(r^n \cos(n\theta)) &= \frac{\partial^2}{\partial r^2}(r^n \cos(n\theta)) + \frac{1}{r} \frac{\partial}{\partial r}(r^n \cos(n\theta)) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}(r^n \cos(n\theta)) \\
 &= n(n-1)r^{n-2} \cos(n\theta) + nr^{n-2} \cos(n\theta) - r^{n-2}n^2 \cos(n\theta) = 0
 \end{aligned}$$

Likewise,  $\Delta(r^n \sin(n\theta)) = 0 \blacksquare$

**Problem 20.15**

Solve the Dirichlet problem

$$\begin{aligned}
 \Delta u &= 0, \quad 0 \leq r < a, \quad -\pi \leq \theta \leq \pi \\
 u(a, \theta) &= \sin^2 \theta.
 \end{aligned}$$

**Solution.**

A solution has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta)$$

where

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 \theta d\theta = 1 \\
 a_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} \sin^2 \theta \cos n\theta d\theta = 0
 \end{aligned}$$

if  $n \neq 2$ . If  $n = 2$  we find  $a_2 = -\frac{1}{2a^2}$ . On the other hand, since  $\sin^2 \theta \sin n\theta$  is odd we have  $b_n = 0$  for all  $n \in \mathbb{N}$ . Thus, solution to the Dirichlet problem is

$$u(r, \theta) = \frac{1}{2} - \frac{r^2}{2a^2} \cos 2\theta \blacksquare$$

**Problem 20.16**

Solve Laplace's equation

$$u_{xx} + u_{yy} = 0$$

outside a circular disk ( $r \geq a$ ) subject to the boundary condition

$$u(a, \theta) = \ln 2 + 4 \cos 3\theta.$$

You may assume that the solution remains bounded as  $r \rightarrow \infty$ .

**Solution.**

Solving the problem the way we did for the inside the circle we find

$$\Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad n = 0, 1, 2, \dots$$

and

$$R_0 = C_0 \ln r + D_0, \quad R_n = C_n r^n + D_n r^{-n}, \quad n \in \mathbb{N}.$$

We use the condition that the solution remains bounded as  $r \rightarrow \infty$  (that is  $C_n = 0$ ) we find

$$u(r, \theta) = \ln 2 + 4 \left(\frac{a}{r}\right)^3 \cos 3\theta \blacksquare$$

## Solutions to Section 21

### Problem 21.1

Determine whether the integral  $\int_0^\infty \frac{1}{1+t^2} dt$  converges. If the integral converges, give its value.

#### Solution.

We have

$$\begin{aligned} \int_0^\infty \frac{1}{1+t^2} dt &= \lim_{A \rightarrow \infty} \int_0^A \frac{1}{1+t^2} dt = \lim_{A \rightarrow \infty} [\arctan t]_0^A \\ &= \lim_{A \rightarrow \infty} \arctan A = \frac{\pi}{2} \end{aligned}$$

So the integral is convergent ■

### Problem 21.2

Determine whether the integral  $\int_0^\infty \frac{t}{1+t^2} dt$  converges. If the integral converges, give its value.

#### Solution.

We have

$$\begin{aligned} \int_0^\infty \frac{t}{1+t^2} dt &= \frac{1}{2} \lim_{A \rightarrow \infty} \int_0^A \frac{2t}{1+t^2} dt = \frac{1}{2} \lim_{A \rightarrow \infty} [\ln(1+t^2)]_0^A \\ &= \frac{1}{2} \lim_{A \rightarrow \infty} \ln(1+A^2) = \infty \end{aligned}$$

Hence, the integral is divergent ■

### Problem 21.3

Determine whether the integral  $\int_0^\infty e^{-t} \cos(e^{-t}) dt$  converges. If the integral converges, give its value.

#### Solution.

Using substitution we find

$$\begin{aligned} \int_0^\infty e^{-t} \cos(e^{-t}) dt &= \lim_{A \rightarrow \infty} \int_1^{e^{-A}} -\cos u du \\ &= \lim_{A \rightarrow \infty} [-\sin u]_1^{e^{-A}} = \lim_{A \rightarrow \infty} [\sin 1 - \sin(e^{-A})] \\ &= \sin 1 \end{aligned}$$

Hence, the integral is convergent ■

**Problem 21.4**

Using the definition, find  $\mathcal{L}[e^{3t}]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

**Solution.**

We have

$$\begin{aligned}\mathcal{L}[e^{3t}] &= \lim_{A \rightarrow \infty} \int_0^A e^{3t} e^{-st} dt = \lim_{A \rightarrow \infty} \int_0^A e^{t(3-s)} dt \\ &= \lim_{A \rightarrow \infty} \left[ \frac{e^{t(3-s)}}{3-s} \right]_0^A \\ &= \lim_{A \rightarrow \infty} \left[ \frac{e^{A(3-s)}}{3-s} - \frac{1}{3-s} \right] \\ &= \frac{1}{s-3}, \quad s > 3 \blacksquare\end{aligned}$$

**Problem 21.5**

Using the definition, find  $\mathcal{L}[t-5]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

**Solution.**

Using integration by parts we find

$$\begin{aligned}\mathcal{L}[t-5] &= \lim_{A \rightarrow \infty} \int_0^A (t-5)e^{-st} dt = \lim_{A \rightarrow \infty} \left\{ \left[ \frac{-(t-5)e^{-st}}{s} \right]_0^A + \frac{1}{s} \int_0^A e^{-st} dt \right\} \\ &= \lim_{A \rightarrow \infty} \left\{ \frac{-(A-5)e^{-sA} + 5}{s} - \left[ \frac{e^{-st}}{s^2} \right]_0^A \right\} \\ &= \frac{1}{s^2} - \frac{5}{s}, \quad s > 0 \blacksquare\end{aligned}$$

**Problem 21.6**

Using the definition, find  $\mathcal{L}[e^{(t-1)^2}]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

**Solution.**

We have

$$\int_0^{\infty} e^{(t-1)^2} e^{-st} dt = \int_0^{\infty} e^{(t-1)^2 - st} dt.$$



Since  $\lim_{t \rightarrow \infty} (t-1)^2 - st = \lim_{t \rightarrow \infty} t^2 \left(1 - \frac{(2+s)}{t} + \frac{1}{t^2}\right) = \infty$ , for any fixed  $s$  we can choose a positive  $C$  such that  $(t-1)^2 - st \geq 0$ . In this case,  $e^{(t-1)^2 - st} \geq 1$  and this implies that  $\int_0^\infty e^{(t-1)^2 - st} dt \geq \int_C^\infty dt$ . The integral on the right is divergent so that the integral on the left is also divergent by the comparison theorem of improper integrals. Hence,  $f(t) = e^{(t-1)^2}$  does not have a Laplace transform ■

**Problem 21.7**

Using the definition, find  $\mathcal{L}[(t-2)^2]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

**Solution.**

We have

$$\mathcal{L}[(t-2)^2] = \lim_{T \rightarrow \infty} \int_0^T (t-2)^2 e^{-st} dt.$$

Using integration by parts with  $u' = e^{-st}$  and  $v = (t-2)^2$  we find

$$\begin{aligned} \int_0^T (t-2)^2 e^{-st} dt &= - \left[ \frac{(t-2)^2 e^{-st}}{s} \right]_0^T + \frac{2}{s} \int_0^T (t-2) e^{-st} dt \\ &= \frac{4}{s} - \frac{(T-2)^2 e^{-sT}}{s} + \frac{2}{s} \int_0^T (t-2) e^{-st} dt. \end{aligned}$$

Thus,

$$\lim_{T \rightarrow \infty} \int_0^T (t-2)^2 e^{-st} dt = \frac{4}{s} + \frac{2}{s} \lim_{T \rightarrow \infty} \int_0^T (t-2) e^{-st} dt$$

Using by parts with  $u' = e^{-st}$  and  $v = t-2$  we find

$$\int_0^T (t-2) e^{-st} dt = \left[ -\frac{(t-2) e^{-st}}{s} + \frac{1}{s^2} e^{-st} \right]_0^T.$$

Letting  $T \rightarrow \infty$  in the above expression we find

$$\lim_{T \rightarrow \infty} \int_0^T (t-2) e^{-st} dt = -\frac{2}{s} + \frac{1}{s^2}, \quad s > 0.$$

Hence,

$$F(s) = \frac{4}{s} + \frac{2}{s} \left( -\frac{2}{s} + \frac{1}{s^2} \right) = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0 \quad \blacksquare$$

**Problem 21.8**

Using the definition, find  $\mathcal{L}[f(t)]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t-1, & t \geq 1 \end{cases}$$

**Solution.**

We have

$$\mathcal{L}[f(t)] = \lim_{T \rightarrow \infty} \int_1^T (t-1)e^{-st} dt.$$

Using integration by parts with  $u' = e^{-st}$  and  $v = t - 1$  we find

$$\lim_{T \rightarrow \infty} \int_1^T (t-1)e^{-st} dt = \lim_{T \rightarrow \infty} \left[ -\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st} \right]_1^T = \frac{e^{-s}}{s^2}, \quad s > 0 \blacksquare$$

**Problem 21.9**

Using the definition, find  $\mathcal{L}[f(t)]$ , if it exists. If the Laplace transform exists then find the domain of  $F(s)$ .

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t-1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

**Solution.**

We have

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_1^2 (t-1)e^{-st} dt = \left[ -\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st} \right]_1^2 \\ &= -\frac{e^{-2s}}{s} + \frac{1}{s^2}(e^{-s} - e^{-2s}), \quad s \neq 0 \blacksquare \end{aligned}$$

**Problem 21.10**

Let  $n$  be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, \quad s > 0.$$

**Solution.**

Let  $u' = e^{-st}$  and  $v = t^n$ . Then  $u = -\frac{e^{-st}}{s}$  and  $v' = nt^{n-1}$ . Hence,

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, \quad s > 0 \blacksquare$$

**Problem 21.11**

For  $s > 0$  and  $n$  a positive integer evaluate the limits

$$(a) \lim_{t \rightarrow 0} t^n e^{-st} \quad (b) \lim_{t \rightarrow \infty} t^n e^{-st}$$

**Solution.**

(a)  $\lim_{t \rightarrow 0} t^n e^{-st} = \lim_{t \rightarrow 0} \frac{t^n}{e^{st}} = \frac{0}{1} = 0.$

(b) Using L'Hôpital's rule repeatedly we find

$$\lim_{t \rightarrow \infty} t^n e^{-st} = \dots = \lim_{t \rightarrow \infty} \frac{n!}{s^n e^{st}} = 0 \blacksquare$$

**Problem 21.12**

Use the linearity property of Laplace transform to find  $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$ . Find the domain of  $F(s)$ .

**Solution.**

We have  $\mathcal{L}[e^{-7t}] = \frac{1}{s+7}$ ,  $s > -7$ ,  $\mathcal{L}[t] = \frac{1}{s^2}$ ,  $s > 0$ , and  $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$ ,  $s > 2$ . Hence,

$$\mathcal{L}[5e^{-7t} + t + 2e^{2t}] = 5\mathcal{L}[e^{-7t}] + \mathcal{L}[t] + 2\mathcal{L}[e^{2t}] = \frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2}, \quad s > 2 \blacksquare$$

**Problem 21.13**

Find  $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$ .

**Solution.**

Since  $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a}$ ,  $s > a$  then

$$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 3e^{2t}, \quad t \geq 0 \blacksquare$$

**Problem 21.14**

Find  $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$ .

**Solution.**

Since  $\mathcal{L}[t] = \frac{1}{s^2}$ ,  $s > 0$  and  $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a}$ ,  $s > a$ , we find

$$\begin{aligned} \mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right) &= -2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= -2t + e^{-t}, \quad t \geq 0 \blacksquare \end{aligned}$$

**Problem 21.15**

Find  $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$ .

**Solution.**

We have

$$\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 2(e^{-2t} + e^{2t}), \quad t \geq 0 \blacksquare$$

**Problem 21.16**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[2e^t + 5]$ .

**Solution.**

$$\mathcal{L}[2e^t + 5] = 2\mathcal{L}[e^t] + 5\mathcal{L}[1] = \frac{2}{s-1} + \frac{5}{s}, \quad s > 1 \blacksquare$$

**Problem 21.17**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{3t-3}H(t-1)]$ .

**Solution.**

$$\mathcal{L}[e^{3t-3}H(t-1)] = \mathcal{L}[e^{3(t-1)}H(t-1)] = e^{-s}\mathcal{L}[e^{3t}] = \frac{e^{-s}}{s-3}, \quad s > 3 \blacksquare$$

**Problem 21.18**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[\sin^2 \omega t]$ .

**Solution.**

$$\mathcal{L}[\sin^2 \omega t] = \mathcal{L}\left[\frac{1 - \cos 2\omega t}{2}\right] = \frac{1}{2}(\mathcal{L}[1] - \mathcal{L}[\cos 2\omega t]) = \frac{1}{2}\left(\frac{1}{s} - \frac{s^2}{s^2 + 4\omega^2}\right), \quad s > 0 \blacksquare$$

**Problem 21.19**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[\sin 3t \cos 3t]$ .

**Solution.**

$$\mathcal{L}[\sin 3t \cos 3t] = \mathcal{L}\left[\frac{\sin 6t}{2}\right] = \frac{1}{2}\mathcal{L}[\sin 6t] = \frac{3}{s^2 + 26}, \quad s > 0 \blacksquare$$

**Problem 21.20**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{2t} \cos 3t]$ .

**Solution.**

$$\mathcal{L}[e^{2t} \cos 3t] = \frac{s-3}{(s-3)^2 + 9}, \quad s > 3 \blacksquare$$

**Problem 21.21**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$ .

**Solution.**

$$\mathcal{L}[e^{4t}(t^2+3t+5)] = \mathcal{L}[e^{4t}t^2] + 3\mathcal{L}[e^{4t}t] + 5\mathcal{L}[1] = \frac{2}{(s-4)^3} + \frac{3}{(s-4)^2} + \frac{5}{s-4}, \quad s > 4 \blacksquare$$

**Problem 21.22**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$ .

**Solution.**

$$\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}] = 2\mathcal{L}^{-1}[\frac{5}{s^2+25}] + 4\mathcal{L}^{-1}[\frac{1}{s-3}] = 2\sin 5t + 4e^{3t}, \quad t \geq 0 \blacksquare$$

**Problem 21.23**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$ .

**Solution.**

$$\mathcal{L}^{-1}[\frac{5}{(s-3)^4}] = \frac{5}{6}\mathcal{L}^{-1}[\frac{3!}{(s-3)^4}] = \frac{5}{6}e^{3t}t^3, \quad t \geq 0 \blacksquare$$

**Problem 21.24**

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$ .

**Solution.**

$$\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}] = e^{9(t-2)}H(t-2) = \begin{cases} 0, & 0 \leq t < 2 \\ e^{9(t-2)}, & t \geq 2 \end{cases} \blacksquare$$

**Problem 21.25**

Using the partial fraction decomposition find  $\mathcal{L}^{-1}[\frac{12}{(s-3)(s+1)}]$ .

**Solution.**

Write

$$\frac{12}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

Multiply both sides of this equation by  $s-3$  and cancel common factors to obtain

$$\frac{12}{s+1} = A + \frac{B(s-3)}{s+1}.$$

Now, find  $A$  by setting  $s = 3$  to obtain  $A = 3$ . Similarly, by multiplying both sides by  $s + 1$  and then setting  $s = -1$  in the resulting equation leads to  $B = -3$ . Hence,

$$\frac{12}{(s-3)(s+1)} = 3 \left( \frac{1}{s-3} - \frac{1}{s+1} \right)$$

Finally,

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{12}{(s-3)(s+1)} \right] &= 3\mathcal{L}^{-1} \left[ \frac{1}{s-3} \right] - 3\mathcal{L}^{-1} \left[ \frac{1}{s+1} \right] \\ &= 3e^{3t} - 3e^{-t}, \quad t \geq 0 \quad \blacksquare \end{aligned}$$

**Problem 21.26**

Using the partial fraction decomposition find  $\mathcal{L}^{-1} \left[ \frac{24e^{-5s}}{s^2-9} \right]$ .

**Solution.**

Write

$$\frac{24}{(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3}$$

Multiply both sides of this equation by  $s - 3$  and cancel common factors to obtain

$$\frac{24}{s+3} = A + \frac{B(s-3)}{s+3}.$$

Now, find  $A$  by setting  $s = 3$  to obtain  $A = 4$ . Similarly, by multiplying both sides by  $s + 3$  and then setting  $s = -3$  in the resulting equation leads to  $B = -4$ . Hence,

$$\frac{24}{(s-3)(s+3)} = 4 \left( \frac{1}{s-3} - \frac{1}{s+3} \right)$$

Finally,

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{24e^{-5s}}{(s-3)(s+3)} \right] &= 4\mathcal{L}^{-1} \left[ \frac{e^{-5s}}{s-3} \right] - 4\mathcal{L}^{-1} \left[ \frac{e^{-5s}}{s+3} \right] \\ &= 4[e^{3(t-5)} - e^{-3(t-5)}]H(t-5), \quad t \geq 0 \quad \blacksquare \end{aligned}$$

**Problem 21.27**

Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 12, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

**Solution.**

Note first that  $g(t) = 12[H(t-1) - H(t-3)]$  so that

$$\mathcal{L}[g(t)] = 12\mathcal{L}[H(t-1)] - 12\mathcal{L}[H(t-3)] = \frac{12(e^{-s} - e^{-3s})}{s}, \quad s > 0.$$

Now taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[g(t)].$$

But  $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$ . Letting  $\mathcal{L}[y] = Y(s)$  we obtain

$$sY(s) - 2 + 4Y(s) = 12\frac{e^{-s} - e^{-3s}}{s}.$$

Solving for  $Y(s)$  we find

$$Y(s) = \frac{2}{s+4} + 12\frac{e^{-s} - e^{-3s}}{s(s+4)}.$$

But

$$\mathcal{L}^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}$$

and

$$\begin{aligned} \mathcal{L}^{-1}\left[12\frac{e^{-s} - e^{-3s}}{s(s+4)}\right] &= 3\mathcal{L}^{-1}\left[(e^{-s} - e^{-3s})\left(\frac{1}{s} - \frac{1}{s+4}\right)\right] \\ &= 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+4}\right] + 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+4}\right] \\ &= 3H(t-1) - 3H(t-3) - 3e^{-4(t-1)}H(t-1) + 3e^{-4(t-3)}H(t-3) \end{aligned}$$

Hence,

$$y(t) = 2e^{-4t} + 3[H(t-1) - H(t-3)] - 3[e^{-4(t-1)}H(t-1) - e^{-4(t-3)}H(t-3)], \quad t \geq 0 \blacksquare$$

**Problem 21.28**

Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

**Solution.**

Taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y''] - 4\mathcal{L}[y] = \mathcal{L}[e^{3t}].$$

But  $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2\mathcal{L}[y]$ . Letting  $\mathcal{L}[y] = Y(s)$  we obtain

$$s^2Y(s) - 4Y(s) = \frac{1}{s-3}.$$

Solving for  $Y(s)$  we find

$$Y(s) = \frac{1}{(s-3)(s-2)(s+2)}.$$

Using partial fraction decomposition

$$\frac{1}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s-2}$$

we find  $A = \frac{1}{5}$ ,  $B = \frac{1}{20}$ , and  $C = -\frac{1}{4}$ . Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s-3)(s-2)(s+2)}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= \frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t}, \quad t \geq 0 \quad \blacksquare \end{aligned}$$

### Problem 21.29

Consider the functions  $f(t) = e^t$  and  $g(t) = e^{-2t}$ ,  $t \geq 0$ . Compute  $f * g$  in two different ways.

- (a) By directly evaluating the integral.  
 (b) By computing  $\mathcal{L}^{-1}[F(s)G(s)]$  where  $F(s) = \mathcal{L}[f(t)]$  and  $G(s) = \mathcal{L}[g(t)]$ .

### Solution.

- (a) We have

$$\begin{aligned} (f * g)(t) &= \int_0^t f(t-s)g(s)ds = \int_0^t e^{(t-s)}e^{-2s}ds \\ &= e^t \int_0^t e^{-3s}ds = \left[\frac{e^{(t-3s)}}{-3}\right]_0^t \\ &= \frac{e^t - e^{-2t}}{3} \end{aligned}$$

- (b) Since  $F(s) = \mathcal{L}[e^t] = \frac{1}{s-1}$  and  $G(s) = \mathcal{L}[e^{-2t}] = \frac{1}{s+2}$  we find  $(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s+2)}\right]$ . Using partial fractions decomposition we find

$$\frac{1}{(s-1)(s+2)} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right).$$



Thus,

$$(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \frac{1}{3}(\mathcal{L}^{-1}[\frac{1}{s-1}] - \mathcal{L}^{-1}[\frac{1}{s+2}]) = \frac{e^t - e^{-2t}}{3} \blacksquare$$

**Problem 21.30**

Consider the functions  $f(t) = \sin t$  and  $g(t) = \cos t$ ,  $t \geq 0$ . Compute  $f * g$  in two different ways.

(a) By directly evaluating the integral.

(b) By computing  $\mathcal{L}^{-1}[F(s)G(s)]$  where  $F(s) = \mathcal{L}[f(t)]$  and  $G(s) = \mathcal{L}[g(t)]$ .

**Solution.**

(a) Using the trigonometric identity  $2 \sin p \cos q = \sin(p+q) + \sin(p-q)$  we find that  $2 \sin(t-s) \cos s = \sin t + \sin(t-2s)$ . Hence,

$$\begin{aligned} (f * g)(t) &= \int_0^t f(t-s)g(s)ds = \int_0^t \sin(t-s) \cos s ds \\ &= \frac{1}{2} \left[ \int_0^t \sin t ds + \int_0^t \sin(t-2s) ds \right] \\ &= \frac{t \sin t}{2} + \frac{1}{4} \int_{-t}^t \sin u du \\ &= \frac{t \sin t}{2} \end{aligned}$$

(b) Since  $F(s) = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$  and  $G(s) = \mathcal{L}[\cos t] = \frac{s}{s^2+1}$  we find

$$(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2} \sin t \blacksquare$$

**Problem 21.31**

Compute  $t * t * t$ .

**Solution.**

By the convolution theorem we have  $\mathcal{L}[t * t * t] = (\mathcal{L}[t])^3 = \left(\frac{1}{s^2}\right)^3 = \frac{1}{s^6}$ . Hence,  $t * t * t = \mathcal{L}^{-1}\left[\frac{1}{s^6}\right] = \frac{t^5}{5!} = \frac{t^5}{120} \blacksquare$

**Problem 21.32**

Compute  $H(t) * e^{-t} * e^{-2t}$ .

**Solution.**

By the convolution theorem we have  $\mathcal{L}[H(t) * e^{-t} * e^{-2t}] = \mathcal{L}[H(t)]\mathcal{L}[e^{-t}]\mathcal{L}[e^{-2t}] = \frac{1}{s} \cdot \frac{1}{s+1} \cdot \frac{1}{s+2}$ . Using the partial fractions decomposition we can write

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2}.$$

Hence,

$$H(t) * e^{-t} * e^{-2t} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \blacksquare$$

**Problem 21.33**

Compute  $t * e^{-t} * e^t$ .

**Solution.**

By the convolution theorem we have  $\mathcal{L}[t * e^{-t} * e^t] = \mathcal{L}[t]\mathcal{L}[e^{-t}]\mathcal{L}[e^t] = \frac{1}{s^2} \cdot \frac{1}{s+1} \cdot \frac{1}{s-1}$ . Using the partial fractions decomposition we can write

$$\frac{1}{s^2(s+1)(s-1)} = -\frac{1}{s^2} - \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1}.$$

Hence,

$$t * e^{-t} * e^t = -t + \frac{e^t}{2} - \frac{e^{-t}}{2} \blacksquare$$

## Solutions to Section 22

### Problem 22.1

Solve by Laplace transform

$$\begin{cases} u_t + u_x = 0 & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

Hint: Method of integrating factor of ODEs.

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) + U_x(x, s) = 0$$

or

$$U_x(x, s) + sU(x, s) = \sin x$$

with boundary condition  $U(0, t) = 0$ . Solving this initial value ODE by the method of integrating (details omitted) we find the unique solution

$$U(x, s) = \frac{1}{s^2 + 1} [s \sin x - \cos x + e^{-sx}].$$

Taking inverse Laplace transform we find

$$u(x, t) = \sin(x - t) - H(t - x) \sin(x - t) \blacksquare$$

### Problem 22.2

Solve by Laplace transform

$$\begin{cases} u_t + u_x = -u & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) + U_x(x, s) = -U(x, s)$$

or

$$U_x(x, s) + (s + 1)U(x, s) = \sin x$$

with boundary condition  $U(0, t) = 0$ . Solving this initial value ODE by the method of integrating factor we find the unique solution

$$U(x, s) = \frac{1}{s^2 + 2s + 2} [(s + 1) \sin x - \cos x + e^{-(s+1)x}].$$

Taking inverse Laplace transform we find

$$u(x, t) = [\sin(x - t) - H(t - x) \sin(x - t)] e^{-t} \blacksquare$$

### Problem 22.3

Solve

$$u_t = 4u_{xx}$$

$$u(0, t) = u(1, t) = 0$$

$$u(x, 0) = 2 \sin \pi x + 3 \sin 2\pi x.$$

Hint: A particular solution of a second order ODE must be found using the method of variation of parameters.

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - 4U_{xx}(x, s) = 0$$

or

$$4U_{xx}(x, s) - sU(x, s) = -2 \sin \pi x - 3 \sin 2\pi x.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{\frac{\sqrt{s}}{2}x} + B(s)e^{-\frac{\sqrt{s}}{2}x} + \frac{2 \sin \pi x}{s + 4\pi^2} + \frac{6 \sin 2\pi x}{s + 16\pi^2}.$$

Next, we apply Laplace transform to the boundary condition obtaining  $U(0, s) = U(1, s) = \mathcal{L}(0) = 0$ . These lead to  $A(s) + B(s) = 0$  and  $A(s)e^{\frac{\sqrt{s}}{2}} + B(s)e^{-\frac{\sqrt{s}}{2}} = 0$ . Solving these equations we find  $A(s) = B(s) = 0$  and the transformed solution becomes

$$U(x, s) = \frac{2 \sin \pi x}{s + 4\pi^2} + \frac{6 \sin 2\pi x}{s + 16\pi^2}.$$

Now, taking inverse Laplace transform we find

$$u(x, t) = 2e^{-4\pi^2 t} \sin \pi x + 6e^{-16\pi^2 t} \sin 2\pi x \blacksquare$$

**Problem 22.4**

Solve by Laplace transform

$$\begin{cases} u_t + u_x = u & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) + U_x(x, s) = U(x, s)$$

or

$$U_x(x, s) + (s - 1)U(x, s) = -\sin x$$

with boundary condition  $U(0, t) = 0$ . Solving this initial value ODE by the method of integrating factor we find the unique solution

$$U(x, s) = \frac{1}{s^2 - 2s + 2} [(s - 1) \sin x - \cos x + e^{-(s-1)x}].$$

Taking inverse Laplace transform we find

$$u(x, t) = [\sin(x - t) - H(t - x) \sin(x - t)]e^t \blacksquare$$

**Problem 22.5**

Solve by Laplace transform

$$\begin{cases} u_t + u_x = t & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = t^2 \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) + U_x(x, s) = \frac{1}{s^2}$$

or

$$U_x(x, s) + sU(x, s) = \frac{1}{s^2}$$

with boundary condition  $U(0, t) = \frac{2}{t^3}$ . Solving this initial value ODE by the method of integrating factor we find the unique solution

$$U(x, s) = \left( \frac{2}{s^3} - \frac{1}{s^2} \right) e^{-x} + \frac{1}{s^2}.$$

Taking inverse Laplace transform we find

$$u(x, t) = t^2 e^{-x} - t e^{-x} + t \blacksquare$$

**Problem 22.6**

Solve by Laplace transform

$$\begin{cases} xu_t + u_x = 0 & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = t \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$xsU(x, s) - xu(x, 0) + U_x(x, s) = 0$$

or

$$U_x(x, s) + xsU(x, s) = 0$$

with boundary condition  $U(0, t) = \frac{1}{t^2}$ . Solving this ODE by the method of separation of variables we find

$$U(x, s) = A(s)e^{-\frac{sx^2}{2}}.$$

Using the boundary condition we find  $A(s) = \frac{1}{s^2}$ . Hence

$$U(x, s) = \frac{e^{-\frac{sx^2}{2}}}{s^2}.$$

Taking inverse Laplace transform we find

$$u(x, t) = \left(t - \frac{1}{2}x^2\right) H\left(t - \frac{1}{2}x^2\right) \blacksquare$$

**Problem 22.7**

Solve by Laplace transform

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, \\ u(0, t) = \sin x, \\ |u(x, t)| < \infty \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$s^2U(x, s) - su(x, 0) - u_t(x, 0) - c^2U_{xx}(x, s) = 0$$

or

$$c^2U_{xx}(x, s) - s^2U(x, s) = 0.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x}.$$

Since  $U(x, s)$  is bounded, we must have  $B(s) = 0$  and in this case we obtain

$$U(x, s) = A(s)e^{-\frac{s}{c}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0, s) = \mathcal{L}(\sin x) = \frac{1}{s^2 + 1}.$$

This leads to  $A(s) = \frac{1}{s^2 + 1}$  and the transformed solution becomes

$$U(x, s) = \frac{e^{-\frac{s}{c}x}}{s^2 + 1}.$$

Thus,

$$u(x, t) = \mathcal{L}^{-1} \left( \frac{e^{-\frac{s}{c}x}}{s^2 + 1} \right) = H \left( t - \frac{x}{c} \right) \sin \left( t - \frac{x}{c} \right) \blacksquare$$

### Problem 22.8

Solve by Laplace transform

$$u_{tt} - 9u_{xx} = 0, \quad 0 \leq x \leq \pi, \quad t > 0$$

$$u(0, t) = u(\pi, t) = 0,$$

$$u_t(x, 0) = 0, \quad u(x, 0) = 2 \sin x.$$

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) - 9U_{xx}(x, s) = 0$$

or

$$9U_{xx}(x, s) - s^2 U(x, s) = -2s \sin x.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{\frac{s}{3}x} + B(s)e^{-\frac{s}{3}x} + \frac{2s \sin x}{s^2 + 9}.$$

Next, we apply Laplace transform to the boundary condition  $u(0, t) = u(\pi, t) = 0$  obtaining

$$U(0, s) = U(\pi, s) = \mathcal{L}(0) = 0.$$

This leads to  $A(s) + B(s) = 0$  and  $A(s)e^s + B(s)e^{-s} = 0$ . Solving these equations we find  $A(s) = B(s) = 0$  and the transformed solution becomes

$$U(x, s) = \frac{2s \sin x}{s^2 + 9}.$$

Using the inverse Laplace transform we find

$$u(x, t) = 2 \sin x \cos 3t \blacksquare$$

### Problem 22.9

Solve by Laplace transform

$$\begin{cases} u_{xy} = 1 & , x > 0, y > 0 \\ u(x, 0) = 1, \\ u(0, y) = y + 1. \end{cases}$$

#### Solution.

First we note that  $u(x, 0) = 1$  implies  $u_x(x, 0) = 0$ . Using Laplace transform in  $y$  we obtain

$$sU_x(x, s) - u_x(x, 0) = \frac{1}{s}$$

or

$$U_x(x, s) = \frac{1}{s^2}.$$

Solving this equation we find

$$U(x, s) = \frac{x}{s^2} + C(s).$$

Now we can apply the BC to obtain

$$U(0, s) = \frac{1}{s^2} + \frac{1}{s} = C(s).$$

Hence,

$$U(x, s) = \frac{1}{s^2}(x + 1) + \frac{1}{s}.$$

Taking the inverse Laplace transform we find

$$u(x, y) = y(x + 1) + 1 \blacksquare$$



**Problem 22.10**

Solve by Laplace transform

$$\begin{cases} u_{tt} = c^2 u_{xx} & , x > 0, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, \\ u_x(0, t) = f(t), \\ |u(x, t)| < \infty. \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) - c^2 U_{xx}(x, s) = 0$$

or

$$c^2 U_{xx}(x, s) - s^2 U(x, s) = 0.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x}.$$

Since  $U(x, s)$  is bounded, we must have  $B(s) = 0$  and in this case we obtain

$$U(x, s) = A(s)e^{-\frac{s}{c}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U_x(0, s) = \mathcal{L}(f(t)) = F(s).$$

This leads to  $A(s) = -\frac{cF(s)}{s}$  and the transformed solution becomes

$$U(x, s) = -\frac{cF(s)}{s}e^{-\frac{s}{c}x}.$$

Using the integration property and the translation property, we find that,

$$u(x, t) = \mathcal{L}^{-1} \left[ -\frac{cF(s)}{s}e^{-\frac{s}{c}x} \right] = -c \int_0^{t-\frac{x}{c}} f(\tau) d\tau.$$

Thus,

$$u(x, t) = \mathcal{L}^{-1} \left( \frac{e^{-\frac{s}{c}x}}{s^2 + 1} \right) = h \left( t - \frac{x}{c} \right) \sin \left( t - \frac{x}{c} \right) \blacksquare$$

**Problem 22.11**

Solve by Laplace transform

$$\begin{cases} u_t - u_x = u & , x > 0, t > 0 \\ u(x, 0) = e^{-5x}, \\ |u(x, t)| < \infty \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - U_x(x, s) = U(x, s)$$

or

$$U_x(x, s) - (s - 1)U(x, s) = -e^{-5x}$$

Solving this ODE by the method of integrating factor we find general solution

$$U(x, s) = \frac{e^{-5x}}{s + 4} + C(s)e^{(s-1)x}$$

Since  $s$  is arbitrary and  $U$  is bounded we must have  $C(s) = 0$ . Hence, we obtain the transformed solution

$$U(x, s) = \frac{e^{-5x}}{s + 4}.$$

Taking inverse Laplace transform we find

$$u(x, t) = e^{-5x}e^{-4t}H(t) \blacksquare$$

**Problem 22.12**

Solve by Laplace transform

$$\begin{cases} u_t - c^2u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = T, \\ u(0, t) = 0, \\ |u(x, t)| < \infty \end{cases}$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - c^2U_{xx}(x, s) = 0$$

or

$$c^2U_{xx}(x, s) - sU(x, s) = -T.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{\sqrt{s}}{c}x} + B(s)e^{\frac{\sqrt{s}}{c}x} + \frac{T}{s}.$$

Since  $U(x, s)$  is bounded in both variables, we must have  $B(s) = 0$  and in this case we obtain

$$U(x, s) = A(s)e^{-\sqrt{s}x} + \frac{T}{s}.$$

Next, we apply Laplace transform to the boundary condition obtaining  $U(0, s) = \mathcal{L}(0) = 0$ . This leads to  $A(s) = -\frac{T}{s}$  and the transformed solution becomes

$$U(x, s) = -\frac{T}{s}e^{-\frac{\sqrt{s}}{c}x} + \frac{T}{s}.$$

Thus,

$$u(x, t) = \mathcal{L}^{-1} \left( -\frac{T}{s}e^{-\frac{\sqrt{s}}{c}x} + \frac{T}{s} \right).$$

One can use a software package to find the expression for  $\mathcal{L}^{-1} \left( \frac{1}{s}e^{-\frac{\sqrt{s}}{c}x} \right)$  ■

### Problem 22.13

Solve by Laplace transform

$$u_t - 3u_{xx} = 0, \quad 0 \leq x \leq 2, \quad t > 0$$

$$u(0, t) = u(2, t) = 0,$$

$$u(x, 0) = 5 \sin(\pi x).$$

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - 3U_{xx}(x, s) = 0$$

or

$$3U_{xx}(x, s) - sU(x, s) = -5 \sin(\pi x).$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{\sqrt{s}}{3}x} + B(s)e^{\frac{\sqrt{s}}{3}x} + \frac{5 \sin(\pi x)}{s + 3\pi^2}.$$

Next, we apply Laplace transform to the boundary condition obtaining  $U(0, s) = U(2, s) = \mathcal{L}(0) = 0$ . These lead to  $A(s) + B(s) = 0$  and  $A(s)e^{-2\frac{\sqrt{s}}{3}} + B(s)e^{2\frac{\sqrt{s}}{3}} = 0$ . Solving these equations we find  $A(s) = B(s) = 0$  and the transformed solution becomes

$$U(x, s) = \frac{5 \sin(\pi x)}{s + 3\pi^2}.$$

Now, taking inverse Laplace transform we find

$$u(x, t) = 5e^{-3\pi^2 t} \sin(\pi x) \blacksquare$$

### Problem 22.14

Solve by Laplace transform

$$u_t - 4u_{xx} = 0, \quad 0 \leq x \leq \pi, \quad t > 0$$

$$u_x(0, t) = u_x(\pi, t) = 0,$$

$$u(x, 0) = 40 \cos \frac{x}{2}.$$

### Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - 4U_{xx}(x, s) = 0$$

or

$$4U_{xx}(x, s) - sU(x, s) = -40 \cos \frac{x}{2}.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{\sqrt{s}}{2}x} + B(s)e^{\frac{\sqrt{s}}{2}x} + \frac{40 \cos \frac{x}{2}}{s + 1}.$$

Next, we apply Laplace transform to the boundary condition obtaining  $U_x(0, s) = U_x(\pi, s) = \mathcal{L}(0) = 0$ . These lead to  $-A(s) + B(s) = 0$  and  $A(s)e^{-\pi\frac{\sqrt{s}}{2}} + B(s)e^{\pi\frac{\sqrt{s}}{2}} = 0$ . Solving these equations we find  $A(s) = B(s) = 0$  and the transformed solution becomes

$$U(x, s) = \frac{40 \cos \frac{x}{2}}{s + 1}.$$

Now, taking inverse Laplace transform we find

$$u(x, t) = 40e^{-t} \cos \frac{x}{2} \blacksquare$$

**Problem 22.15**

Solve by Laplace transform

$$u_{tt} - 4u_{xx} = 0, \quad 0 \leq x \leq 2, \quad t > 0$$

$$u(0, t) = u(2, t) = 0,$$

$$u_t(x, 0) = 0, \quad u(x, 0) = 3 \sin \pi x.$$

**Solution.**

Applying Laplace transform to both sides of the equation we obtain

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) - 4U_{xx}(x, s) = 0$$

or

$$4U_{xx}(x, s) - s^2 U(x, s) = -3s \sin \pi x.$$

This is a second order linear ODE in the variable  $x$  and positive parameter  $s$ . Its general solution is

$$U(x, s) = A(s)e^{-\frac{s}{2}x} + B(s)e^{\frac{s}{2}x} + \frac{3s \sin \pi x}{s^2 + 4\pi^2}.$$

Next, we apply Laplace transform to the boundary condition  $u(0, t) = u(2, t) = 0$  obtaining

$$U(0, s) = U(2, s) = \mathcal{L}(0) = 0.$$

This leads to  $A(s) + B(s) = 0$  and  $A(s)e^{-s} + B(s)e^s = 0$ . Solving these equations we find  $A(s) = B(s) = 0$  and the transformed solution becomes

$$U(x, s) = \frac{3s \sin \pi x}{s^2 + 4\pi^2}.$$

Using the inverse Laplace transform we find

$$u(x, t) = 3 \sin \pi x \cos 2\pi t \quad \blacksquare$$

## Solutions to Section 23

### Problem 23.1

Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -1 \leq x \leq 1$$

extended to be periodic of period 2.

**Solution.** Using integration by parts we find

$$\begin{aligned} c_n &= \frac{1}{2} \int_{-1}^1 x e^{-in\pi x} dx \\ &= \frac{1}{2} \left[ \left( \frac{ix}{n\pi} \right) e^{-in\pi x} \Big|_{-1}^1 - \int_{-1}^1 \left( \frac{i}{n\pi} \right) e^{-in\pi x} dx \right] \\ &= \frac{1}{2} \left[ \left( \frac{i}{n\pi} \right) e^{-in\pi} + \left( \frac{i}{n\pi} \right) e^{in\pi} \right] \\ &\quad + \frac{1}{2} \left[ \frac{1}{(n\pi)^2} e^{-in\pi} - \frac{1}{(n\pi)^2} e^{in\pi} \right] \\ &= \frac{1}{2} \left[ \frac{i(-1)^n}{n\pi} + \frac{i(-1)^n}{n\pi} + \frac{1}{(n\pi)^2} (-1)^n - \frac{1}{(n\pi)^2} (-1)^n \right] \\ &= \frac{(-1)^n i}{n\pi} \blacksquare \end{aligned}$$

### Problem 23.2

Let

$$f(x) = \begin{cases} 0 & -\pi < x < \frac{-\pi}{2} \\ 1 & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & \pi < x < \pi \end{cases}$$

be  $2\pi$ -periodic. Find its complex series representation.

**Solution.**

We have

$$c_0 = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2} dx = \frac{1}{2}$$

and

$$c_n = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2} e^{-inx} dx = \frac{1}{2in\pi} (e^{\frac{in\pi}{2}} - e^{-\frac{in\pi}{2}})$$

for  $n = \pm 1, \pm 2, \dots$ . These coefficients reduce to the real values

$$c_n = \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right), \quad n = \pm 1, \pm 2, \dots$$

Note that  $c_{-n} = c_n$ . Thus, the complex series representation of  $f$  is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) (e^{inx} + e^{-inx}) \blacksquare$$

**Problem 23.3**

Find the complex Fourier series of the  $2\pi$ -periodic function  $f(x) = e^{ax}$  over the interval  $(-\pi, \pi)$ .

**Solution.**

We have for  $n = 0, \pm 1, \pm 2, \dots$

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx \\ &= \frac{1}{2\pi} \left[ \frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \cdot \frac{1}{a-in} [e^{(a-in)\pi} - e^{(a-in)(-\pi)}] \\ &= \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)} \end{aligned}$$

Hence, the complex Fourier series of  $f(x)$  is

$$f(x) = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{(a^2+n^2)} e^{inx} \blacksquare$$

**Problem 23.4**

Find the complex Fourier series of the  $2\pi$ -periodic function  $f(x) = \sin x$  over the interval  $(-\pi, \pi)$ .

**Solution.**

We have

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x e^{-inx} dx \\ &= \frac{1}{2\pi} \left[ \frac{e^{in\pi} - e^{-in\pi}}{n^2 - 1} \right] = 0 \end{aligned}$$

for  $n \neq 1$  or  $n \neq -1$ . Thus,

$$c_{-1} = \frac{-1}{2i} \text{ and } c_1 = \frac{1}{2i}.$$

Hence, the complex Fourier series of  $f(x)$  is

$$f(x) = \frac{e^{ix} - e^{-ix}}{2i} \blacksquare$$

**Problem 23.5**

Find the complex Fourier series of the  $2\pi$ -periodic function defined

$$f(x) = \begin{cases} 1 & 0 < x < T \\ 0 & T < x < 2\pi \end{cases}$$

**Solution.**

We have

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} dx = \frac{1}{2\pi} \int_0^T e^{-inx} dx \\ &= \frac{1}{2\pi n} [e^{-inT} - 1] \end{aligned}$$

for  $n \neq 0$ . For  $n = 0$  we find

$$c_0 = \frac{1}{2\pi} \int_0^T dt = \frac{T}{2\pi}.$$

Hence, the complex Fourier series of  $f(x)$  is

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \left\{ T + \sum_{n=-\infty}^{-1} \frac{i}{n} [e^{-int} - 1] e^{int} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{i}{n} [e^{-int} - 1] e^{int} \right\} \end{aligned}$$

**Problem 23.6**

Let  $f(x) = x^2$ ,  $-\pi < x < \pi$ , be  $2\pi$ -periodic.

- Calculate the complex Fourier series representation of  $f$ .
- Using the complex Fourier series found in (a), recover the real Fourier series representation of  $f$ .



**Solution.**

(a) Using integration by parts we find

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\ &= \frac{1}{2\pi} \left[ \frac{x^2 i}{n} e^{-inx} \Big|_{-\pi}^{\pi} - \frac{2i}{n} \int_{-\pi}^{\pi} x e^{-inx} dx \right] \\ &= \frac{1}{2\pi} \left[ 0 + \left( \frac{2x}{n^2} - \frac{2i}{n^3} \right) e^{-inx} \Big|_{-\pi}^{\pi} \right] \\ &= \frac{2}{n^2} (-1)^n \end{aligned}$$

for  $n \neq 0$  and

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}.$$

Hence, the complex Fourier series of  $f(x)$  is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2} (-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}.$$

(b) We have  $\frac{a_0}{2} = c_0 = \frac{\pi^2}{3}$  and

$$a_n = c_n + c_{-n} = \frac{4}{n^2} (-1)^n \text{ and } b_n = 0.$$

Hence, the real Fourier series representation of  $f$  is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx \blacksquare$$

**Problem 23.7**

Let  $f(x) = \sin n\pi x$ ,  $-\frac{1}{2} < x < \frac{1}{2}$ , be of period 1.

(a) Calculate the coefficients  $a_n$ ,  $b_n$  and  $c_n$ .

(b) Find the complex Fourier series representation of  $f$ .

**Solution.**

(a) We have

$$a_0 = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x dx = -\frac{2}{\pi} \left[ \cos \frac{\pi}{2} - \cos -\frac{\pi}{2} \right] = 0$$

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x \cos 2n\pi x dx = 0$$

$$b_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x \sin 2n\pi x dx = \frac{8(-1)^n n}{\pi - 4n^2\pi}$$

where we used a computer software to evaluate  $b_n$ .  
Now to find  $c_n$ 's we have

$$c_0 = \frac{a_0}{2} = 0$$

and for  $n \in \mathbb{N}$  we have

$$c_n = \frac{a_n - ib_n}{2} = \frac{4(-1)^n n}{i(\pi - 4n^2\pi)}$$

and

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{4(-1)^n in}{\pi - 4n^2\pi}$$

(b) The complex Fourier representation of  $f(x)$  is

$$f(x) = \frac{4}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{i(1 - 4n^2)} e^{2n\pi ix} \blacksquare$$

### Problem 23.8

Let  $f(x) = 2 - x$ ,  $-2 < x < 2$ , be of period 2.

- (a) Calculate the coefficients  $a_n$ ,  $b_n$  and  $c_n$ .  
(b) Find the complex Fourier series representation of  $f$ .

### Solution.

(a) We have

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 (2 - x) dx = 4 \\ a_n &= \frac{1}{2} \int_{-2}^2 (2 - x) \cos\left(\frac{n\pi}{2}x\right) dx = 0 \\ b_n &= \frac{1}{2} \int_{-2}^2 (2 - x) \sin\left(\frac{n\pi}{2}x\right) dx = \frac{4(-1)^n}{n\pi} \end{aligned}$$

where we used a computer software to evaluate  $b_n$ .

Now to find  $c_n$ 's we have

$$c_0 = \frac{a_0}{2} = 2$$

and for  $n \in \mathbb{N}$  we have

$$c_n = \frac{a_n - ib_n}{2} = \frac{2(-1)^{n+1}i}{n\pi}$$

and  $c_{-n} = -c_n$ .

(b) The complex Fourier representation of  $f(x)$  is

$$f(x) = 2 + \sum_{n=-\infty}^{-1} \frac{2(-1)^{n+1}i}{n\pi} e^{(\frac{in\pi}{2}x)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}i}{n\pi} e^{(\frac{in\pi}{2}x)} \blacksquare$$

**Problem 23.9**

Suppose that the coefficients  $c_n$  of the complex Fourier series are given by

$$c_n = \begin{cases} \frac{2}{i\pi n} & \text{if } |n| \text{ is odd} \\ 0 & \text{if } |n| \text{ is even.} \end{cases}$$

Find  $a_n$ ,  $n = 0, 1, 2, \dots$  and  $b_n$ ,  $n = 1, 2, \dots$ .

**Solution.**

We first find the  $a_n$ . For  $n = 0$  we have  $a_0 = 2c_0 = 0$ . For  $n \in \mathbb{N}$  we have

$$a_n = c_n + c_{-n} = 0.$$

Next, we find the coefficients  $b_n$ . We have for  $|n|$  odd

$$b_n = i \frac{4}{in\pi} = \frac{4}{n\pi}$$

and for  $|n|$  even  $b_n = 0$  ■

**Problem 23.10**

Recall that any complex number  $z$  can be written as  $z = Re(z) + iIm(z)$  where  $Re(z)$  is called the real part of  $z$  and  $Im(z)$  is called the **imaginary part**. The **complex conjugate** of  $z$  is the complex number  $\bar{z} = Re(z) - iIm(z)$ . Using these definitions show that  $a_n = 2Re(c_n)$  and  $b_n = -2Im(c_n)$ .

**Solution.**

Note that for any complex number  $z$  we have  $z + \bar{z} = 2Re(z)$  and  $z - \bar{z} = -2iIm(z)$ . Thus,

$$c_n + \bar{c}_n = a_n$$

which means that  $a_n = 2Re(c_n)$ . Likewise, we have

$$c_n - \bar{c}_n = ib_n$$

That is  $ib_n = -2iIm(c_n)$ . Hence,  $b_n = -2Im(c_n)$  ■

**Problem 23.11**

Suppose that

$$c_n = \begin{cases} \frac{i}{2\pi n} [e^{-inT} - 1] & \text{if } n \neq 0 \\ \frac{T}{2\pi} & \text{if } n = 0. \end{cases}$$

Find  $a_n$  and  $b_n$ .

**Solution.**

We have

$$a_0 = 2c_0 = \frac{T}{\pi}$$

Now note that

$$2c_n = \frac{1}{\pi n} [\sin(nT) + i(\cos(nT) - 1)].$$

Hence,

$$a_n = 2\operatorname{Re}(c_n) = \frac{1}{\pi n} \sin(nT) \text{ and } b_n = \frac{1 - \cos(nT)}{n\pi} \blacksquare$$

**Problem 23.12**

Find the complex Fourier series of the function  $f(x) = e^x$  on  $[-2, 2]$ .

**Solution.**

We have

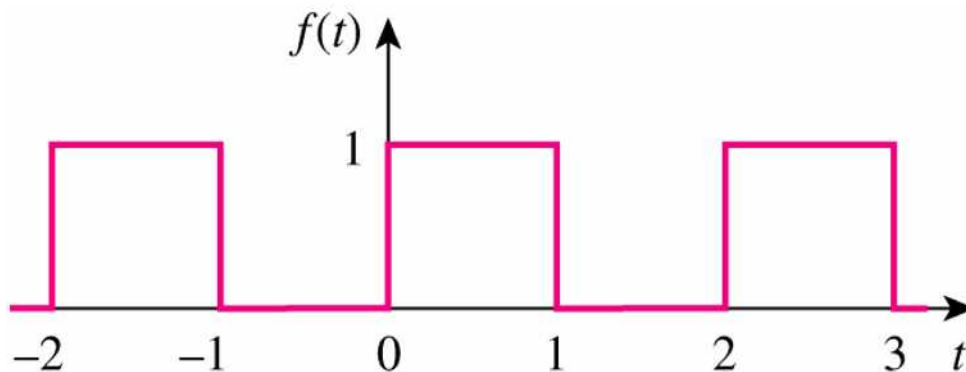
$$\begin{aligned} c_n &= \frac{1}{4} \int_{-2}^2 e^x e^{-\frac{in\pi x}{2}} dx \\ &= \frac{1}{4} \left. \frac{e^{x(1 - \frac{in\pi}{2})}}{1 - \frac{in\pi}{2}} \right|_{-2}^2 \\ &= \frac{i \sin(2 - in\pi)}{2 - in\pi} \end{aligned}$$

The complex Fourier series is

$$f(x) = i \sum_{n=-\infty}^{\infty} \frac{i \sin(2 - in\pi)}{2 - in\pi} e^{\frac{in\pi}{2}x} \blacksquare$$

**Problem 23.13**

Consider the wave form



- (a) Write  $f(x)$  explicitly. What is the period of  $f$ .  
 (b) Determine  $a_0$  and  $a_n$  for  $n \in \mathbb{N}$ .  
 (c) Determine  $b_n$  for  $n \in \mathbb{N}$ .  
 (d) Determine  $c_0$  and  $c_n$  for  $n \in \mathbb{N}$ .

**Solution.**

(a) We have

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

and  $f(t+2) = f(t)$  for all  $t \in \mathbb{R}$ .

(b) We have

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = \int_0^2 dx = \int_0^1 dx = 1$$

$$a_n = \int_0^1 \cos n\pi x dx = \frac{\sin n\pi}{n\pi} = 0.$$

(c) We have

$$b_n = \int_0^1 \sin n\pi x dx = \frac{1 - \cos n\pi}{n\pi} = \frac{1 - (-1)^n}{n\pi}.$$

Hence,

$$b_n = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(d) We have  $c_0 = \frac{a_0}{2} = \frac{1}{2}$  and for  $n \in \mathbb{N}$  we have

$$c_n = \frac{a_n - ib_n}{2} = \begin{cases} -\frac{i}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \blacksquare$$

**Problem 23.14**

If  $z$  is a complex number we define  $\sin z = \frac{1}{2}(e^{iz} - e^{-iz})$ . Find the complex form of the Fourier series for  $\sin 3x$  without evaluating any integrals.

**Solution.**

We have

$$\sin 3x = \frac{1}{2}(e^{3ix} - e^{-3ix}) \blacksquare$$

**Problem 23.15**

Find  $c_n$  for the  $2\pi$ -periodic function

$$f(x) = \begin{cases} 1 & \text{if } s \leq x \leq s+h \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

**Solution.**

We have

$$c_n = \frac{2\pi^{-\pi}}{\int_{-\pi}^{\pi}} f(x)e^{-inx} dx = \frac{2\pi^{s+h}}{\int_s} e^{-inx} dx = e^{-ins} \left( \frac{1 - e^{-inh}}{2\pi in} \right) \blacksquare$$

## Solutions to Section 24

### Problem 24.1

Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

### Solution.

We have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx = \int_{-1}^1 e^{-i\xi x} dx = \int_{-1}^1 \cos \xi x dx - i \int_{-1}^1 \sin \xi x dx.$$

The second integral is zero since the integrand is odd. Hence,

$$\hat{f}(\xi) = \begin{cases} 2\frac{\sin \xi}{\xi} & \text{if } \xi \neq 0 \\ 2 & \text{if } \xi = 0 \blacksquare \end{cases}$$

### Problem 24.2

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and initial condition

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x).$$

### Solution.

Let  $\hat{u}(\xi, t)$  be the Fourier transform of  $u$  in  $x$ . Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in  $t$

$$\frac{\partial \hat{u}}{\partial t} + i\xi c \hat{u} = 0$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi) \blacksquare$$

### Problem 24.3

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x).$$

**Solution.**

By performing the Fourier transform of  $u$  in  $x$ , we reduce the PDE problem into an ODE problem in the variable  $t$ :

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2 \xi^2 \hat{u}$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi)$$

$$\hat{u}_t(\xi, 0) = \hat{g}(\xi) \blacksquare$$

**Problem 24.4**

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < L$$

$$u(x, 0) = 0$$

$$u(x, L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

**Solution.**

Performing Fourier Transform in  $x$  for the PDE we obtain the second order PDE in  $y$

$$\hat{u}_{yy} = \xi^2 \hat{u}$$

$$\hat{u}(\xi, 0) = 0, \quad \hat{u}(\xi, L) = \frac{2 \sin \xi a}{\xi} \blacksquare$$

**Problem 24.5**

Find the Fourier transform of  $f(x) = e^{-|x|^\alpha}$ , where  $\alpha > 0$ .



**Solution.**

We have

$$\begin{aligned}
 \hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-|x|\alpha} e^{-i\xi x} dx \\
 &= \int_{-\infty}^0 e^{x\alpha} e^{-i\xi x} dx + \int_0^{\infty} e^{-x\alpha} e^{-i\xi x} dx \\
 &= \int_0^{\infty} e^{-x\alpha} e^{i\xi x} dx + \int_0^{\infty} e^{-x\alpha} e^{-i\xi x} dx \\
 &= \int_0^{\infty} e^{-x(\alpha-i\xi)} dx + \int_0^{\infty} e^{-x(\alpha+i\xi)} dx \\
 &= -\frac{e^{-x(\alpha-i\xi)}}{\alpha-i\xi} \Big|_0^{\infty} - \frac{e^{-x(\alpha+i\xi)}}{\alpha+i\xi} \Big|_0^{\infty} \\
 &= \frac{1}{\alpha-i\xi} + \frac{1}{\alpha+i\xi} = \frac{2\alpha}{\alpha^2 + \xi^2} \blacksquare
 \end{aligned}$$

**Problem 24.6**

Prove that

$$\mathcal{F}[e^{-x}H(x)] = \frac{1}{1+i\xi}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

**Solution.**

We have

$$\begin{aligned}
 \mathcal{F}[e^{-x}H(x)] &= \int_{-\infty}^{\infty} e^{-x}H(x)e^{-i\xi x} dx \\
 &= \int_0^{\infty} e^{-x(1+i\xi)} dx = -\frac{e^{-x(1+i\xi)}}{1+i\xi} \Big|_0^{\infty} = \frac{1}{1+i\xi} \blacksquare
 \end{aligned}$$

**Problem 24.7**

Prove that

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = 2\pi e^{\xi}H(-\xi).$$

**Solution.**

Using the duality property, we have

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = \mathcal{F}[\mathcal{F}[e^{-\xi}H(\xi)]] = 2\pi e^{\xi}H(-\xi) \blacksquare$$

**Problem 24.8**

Prove

$$\mathcal{F}[f(x - \alpha)] = e^{-i\xi\alpha} \hat{f}(\xi).$$

**Solution.**

We have

$$\begin{aligned} \mathcal{F}[f(x - \alpha)] &= \int_{-\infty}^{\infty} f(x - \alpha) e^{-i\xi x} dx \\ &= e^{-i\xi\alpha} \int_{-\infty}^{\infty} f(u) e^{-i\xi u} du \\ &= e^{-i\xi\alpha} \hat{f}(\xi) \end{aligned}$$

where  $u = x - \alpha$  ■**Problem 24.9**

Prove

$$\mathcal{F}[e^{i\alpha x} f(x)] = \hat{f}(x - \alpha).$$

**Solution.**

We have

$$\mathcal{F}[e^{i\alpha x} f(x)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} e^{ix(\alpha - \xi)} f(x) e^{-i\xi x} dx = \hat{f}(\xi - \alpha) \blacksquare$$

**Problem 24.10**

Prove the following

$$\mathcal{F}[\cos(\alpha x) f(x)] = \frac{1}{2} [\hat{f}(\xi + \alpha) + \hat{f}(\xi - \alpha)]$$

$$\mathcal{F}[\sin(\alpha x) f(x)] = \frac{1}{2} [\hat{f}(\xi + \alpha) - \hat{f}(\xi - \alpha)]$$

**Solution.**

We will just prove the first one. We have

$$\begin{aligned} \mathcal{F}[\cos(\alpha x) f(x)] &= \mathcal{F}\left[\frac{f(x) e^{i\alpha x}}{2} + f(x) \frac{e^{-i\alpha x}}{2}\right] \\ &= \frac{1}{2} [\mathcal{F}[f(x) e^{i\alpha x}] + \mathcal{F}[f(x) e^{-i\alpha x}]] \\ &= \frac{1}{2} [\hat{f}(x - \alpha) + \hat{f}(x + \alpha)] \blacksquare \end{aligned}$$

**Problem 24.11**

Prove

$$\mathcal{F}[f'(x)] = (i\xi)\hat{f}(\xi).$$

**Solution.**

Using the definition and integration by parts we find

$$\begin{aligned}\mathcal{F}[f'(x)] &= \int_{-\infty}^{\infty} f'(x)e^{-i\xi x} dx \\ &= f(x)e^{-i\xi x} \Big|_{-\infty}^{\infty} + (i\xi) \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx \\ &= f(x) \cos \xi x - i f(x) \sin \xi x + (i\xi)\hat{f}(\xi) = (i\xi)\hat{f}(\xi)\end{aligned}$$

where we used the fact that  $\lim_{x \rightarrow \infty} f(x) = 0$  ■**Problem 24.12**Find the Fourier transform of  $f(x) = 1 - |x|$  for  $-1 \leq x \leq 1$  and 0 otherwise.**Solution.**

We have

$$\begin{aligned}\hat{f}(\xi) &= \int_{-1}^1 (1 - |x|)e^{-i\xi x} dx \\ &= 2 \int_0^1 (1 - x)e^{-i\xi x} dx \\ &= 2 \int_0^1 (1 - x) \cos \xi x dx \\ &= \frac{2}{\xi^2}(1 - \cos \xi) \blacksquare\end{aligned}$$

**Problem 24.13**

Find, using the definition, the Fourier transform of

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

**Solution.**

We have

$$\begin{aligned}
 \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx \\
 &= - \int_{-a}^0 e^{-\xi x} dx + \int_0^a e^{-\xi x} dx \\
 &= \frac{1}{i\xi}(1 - e^{i\xi a}) + \frac{1}{i\xi}(1 - e^{-i\xi a}) \\
 &= \frac{2}{i\xi}(1 - \cos \xi a)
 \end{aligned}$$

Here we use Euler's formula  $e^{\pm i\xi a} = \cos \xi a \pm i \sin \xi a$  ■

**Problem 24.14**

Find the inverse Fourier transform of  $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$ .

**Solution.**

Using (5') we find

$$\mathcal{F}^{-1}[\hat{f}(\xi)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \blacksquare$$

**Problem 24.15**

Find  $\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right)$ .

**Solution.**

From Example 24.1, we find

$$\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right) = e^{-ax}, \quad x \geq 0.$$

## Solutions to Section 25

### Problem 25.1

Solve, by using Fourier transform

$$\begin{aligned}\Delta u &= 0 \\ u_y(x, 0) &= f(x) \\ \lim_{x^2+y^2 \rightarrow \infty} u(x, y) &= 0.\end{aligned}$$

### Solution.

Using the Fourier transform method, we begin by taking the transform of the PDE in  $x$ . The result is

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0.$$

The solution of the ODE in  $y$  is

$$\hat{u}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}.$$

Applying the boundary condition

$$\lim_{x^2+y^2 \rightarrow \infty} u(x, y) = 0$$

we can write

$$\hat{u}(\xi, y) = C(\xi)e^{-|\xi|y}$$

where  $C(\xi)$  is some constant distinct from  $A(\xi)$  or  $B(\xi)$ . Applying the first boundary condition, we get

$$\hat{u}_y(\xi, 0) = -|\xi|C(\xi)e^{-|\xi|y} \Big|_{y=0} = -|\xi|C(\xi) = \hat{f}(\xi).$$

Thus,

$$C(\xi) = -\frac{\hat{f}(\xi)}{|\xi|}$$

and

$$\hat{u}(\xi, y) = -\frac{\hat{f}(\xi)}{|\xi|}e^{-|\xi|y}.$$

If we leave this in terms of a convolution integral, we obtain

$$u(x, t) = f(x) * \mathcal{F}^{-1}\left[-\frac{1}{|\xi|}e^{-|\xi|y}\right] \blacksquare$$

**Problem 25.2**

Solve, by using Fourier transform

$$u_t + cu_x = 0$$

$$u(x, 0) = e^{-\frac{x^2}{4}}.$$

**Solution.**

Let  $\hat{u}(\xi, t)$  be the Fourier transform of  $u$  in  $x$ . Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE

$$\frac{d\hat{u}}{dt} + i\xi c\hat{u} = 0$$

$$\hat{u}(\xi, 0) = \frac{1}{\sqrt{\pi}}e^{-\xi^2}.$$

Solution of the ODE gives

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{\pi}}e^{-\xi^2}e^{-i\xi ct}.$$

Thus,

$$u(x, t) = \mathcal{F}^{-1}[u(\xi, t)] = e^{-\frac{(x-ct)^2}{4}} \blacksquare$$

**Problem 25.3**

Solve, by using Fourier transform

$$u_t = ku_{xx} - \alpha u, \quad x \in \mathbb{R}$$

$$u(x, 0) = e^{-\frac{x^2}{\gamma}}.$$

**Solution.**

Let  $\hat{u}(\xi, t)$  be the Fourier transform of  $u$  in  $x$ . Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in  $t$

$$\frac{\partial \hat{u}}{\partial t} = -(k\xi^2 + \alpha)\hat{u}$$

$$\hat{u}(\xi, 0) = \sqrt{\frac{\gamma}{4\pi}}e^{-\gamma\frac{\xi^2}{4}}.$$

Solution of the ODE in  $t$  gives

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0)e^{-(k\xi^2 + \alpha)t}.$$

Thus,

$$\hat{u}(\xi, t) = \sqrt{\frac{\gamma}{4\pi}} e^{-\xi^2(kt + \frac{\gamma}{4})} e^{-\alpha t}.$$

Taking inverse Fourier transform we find

$$\begin{aligned} u(x, t) &= \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \mathcal{F}^{-1}[e^{-\xi^2(kt + \frac{\gamma}{4})}] \\ &= \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \cdot \sqrt{\frac{\pi}{kt + \gamma/4}} \cdot e^{-\frac{x^2}{4(kt + \gamma/4)}} \\ &= \sqrt{\gamma} 4kt + \gamma e^{-\frac{x^2}{4kt + \gamma}} e^{-\alpha t} \blacksquare \end{aligned}$$

#### Problem 25.4

Solve the heat equation

$$u_t = k u_{xx}$$

subject to the initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

#### Solution.

The solution is

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-s)^2}{4kt}} ds \blacksquare$$

#### Problem 25.5

Use Fourier transform to solve the heat equation

$$u_t = u_{xx} + u, \quad -\infty < x < \infty < t > 0$$

$$u(x, 0) = f(x).$$

#### Solution.

Let  $\hat{u}(\xi, t)$  be the Fourier transform of  $u$  in  $x$ . Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in  $t$

$$\frac{\partial \hat{u}}{\partial t} = -(\xi^2 - 1)\hat{u}$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

Solution of the ODE in  $t$  gives

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0) e^{-(\xi^2 - 1)t}.$$

Thus,

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-\xi^2 t} e^t.$$

Taking inverse Fourier transform we find

$$\begin{aligned} u(x, t) &= e^t \mathcal{F}^{-1}[e^{-\xi^2 t}] \\ &= e^{-\alpha t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \blacksquare \end{aligned}$$

**Problem 25.6**

Prove that

$$\int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi = \frac{2y}{x^2 + y^2}.$$

**Solution.**

We have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi &= \int_{-\infty}^0 e^{\xi y} e^{i\xi x} d\xi + \int_0^{\infty} e^{-\xi y} e^{i\xi x} d\xi \\ &= \frac{1}{y + ix} e^{\xi(y+ix)} \Big|_{-\infty}^0 + \frac{1}{-y + ix} e^{\xi(-y+ix)} \Big|_0^{\infty} \\ &= \frac{1}{y + ix} + \frac{1}{-y + ix} = \frac{2y}{x^2 + y^2} \blacksquare \end{aligned}$$

**Problem 25.7**

Solve Laplace's equation in the half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty$$

subject to the boundary condition

$$u(x, 0) = f(x), \quad |u(x, y)| < \infty.$$

**Solution.**

Performing Fourier Transform in  $x$  for the PDE we obtain the second order PDE in  $y$

$$\hat{u}_{yy} = \xi^2 \hat{u}.$$

The general solution is given by

$$\hat{u}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}.$$

To ensure boundedness we must have



$$A(\xi) = 0 \text{ for } \xi > 0 \text{ or } B(\xi) = 0 \text{ for } \xi < 0.$$

Hence,

$$\hat{u}(\xi, y) = C(\xi)e^{-|\xi|y}.$$

Using the boundary condition  $\hat{u}(\xi, 0) = \hat{f}(\xi)$  we find  $C(\xi) = \hat{f}(\xi)$ . Hence,

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-|\xi|y}.$$

Taking inverse Fourier transform we find

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-|\xi|y} e^{i\xi x} d\xi \\ &= \frac{1}{2\pi} f(x) * \left[ \frac{2y}{x^2 + y^2} \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2y}{(x - \xi)^2 + y^2} d\xi \blacksquare \end{aligned}$$

### Problem 25.8

Use Fourier transform to find the transformed equation of

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where  $\alpha, \beta > 0$ .

### Solution.

Using the properties of Fourier transform we find

$$\hat{u}_{tt} + (\alpha + \beta)\hat{u}_t + \alpha\beta\hat{u} = -c^2\xi^2\hat{u} \blacksquare$$

### Problem 25.9

Solve the initial value problem

$$u_t + 3u_x = 0$$

$$u(x, 0) = e^{-x}$$

using the Fourier transform.

### Solution.

The answer is (see notes)

$$u(x, t) = e^{-(x-3t)} \blacksquare$$

**Problem 25.10**

Solve the initial value problem

$$\begin{aligned}u_t &= k u_{xx} \\ u(x, 0) &= e^{-x}\end{aligned}$$

using the Fourier transform.

**Solution.**

The answer is (see notes)

$$u(x, t) = e^{-(x-kt)} \blacksquare$$

**Problem 25.11**

Solve the initial value problem

$$\begin{aligned}u_t &= k u_{xx} \\ u(x, 0) &= e^{-x^2}\end{aligned}$$

using the Fourier transform.

**Solution.**

The answer is (see notes)

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 - \frac{(x-s)^2}{4kt}} ds \blacksquare$$

**Problem 25.12**

Solve the initial value problem

$$\begin{aligned}u_t + c u_x &= 0 \\ u(x, 0) &= x^2\end{aligned}$$

using the Fourier transform.

**Solution.**

The answer is (see notes)

$$u(x, t) = (x - ct)^2 \blacksquare$$

**Problem 25.13**

Solve, by using Fourier transform

$$\begin{aligned}\Delta u &= 0 \\ u_y(x, 0) &= f(x) \\ \lim_{x^2+y^2 \rightarrow \infty} u(x, y) &= 0.\end{aligned}$$

**Solution.**

Using the Fourier transform method, we begin by taking the transform of the PDE in  $x$ . The result is

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0.$$

The solution of the ODE in  $y$  is

$$\hat{u}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}.$$

Applying the boundary condition

$$\lim_{x^2+y^2 \rightarrow \infty} u(x, y) = 0$$

we can write

$$\hat{u}(\xi, y) = C(\xi)e^{-|\xi|y}$$

where  $C(\xi)$  is some constant distinct from  $A(\xi)$  or  $B(\xi)$ . Applying the first boundary condition, we get

$$\hat{u}_y(\xi, 0) = -|\xi|C(\xi)e^{-|\xi|y} \Big|_{y=0} = -|\xi|C(\xi) = \hat{f}(\xi).$$

Thus,

$$C(\xi) = -\frac{\hat{f}(\xi)}{|\xi|}$$

and

$$\hat{u}(\xi, y) = -\frac{\hat{f}(\xi)}{|\xi|} e^{-|\xi|y}.$$

If we leave this in terms of a convolution integral, we obtain

$$u(x, t) = f(x) * \mathcal{F}^{-1}\left[-\frac{1}{|\xi|} e^{-|\xi|y}\right] \blacksquare$$