A First Course on Variational Methods in Structural Mechanics and Engineering

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This book is dedicated to the memory of Juan Carlos Simo: a truly exceptional scholar, mentor, and friend

$$\delta\Pi := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Pi(\boldsymbol{\varphi} + \epsilon\,\delta\boldsymbol{\varphi}) = 0$$

Preface

This text was designed for a second level course in Mechanics that I developed starting in Spring 2009. At that time the engineering mechanics curriculum at Berkeley was reorganized to combine a standalone Sophomore level statics course with a Junior level strength of materials (engineering mechanics) course. In the design of the new single course, most of the topics of the original two courses were retained but the coverage and sophistication of the presentation was reduced. With only one course in (solid) mechanics, it was felt that a follow-on course was essential for the students. One option was to identify the material that was no longer being taught in our new single combined course and to assemble it into a follow-on course. However, due to the design of the single course, this did not present a topical outline which was very appealing. Thus we opted instead to create a course from scratch which we felt would complement our combined statics and strength course. It would build upon and extend the material the students has already learned but it would explicitly not revisit the topics in more technical detail, rather it would expand their understanding of mechanics and engineering problem solving. It would prepare them for advanced studies.

This book covers essential topics in variational methods using structural mechanics as the application area. At selected junctures the reader is also exposed to how the analysis concepts can be applied to other areas of engineering such as piping flow, thermal networks, ground water diffusion, and advective pollutant transport to name several. The book begins with a quick overview of the elementary relations governing tension-compression bars (rods), torsion bars, and beams – all within the setting of elasticity. This review considers the solution of such problems from a differential equation viewpoint and introduces the reader to their solution using MATLAB's `bvp4c` functionality. Computer solution of problems is a central aspect of this book and this exercise is designed to introduce the reader to one option for the solution of standard problems involving these basic systems. The basic systems are all one-dimensional and this, of course, is rather limiting since many engineered systems are multi-dimensional in nature. To that end, the book next treats trusses and networks in a systematic way. Notions of kinematics are separated from those of equilibrium and material response. The concepts developed are intended to introduce the student to the canonical linear structure associated with many numerical solution methods. Careful attention is paid to issues of boundary conditions and the concept of static condensation. Problems in this section, as is true throughout the book, are both pencil and paper as well as programming oriented. The reader writes their own programs to solve problems.

With this material in hand the book turns to energy, energy methods, and the principle of stationary potential energy as a unifying concept in engineering. The principle of stationary potential energy is a powerful tool to organize our understanding of conservative systems and its exploitation is a common analysis methodology. The emphasis here is on the method of Ritz. Problems are presented and solved both by hand and via the construction of computer programs. Issues of approximation and error are discussed and a clear connection to the truss program written in the earlier section is made. The material is then extended to the problem of structural instability, viz., buckling. Again problems are treated via pencil and paper as well as via the writing of computer programs.

The methods developed to this point are all designed for conservative systems. As not all systems are conservative we also introduce the concept of virtual work (weak form descriptions). The equivalence of the principle of virtual work to the concept of stationary potential energy is developed but the reader is also shown how the principle applies to a non-conservative system. The final part of the book deals with two-dimensional problems, those that are described by partial differential equations. The method of Ritz is applied as well as the principle of virtual work and applications are made to problems beyond structural engineering.

As noted, computer programing is a central tool for engineering analysis and throughout the material is developed with this in mind. The end of chapter problems all include programming questions. Beyond that, Appendix C provides at set of directed computer laboratory exercises. These are designed to be completed most weeks during a standard 14 week semester at a single three hour laboratory session. At Berkeley, we cover the entire contents of this book using two hours of lecture per week plus one three hour computer laboratory session; a sample syllabus is given in Appendix D. The pace is modest and unhurried. Thus if desired, there is room to augment the material covered with other application areas. Likewise the book can be comfortably covered in a single quarter, if one provides three hours of lecture per week.

As mentioned earlier, the writing of this text began in Spring 2009. At that time, Dr. Tsuyoshi Koyama took careful lecture notes of the original offering of the course and typeset them. He also organized and conducted the computational laboratory. I am very grateful for his excellent efforts. Since that first offering the course has evolved and the notes were updated into the present form. My colleague Dr. Shaofan Li has also taught the course several times and I am very appreciative of his feedback as well as that from the course's graduate student instructors (TAs), who direct the computer laboratory sessions.

Notwithstanding, improvement is always possible. If you find any errors or have suggestions for improvements, please let me know.

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Contents

Review of Engineering Mechanics

1

Elementary engineering mechanics or strength of materials deals with approximate theories that allow one to easily compute the mechanical behavior of simple slender bodies under the action of axial forces, transverse forces and moments, as well as axial torques. In this introductory chapter we will briefly review the essential elements of these theories restricted to the special case of linear elastic materials. For our purposes we will be interested in a formulation of the governing equations in a manner suitable for both statically determinate and indeterminate problems.[1]

[1] Recall statically determinate problems are ones for which the internal forces and moments can be determined from statics alone without the need for kinematic considerations.

1.1 Tension-compression bars

A tension-compression bar is a slender bar subject exclusively to axial forces. These forces can be applied to the ends of the bar as well as distributed along the length of the bar. Additionally the bar can be subject to kinematic boundary conditions on its axial motion. To formalize the governing equations, consider the bar shown in Fig. 1.1. Our axial coordinate will be x and the bar will be subjected to a distributed axial force $b(x)$. The dimensions of $b(x)$ are force per unit length. The bar has a cross-sectional area $A(x)$ and is assumed linear elastic with Young's modulus $E(x)$.

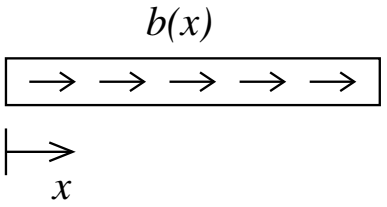


Fig. 1.1 Canonical tension-compression bar.

If we consider a section cut at an arbitrary location x along the bar, then we have internal (or resultant) forces $R(x)$ acting of the faces of the section cut. Using a differential element of length Δx as shown in Fig. 1.2, one sees by force equilibrium in the x-direction that

$$R(x+\Delta x) - R(x) + \int_x^{x+\Delta x} b(x)\,dx = 0\,.$$

Dividing through by Δx and then taking the limit as Δx goes to zero yields the differential equation of equilibrium for the bar:

$$\boxed{\frac{dR}{dx} + b = 0\,.}$$

The internal force $R(x)$ is related to the stresses in the bar by the relation

$$\boxed{R(x) = \int_{A(x)} \sigma\,dA = \sigma \cdot A\,,}$$

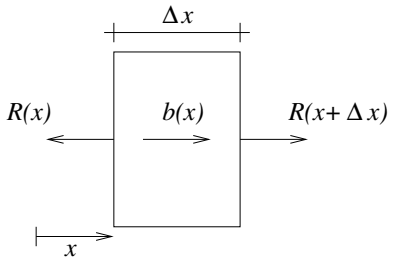


Fig. 1.2 Differential construction for equilibrium in a tension-compression bar.

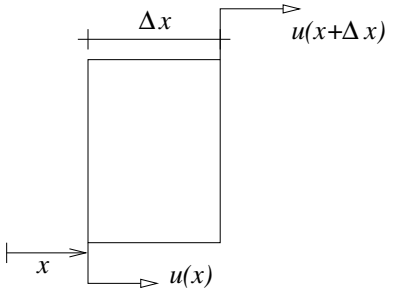


Fig. 1.3 Differential construction for strain-displacement relation in a tension-compression bar.

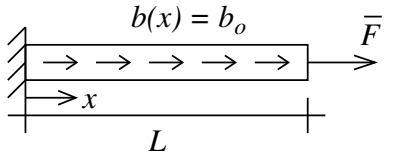


Fig. 1.4 Bar with a constant distributed load $b(x) = b_o$ and constant AE with an end-load $\bar{F}$.

where the second equality follows only in the case where σ is a constant on the cross-section (i.e., when the material is homogeneous on the cross-section).

The primary kinematic assumption for the axial deformation of slender bars is that plane sections remain plane under deformation. A differential argument applied to a section of the bar (see Fig. 1.3) shows that the normal strain in the x-direction is given by

$$\boxed{\varepsilon(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{du}{dx}\,,}$$

where $\varepsilon(x)$ is the normal strain in the bar and $u(x)$ is the displacement field in the bar. This relation is the primary kinematic relation for axial deformation in slender bars. It is also known as the strain-displacement relation.

To close the system of equations, one needs an expression for the material response – a constitutive relation. In this text, we will restrict our attention to linear elastic response and hence our governing constitutive relation will be

$$\boxed{\sigma = E\varepsilon\,,}$$

where E is the Young's modulus for the material. In total, one has 4 linear equations in 4 unknowns $u(x)$, $\varepsilon(x)$, $R(x)$, and $\sigma(x)$. A convenient re-writing of these equations involves the repeated substitution of one into the other until one is left with a single (differential) equation for the displacement field:

$$\boxed{\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0\,.} \tag{1.1}$$

For a given problem this relation can be integrated twice to yield the displacement field. Knowing the displacement field, the strains are easily computed via differentiation, the stresses by multiplication, and similarly for the forces.

Remarks:

(1) In the case where the Young's modulus is not a constant on the cross-section, then the term AE can be replaced by $(AE)_{\text{eff}} = \int_{A(x)} E\,dA$ in eqn (1.1).

Example 1.1

Bar with a constant distributed load and an end-load

Consider the bar shown in Fig. 1.4 and determine the displacement field in the bar.

Solution: Since AE is a constant the governing differential equation reduces to a second order ordinary differential equation with constant coefficients:

$$AE\frac{d^2u}{dx^2} + b_o = 0\,.$$

This equation can be integrated twice to yield

$$\begin{aligned} AE\frac{du}{dx} &= -b_o x + C_1 \\ AEu &= -\frac{1}{2}b_o x^2 + C_1 x + C_2\,. \end{aligned}$$

The integration constants C_1 and C_2 can be determined from the boundary conditions. At $x = 0$ the displacement is zero, thus $u(0) = 0$ and $C_2 = 0$. At $x = L$ the applied force is $\bar{F}$ and hence $R(L) = \bar{F}$. In terms of the displacements, $R = \sigma A = AE\varepsilon = AEdu/dx$. Hence,

$$AE\frac{du}{dx}(L) = \bar{F}\,,$$

which implies that $C_1 = \bar{F} + b_o L$. Thus the final result is

$$u(x) = \frac{1}{AE}\left(-\frac{1}{2}b_o x^2 + (\bar{F} + b_o L)x\right)\,.$$

Remarks:

(1) This method of solving the differential equation to determine the system response works independent of whether or not the problem is statically determinate or indeterminate – here the problem was determinate.

(2) Once the displacement field is known all other quantities of interest are easily computed: $\varepsilon = du/dx$, $\sigma = E\varepsilon$, and $R = \sigma A$.

For a more comprehensive discussion and further examples of the tension-compression bar relations see Chapter 2 in S. Govindjee *Engineering Mechanics of Deformable Solids*, Oxford University Press, Oxford (2013).

1.2 Torsion bars

A torsion bar is a slender bar subjected to distributed torques about its long-axis as well as similarly oriented end-torques. The overall structure of the governing equations for torsion is virtually identical to that for tension-compression bars.

Consider the bar shown in Fig. 1.5. The bar is subject to distributed torques $t(z)$ with dimensions of force times length per unit length. If we make a section cut as some location z along the length of the bar the cut faces have resultant (internal) torques $T(z)$ acting on them. Using a differential argument, as in Sec. 1.1, it is easy to show that the governing differential equation of equilibrium for a bar in torsion is

$$\boxed{\frac{dT}{dz} + t = 0\,.}$$

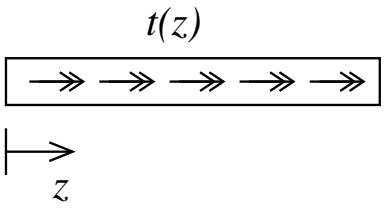


Fig. 1.5 Torsion bar subjected to a distributed torque $t(z)$.

The internal resultant torques $T(z)$ are related to the shear stresses ($\tau = \sigma_{z\theta}$) on the cross-sectional cuts via

$$\boxed{T(z) = \int_{A(z)} r\tau \, dA \,,}$$

where $A(z)$ is the cross-sectional area of the bar and $\tau(r, z)$ is the shear stress acting on the section cut. Using the plane sections remain plane assumption along with the assumption that the cross-sections rotate as rigid bodies, one can show that the fundamental kinematic relation is given by

$$\boxed{\gamma = r\frac{d\phi}{dz} \,,}$$

where $\gamma(r, z)$ is the engineering shear strain $2\varepsilon_{z\theta}$ and $\phi(z)$ is the rotation field for the bar – i.e. the amount of rotation of each cross-section.[2] The constitutive relation that is relevant to the present situation is

$$\boxed{\tau = G\gamma \,,}$$

where G is the shear modulus – note that as before, we restrict ourselves to the linear elastic case.

[2]This relation derives from assumptions that are only reasonable for bars with circular cross-sections.

The four basic equations can be combined into a single equation by repeated substitution. If we introduce the kinematic relation into the constitutive relation, we find that $\tau = Gr d\phi/dz$. Inserting this expression into the resultant definition, one finds that $T = GJ d\phi/dz$, where $J = \int_A r^2 \, dA$ is the polar moment of inertia for the cross-section. Combining this last result with the equilibrium relation gives

$$\boxed{\frac{d}{dz}\left(GJ\frac{d\phi}{dz}\right) + t = 0 \,.} \tag{1.2}$$

Remarks:

(1) This equation is mathematically identical to the one for tension-compression bars. Note that in the present setting GJ takes the place of AE, ϕ takes the place of u, z takes the place of x, t takes the place of b, and T takes the place of R.

(2) To solve structural analysis problems in torsional systems, statically determinate or indeterminate, one can integrate our final ordinary differential equation twice to find $\phi(z)$. Constants of integration are determined from the boundary conditions and all other quantities of interest can be determined from ϕ via the operations of differentiation and multiplication.

(3) In the case that the shear modulus is not a constant on the cross-section, then the product GJ can be replaced by $(GJ)_{\text{eff}} = \int_{A(z)} Gr^2 \, dA$ in eqn (1.2).

Example 1.2

Statically indeterminate torsion bar with a point load

Consider the torsion bar shown in Fig. 1.6. It is subjected to a point torque T_o at mid-span and has a constant GJ. Find the rotation field for the bar.

Solution: The appropriate distributed torque expression for a point torque is a Dirac delta function (see Appendix B):

$$t(z) = T_o \delta\left(z - \frac{L}{2}\right) .$$

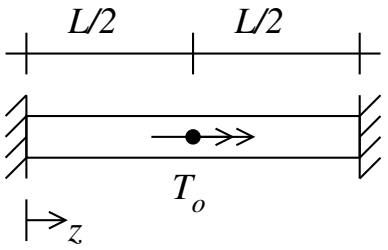


Fig. 1.6 Doubly built-in torsion bar with a point torque T_o at mid-span and constant GJ.

Since GJ is constant we have

$$\begin{aligned} GJ\frac{d^2\phi}{dz^2} &= -T_o \delta\left(z - \frac{L}{2}\right) \\ GJ\frac{d\phi}{dz} &= -T_o H\left(z - \frac{L}{2}\right) + C_1 \\ GJ\phi &= -T_o \left\langle z - \frac{L}{2}\right\rangle + C_1 z + C_2 . \end{aligned}$$

Note that $H(\cdot)$ is the Heaviside step function and $\langle\cdot\rangle$ are the Macauley brackets (they evaluate to their argument if the argument is positive, else they evaluate to zero).[3] The boundary conditions for the bar are $\phi(0) = \phi(L) = 0$:

$$\phi(0) = 0 \quad \Rightarrow \quad 0 = -T_o \underbrace{\left\langle -\frac{L}{2}\right\rangle}_{=0} + C_1 \cdot 0 + C_2 \quad \Rightarrow \quad 0 = C_2 .$$

[3] $\int \delta(x)\,dx = H(x) + C$, $\int H(x)\,dx = \langle x\rangle + C$, and $\int \langle x\rangle^n\,dx = \frac{1}{n+1}\langle x\rangle^{n+1} + C$.

and

$$\phi(L) = 0 \quad \Rightarrow \quad 0 = -T_o \left\langle L - \frac{L}{2}\right\rangle + C_1 L \quad \Rightarrow \quad C_1 = T_o/2 .$$

Thus,

$$\boxed{\phi(z) = \frac{1}{GJ}\left(-T_o \left\langle z - \frac{L}{2}\right\rangle + \frac{1}{2}T_o z\right) .}$$

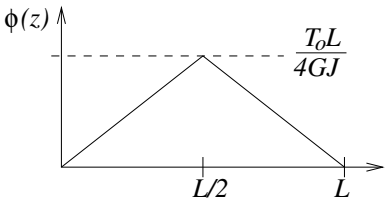


Fig. 1.7 Rotation field for Example 1.2.

Remarks:

(1) A plot of the solution is shown in Fig. 1.7. Note that the first term in the Macauley brackets only gives a non-zero result for $z > L/2$.

For a more comprehensive discussion and further examples of the torsion bar relations see Chapter 7 in S. Govindjee *Engineering Mechanics of Deformable Solids*, Oxford University Press, Oxford (2013).

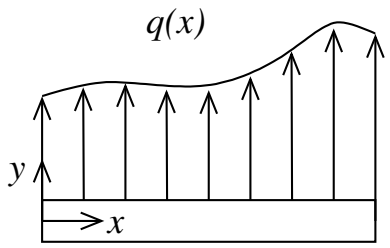


Fig. 1.8 Beam subjected to a distributed load $q(x)$.

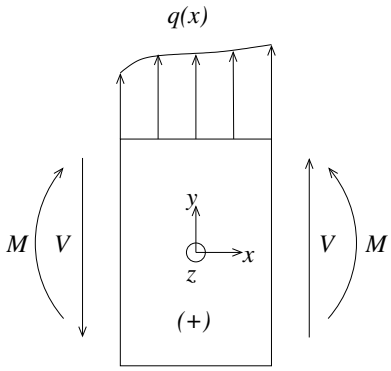


Fig. 1.9 Assumed positive sign convention for distributed loads, internal shear forces, and internal bending moments.

1.3 Beams

A slender bar acts as a beam when it is subjected to transverse loads and/or moments perpendicular to its long axis. For our purposes we will be concerned with beam bending in the x-y plane.

Consider the beam shown in Fig. 1.8. The beam is subjected to a distributed load $q(x)$ with dimensions of force per unit length. If we make a section cut at some location x along the length of the beam, then the cut faces will have resultant (internal) shear forces $V(x)$ and bending moments $M(x)$ acting on them. Fig. 1.9 shows our sign convention for *positive* values; note further, we assume that the beam deflection $v(x)$ is positive in the positive y-direction. Using a differential element argument one can show that force equilibrium in the y-direction and moment equilibrium about the z-axis, respectively, imply that

$$\boxed{\begin{aligned} \frac{dV}{dx} + q &= 0 \\ \frac{dM}{dx} + V &= 0\,. \end{aligned}}$$

These two equations can be combined into a single differential equation of equilibrium by eliminating the shear force to give

$$\boxed{\frac{d^2M}{dx^2} = q\,.}$$

The internal bending moment $M(x)$ is related to the normal stress ($\sigma = \sigma_{xx}$) on the cross-section by

$$\boxed{M = \int_{A(x)} -y\sigma \, dA\,.}$$

Using a plane-sections remain plane and normals remain normal assumption, one can shown that the fundamental kinematic relation for a beam is given by

$$\boxed{\varepsilon = -y\kappa(x)\,,}$$

where $\varepsilon(x,y)$ is the bending strain (ε_{xx}), $\kappa(x) = d\theta/dx$ is the bending curvature, and $\theta(x) = dv/dx$ is the cross-section rotation field.[4] The constitutive relation that is relevant to the present situation is

$$\boxed{\sigma = E\varepsilon\,,}$$

where E is the Young's modulus.

The basic equations can be combined into a single governing differential equation by repeated substitution. If we introduce the kinematic relation into the constitutive relation, we find that $\sigma = -yE\kappa$. Inserting this into the moment resultant definition, one finds that $M = EI\kappa$, where $I = \int_A y^2 \, dA$ is the moment of inertia of the cross-section (about

[4] Note that one can also write $\kappa = d^2v/dx^2$.

the z-axis). Combining this last result with the equilibrium relations gives

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = q\,.} \tag{1.3}$$

Remarks:

(1) The governing equation here is a fourth order ordinary differential equation. In the case that EI is not a function of x, this is an ordinary differential equation with constant coefficients.

(2) The solution of the governing equation for the deflection $v(x)$ for statically determinate or indeterminate problems requires one to integrate four times. This will produce four constants of integration. For their determination, one will need four boundary conditions. Typically, two are taken from the left and two are taken from the right. These will normally be given in terms of the deflection directly, the rotation $\theta = dv/dx$, the bending moment $M = EId^2v/dx^2$, or the shear force $V = -EId^3v/dx^3$. Once the deflection field has been determined, then all other quantities of interest can be found from $v(x)$ via differentiation and multiplication.

(3) In the case that the Young's modulus is not a constant on the cross-section, the product EI can be replaced by $(EI)_{\text{eff}} = \int_{A(x)} Ey^2\,dA$ in eqn (1.3).

Example 1.3

Cantilever beam with an end-force

Consider the cantilever beam shown in Fig. 1.10. It is subjected to an end-force P at $x = L$ and has a constant EI. Find the deflection field for the beam.

Fig. 1.10 Cantilever beam with an end-force P at $x = L$ and constant EI.

Solution: The distributed load in this problem is zero; i.e. $q = 0$. Since EI is constant we have

$$\begin{aligned}
EI\frac{d^4v}{dx^4} &= 0\\
EI\frac{d^3v}{dx^3} &= C_1\\
EI\frac{d^2v}{dx^2} &= C_1x + C_2\\
EI\frac{dv}{dx} &= \frac{1}{2}C_1x^2 + C_2x + C_3\\
EIv &= \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4\,.
\end{aligned}$$

The boundary conditions for the beam are $v(0) = 0$, $\theta(0) = dv/dx(0) = 0$, $M(L) = EId^2v/dx^2(L) = 0$, and $V(L) = -EId^3v/dx^3(L) = P$. The first two boundary conditions give:

$$v(0) = 0 \quad \Rightarrow \quad 0 = \frac{1}{6}C_1 \cdot 0 + \frac{1}{2}C_2 \cdot 0 + C_3 \cdot 0 + C_4 \quad \Rightarrow \quad 0 = C_4$$

and

$$\frac{dv}{dx}(0) = 0 \quad \Rightarrow \quad 0 = \frac{1}{2}C_1 \cdot 0 + C_2 \cdot 0 + C_3 \quad \Rightarrow \quad 0 = C_3\,.$$

The condition on the end-shear gives

$$-EI\frac{d^3v}{dx^3}(L) = P \quad \Rightarrow \quad P = -C_1\,.$$

Lastly, the moment condition gives

$$EI\frac{d^2v}{dx^2}(L) = 0 \quad \Rightarrow \quad 0 = -PL + C_2 \quad \Rightarrow \quad C_2 = PL\,.$$

Thus,

$$\boxed{v(x) = \frac{Px^2}{2EI}\left(L - \frac{x}{3}\right)\,.}$$

Remarks:

(1) This procedure is applicable to both determinate and indeterminate problems. The example shown was determinate.

For a more comprehensive discussion and further examples of the beam bending relations see Chapter 8 in S. Govindjee *Engineering Mechanics of Deformable Solids*, Oxford University Press, Oxford (2013).

1.4 A purely structural perspective

In Sec. 1.1–1.3 we have presented the classical equations learned in elementary mechanics courses. The presentation follows the typical development and the underpinning assumption of each theory is a kinematic assumption on the three dimensional motion of the system in question. However, from a purely structural perspective, leaving aside three dimensional issues and issues of the detailed response of the material on the cross-section, one can abstract complete theories from the equations presented that only involve functions of the axial coordinate. These relations are already present in what was reviewed but for added clarity we summarize them here. The main principle is to select a set of equations that only depend upon a single coordinate and constitute a complete system of equations.

1.4.1 Tension-compression bars

For tension-compression bars, this can be achieved by constructing a system of equations in terms of the internal forces and the displacements as:

$$\begin{aligned} \frac{dR}{dx} + b &= 0 \\ \frac{du}{dx} &= \frac{R}{AE} \,. \end{aligned}$$

Alternately, in vector form we have

$$\boxed{\frac{d}{dx}\begin{pmatrix} u \\ R \end{pmatrix} = \begin{pmatrix} R/AE \\ -b \end{pmatrix} .}$$

In this system of two coupled ordinary differential equations, we have two unknown fields $u(x)$ and $R(x)$. The remaining terms are the applied loading $b(x)$ and the property combination $A(x)E(x)$, which is sometimes called the axial stiffness or axial rigidity. As written this is a first order system. If one combines these two relations, then one recovers the second order differential equation from Sec. 1.1. Note that nowhere in these equations do we encounter terms that depend on the y or z coordinates.

1.4.2 Torsion bars

For torsion bars, this can be achieved by constructing a system of equations in terms of the internal torques and the rotations as:

$$\begin{aligned} \frac{dT}{dx} + t &= 0 \\ \frac{d\phi}{dz} &= \frac{T}{GJ} \,. \end{aligned}$$

Alternately, in vector form we have

$$\boxed{\frac{d}{dz}\begin{pmatrix} \phi \\ T \end{pmatrix} = \begin{pmatrix} T/GJ \\ -t \end{pmatrix} .}$$

In this system of two coupled ordinary differential equations, we have two unknown fields $\phi(x)$ and $T(x)$. The remaining terms are the applied loading $t(z)$ and the property combination $G(z)J(z)$, which is sometimes called the torsional stiffness or torsional rigidity. As written this is a first order system. If one combines these two relations, then one recovers the second order differential equation from Sec. 1.2. Note that nowhere in these equations do we encounter terms that depend on the r or θ coordinates.

1.4.3 Beams

For beams, the same can be achieved by constructing a system of equations in terms of the internal forces and moments as well as the deflections and rotations as:

$$\begin{aligned} \frac{dV}{dx} + q &= 0 \\ \frac{dM}{dx} + V &= 0 \\ \frac{d\theta}{dx} &= \frac{M}{EI} \\ \frac{dv}{dx} &= \theta\,. \end{aligned}$$

Alternately, in vector form we have

$$\boxed{\frac{d}{dx}\begin{pmatrix} v \\ \theta \\ M \\ V \end{pmatrix} = \begin{pmatrix} \theta \\ M/EI \\ -V \\ -q \end{pmatrix}\,.}$$

In this system of four coupled ordinary differential equations, we have four unknown fields $v(x)$, $\theta(x)$, $M(x)$, and $V(x)$. The remaining terms are the distributed load $q(x)$ and the property combination $E(x)I(x)$, which is sometimes called the bending stiffness or the flexural rigidity. As written this is a first order system. If one combines these four relations, then one recovers the fourth order differential equation from Sec. 1.3. Note that nowhere in these equations do we encounter terms that depend the y or z coordinates.

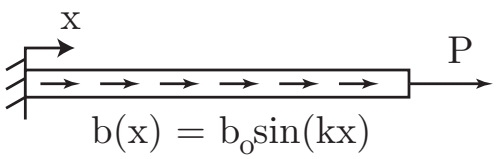
Remarks:

(1) The utility of expressing the governing system of equations in these matrix formats is that they are then seen to fit the abstract format $d\boldsymbol{y}/dx = \boldsymbol{f}(\boldsymbol{y}, x)$ for which effective numerical schemes are known to exist.

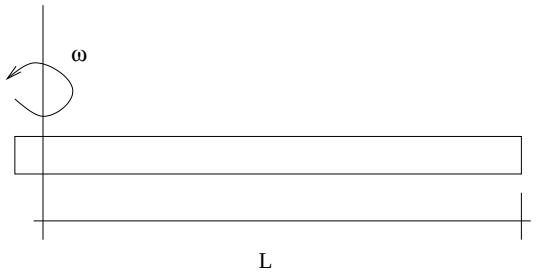
Exercises

(1.1) Using the governing equation for the axial deformation of a bar, argue why the displacement field must be linear (independent of the boundary conditions for the bar) in the absence of any distributed body forces; i.e. for the case where $b(x) = 0$. Assume AE is constant.

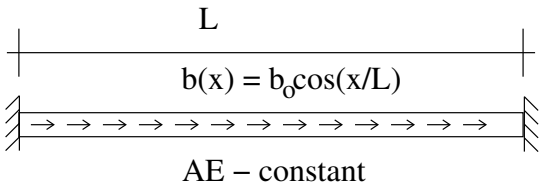
(1.2) Consider the linear elastic tension-compression bar shown. The bar is subjected to an end-force P and a distributed load $b(x) = b_o \sin(kx)$, where b_o and k are given constants. Assume that AE is constant and determine the displacement field of the bar.



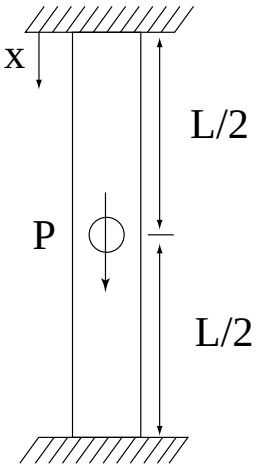
(1.3) Consider a bar of length L with constant EA and constant density ρ. The bar is supported by a fixed pivot and spun about it at angular frequency ω. Doing so produces a distributed body force of $b(x) = A\rho\omega^2 x$, where x is measured from the pivot. Find the maximum and minimum strains and their locations. Write down the governing ordinary differential equation, then solve.



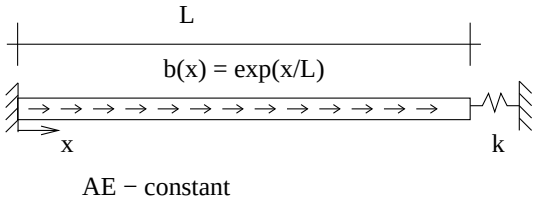
(1.4) For the bar shown determine the displacement field $u(x)$.



(1.5) You are given a prismatic bar with constant cross-sectional area A, Young's Modulus E, and length L. Determine the reaction force at the top of the bar due to the load P.



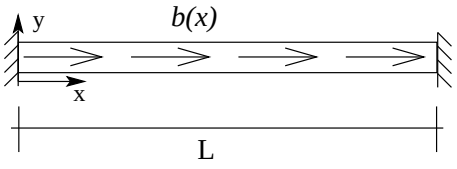
(1.6) The bar shown is built-in at the left and supported by a spring with spring constant k at the right. List the boundary conditions at $x = 0$ and $x = L$. Find the expression for $u(x)$.



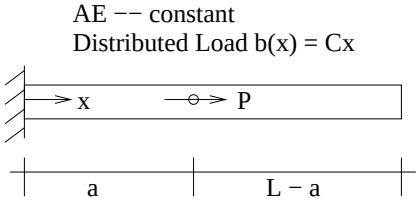
(1.7) Consider an elastic bar with constant Young's modulus, E, and constant cross sectional area, A. The bar is built-in at both ends and subject to a spatially varying distributed axial load

$$b(x) = b_o \sin(\frac{2\pi}{L}x)\,,$$

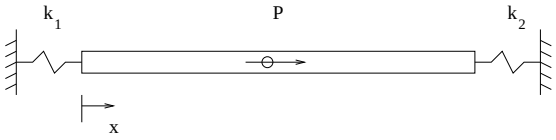
where b_o is a constant with dimensions of force per unit length. Determine the largest (in magnitude) **compressive** internal force. Write down the governing ordinary differential equation, then solve.



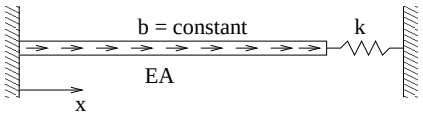
(1.8) For the linear elastic bar shown. Determine the axial displacement as a function of x. Note that there is a distributed load and a point load. The point load should be modeled using a Dirac delta function. Write down the governing ordinary differential equation, then solve.



(1.9) Consider an elastic bar with length L and constant AE that is subjected to a point force at $x = a$. Find $u(x)$.



(1.10) Find the axial deflection $u(x)$ in the bar shown. The bar has length L and a constant AE. Write down the governing ordinary differential equation, then solve.

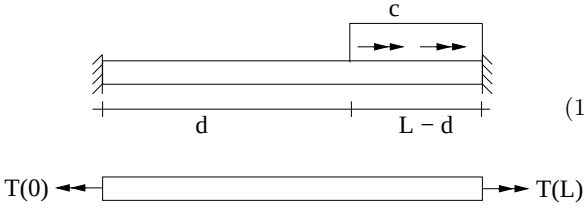


(1.11) Consider the bar shown in Fig. 1.4. Assume that in addition to the loads shown, that it is subject to a temperature change ΔT. Formulate the governing differential equations in first order form along with expressions for the boundary conditions. Assume that the material response is given by $\sigma = E(\varepsilon - \alpha\Delta T)$, where α is the coefficient of thermal expansion.

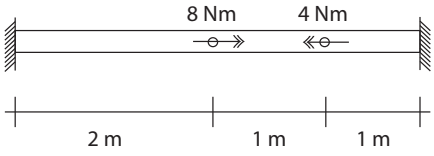
(1.12) An elastic solid circular bar of length L with polar moment of inertia J and shear modulus G is built-in at both ends and subject to a system of distributed torques:

$$t(z) = \begin{cases} 0 & z < d \\ c & z \geq d \end{cases}$$

Determine the support torques $T(0)$ and $T(L)$ at the two ends of the bar. Solve this problem using the governing ordinary differential equation.

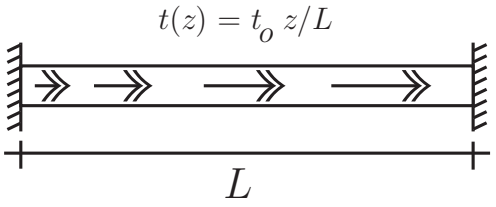


(1.13) For the round torsion bar show, with constant GJ, find the maximum torque (magnitude) and its location.

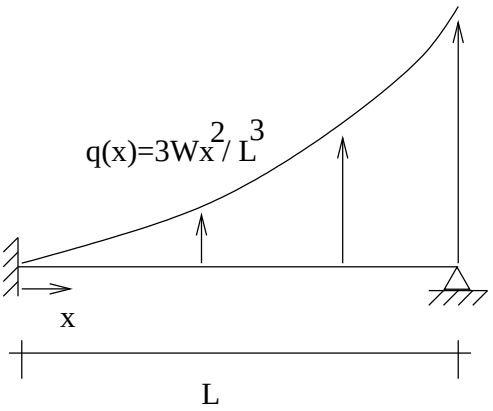


(1.14) Consider a circular bar which is built-in at both ends and loaded by a linear distributed load, $t(z) = t_o\, z$. By solving the governing second order ordinary differential equation find the value of t_o needed to induce a rotation $\hat{\theta}$ at the mid-point of the bar. Assume GJ is a constant.

(1.15) Consider the linear elastic (circular) bar shown below. Determine the rotation field, $\phi(z)$, for the bar. Assume GJ to be constant.

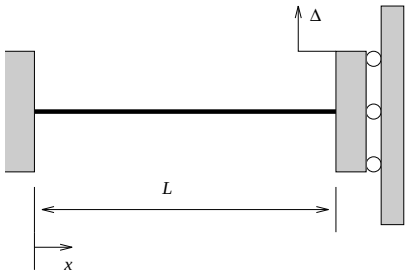


(1.16) In the statically indeterminate beam shown, find the reactions at the wall by integrating the differential equation for the deflection of the beam.

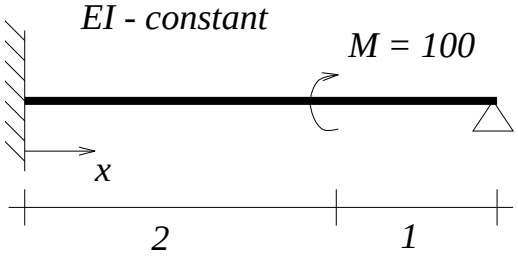


(1.17) For the beam shown below, with imposed deflection Δ, find the deflection curve $v(x)$ and the location and magnitude of the maximum bending stress. Assume EI is constant and the maximum distance

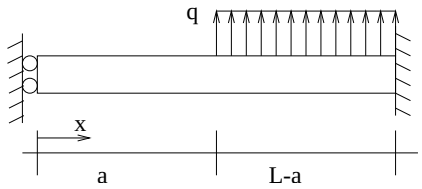
from the neutral axis to the outer fibers of the beam is c. Use the governing ordinary differential equation to solve this problem.



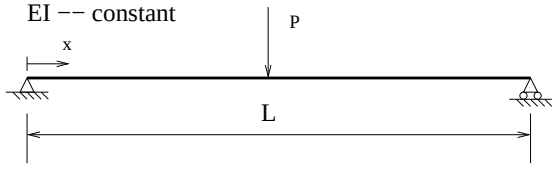
(1.18) The beam shown below is loaded by a point moment at $x = 2$; find the maximum internal moment (in absolute value – i.e. independent of sign) by first solving the governing fourth order differential equation. [Hint: to find the maximum value just plot your answer.]



(1.19) Find the equation for the deflection of the beam shown. Assume a constant value for EI. Use the given coordinate system.



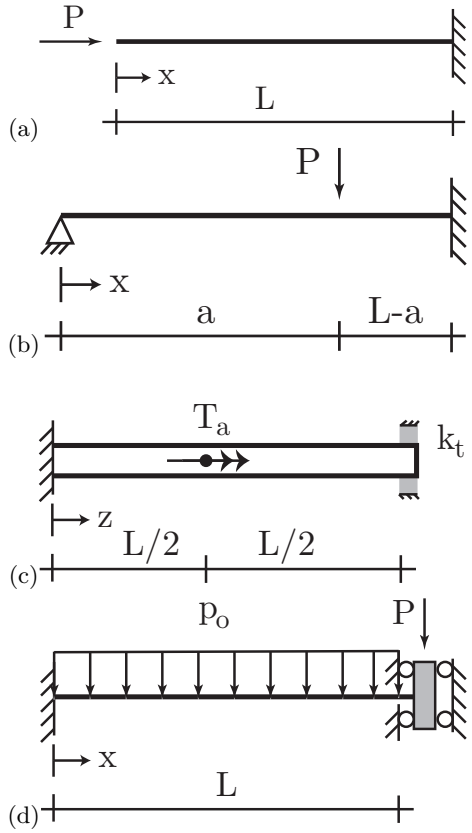
(1.20) Consider a simply supported beam with a transverse load P applied at its mid-span.



Determine the deflection of the beam using the governing ordinary differential equation as follows:

(a) State the relevant boundary conditions.
(b) State the relevant distributed load function.
(c) Solve the relevant ODE for the beam's deflection.

(1.21) For each system below, (i) state the relevant boundary conditions in the form needed to solve for the motion of each when using the governing ordinary differential equation, (ii) provide the proper expression for the distributed loads, if relevant.

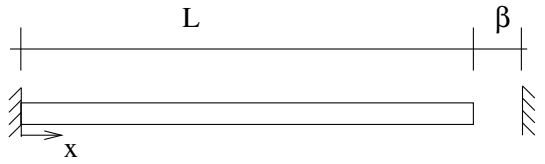


MATLAB Exercises

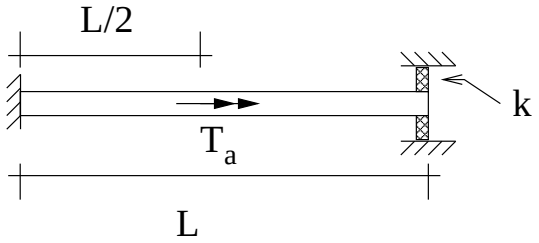
(1.22) Write a MATLAB program using `bvp4c` that solves Exercise 1.11 for the case of $b_o = 0$, $\bar{F} = 0$, $A = 100$ mm^2, $E = 210 \times 10^3$ N/mm^2, $L = 1000$ mm, $\alpha = 6 \times 10^{-6}$ 1/C, and $\Delta T = 300$ C. You should check your computation against a hand solution.

(1.23) Re-solve Exercise 1.22 where the boundary condition at $x = L$ has been changed to $u(L) = 0$ (instead of being force free).

(1.24) The bar shown below is built-in at the left. On the right, it is free to expand, until a distance β. At that point it encounters a rigid wall. The bar is subject to change in temperature which may or may not induce contact with the wall. Write a MATLAB program that uses `bvp4c` and can handle the case of contact or no contact without the need for re-coding. Assume $A = 100$ mm^2, $E = 210\times10^3$ N/mm^2, $L = 1000$ mm, $\alpha = 6\times10^{-6}$ 1/C, and $\beta = 3$ mm. Test your program with the temperature changes $\Delta T = \{100, 500, 600\}$ C. These represent no contact, just touching, and contact, respectively.



(1.25) An elastic circular bar is fixed at one end and attached to a spring support at the other end. The torsional stiffness of the spring support is $k = 50$ N m/rad. If a concentrated torque of magnitude $T_a = 500$ N m is applied in the center of the bar, what is the rotation at the end of the bar, $\phi(L)$, where $L = 300$ mm? Assume a shear modulus $G = 10$ kN/mm^2 and polar moment of inertia $J = 2000$ mm^4. Use the built-in MATLAB function `bvp4c` to solve this problem. Make sure to double check your code by looking at the limits of zero and very high spring stiffness compared to a hand computation.

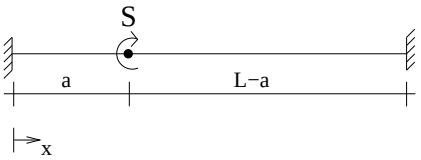


(1.26) Consider a solid round elastic bar with constant shear modulus, $G = 140$ kN/mm^2, and cross sectional area, $A = 40$ mm^2. The bar is built-in at both ends and subject to a spatially varying distributed torsional load

$$t(x) = p\sin(\frac{2\pi}{L}x)\,,$$

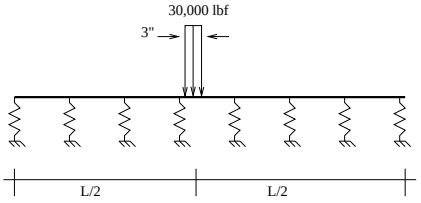
where $p = 50$ N $\cdot$ mm/mm and $L = 1000$ mm. Solve for the system response and determine the location and magnitude of the maximum internal torque in the bar. Use the built-in MATLAB function `bvp4c` to solve this problem.

(1.27) Consider an elastic beam of length $L = 10$ ft with constant Young's modulus $E = 30 \times 10^6$ psi, and cross sectional area moment of inertia $I = 256$ in^4. The beam is subject to a point moment at $x = a = 5$ ft. Determine the torsional stiffness at $x = a$; i.e. determine $k_T = S/\theta(a)$. Use the built-in MATLAB function `bvp4c` to solve this problem.



[Remark: S is intensionally unspecified. Why?]

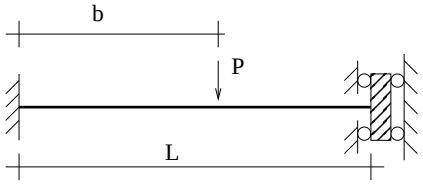
(1.28) Consider a beam supported by a distributed spring foundation (e.g. a railroad rail or grade beam). Such spring supports are known as *Winkler foundations*[5] . Assume the beam is 100 ft long with a Young's modulus of $E = 30 \times 10^6$ psi and a cross sectional area moment of inertia $I = 77.4$ in^4. Assume a foundation stiffness of $k = 100$ lb/in^2 and determine the maximum positive and negative moments in the beam for a 30×10^3 lb load distributed over 3 in at the beam's center. Note that for this problem the governing equation is given by $EIv(x)'''' = q(x)$, where $q(x) = q_{\text{applied}}(x) - kv(x)$. For boundary conditions, assume zero moment and shear. Use the built-in MATLAB function `bvp4c` to solve this problem.



(1.29) Consider an elastic beam of length $L = 10$ ft with constant Young's modulus $E = 30 \times 10^6$ psi, and cross sectional area moment of inertia $I = 256$ in^4. The beam is subject to a point force P at $x = b =$

[5] If you are interested in historical matters, you can find Emil Winkler's original developments in his 1867 book on Elasticity and Strength on Google Books. Search on "Die Lehre von der Elasticitaet und Festigkeit" pages 182-184.

6 ft. Determine the transverse stiffness at $x = b$; i.e. determine $k = P/v(b)$. Use the built-in MATLAB function `bvp4c` to solve this problem.



[Remark: P is intentionally unspecified. Why?]

Trusses and Networks

2

2.1 Analysis of truss structures

So far we have looked at the solution of basic one dimensional systems under various types of loadings. Real engineering systems, of course, come in a much greater variety. We wish to look at an extension of our basic notions to examine systems with greater geometric complexity while still being composed of elementary parts. Our central example will be trusses but the methodologies developed also allow one to study piping systems for water delivery, traffic networks for flow properties, as well as other network systems that appear in fields ranging even to electrical and mechanical engineering. We will see that the basic concepts of kinematics, equilibrium, and constitutive relations also have counterparts in these more complex settings. Lastly, we will encounter mathematical constructs that are central to many areas of engineering and science. In the context of trusses we will be able ascribe concrete physical meaning to them and this will help us later on when we encounter them in more abstract settings.

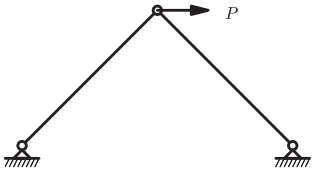


Fig. 2.1 Statically indeterminate truss with 4 unknown reactions.

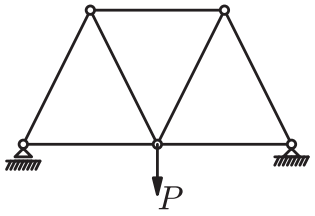


Fig. 2.2 Statically determinate truss with 3 unknown reactions.

2.1.1 Introduction

A truss is a mechanical system composed of tension-compression bars that are connected together by frictionless hinges, or nodes, and is loaded only at its nodes. Thus, truss structures are mechanical structures in which we make the following restrictions or assumptions in the theory:

- Tension-compression bars are the individual components which constitute the structure.
- All joints, or nodes, are friction free pin connections.
- Applied loads can only occur at nodes.

Some examples are shown in Figs. 2.1-2.3. An important consequence of these assumptions is that internal forces in the bars are co-linear with the bars and are constants for each bar in the truss; consequently the stresses and strains are also constants. Further, under small strain assumptions, the axial displacements will be linear.

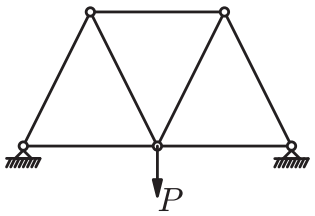


Fig. 2.3 Statically indeterminate truss with 4 unknown reactions.

Simple trusses

An important sub-category of trusses are simple trusses. Simple trusses are trusses that can be built-up sequentially from a single triangular

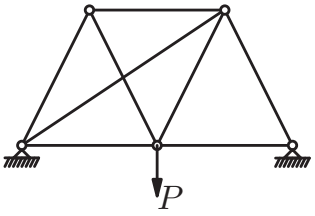


Fig. 2.4 Statically indeterminate non-simple truss with 4 unknown reactions.

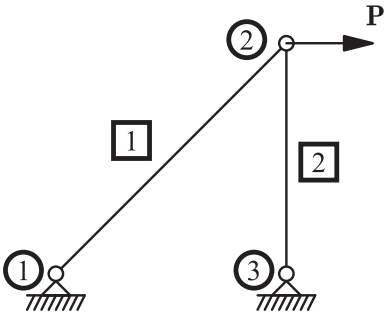


Fig. 2.5 2 bar truss structure.

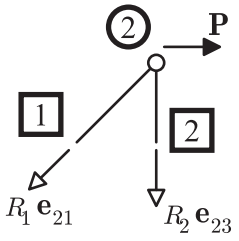


Fig. 2.6 Equilibrium at Node 2.

truss, through the step-by-step addition of two bars and one joint to the current configuration of the truss, where the two new bars in each step are precluded from being co-linear. Figures 2.2 and 2.3 are examples of simple trusses. Figure 2.4 shows an example of a non-simple truss. A fundamental advantage of simple trusses is that one can determine whether or not they are statically determinate via a simple counting procedure, which we will discuss later.

2.1.2 Governing equations

Our interest will be in developing an analysis scheme that will work for understanding the behavior of both determinate and indeterminate trusses. Our plan of attack will be to develop the relevant equilibrium, kinematic, and constitutive relations for general trusses and then combine the expressions to produce a single equation capable of describing the behavior of the system, determinate or indeterminate, just as we did in Chapter 1 for the basic one-dimensional problems of tension-compression bars, torsion rods, and beams. Our final result will be a set of equilibrium equations solely in terms of the displacements of the nodes of the truss. Thus it will be quite analogous, for example, to the relation $AEd^2u/dx^2 + b = 0$.

Equilibrium

To develop the equilibrium equations for the truss, let us examine the truss system in Fig. 2.5. We will begin by defining a unit vector pointing from node i to node j in the following manner,

$$\boldsymbol{e}_{ij} \quad := \quad \frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|}.$$

As a result of the definition, one can also observe the relation

$$\boldsymbol{e}_{ji} \quad = \quad -\boldsymbol{e}_{ij}.$$

For example the unit vector pointing from node 1 to node 2 is denoted as

$$\boldsymbol{e}_{12} \quad := \quad \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{L_1} \ ,$$

where L_1 is the length of bar 1, and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the positions of nodes 1 and 2.

For each node in the truss, one can write an equilibrium equation. For example, considering node 2, the balance of forces can be depicted as shown in Fig. 2.6. R_1 and R_2 denote the axial forces in bars 1 and 2, respectively. The nodal equilibrium equation is,

$$\boxed{R_1\boldsymbol{e}_{21} + R_2\boldsymbol{e}_{23} + \boldsymbol{P} = 0 \ .} \tag{2.1}$$

Remarks:

(1) Equation (2.1) is a vector equation and thus represents two scalar equations – balance of forces in the horizontal and vertical directions.

(2) Note that our convention is to always assume that forces are tensile. If they are actually compressive, then they will come out to be negative at the end of the analysis.

If we now consider equilibrium at node 1, the balance of forces can be depicted as shown in Fig. 2.7. R_1 denotes the axial force in bar 1 and $\boldsymbol{F}_{r1}$ is the reaction force at node 1. Thus, the nodal equilibrium equation is

$$\boxed{R_1 \boldsymbol{e}_{12} + \boldsymbol{F}_{r1} = 0 \ .}$$

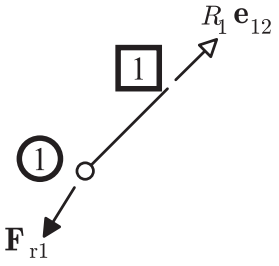


Fig. 2.7 Equilibrium at Node 1

The last node in the truss is node 3. The balance of forces at this node is depicted in Fig. 2.8. R_2 denotes the axial force in bar 2 and $\boldsymbol{F}_{r3}$ is the reaction force at node 3. The nodal equilibrium equation is

$$\boxed{R_2 \boldsymbol{e}_{32} + \boldsymbol{F}_{r3} = 0 \ .}$$

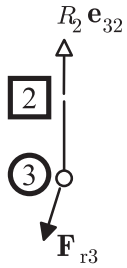


Fig. 2.8 Equilibrium at Node 3

Gathering the nodal equations of equilibrium, one has 3 vector (6 scalar) equations:

$$\begin{array}{rrcll} \boldsymbol{e}_{12} R_1 & & + & \boldsymbol{F}_{r1} & = 0, \\ \boldsymbol{e}_{21} R_1 & + \boldsymbol{e}_{23} R_2 & + & \boldsymbol{P} & = 0, \\ & \boldsymbol{e}_{32} R_2 & + & \boldsymbol{F}_{r3} & = 0, \end{array}$$

which can be written as a single equation:

$$\begin{aligned} & \begin{bmatrix} \boldsymbol{e}_{12} & \boldsymbol{0} \\ \boldsymbol{e}_{21} & \boldsymbol{e}_{23} \\ \boldsymbol{0} & \boldsymbol{e}_{32} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}_{r1} \\ \boldsymbol{P} \\ \boldsymbol{F}_{r3} \end{bmatrix} = \boldsymbol{0}, \\ \Leftrightarrow \quad & \begin{bmatrix} \boldsymbol{e}_{21} & \boldsymbol{0} \\ \boldsymbol{e}_{12} & \boldsymbol{e}_{32} \\ \boldsymbol{0} & \boldsymbol{e}_{23} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{r1} \\ \boldsymbol{P} \\ \boldsymbol{F}_{r3} \end{bmatrix}, \\ \Leftrightarrow \quad & \boldsymbol{A}^T \boldsymbol{R} = \boldsymbol{F}, \end{aligned}$$

where we have defined,

$$\begin{aligned} \boldsymbol{R} &:= \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_{2\times 1}, \\ \boldsymbol{F} &:= \begin{bmatrix} \boldsymbol{F}_{r1} \\ \boldsymbol{P} \\ \boldsymbol{F}_{r3} \end{bmatrix}_{6\times 1}, \\ \boldsymbol{A} &:= \begin{bmatrix} \boldsymbol{e}_{21}^T & \boldsymbol{e}_{12}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix}_{2\times 6}. \end{aligned}$$

$\boldsymbol{R}$ is the vector of internal bar forces, $\boldsymbol{F}$ is the vector of external nodal forces (applied forces as well as reaction forces), and $\boldsymbol{A}$ is called the compatibility matrix. The rows of $\boldsymbol{A}$ correspond to bars, one row for each bar, and the block columns correspond to the nodes, one block column for each node. This view point allows for a systematic element-by-element construction of the compatibility matrix $\boldsymbol{A}$. One sequentially loops over the number of bars, constructing one row at a time until the full matrix has been determined. Note that the vectors $\boldsymbol{e}_{ij}$ are assumed to be column vectors.

Remarks:

(1) The expression $\boldsymbol{A}^T\boldsymbol{R} = \boldsymbol{F}$ is the equilibrium relation for the truss; i.e. it is the analog to $dR/dx+b = 0$, $dT/dz+t = 0$ and $d^2M/dx^2 = q$. $\boldsymbol{F}$ plays the role of the applied loads; $\boldsymbol{R}$ plays the role of the internal resultant fields; and $\boldsymbol{A}$ plays the role of the differential operator.

(2) The individual scalar rows in $\boldsymbol{A}^T\boldsymbol{R} = \boldsymbol{F}$ represent force equilibrium in the horizontal and vertical directions at the nodes of the truss.

(3) It should be observed that $\boldsymbol{A}$ will in general not be square.

(4) The unknown forces appear in both $\boldsymbol{R}$ and certain entries of $\boldsymbol{F}$. This slightly complicates the solution of these equations and the determination of whether or not the system of equations is statically determinate or not.

(5) To write $\boldsymbol{A}^T$ in expanded form, for our example, one simply needs expressions for the unit (column) vectors connecting the nodes. If we assume that bar 1 makes an angle of $\pi/3$ with respect to the horizontal, then $\boldsymbol{e}_{12} = (1/2, \sqrt{3}/2)^T$ and $\boldsymbol{e}_{32} = (0, 1)^T$, implying that

$$\boldsymbol{A}^T = \begin{bmatrix} -1/2 & 0 \\ -\sqrt{3}/2 & 0 \\ 1/2 & 0 \\ \sqrt{3}/2 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Static determinacy and indeterminacy

As with tension-compression bars, torsion rods, and beams, trusses can be either statically determinate or indeterminate. Deciding whether a truss is statically determinate or indeterminate comes down to examining the equilibrium equations, $\boldsymbol{A}^T\boldsymbol{R} = \boldsymbol{F}$, and trying to decide if one can uniquely solve them for all the unknowns: *If all the unknowns in these equations can be uniquely solved for, then the system is considered to be statically determinate. If the unknowns can not be uniquely solved for, then the system is considered to be statically indeterminate.*

The equilibrium equations, as given, are a bit tricky to analyze since there are unknown quantities on both the left- and right-hand sides of

the equation. So before looking at them, let us first consider the classic linear algebra question of solving the linear equations $\boldsymbol{M}\boldsymbol{x} = \boldsymbol{b}$ for the vector $\boldsymbol{x} \in \mathbb{R}^n$ when the matrix $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ and the vector $\boldsymbol{b} \in \mathbb{R}^m$ are given. The *Roché-Capelli Theorem* of linear algebra tells us whether or not this system of equations can be uniquely solved for $\boldsymbol{x}$. The theorem considers two matricies, $\boldsymbol{M}$ and the *augmented* matrix $[\boldsymbol{M}\ \boldsymbol{b}]$, which is simply the matrix $\boldsymbol{M}$ with an additional column composed of the entries of $\boldsymbol{b}$. The theorem enumerates three possibilities:

(1) If $\text{rank}(\boldsymbol{M}) \neq \text{rank}([\boldsymbol{M}\ \boldsymbol{b}])$, then the system of equations has no solution.

(2) If $\text{rank}(\boldsymbol{M}) = \text{rank}([\boldsymbol{M}\ \boldsymbol{b}]) = n$, then the system of equations has a unique solution.

(3) If $\text{rank}(\boldsymbol{M}) = \text{rank}([\boldsymbol{M}\ \boldsymbol{b}]) < n$, then the system of equations has an infinite number of solutions.

Given a system of equations, it is a simple matter to determine the rank of the relevant matrices; for example, one can convert the matrices to (row) echelon form and count the number of non-zero rows.

In our case the vector of unknowns is composed of the bar forces, $\boldsymbol{R}$, and any unknown reaction force components, which we collectively denote as $\boldsymbol{F}_r$:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{F}_r \end{bmatrix} .$$

The vector $\boldsymbol{b}$ is equal to the original vector $\boldsymbol{F}$ where every unknown reaction force component has been set to zero. Lastly the matrix $\boldsymbol{M} = [\boldsymbol{A}^T\ \boldsymbol{C}]$, where the matrix $\boldsymbol{C}$ accounts for the fact that the unknown reaction force components have been moved to the left-hand side of the equation. The number of rows in $\boldsymbol{C}$ matches the number of equilibrium equations, and its number of columns is equal to the number of unknown reaction force components. Most of the entries of $\boldsymbol{C}$ are zero. However, if a particular reaction force component contributes to a given scalar equilibrium equation, then in that row one sets the corresponding column entry to minus one (-1). Once constructed one can apply the *Roché-Capelli Theorem* to see if the system of equations has a unique solution and is thus statically determinate.

The use of the *Roché-Capelli Theorem* works for all trusses, whether they are simple trusses or not. However in the case of simple trusses, a simpler procedure is available to determine if a truss is statically determinate or not. The procedure simply counts the number of available equilibrium equations and compares the count to the number of unknown forces. In general, for simple trusses compare the two quantities,

$$b + r\,, \tag{2.2}$$

and

$$n \times d\,, \tag{2.3}$$

where

b : Number of bars in the truss
d : Space dimension of the problem ($d = 2$ for planar trusses) and 3D($d = 3$ for space trusses)
n : Number of nodes in the truss
r : Number of support reaction forces

Note that $b+r$ in eqn (2.2) represents the total number of force unknowns in the problem and $n \times d$ in eqn (2.3) represents the total number of available equilibrium equations in the problem. The criterion for ascertaining determinacy versus indeterminacy is given as:

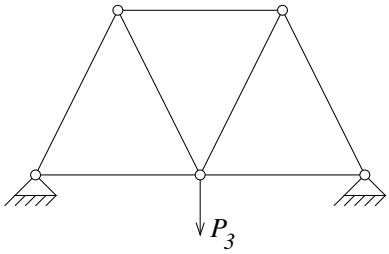


Fig. 2.9 Statically indeterminate truss structure with 7 elements.

$$\begin{aligned} b + r > n \times d &: \text{Indeterminate of degree } (b+r) - (n \times d) \\ = &: \text{Determinate (but possibly unstable)} \\ < &: \text{Indeterminate (unstable)} \end{aligned}$$

The three cases correspond directly to the three cases of the *Roché-Capelli Theorem* but in an quicker to ascertain manner. Beware, however, this counting methodology is only guaranteed to work for simple trusses.

Example 2.1

Seven bar truss

Consider the 7 bar truss shown in Fig. 2.9 and find the equilibrium equations for the truss.

Solution: To construct the equilibrium equations for the truss, one needs to number the nodes and the bars. This is arbitrary but necessary. The numbering scheme we will use is shown in Fig. 2.10, where the supports have been replaced by the (unknown) support reactions which they provide. Once the numbering scheme has been established, the equilibrium equations are easily formed by assembling the compatibility matrix (in transpose form). Each row represents one node and each column corresponds to a bar. Begin with the first row (node) and add entries in the columns associated with the bars that connect to it – in this case bars 1 and 2. Thus we get entries in columns 1 and 2. The first subscript of the $\boldsymbol{e}_{\sqcup\sqcup}$ entry will be the node the corresponding bar connects to and the second subscript will be the row number. All other row entries will be zero. Likewise, if there are any applied forces or support reactions on the node, then an entry needs to be added to the right-hand side load vector. For the truss shown the result will be

$$\begin{bmatrix} \boldsymbol{e}_{21} & \boldsymbol{e}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{e}_{12} & \mathbf{0} & \boldsymbol{e}_{32} & \boldsymbol{e}_{42} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e}_{13} & \boldsymbol{e}_{23} & \mathbf{0} & \boldsymbol{e}_{43} & \boldsymbol{e}_{53} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_{24} & \boldsymbol{e}_{34} & \mathbf{0} & \boldsymbol{e}_{54} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_{35} & \boldsymbol{e}_{45} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{r1} \\ \mathbf{0} \\ \boldsymbol{P}_3 \\ \mathbf{0} \\ \boldsymbol{F}_{r5} \end{bmatrix}$$

$$\Leftrightarrow \quad \boldsymbol{A}^T \boldsymbol{R} = \boldsymbol{F} \ .$$

For this structure, the expanded size of the matrix $\boldsymbol{A}^T$ is 10×7. The 10 comes from $n = 5$ nodes times $d = 2$ and the 7 comes from $b = 7$ bars. This implies that the system of equations expressing equilibrium is 10 scalar equations in 7 unknown bar force variables. The total number of unknowns is actually larger, since $r = 4$ (scalar) entries in $\boldsymbol{F}$ are also unknown; these correspond to the 4 scalar components of the support reactions. Thus we have 10 equilibrium equations for the truss but there are 11 unknown force quantities in the problem. Since the truss is simple, we can conclude that the system is statically indeterminate.

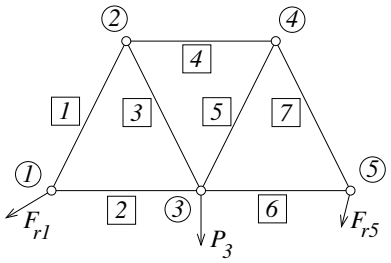


Fig. 2.10 Statically indeterminate truss structure with 7 elements (with support forces shown).

Remarks:

(1) If one adjusts the structure to allow, for example, node 5 to freely move in the horizontal direction, then there will only be $r = 3$ (scalar) support reactions and the system will be statically determinate; i.e. one will have 10 equations in 10 unknowns.

(2) In the determinate case, from the solution to $\boldsymbol{R}$, we also immediately know the bar stresses by division by the bar areas and the bar strains by subsequent division by the bar moduli. Determining the displacements, however, is a bit trickier and we will take that up next by considering the kinematics of truss systems.

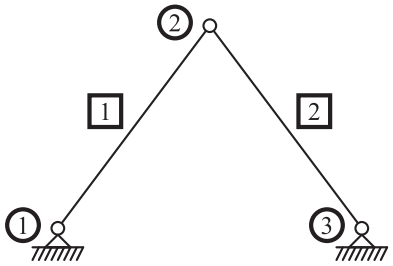


Fig. 2.11 2 bar truss structure.

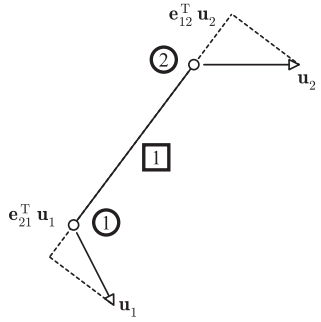


Fig. 2.12 Compatibility for element 1.

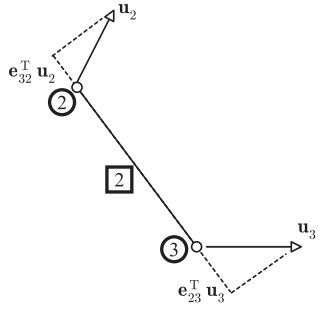


Fig. 2.13 Compatibility for element 2.

Kinematics

In considering the kinematics of a truss, we are interested in finding the relation between the nodal displacements and the bar strains. Consider the structure in Fig. 2.11. One can relate the change in length, ΔL_i, of bar i with the strain ε_i through the relations

$$\begin{aligned} \varepsilon_1 &= \frac{\Delta L_1}{L_1}, \\ \varepsilon_2 &= \frac{\Delta L_2}{L_2}. \end{aligned}$$

Note that the bar strains are constant for each bar since the loads are restricted to the nodes. Here, L_1 and L_2 are defined as the undeformed bar lengths. This relation can be reorganized in matrix notation as,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & \\ & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} \Delta L_1 \\ \Delta L_2 \end{bmatrix}$$

or more compactly as

$$\boldsymbol{\varepsilon} = \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{\Delta L} \ ,$$

where $\lceil 1/L \rfloor$ is understood to be a diagonal matrix with one-over-the length of each bar on the diagonal[1], $\boldsymbol{\varepsilon}$ is a vector of bar strains (one entry per bar), and $\boldsymbol{\Delta L}$ is a vector of changes of lengths of the bars (one entry per bar) . This gives us the relation between the bar strains and the changes in length of the bars. To convert this to the desired relation between bar strains and nodal displacements, we will need to develop a relation between $\boldsymbol{\Delta} L$ and the displacements of the nodes.

Consider, first, bar 1 as shown in Fig. 2.12. The relation between the change in length ΔL_1 and the displacements of the nodes at the ends $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ is obtained as,

$$\Delta L_1 = \boldsymbol{e}_{21}^T \boldsymbol{u}_1 + \boldsymbol{e}_{12}^T \boldsymbol{u}_2 \, .$$

The change in length is computed by projecting the nodal displacement vector at each end onto the unit vector parallel to the bar.

Consider now bar 2 as shown in Fig. 2.13. The relation between the change in length ΔL_2 and the displacements of the nodes at the ends $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ is obtained as,

$$\Delta L_2 = \boldsymbol{e}_{32}^T \boldsymbol{u}_2 + \boldsymbol{e}_{23}^T \boldsymbol{u}_3 \, .$$

The change in length is computed, again, by projecting the nodal displacement vector at each end onto the unit vector parallel to the bar.

[1]The notation $\boldsymbol{M} = \lceil f \rfloor$ is defined to mean a diagonal matrix with entries $M_{ii} = f_i$.

The two equations can be organized in matrix notation as:

$$\begin{bmatrix} \Delta L_1 \\ \Delta L_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_{21}^T & \boldsymbol{e}_{12}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{bmatrix}$$

or

$$\boldsymbol{\Delta L} = \boldsymbol{A}\boldsymbol{u} \ ,$$

where the matrix $\boldsymbol{A}$ is the same matrix that appears in the equilibrium equations. Combining $\boldsymbol{\varepsilon} = \lceil 1/L \rfloor \boldsymbol{\Delta L}$ and $\boldsymbol{\Delta L} = \boldsymbol{A}\boldsymbol{u}$, one obtains a single equation relating the vector of bar strains $\boldsymbol{\varepsilon}$ with the vector of nodal displacements:

$$\boxed{\boldsymbol{\varepsilon} = \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u} \ .}$$

Remarks:

(1) It should be emphasized that the rows of the compatibility matrix correspond to each bar and the block columns correspond to each node

$$\begin{array}{r} \\ \text{bar } 1 \rightarrow \\ \text{bar } 2 \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} \text{node } 1 & \text{node } 2 & \text{node } 3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \boldsymbol{e}_{21}^T & \boldsymbol{e}_{12}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix} \end{array}$$

Thus the size of $\boldsymbol{A}$ is b-by-$(d \cdot n)$.

Constitutive relation

The relationship between the stress σ and strain ε in bar i is given as, $\sigma_i = E_i \varepsilon_i$, where E_i is the Young's modulus of bar i. One can organize this in matrix notation as,

$$\boxed{\boldsymbol{\sigma} = \lceil E \rfloor \boldsymbol{\varepsilon} \ ,}$$

where $\lceil E \rfloor$ is a diagonal matrix with the Young's modulus of each bar as the diagonal components and the bold $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are vectors containing the stresses and strains of the bars, respectively.

Resultant definition

The relationship between the bar force R and bar stress σ in bar i is given as, $R_i = A_i \sigma_i$, where A_i is the cross-sectional area of the bar. One can organized this in matrix notation as

$$\boxed{\boldsymbol{R} = \lceil A \rfloor \boldsymbol{\sigma} \ ,}$$

where $\lceil A \rfloor$ is a diagonal matrix with the cross-sectional area of each bar as the diagonal components.

2.1.3 Problem setting

If we now collect all of our results, we will see that we have a system of equations that is quite analogous to those governing the basic problems of tension-compression bars, torsion rods, and beams. In particular we have:

- Equilibrium

$$\boldsymbol{A}^T \boldsymbol{R} = \boldsymbol{F}\,,$$

- Compatibility/Strain-displacement relation

$$\boldsymbol{\varepsilon} = \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u}\,,$$

- Constitutive relation

$$\boldsymbol{\sigma} = \lceil E \rfloor \boldsymbol{\varepsilon}\,,$$

- Resultant definition

$$\boldsymbol{R} = \lceil A \rfloor \boldsymbol{\sigma}\,.$$

Thus our system of equations has an identical structure to the problems we are already familiar with. This indicates that if we want, we can combine them as we did before to create a single equilibrium equation in terms of the displacements of the system and this system can be used to solve both determinate and indeterminate problems once the boundary conditions are known.

Equilibrium in terms of the nodal displacements

To create the equilibrium equation in terms of the nodal displacements we can proceed as before; viz., we can substitute the strain-displacement relations into the constitutive relation, into the resultant definition, and finally into the equilibrium equation. Thus the stresses in terms of displacements is given as:

$$\boldsymbol{\sigma} = \lceil E \rfloor \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u} = \left\lceil \frac{E}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u}$$

and the internal resultants in terms of the displacements are given as:

$$\boldsymbol{R} = \lceil A \rfloor \left\lceil \frac{E}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u} = \left\lceil \frac{AE}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u}\,.$$

Substitution into the equilibrium equation yields the final result:

$$\boxed{\boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}\boldsymbol{u} = \boldsymbol{F}\,.}$$

This is the equilibrium equation for a truss in terms of the nodal displacement vector $\boldsymbol{u}$.

Remarks:

(1) $\boldsymbol{K} := \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}$ is called the stiffness matrix. It is important to note that this matrix is symmetric. This can be shown through the following manipulation:

$$\boldsymbol{K}^T = \left(\boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A} \right)^T = \boldsymbol{A}^T \lceil AE/L \rfloor^T (\boldsymbol{A}^T)^T = \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A} ,$$

where we have used the fact that $\lceil AE/L \rfloor^T = \lceil AE/L \rfloor$ since it is a diagonal matrix.

(2) $\boldsymbol{A}$ is the *compatibility matrix*.

(3) $\boldsymbol{K}$ is a $(d \cdot n)$-by-$(d \cdot n)$ square matrix, and $\boldsymbol{u}$ and $\boldsymbol{F}$ are vectors of size $(d \cdot n)$-by-1.

(4) It customary in structural engineering to write $\boldsymbol{d}$ for $\boldsymbol{u}$. Employing this notation, the equilibrium equation in terms of nodal displacements is often written as

$$\boxed{\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \, .}$$

Example 2.2

Computation of the stiffness matrix

Compute the stiffness matrix $\boldsymbol{K}$ of the structure in Fig. 2.14.

Solution: The nodes and bars are numbered as shown. The compatibility matrix for the structure is

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{e}_{31}^T & \boldsymbol{0} & \boldsymbol{e}_{13}^T \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix} .$$

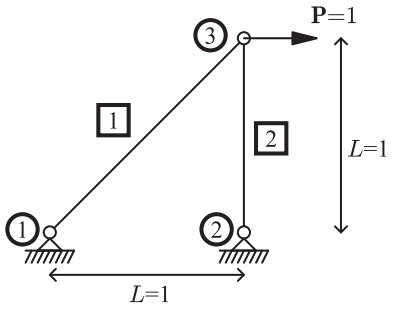


Fig. 2.14 2 bar truss structure.

The stiffness matrix is then obtained through the computation:

$$\begin{aligned}
\boldsymbol{K} &= \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A} \\
&= \begin{bmatrix} \boldsymbol{e}_{31} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{32} \\ \boldsymbol{e}_{13} & \boldsymbol{e}_{23} \end{bmatrix} \begin{bmatrix} \left(\frac{AE}{L}\right)_1 & \boldsymbol{0} \\ \boldsymbol{0} & \left(\frac{AE}{L}\right)_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{31}^T & \boldsymbol{0} & \boldsymbol{e}_{13}^T \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{e}_{31}\left(\frac{AE}{L}\right)_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{32}\left(\frac{AE}{L}\right)_2 \\ \boldsymbol{e}_{13}\left(\frac{AE}{L}\right)_1 & \boldsymbol{e}_{23}\left(\frac{AE}{L}\right)_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{31}^T & \boldsymbol{0} & \boldsymbol{e}_{13}^T \\ \boldsymbol{0} & \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{e}_{31}\left(\frac{AE}{L}\right)_1 \boldsymbol{e}_{31}^T & \boldsymbol{0} & \boldsymbol{e}_{31}\left(\frac{AE}{L}\right)_1 \boldsymbol{e}_{13}^T \\ \boldsymbol{0} & \boldsymbol{e}_{32}\left(\frac{AE}{L}\right)_2 \boldsymbol{e}_{32}^T & \boldsymbol{e}_{32}\left(\frac{AE}{L}\right)_2 \boldsymbol{e}_{23}^T \\ \boldsymbol{e}_{13}\left(\frac{AE}{L}\right)_1 \boldsymbol{e}_{31}^T & \boldsymbol{e}_{23}\left(\frac{AE}{L}\right)_2 \boldsymbol{e}_{32}^T & \boldsymbol{e}_{23}\left(\frac{AE}{L}\right)_2 \boldsymbol{e}_{23}^T + \boldsymbol{e}_{13}\left(\frac{AE}{L}\right)_1 \boldsymbol{e}_{13}^T \end{bmatrix} ,
\end{aligned}$$

where

$$\boldsymbol{e}_{32} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \boldsymbol{e}_{31} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\boldsymbol{e}_{32}\boldsymbol{e}_{32}^T = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{e}_{31}\boldsymbol{e}_{31}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

To simplify matters a little bit let us assume that each bar has the same Young's modulus and cross-sectional area. Further, let us define the matrices

$$\begin{aligned} \boldsymbol{k}_1 &:= \left(\frac{EA}{L}\right)_1 \boldsymbol{e}_{31}\boldsymbol{e}_{31}^T = \frac{EA}{\sqrt{2}L} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \\ \boldsymbol{k}_2 &:= \left(\frac{EA}{L}\right)_2 \boldsymbol{e}_{32}\boldsymbol{e}_{32}^T = \frac{EA}{L} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

With these definitions, one has the following structure for the stiffness matrix $\boldsymbol{K}$:

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1 & \boldsymbol{0} & -\boldsymbol{k}_1 \\ \boldsymbol{0} & \boldsymbol{k}_2 & -\boldsymbol{k}_2 \\ -\boldsymbol{k}_1 & -\boldsymbol{k}_2 & \boldsymbol{k}_1 + \boldsymbol{k}_2 \end{bmatrix}.$$

The individual matrices (the so-called truss bar stiffness matrices) $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are symmetric, and the entire stiffness matrix $\boldsymbol{K}$ is symmetric. If we fully expand the stiffness matrix (to have scalar rows and columns) we have

$$\boldsymbol{K} = \frac{AE}{L} \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} \end{bmatrix}.$$

Boundary conditions

To solve for the vector of nodal displacements $\boldsymbol{u}$, which has a total of $d \cdot n$ unknowns (degrees-of-freedom), one must specify the boundary conditions. This situation is identical to the setup for the previously introduced mechanical problems governed by differential equations. When boundary conditions on the displacements are specified (e.g., in the case

of a pin support the displacements are restrained to be zero) the value of the displacement for the corresponding degree-of-freedom u_i becomes a known value but the corresponding entry in the right-hand side F_i is unknown. For the other degrees-of-freedom where the displacement value is not specified, the degree-of-freedom u_i is free to move and consequently must be solved for; the corresponding entry in the force vector is known. Based on these delineations, one can separate the set of degrees-of-freedom (a total of $d \cdot n$) into two disjoint groups:

- id_f: The degrees-of-freedom which are (F)REE TO MOVE,
- id_d: The degrees-of-freedom where the (D)ISPLACEMENTS ARE KNOWN.

It should be noted that at the degrees-of-freedom associated with the set id_f the value of the applied force F_i is known and at the degrees-of-freedom which are in the set id_d the value of the applied forces F_i are unknown. These unknown forces are exactly the reaction forces at the supports.

Using these two disjoint sets of indices, one can group the nodal displacement degrees-of-freedom into two vectors $\boldsymbol{u}_f$ and $\boldsymbol{u}_d$. Similarly the entries of the force vector can also be grouped into two vectors $\boldsymbol{F}_f$ and $\boldsymbol{F}_d$. To clarify this concept, assume that id contains all the degrees-of-freedom $id = \{1, 2, 3, 4, 5, 6\}$ where we have assumed the problem has a total of 6 degrees-of-freedom. Let us assume that id_f and id_d are defined as

$$\begin{aligned} id_f &:= \{3, 5, 6\}, \\ id_d &:= \{1, 2, 4\} \ . \end{aligned}$$

Then,

$$\boldsymbol{u}_f = \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix}, \qquad \boldsymbol{u}_d = \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix}, \qquad \boldsymbol{F}_f = \begin{bmatrix} F_3 \\ F_5 \\ F_6 \end{bmatrix}, \qquad \boldsymbol{F}_d = \begin{bmatrix} F_1 \\ F_2 \\ F_4 \end{bmatrix} \ .$$

Let us additionally define the vectors $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{F}}$ as,

$$\widehat{\boldsymbol{u}} := \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_d \end{bmatrix}, \qquad \widehat{\boldsymbol{F}} := \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_d \end{bmatrix} \ .$$

These two vectors are permuted versions of $\boldsymbol{u}$ and $\boldsymbol{F}$, i.e., the entries of the vectors have been reordered as:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \xrightarrow{\text{permute entries}} \widehat{\boldsymbol{u}} = \begin{bmatrix} u_3 \\ u_5 \\ u_6 \\ u_1 \\ u_2 \\ u_4 \end{bmatrix}$$

We can permute the entries of the stiffness matrix $\boldsymbol{K}$ to respect this permutation and obtain its permuted version $\widehat{\boldsymbol{K}}$

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \xrightarrow{\text{permute entries}}$$

$$\widehat{\boldsymbol{K}} = \begin{bmatrix} K_{33} & K_{35} & K_{36} & K_{31} & K_{32} & K_{34} \\ K_{53} & K_{55} & K_{56} & K_{51} & K_{52} & K_{54} \\ K_{63} & K_{65} & K_{66} & K_{61} & K_{62} & K_{64} \\ K_{13} & K_{15} & K_{16} & K_{11} & K_{12} & K_{14} \\ K_{23} & K_{25} & K_{26} & K_{21} & K_{22} & K_{24} \\ K_{43} & K_{45} & K_{46} & K_{41} & K_{42} & K_{44} \end{bmatrix}$$

which relates $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{F}}$ by

$$\widehat{\boldsymbol{K}}\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{F}} \ .$$

Just as $\boldsymbol{u}_f$ and $\boldsymbol{u}_d$ partition $\widehat{\boldsymbol{u}}$, we can partition $\widehat{\boldsymbol{K}}$ accordingly:

$$\widehat{\boldsymbol{K}} \quad = \quad \begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fd} \\ \boldsymbol{K}_{df} & \boldsymbol{K}_{dd} \end{bmatrix} \ .$$

The subscript f denotes that the rows or columns correspond to entries coming from the set id_f and the subscript d denotes that the rows or columns correspond to entries coming from the set id_d. Thus for our 6 degree-of-freedom example,

$$\boldsymbol{K}_{ff} \quad = \quad \begin{bmatrix} K_{33} & K_{35} & K_{36} \\ K_{53} & K_{55} & K_{56} \\ K_{63} & K_{65} & K_{66} \end{bmatrix}$$

and

$$\boldsymbol{K}_{df} \quad = \quad \begin{bmatrix} K_{13} & K_{15} & K_{16} \\ K_{23} & K_{25} & K_{26} \\ K_{43} & K_{45} & K_{46} \end{bmatrix} \ .$$

It should be emphasized that physically there is no difference between the expressions $\widehat{\boldsymbol{K}}\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{F}}$ and $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$. They are just permuted versions of each other. Thus one can also solve the mechanical problem using the permuted form. Employing the partitioned form, one has

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fd} \\ \boldsymbol{K}_{df} & \boldsymbol{K}_{dd} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_d \end{bmatrix} \quad = \quad \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_d \end{bmatrix} .$$

Employing the fact that $\boldsymbol{u}_d$ is known, we can solve for the unknowns $\boldsymbol{u}_f$ and $\boldsymbol{F}_d$ in the following two steps:

(1) Use the first (block) row to determine $\boldsymbol{u}_f$ as:

$$\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}(\boldsymbol{F}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d) \ .$$

(2) Evaluate $\boldsymbol{F}_d$ from the second (block) row

$$\boldsymbol{F}_d = \boldsymbol{K}_{df}\boldsymbol{u}_f + \boldsymbol{K}_{dd}\boldsymbol{u}_d \ ,$$

since $\boldsymbol{u}_f$ is now a known quantity from step 1.

(3) This procedure is commonally known as *static condensation.*

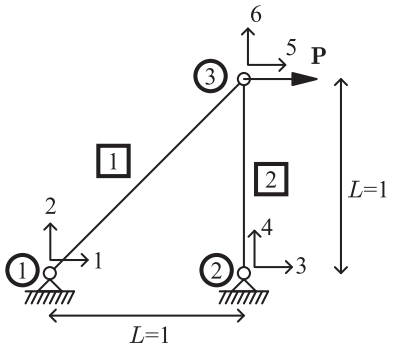


Fig. 2.15 2 bar truss structure with DOF labeling; assume $P = 1$.

Example 2.3

Solution of direct-stiffness equations

Solve for the displacements and reaction forces of the structure shown in Fig. 2.15. The compatibility matrix $\boldsymbol{A}$ and stiffness matrix $\boldsymbol{K}$ for this structure have been computed in Example 2.2.

Solution: First determine the sets id_f and id_f. Since node 3 is free to move in the x and y directions, these are the (F)ree degrees-of-freedom. The remaining degrees of freedom correspond to the known (D)isplacement degrees-of-freedom:

$$\begin{aligned} id_f &:= \{5,6\}, \\ id_d &:= \{1,2,3,4\} \ . \end{aligned}$$

From this identification, one has,

$$\boldsymbol{u}_d = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{F}_f = \begin{bmatrix} F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ ,$$

and

$$\boldsymbol{K}_{ff} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 \end{bmatrix} .$$

Note $\boldsymbol{K}_{ff}$ is obtained directly from the full $\boldsymbol{K}$ matrix by extracting the rows and columns corresponding to the id_f indices; i.e. one extracts rows 5,6 and columns 5,6. Since $\boldsymbol{u}_d = \boldsymbol{0}$, the linear system we must solve for $\boldsymbol{u}_f$ is

$$\begin{aligned} & \boldsymbol{F}_f = \boldsymbol{K}_{ff}\boldsymbol{u}_f \\ \Leftrightarrow & \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+1 \end{bmatrix}\begin{bmatrix} u_5 \\ u_6 \end{bmatrix} \\ \Rightarrow & \begin{bmatrix} u_5 \\ u_6 \end{bmatrix} = \frac{L}{EA}\begin{bmatrix} 1+2\sqrt{2} \\ -1 \end{bmatrix} \ . \end{aligned}$$

Since $\boldsymbol{u}_f$ is now known, one can compute the support reactions $\boldsymbol{F}_d$,

$$\begin{aligned}
\boldsymbol{F}_d &= \boldsymbol{K}_{df}\boldsymbol{u}_f + \boldsymbol{K}_{dd}\boldsymbol{u}_d \\
&= \boldsymbol{K}_{df}\boldsymbol{u}_f \\
&= \frac{AE}{L}\begin{bmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \frac{L}{EA}\begin{bmatrix} 1+2\sqrt{2} \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Remarks:

(1) Knowing the displacement vector $\boldsymbol{u}$ one can also compute the changes in lengths, strains, stresses, and forces in the bars through the expressions:

$$\begin{aligned}
\boldsymbol{\Delta L} &= \boldsymbol{A}\boldsymbol{u}, \\
\boldsymbol{\varepsilon} &= \lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{u}, \\
\boldsymbol{\sigma} &= \lceil E/L \rfloor \boldsymbol{A}\boldsymbol{u}, \\
\boldsymbol{R} &= \lceil EA/L \rfloor \boldsymbol{A}\boldsymbol{u}\ .
\end{aligned}$$

Solution procedure

The procedure of solving truss structures can be summarized as follows:

(1) Determine the geometry and boundary conditions (loading and supports).

(2) Form the matrices $\boldsymbol{A}$, $\lceil E \rfloor$, $\lceil A \rfloor$, $\lceil 1/L \rfloor$, and compute $\boldsymbol{K} = \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}$.

(3) Identify the (F)ree degrees-of-freedom and known (D)isplacement degrees-of-freedom, to extract $\boldsymbol{K}_{ff}$ and form $\boldsymbol{F}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d$.

(4) Solve for the displacements $\boldsymbol{u}_f$ through the equation $\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d$.

(5) Evaluate $\boldsymbol{F}_d = \boldsymbol{K}_{df}\boldsymbol{u}_f + \boldsymbol{K}_{dd}\boldsymbol{u}_d$ for the support reactions.

(6) Post-process the results for other desired quantities such as bar forces, stresses, and strains.

2.2 Analysis of flow networks

As an example of other network like structures that can be analyzed with a similar method, let us look at the problem of fluid flow in a piping network. Such systems are made from pipes with varying diameters and subject to in-flow/out-flow conditions and/or imposed fluid pressures. Shown in Fig. 2.16 is an example network composed of 7 pipes with possibly varying resistances to fluid flow. The system of pipes is connected at 6 nodes or pipe junctions. Knowing the specified values and the pipe geometry the standard question is to determine the flow in each pipe as well as the fluid pressure at the nodes. For our analysis we will make the assumption that the fluid flow is viscous and laminar (i.e. one has what is known as Poiseuille flow in the pipes which we will assume to be circular).

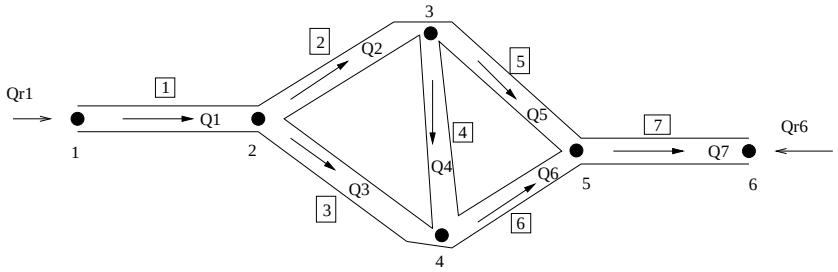


Fig. 2.16 Example piping network with imposed pressures at nodes 1 and 6.

The important physical variables in such problems are:

(1) The lengths of the pipes, L_i.

(2) The areas of the pipes, $A_i = \pi d_i^2/4$, where d_i are the pipe diameters.

(3) The resistances to fluid flow in the pipes per unit length, R_i. For viscous laminar flow in circular pipes $R_i = 32\nu/d_i^2$, where ν is the fluid's *kinematic* viscosity.

(4) The pressure at each node, P_i, which we will gather into a vector of nodal pressures, $\boldsymbol{P}$. The dimensions of pressure are force per unit area.

(5) The fluid flux in each pipe, q_i, which we will gather into a vector of pipe fluxes, $\boldsymbol{q}$. The dimensions of fluid flux are mass per unit area per unit time.

(6) The resultant rate of fluid flow in each pipe, Q_i, which we will gather into a vector of pipe flows, $\boldsymbol{Q}$. The dimensions of fluid flow in a pipe are mass per unit time.[2]

[2]If one chooses μ viscosity, instead of kinematic viscosity, in the definition of the resistances, then the flux will be in dimensions of fluid velocity and the flow will be in dimensions of volume per unit time.

2.2.1 Conservation of mass

In this type of problem the important conserved quantity is mass; i.e. mass conservation will serve as the counter part to conservation of momentum (force equilibrium). As a sign convention we will assume that the fluid flow is positive in the direction of increasing *nodal numbering*. The actual direction of the fluid flow will then be determined by the sign of the solution. If we consider for example node 2 of the network shown in Fig. 2.16, then the balance of mass tells us

$$Q_1 - Q_2 - Q_3 = 0\,,$$

where we have respected our sign convention. Similar balance equations can be written for all the other nodes in the system to give six mass/volume balance equations:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 7} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{pmatrix}_{7\times 1} + \begin{pmatrix} Q_{r1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_{r6} \end{pmatrix}_{6\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{6\times 1}$$

As before, we will move the vector associated with the imposed and/or unknown external nodal fluid flows to the right-hand side and then absorb the minus sign into the matrix. In compact form we can then write

$$\boldsymbol{A}^T\boldsymbol{Q} = \boldsymbol{Q}_r\,,$$

where

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Note that each row of $\boldsymbol{A}^T$ corresponds to a node in the network and each column a pipe. If the pipe carries fluid away from the node the sign of the entry is positive else it is negative.

2.2.2 Compatibility

Compatibility in this context is the relation between the nodal pressures and the pressure gradients in individual pipes. We will define a positive change in pressure for a given pipe ΔP_i as the difference between the nodal pressure at the node with higher index from that with lower index. Thus for our example network

$$\begin{pmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \Delta P_5 \\ \Delta P_6 \\ \Delta P_7 \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{pmatrix} .$$

In compact form this can be written as

$$\boldsymbol{\Delta P} = -\boldsymbol{A P} \,,$$

where we have the same $\boldsymbol{A}$ matrix as in the expression for mass conservation. The gradient of pressure in each pipe is $P_i' = \Delta P_i / L_i$. Thus we have as a final result that

$$\boldsymbol{P'} = -\left\lceil \frac{1}{L} \right\rfloor \boldsymbol{A P} \,.$$

This equation is the counter-part to the relation between bar strains and nodal displacements.

2.2.3 Constitutive relation

The flux of fluid in a pipe is related to the negative pressure gradient as $q_i = -(1/R_i)P_i'$. As a result, we can write

$$\boldsymbol{q} = -\left\lceil \frac{1}{R} \right\rfloor \boldsymbol{P'} \,.$$

2.2.4 Resultant relation

The total flow in a given pipe is related to the fluid flux by multiplying by the pipe's cross-sectional area. This gives

$$\boldsymbol{Q} = \lceil A \rfloor \, \boldsymbol{q} \,.$$

2.2.5 Problem setting

If we now collect our results, we will see that we have a system of equations that is quite analogous to the case of trusses:

- Mass conservation

$$\boldsymbol{A}^T \boldsymbol{Q} = \boldsymbol{Q}_r \,,$$

- Compatibility/Pressure gradient-pressure relation

$$\boldsymbol{P'} = -\left\lceil \frac{1}{L} \right\rfloor \boldsymbol{AP}\,,$$

- Constitutive relation

$$\boldsymbol{q} = -\left\lceil \frac{1}{R} \right\rfloor \boldsymbol{P'}\,,$$

- Resultant definition

$$\boldsymbol{Q} = \lceil A \rfloor \boldsymbol{q}\,.$$

Thus our system of equations has an identical structure to the problems we are already familiar with. We can proceed as before and create a single balance equation (conservation of mass) in terms of the nodal pressures.

Conservation of mass in terms of the nodal pressures

To create the conversation equation in terms of the nodal pressures we can proceed as before; viz., we can substitute the compatibility relation into the constitutive relation, then into the resultant definition, and finally into the equilibrium equation. Thus fluxes in terms of pressures is given as:

$$\boldsymbol{q} = -\left\lceil \frac{1}{R} \right\rfloor (-1) \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{AP} = \left\lceil \frac{1}{RL} \right\rfloor \boldsymbol{AP}$$

and the resultant flows in terms of the pressures are given as:

$$\boldsymbol{Q} = \lceil A \rfloor \left\lceil \frac{1}{RL} \right\rfloor \boldsymbol{AP} = \left\lceil \frac{A}{RL} \right\rfloor \boldsymbol{AP}\,.$$

Substitution into the mass conservation equation yields the final result:

$$\boxed{\boldsymbol{A}^T \left\lceil \frac{A}{RL} \right\rfloor \boldsymbol{AP} = \boldsymbol{Q}_r\,.}$$

This is the mass conservation equation for a pipe network in terms of the nodal pressure vector $\boldsymbol{P}$. The process of solution is exactly as with the truss. Some of the unknowns are in $\boldsymbol{P}$ and some in $\boldsymbol{Q}_r$. As before one can split the knowns from the unknowns by reordering. Then one first solves for the unknown pressures in terms of the known pressures and the known input/output flows. Following that, one can evaluate for the unknown flows at locations where the nodal pressures are given.

2.3 Results for other networks

The method of analysis we have introduced is quite general and as mentioned above can be applied to many other types of systems. Without derivation, we list the results for a few other systems.

2.3.1 Electrical resistance networks

In electrical resistance networks/circuits, we have a very similar result. The conservation of current (charge) can be written in the form $\boldsymbol{A}^T\boldsymbol{I} = \boldsymbol{I}_s$, where $\boldsymbol{I}$ is a vector of branch currents and $\boldsymbol{I}_s$ is a vector of junction current sources. $\boldsymbol{A}$ is as in the piping network case. The voltage gradients in the branches of the circuit are given by $\boldsymbol{V}' = -\lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{V}$ – quite similar to the flow network relation for pressure gradients. Ohm's law is the constitutive relation and it tells us that the current density $\boldsymbol{i} = -\lceil \sigma \rfloor \boldsymbol{V}'$, where σ_i are the electrical specific conductances of the branches. The current density is related to the branch currents by $\boldsymbol{I} = \lceil A \rfloor \boldsymbol{i}$. These relations can be combined to give the current conservation relations in terms of the junction voltages:

$$\boldsymbol{A}^T \left\lceil \frac{1}{R} \right\rfloor \boldsymbol{A}\boldsymbol{V} = \boldsymbol{I}_s \,, \tag{2.4}$$

where $R_i = L_i/\sigma_i A_i$ are the branch resistances. Once the junction voltages have been computed, the branch currents are also easily computed.[3]

2.3.2 Thermal conduction networks

In thermal conduction networks, one has a network of conductors that transport thermal energy. In this setting the conservation of energy can be written in the form $\boldsymbol{A}^T\boldsymbol{Q} = \boldsymbol{Q}_t$, where $\boldsymbol{Q}$ is a vector of heat fluxes and $\boldsymbol{Q}_t$ is a vector of nodal heat sources. Again $\boldsymbol{A}$ is as before. The temperature gradients in the network are given by $\boldsymbol{T}' = -\lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{T}$ similar to our other examples. The entries L_i are the lengths of the conductors. The constitutive law for heat conduction is Fourier's Law $\boldsymbol{q} = -\lceil k \rfloor \boldsymbol{T}'$, where k_i are the conductivities of the conductors and $\boldsymbol{q}$ are the specific heat fluxes (flux per unit area). These are related to the heat fluxes via $\boldsymbol{Q} = \lceil A \rfloor \boldsymbol{q}$, where A_i are the conductor areas. Assembling, these relations yields a system of equations for conservation of energy in terms of the nodal temperatures:

$$\boldsymbol{A}^T \left\lceil \frac{Ak}{L} \right\rfloor \boldsymbol{A}\boldsymbol{T} = \boldsymbol{Q}_t \,. \tag{2.5}$$

Once the nodal temperatures have been determined other quantities of interest are easily determined using the individual relations.[4]

2.3.3 Other cases

These are only a few of the many possible cases. Other examples governed by similar systems are porous flow systems, elastic cable networks,

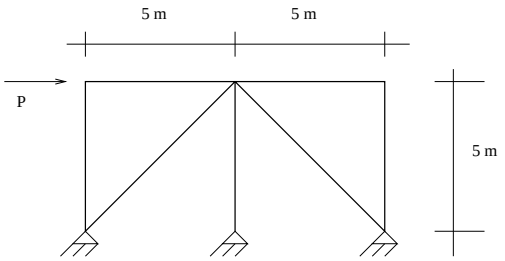
[3]Common (consistent) units for the electrical quantities are: current in amps, voltage in volts, specific conductance in Siemens per meter, and resistance in Ohms (=1/Siemens).

[4]A common (consistent) units for the thermal quantities are: temperature in Kelvin or Celsius, heat flux in Watts, specific heat flux in Watts per meter squared, and conductivity in Watts per meter per degree Kelvin/Celsius.

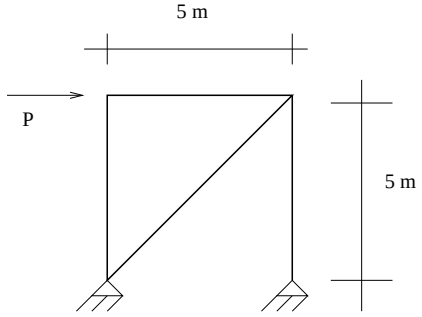
torsion rod systems, and contaminant diffusion networks – to name just a few more.

Exercises

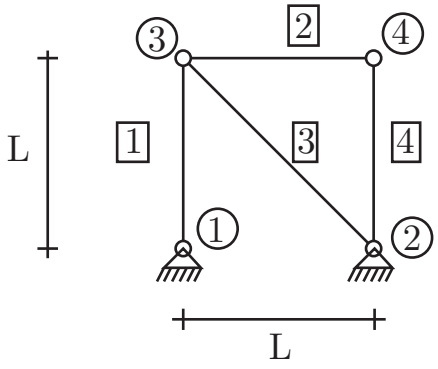
(2.1) For the truss shown construct the $\boldsymbol{A}^T$ matrix and the system's equilibrium equations.



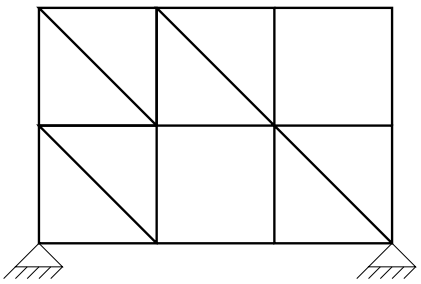
(2.2) For the truss shown construct the $\boldsymbol{A}^T$ matrix and the system's equilibrium equations.



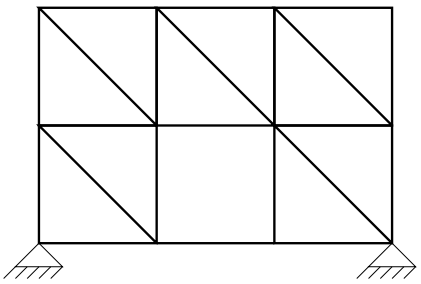
(2.3) For the 4-bar truss shown, determine the 4×8 compatibility matrix $\boldsymbol{A}$. Use the provided numbering scheme.



(2.4) For the simple truss shown, determine its degree of indeterminacy.



(2.5) For the simple truss shown, determine its degree of indeterminacy.



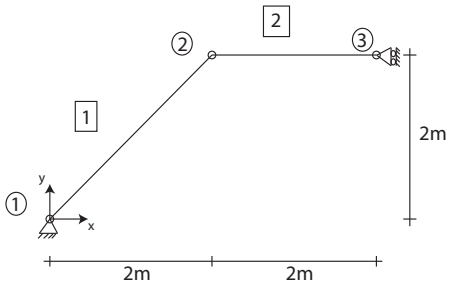
(2.6) For the truss shown, find:

(a) The 2×6 compatibility matrix $\boldsymbol{A}$.

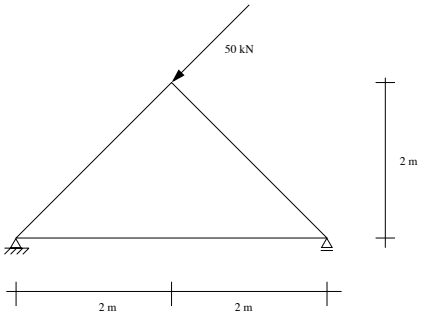
(b) Assuming that the nodal displacements are

$$\boldsymbol{u}_1 = \boldsymbol{0}, \quad \boldsymbol{u}_2 = 2\boldsymbol{e}_x + 2\boldsymbol{e}_y \text{ (mm)}, \quad \boldsymbol{u}_3 = 2\boldsymbol{e}_y \text{ (mm)},$$

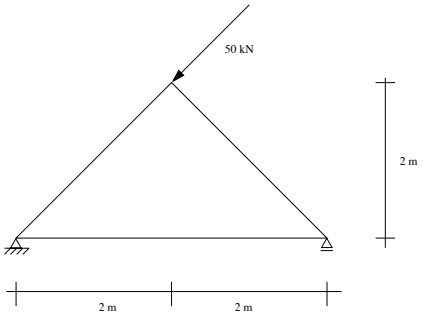
what are the bar strains?



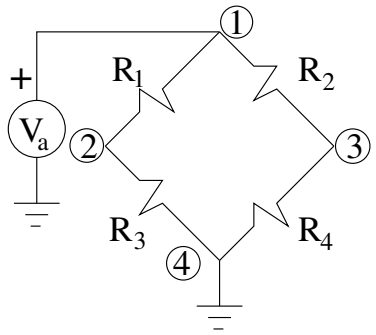
(2.7) Consider the truss shown below. Write out the governing equilibrium equations for the truss in the form $\boldsymbol{A}^T\boldsymbol{R} = \boldsymbol{F}$.



(2.8) Consider the truss shown below. Write out the 6x6 stiffness matrix for this truss. Assume that the horizontal bar has an $AE = 180$ MN and that the other two have an $AE = 200$ MN. Using the given displacement conditions, write out the reduced 3x3 system of equations that needs to be solved to find $\boldsymbol{u}_f$.



(2.9) Consider the (Wheatstone) bridge circuit shown below.



(a) Determine the $\boldsymbol{A}$ matrix.

(b) Determine the matrix $\boldsymbol{K} = \boldsymbol{A}^T \lceil 1/R \rfloor \boldsymbol{A}$.

(c) Identify the two unknown voltages and two unknown current "sources".

(d) Solve the governing equations for the two unknown voltages.

(e) Now assume that $R_1 = R_3 = R_4$ and that $R_2 = (1 + \epsilon)R_4$, where $|\epsilon| \ll 1$; i.e. the resistance in branch is 2 is almost the same as the others. Show that under these conditions, that $V_o = V_3 - V_2 \approx -\frac{V_a}{4}\epsilon$.

[Remark: This circuit is a common way of detecting small changes in a given resistor. It has wide applications, for example in the measurement of mechanical strain using resistive foil-gauges.]

(2.10) The governing equations for a 6 degree of freedom system have been assembled to yield the following linear system of equations:

$$\left[\begin{array}{cc|cccc} 1 & 2 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \left(\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array}\right) = \left(\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array}\right)$$

From the boundary conditions, it is known that $c_3 = c_4 = c_5 = c_6 = 2$ and that $F_1 = F_2 = 1$.

(a) Find c_1 and c_2.

(b) Find F_3, F_4, F_5, and F_6.

(2.11) Consider the following system of equations and solve for the unknowns u_1, u_2, u_3, F_4, and F_5.

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 2 \\ 0 & \frac{1}{2} & 0 & 1 & 2 \\ 0 & 0 & \frac{1}{3} & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{array}\right] \left(\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ 1 \\ \frac{1}{2} \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ 3 \\ F_4 \\ F_5 \end{array}\right)$$

MATLAB Exercises

(2.12) Consider the truss shown below. Assume the bars are steel $E = 30 \times 10^6$ psi. The vertical bars have a diameter of 0.5 in, the horizontal 0.75 in, and the diagonal 1.0 in. Using the numbering scheme shown:

(a) Find the compatibility matrix for the truss.

(b) Find the diagonal matrix $\lceil AE/L \rfloor$.

(c) Assuming $u_{2x} = 0.1$ in, $u_{2y} = 0.0$ in, $u_{3x} = -0.1$ in, $u_{3y} = 0.1$ in,

(i) Find the strains in the 5 bars.

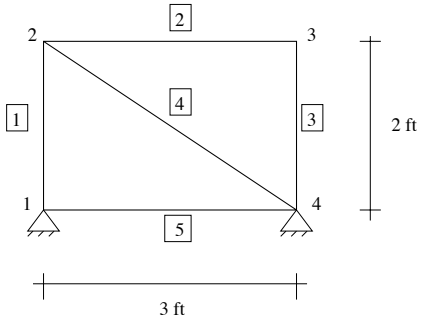
(ii) What are the stresses in the 5 bars?

(iii) What are the internal forces in the 5 bars?

(iv) What forces must have been applied to nodes 2 and 3?

(v) What are the support reactions at nodes 1 and 4?

[Note you should not have any need to use the $\boldsymbol{K}$ matrix to solve this problem.]



(2.13) Consider the truss shown below. Assume the bars are steel $E = 30 \times 10^6$ psi. The vertical bars have a diameter of 0.5 in, the horizontal 0.75 in, and the diagonal 1.0 in. Using the numbering scheme shown:

(a) Find the compatibility matrix for the truss.

(b) Find the diagonal matrix $\lceil AE/L \rfloor$.

(c) Assuming $u_{1x} = -0.1$ in, $u_{1y} = 0.1$ in, $u_{3x} = 0.1$ in, $u_{3y} = 0.0$ in.

(i) Find the strains in the 5 bars.

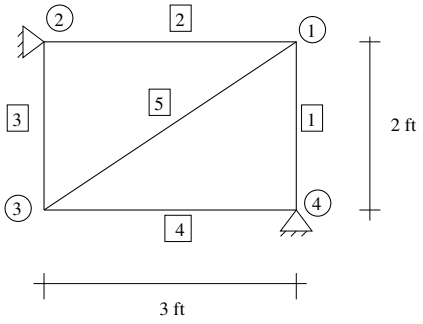
(ii) What are the stresses in the 5 bars?

(iii) What are the internal forces in the 5 bars?

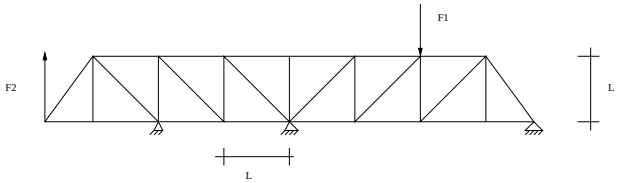
(iv) What forces must have been applied to nodes 1 and 3?

(v) What are the support reactions at nodes 2 and 4?

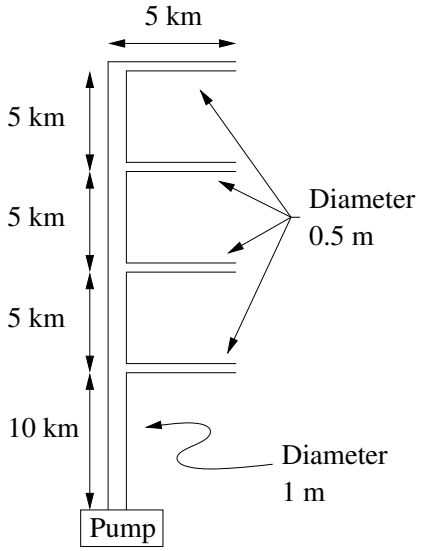
[Note you should not have any need to use the $\boldsymbol{K}$ matrix to solve this problem.]



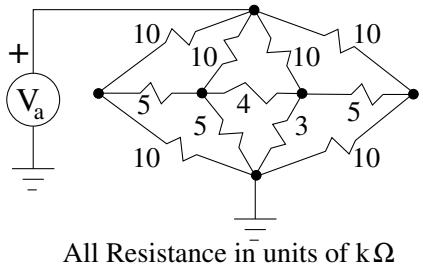
(2.14) Consider the truss shown below. Assume that $L = 2$ ft, the bars are all solid round stock mild steel, the upper bars have diameter 0.75 in, the lower bars have diameter 1.0 in and the diagonal and vertical bars have diameter 0.5 in. Assume that $F_1 = 3.0F_2 > 0$. Find the smallest value of F_2 for which a bar in the truss reaches the yield stress $\sigma_Y = 40$ ksi. What is the deflection at the two applied loads at this moment?



(2.15) Consider the piping system shown below. The system is to carry an oil with density $\rho = 950$ kg/m^3 and kinematic viscosity $\nu = 10^{-4}$ m^2/s. The pump provides oil at a constant pressure P_{pump}. The pressure at the end of each line is taken to be 0 (gauge pressure). What is the maximum allowed pump pressure to ensure that the flow remains laminar (non-turbulent)? In what section will the system first transition from laminar to turbulent flow. Assume turbulence sets in at a Reynold's number of 2100, where $\text{Re} = \frac{\rho v D}{\mu} = \frac{v D}{\nu}$.



(2.16) For the electrical circuit shown below, determine all the junction voltages and all the branch currents. Assume $V_a = 10$ V and that the resistances are given in kΩ.



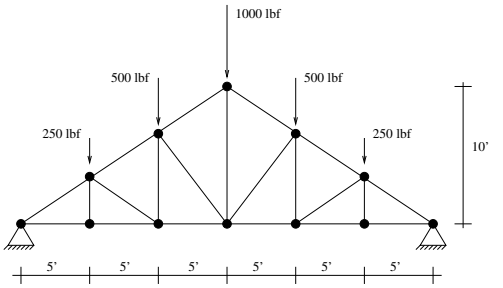
(2.17) Consider the Pratt roof truss shown below. Assume that the members are 2x4 wood sections and are connected by pin joints.

(a) What is the largest magnitude compressive force in the truss and where does it occur?

(b) What is the largest magnitude tensile force in the truss and where does it occur?

(c) If the truss is made statically determinate by changing the right support to a horizontal roller, how does your answer change?

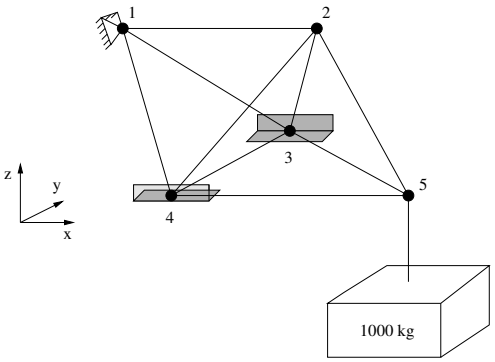
[Note: (1) A 2x4 is not 2 inches by 4 inches! (2) You will need to assume a reasonable value for Young's modulus of construction lumber. (3) For Part 2.17c the values of A and E should play no role – test your program to verify.]



(2.18) Consider the space truss as shown. Assume node 1 is fixed. Assume that nodes 3 and 4 are free to roll in the x-direction but are otherwise constrained and that the load at node 5 is in the negative z-direction. Let $A = 1000$ mm^2, $E = 210\times10^3$ N/mm^2, and assume Earth's gravitational acceleration acts in the negative z-direction. The coordinates of the nodes are

$$\begin{aligned}
\boldsymbol{x}_1 &= 500\boldsymbol{e}_y + 500\boldsymbol{e}_z && \text{(mm)}\\
\boldsymbol{x}_2 &= 700\boldsymbol{e}_x + 700\boldsymbol{e}_y + 500\boldsymbol{e}_z && \text{(mm)}\\
\boldsymbol{x}_3 &= 500\boldsymbol{e}_x + 800\boldsymbol{e}_y && \text{(mm)}\\
\boldsymbol{x}_4 &= 500\boldsymbol{e}_x + 100\boldsymbol{e}_y && \text{(mm)}\\
\boldsymbol{x}_5 &= 1200\boldsymbol{e}_x + 100\boldsymbol{e}_y && \text{(mm)}\,.
\end{aligned}$$

Find the x, y, z-displacement of node 5.



Power, Work, and Conservation

3

In Chapters 1 and 2, we have approached the formulation and solution to various problems by first investigating the conservation/balance equations that govern the systems. This was then coupled with knowledge of the variables that describe the state of the system and their connection to the conserved quantities. The result in all cases was an equation (or system of equations) that represented the conservation/balance equation in terms of the variables that described the state of the system. This process is very effective and the results are quite useful in a wide variety of contexts. There are however very powerful alternate formulations of the problems that can in certain circumstance provide better results in terms of easy of use or efficiency in determining a systems response. The entry to these alternate formulations is the concept of energy. In this chapter we will first review some basic concepts of energy in mechanical systems and then look at how one can exploit the concept of conservation of energy. This will be our introduction to energy methods. Then in Chapter 4, we will look at more sophisticated look at energy methods and in particular the powerful concept of stationary potential energy.

3.1 Work and Power

3.1.1 Power

The power associated with a force (load) is the dot product (inner product) of that force with the velocity that it causes. Consider, for example, the beam shown in Fig. 3.1. The power of the force, $\boldsymbol{F}$, is defined as the work is does per unit time; viz.,

$$\begin{aligned} \mathcal{P} &:= (\boldsymbol{F}\ [\text{Force}]) \cdot (\boldsymbol{v}\ [\text{Length/Time}]) \\ &= \boldsymbol{F} \cdot \boldsymbol{v}\ [\text{Force} \cdot \text{Length/Time}] \\ &= \sum_i F_i v_i\ . \end{aligned}$$

The last expression emphasizes the point that work per unit time is nonzero only when the directions of the force and velocity line up, i.e., are in the same direction. Consider the case shown in Fig. 3.2. Although the applied force exerts a load onto the wall, this does not contribute to the power. The wall ensures a zero velocity in the direction orthogonal

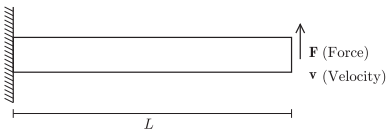


Fig. 3.1 Force applied to a cantilever and corresponding velocity.

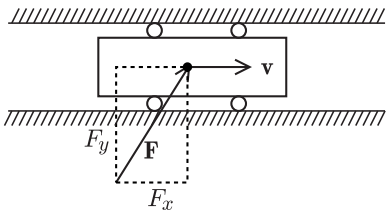


Fig. 3.2 Slider in a slot.

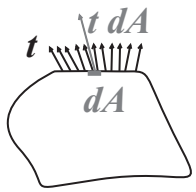


Fig. 3.3 Distributed load $\boldsymbol{t}$ (force per unit area) over the surface of a body.

to the wall and this implies that the force orthogonal to the wall will not perform any work. It should be emphasized, that the power of the load is the time rate of change of work performed by the loading system.

In the case of a distributed load, $\boldsymbol{t}$, over a surface as is shown in Fig. 3.3, we can consider small patches of area as being acted upon by a force equivalent to the distributed load times the area of the patch. Thus one can write the power as:

$$\begin{aligned} \mathcal{P} &:= \int_A (\boldsymbol{t}\ dA) \cdot \boldsymbol{v} \\ &= \int_A \boldsymbol{t} \cdot \boldsymbol{v}\ dA\ . \end{aligned}$$

The quantity $\boldsymbol{t}\ dA$ gives the amount of force applied on a patch with area dA. The inner product of the force with the velocity $\boldsymbol{v}$ gives the local power, which must be summed over the surface where the distributed load is applied to give the total power of the distributed load.

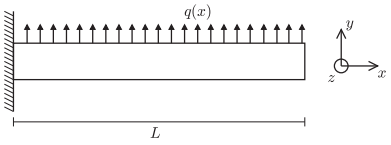


Fig. 3.4 Distributed load over a beam.

Example 3.1

Power for beam with a distributed load

Consider a distributed load $q(x)$ defined along a beam as shown in Fig. 3.4. Assume the velocity $\boldsymbol{v}(x) = v_y \boldsymbol{e}_y$ of the beam is known and determine a relation for the power of the load.

Solution: In this case, we need to observe that the distributed load is $q\boldsymbol{e}_y$, then

$$\begin{aligned} \mathcal{P} &:= \int_0^L (q(x)\ dx\ \boldsymbol{e}_y) \cdot (v_y(x)\ \boldsymbol{e}_y) \\ &= \int_0^L q(x)\ v_y(x)\ dx\ . \end{aligned}$$

The term $(q(x)dx\,\boldsymbol{e}_y)$ represents the total force on a segment of the beam of length dx and $v_y(x)\boldsymbol{e}_y$ is the given velocity.

If loads are given per unit volume, i.e. are distributed body forces $\boldsymbol{b}$, then the power is given as:

$$\begin{aligned} \mathcal{P} &:= \int_V (\boldsymbol{b}\ dV) \cdot \boldsymbol{v} \\ &= \int_V \boldsymbol{b} \cdot \boldsymbol{v}\ dV\ . \end{aligned}$$

The quantity $\boldsymbol{b}\ dV$ is the force applied on a small volume dV. The inner product of this force with the velocity $\boldsymbol{v}$ gives the local power of the load on dV. Summing over the whole volume, i.e. integration, gives the total power of the distributed body force.

3.1.2 Work

Consider a load applied to a point mass over a time interval $[t_1, t_2]$. The power of the load will be a function of time $\mathcal{P}(t)$ and the time integral of the power will represent the work done by the load on the mass; i.e., the work done will be given by:

$$\begin{aligned} \mathcal{W} &:= \int_{t_1}^{t_2} \mathcal{P}\ dt \\ &\xrightarrow{\text{Point force case}} \int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}\ dt \\ &= \int_{t_1}^{t_2} \boldsymbol{F}\cdot\frac{\partial \boldsymbol{x}}{\partial t}\ dt \\ &= \int_{\boldsymbol{x}_P}^{\boldsymbol{x}_Q} \boldsymbol{F}\cdot\ d\boldsymbol{x}\ . \end{aligned}$$

The last expressions denotes a line integral over a path, say Γ, with starting and ending points $\boldsymbol{x}_P = \boldsymbol{x}(t_1)$ and $\boldsymbol{x}_Q = \boldsymbol{x}(t_2)$. In general, the work will depend on the path taken. Path dependence can be understood by considering Fig. 3.5. Shown are two paths Γ and Γ^* from $\boldsymbol{x}_P = \boldsymbol{x}(t_1)$ to $\boldsymbol{x}_Q = \boldsymbol{x}(t_2)$. If the work is path dependent, then in general,

$$\begin{aligned} \mathcal{W} &\neq \mathcal{W}^*, \\ \int_\Gamma \boldsymbol{F}\cdot\ d\boldsymbol{x} &\neq \int_{\Gamma^*} \boldsymbol{F}\cdot\ d\boldsymbol{x} \end{aligned}$$

and the force system is called *non-conservative*.

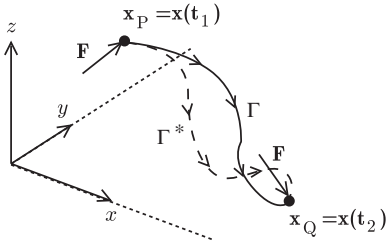


Fig. 3.5 Work along two different paths from $\boldsymbol{x}_P$ to $\boldsymbol{x}_Q$, Γ and Γ^*.

Example 3.2

Moving a mass over a frictional surface

Consider a mass sitting on a table with a frictional surface as shown in Fig. 3.6. The mass starts at point A and is moved by a force to point B. If the mass is moved on path 1, the work required will be less than if the mass is move on path 2, since path 2 is longer and the motion is resisted by frictional sliding. The (total) force system acting on the mass is non-conservative. In particular, the frictional forces that act on the mass are non-conservative; they are path dependent.

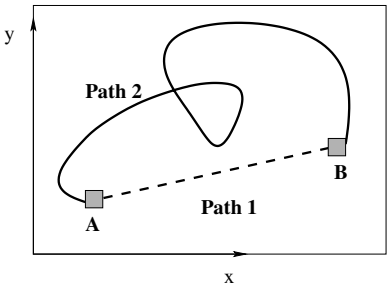


Fig. 3.6 Mass on a frictional surface with two paths of motion from point A to point B.

3.1.3 Conservative Forces

A very important class of force systems are those where the work done by the forces is path *independent*. Such force systems are called *conservative*.

In the one-dimensional case, forces $F(x)$ which can be expressed as the derivative of an another function will be conservative; i.e. forces where one can write

$$F(x) = \frac{df(x)}{dx} \ ,$$

for some function $f(x)$. The fact that this property of a force system gives a force system whose work is path independent can be seen through the following simple manipulation:

$$\begin{aligned} \mathcal{W} &= \int_{x_P}^{x_Q} F(x)\ dx = \int_{x_P}^{x_Q} \frac{df}{dx}\ dx \\ &= f(x_Q) - f(x_P)\ . \end{aligned}$$

Thus the work involved only depends on the end-point values and not on the path taken in going from x_P to x_Q.

In the multi-dimensional case, forces $\boldsymbol{F}(\boldsymbol{x})$ which can be expressed in terms of the gradient of a scalar function:

$$\boldsymbol{F}(\boldsymbol{x}) = \nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \frac{\partial f}{\partial x}\boldsymbol{e}_x + \frac{\partial f}{\partial y}\boldsymbol{e}_y + \frac{\partial f}{\partial z}\boldsymbol{e}_z$$

are conservative forces. When a force has this property,

$$\mathcal{W} = \int_\Gamma \boldsymbol{F} \cdot\ d\boldsymbol{x}$$

is path independent.

Remarks:

(1) In the fields of mechanics and physics, the preference of defining conservative forces is not through the expression

$$\boldsymbol{F} = \nabla f$$

but rather through the expression

$$\boldsymbol{F} = -\nabla V\ ;$$

i.e., we arbitrarily relabel f as $-V$. The function V is called the potential energy of the load/force. The path independence of the work is observed through the following manipulation:

$$\begin{aligned} \mathcal{W} &= \int_{\boldsymbol{x}_P}^{\boldsymbol{x}_Q} \boldsymbol{F} \cdot\ d\boldsymbol{x} \\ &= \int_{\boldsymbol{x}_P}^{\boldsymbol{x}_Q} -\nabla V \cdot\ d\boldsymbol{x} \\ &= \int_{\boldsymbol{x}_P}^{\boldsymbol{x}_Q} -\frac{\partial V}{\partial \boldsymbol{x}} \cdot\ d\boldsymbol{x} \\ \mathcal{W} &= -\left[V(\boldsymbol{x}_Q) - V(\boldsymbol{x}_P)\right]\ . \end{aligned}$$

This shows that the work only depends on the end-point values of the potential energy and not on the path that joins the two end-points.

Example 3.3

Gravity

A simple example of a conservative force is a gravitational force. Consider the setup in Fig. 3.7. The potential V for the gravitational loading system can be defined as

$$V \;:=\; z\,Mg\;,$$

where g is the gravitational acceleration. To check that this potential gives the correct force, one only has to take the (negative) gradient of the potential:

$$\begin{aligned} \boldsymbol{F} &= -\nabla V \\ &= -Mg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= -Mg\boldsymbol{e}_z\;, \end{aligned}$$

which is the desired result, viz. a constant force of magnitude Mg directed in the negative z-direction.

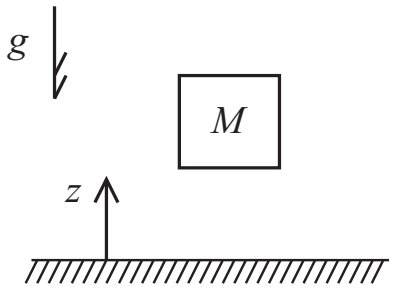


Fig. 3.7 Mass under the action of gravitational forces.

Remarks:

(1) In the expression for the potential V, the addition or subtraction of a constant value does not make a difference since the force is defined through its gradient; the taking of derivatives eliminates the constant. This implies that the absolute value of the potential energy is irrelevant. The two expressions for V:

$$\begin{aligned} V &= Mgz, \\ V &= Mgz + C, \end{aligned}$$

are physically identical in meaning. The only physically relevant item is the difference in the potential energy between two states, i.e., gradients.

(2) Given a force field $\boldsymbol{F}(x, y)$, one can use a classical result from multivariable calculus to determine if the force field emanates from a potential; viz., $\boldsymbol{F}$ is the gradient of a single-valued scalar potential function, if and only if $\nabla \times \boldsymbol{F} = 0$.

3.1.4 Conservative Mechanical System

A system in which the energy of the system is constant (conserved) is called a *Conservative Mechanical System* (i.e. a system without energy dissipation). It should be emphasized that this is an independent definition from that of conservative forces. However, systems comprised of non-conservative forces will not be conservative.

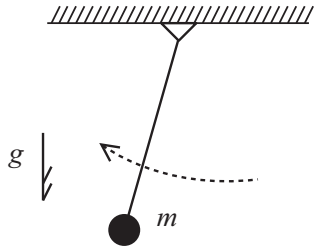


Fig. 3.8 Gravitational pendulum (conservative system).

Example 3.4

Ideal pendulum

Figure 3.8 shown an ideal pendulum. Ideal implies that it experiences no air resistance and swings on a frictionless pivot. With these assumptions, one observes that the

$$\text{Total Energy} = \text{Potential Energy} + \text{Kinetic Energy}$$

at all times and is a constant. This implies that the system, the pendulum together with the loading system, constitutes a conservative system.

In static mechanical systems:

$$\text{A Conservative Loading System} + \text{An Elastic Body} \Rightarrow \text{A Conservative Mechanical System.}$$

For such systems we can apply two very important and useful concepts:

(1) We can directly exploit the fact that the system's total energy is constant.

(2) We can employ the *Principle of Stationary Potential Energy*, which is a special way of writing force equilibrium. (This will be the topic of Chapter 4.)

A basic example of this type of conservative system is an elastic spring fixed at one end and statically loaded at the other by a mass under gravitational forces.

3.2 Conservation of Energy

The direct exploitation of the concept of conservation of energy can be made in systems subject to single loads. In such systems it is often easy to compute the work performed by the load and likewise the work stored in the system. This is especially so when the system is composed of elastic elements. In the next sub-sections, we review the basic relations for computing work stored and work input for linear elastic systems.

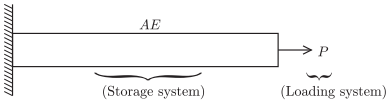


Fig. 3.9 Tension-compression bar loaded at one end.

Tension-compression bars

Consider the mechanical system of Fig. 3.9, where a linear elastic tension-compression bar is loaded at $x = L$ by a *conservative* force P. Since there is no energy dissipation in the system, one has by conservation of energy:

$$\mathcal{W}_{\text{stored}} = \mathcal{W}_{\text{in}},$$

where $\mathcal{W}_{\text{stored}}$ is the energy stored in the elastic tension-compression bar and $\mathcal{W}_{\text{in}}$ is the work performed by the load. Since the mechanical system

is linear, the deflection Δ at the tip is a linear function of the end-load P. Thus the work done by the load is:

$$\boxed{\mathcal{W}_{\text{in}} = \frac{1}{2} P\Delta \ .}$$

This relationship is shown in Fig. 3.10 as the area under the force-deflection curve. The elastic tension-compression bar serves as an energy storage system, where the energy is stored as elastic energy. The energy stored can be computed through the integration of the strain energy per unit volume over the entire body:

$$\begin{aligned} \mathcal{W}_{\text{stored}} &= \int_V \frac{1}{2} E\varepsilon^2 \, dV \\ &= \int_L \int_A \frac{1}{2} E \left(\frac{\Delta}{L}\right)^2 \, dV \\ &= \boxed{\frac{1}{2}\frac{AE}{L}\Delta^2 \qquad \left(= \frac{1}{2}\frac{P^2 L}{AE}\right) \ .} \end{aligned}$$

The integration here is simple since the strain is constant in the bar. This is a consequence of the absence of distributed loads.

Equating the stored energy with the work done by the load yields:

$$\begin{aligned} & \mathcal{W}_{\text{in}} = \mathcal{W}_{\text{stored}} \\ \Leftrightarrow & \frac{1}{2} P\Delta = \frac{1}{2}\frac{EA}{L}\Delta^2 \\ \Leftrightarrow & \quad P = \frac{AE}{L}\Delta, \end{aligned}$$

or

$$\Delta \ = \ \frac{PL}{AE} \ .$$

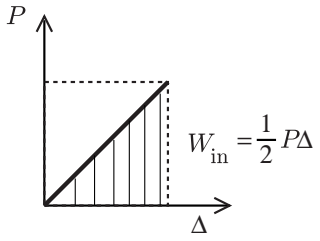


Fig. 3.10 Work input for a linear system is the area under the force-deflection curve.

Remarks:

(1) There is only 1 equation (the conservation of energy). Thus one can only solve for 1 unknown (the end displacement Δ) using such a technique. Notwithstanding, this can be a very effective analysis methodology in many situations.

(2) The system was kinematically determinate, which allowed us to compute the bar strain and directly $\mathcal{W}_{\text{stored}}$. The same also holds for statically determinate systems.

(3) The concept of exploiting conservation of energy also extends to much more complex situations and this is where it becomes most powerful.

Example 3.5

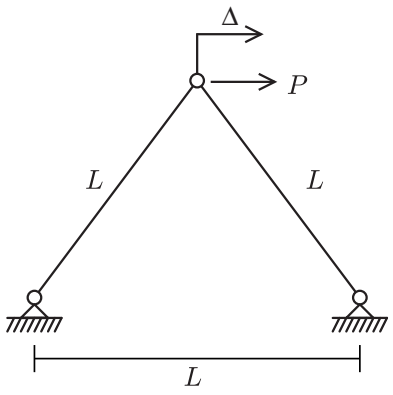


Fig. 3.11 2 bar truss system.

Two bar truss

Consider the 2-bar truss in Fig. 3.11. Find Δ the horizontal motion at the load point.

Solution: The expression for the work done by the load and the stored energy in the truss bars are:

$$\begin{aligned} \mathcal{W}_{\rm in} &= \frac{1}{2}P\Delta, \\ \mathcal{W}_{\rm s} &= \mathcal{W}_{s1} + \mathcal{W}_{s2} \\ &= \frac{1}{2}\frac{P_1^2 L}{AE} + \frac{1}{2}\frac{P_2^2 L}{AE} \\ &= \frac{P^2 L}{AE}\ . \end{aligned}$$

Here, we have exploited the statics result that $P_1 = -P_2 = P$. Equating the expressions, $\mathcal{W}_{\rm in} = \mathcal{W}_{\rm s}$, yields:

$$\Delta = \frac{2PL}{AE}\ .$$

Remarks:

(1) In coming to this result we have further exploited the fact that energy is an additive quantity. The work stored is the sum of the work stored in the individual components of the system.

(2) The system was statically determinate and this allowed us to compute the bar forces directly in order to evaluate $\mathcal{W}_{\rm s}$.

(3) If the bars were inhomogeneous one would need to use $(AE)_{\rm eff}$.

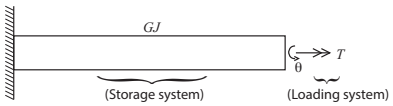


Fig. 3.12 Torsion bar with end-torque.

Torsion

Consider a linear elastic torsion bar as shown in Fig. 3.12. The expression for the work done by the load is given by

$$\boxed{\mathcal{W}_{\rm in} = \frac{1}{2}T\theta.}$$

This work expression follows by noting that for linear systems the response must be linear, as shown in Fig. 3.13, and the work is simply the area under the torque-rotation curve. The expression for the work stored in torsion is given by integrating the strain energy density in the

bar:

$$\begin{aligned}
\mathcal{W}_{\mathrm{s}} &= \int_V \frac{1}{2} G\gamma^2 \, dV \\
&= \int_L \int_A \frac{1}{2} G \left(\varphi' \, r\right)^2 \, dV \\
&= \int_L \frac{1}{2} GJ \left(\varphi'\right)^2 \, dz \\
&= \int_L \frac{1}{2} GJ \left(\frac{\theta}{L}\right)^2 \, dz \\
&= \boxed{\frac{1}{2}\frac{GJ}{L}\theta^2 \qquad \left(= \frac{1}{2}\frac{T^2 L}{GJ}\right)} \, .
\end{aligned}$$

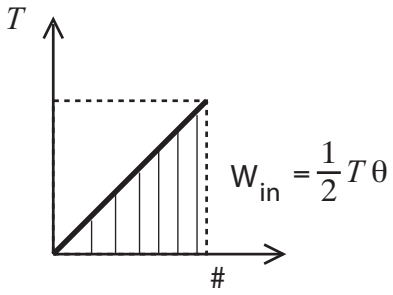


Fig. 3.13 Work input for a linear system in torsion is the area under the torque-rotation curve.

Example 3.6

Torsion rod

As an example exploitation of these results, we can use them to determine the torque-rotation relation of the bar shown in Fig. 3.12. Equating the stored energy with the work done by the load yields:

$$\begin{aligned}
&\mathcal{W}_{\mathrm{in}} = \mathcal{W}_{\mathrm{stored}} \\
\Leftrightarrow &\frac{1}{2} T\theta = \frac{1}{2}\frac{GJ}{L}\theta^2 \\
\Leftrightarrow \quad &T = \frac{GJ}{L}\theta,
\end{aligned}$$

or

$$\theta \;=\; \frac{TL}{GJ} \, .$$

Remarks:

(1) In linear elastic problems where the load is a torque, the work done by the torque will always be $T\theta/2$, where θ is the rotation at the load point in the direction of the torque.

(2) The expressions given for the stored work are appropriate for linear elastic homogeneous systems. For inhomogeneous systems one would need to use $(GJ)_{\mathrm{eff}}$.

Bending

For systems responding in bending the elastic energy is stored by both normal strains (bending strains) and shear strains (direct shear). Let us

consider the two contributions separately. The expression for the energy stored in the beam by bending strains $\varepsilon = \varepsilon_{xx}$ is:

$$\begin{aligned}
\mathcal{W}_{\text{s}} &= \int_V \frac{1}{2} E\varepsilon^2 \, dV \\
&= \int_L \int_A \frac{1}{2} E \left(-y\kappa\right)^2 \, dV \\
&= \int_L \frac{1}{2} E\kappa^2 \left[\int_A y^2 \, dA\right] \, dx \\
&= \boxed{\int_L \frac{1}{2} EI\kappa^2 \, dx \; .}
\end{aligned}$$

To come to a more concrete expression we need to consider particular examples.

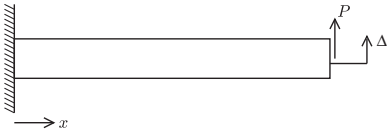


Fig. 3.14 Cantilever beam with an end-load.

Example 3.7

Cantilever beam

Consider the system shown in Fig. 3.14 and find the relation between P and Δ at the end of the beam.

Solution: For this case, the curvature κ varies linearly. Since the shear force in the beam (from statics) is a constant $V(x) = P$, the moment is linear and given by the expression $M(x) = P(L - x)$. From the beam constitutive relation $M = EI\kappa$, one obtains the expression for the curvature κ,

$$\kappa = \frac{M}{EI} = \frac{P}{EI}(L - x) \; .$$

Substituting this relation into the expression for the stored energy yields,

$$\begin{aligned}
\mathcal{W}_{\text{s}} &= \int_L \frac{1}{2} EI \left(\frac{P}{EI}(L - x)\right)^2 \, dx \\
&= \frac{P^2}{2EI} \int_L (L - x)^2 \, dx \\
&= \frac{P^2 L^3}{6EI} \; .
\end{aligned}$$

Equating the stored energy with the work done by the load yields:

$$\begin{aligned}
\mathcal{W}_{\text{in}} &= \mathcal{W}_{\text{stored}} \\
\Leftrightarrow \frac{1}{2} P\Delta &= \frac{P^2 L^3}{6EI} \\
\Leftrightarrow \quad \Delta &= \frac{PL^3}{3EI} \; ,
\end{aligned}$$

or

$$P = \frac{3EI}{L^3}\Delta \; .$$

Remarks:

(1) This is the result one would obtain by integrating the governing differential equation $EIv'''' = q$. Thus one can interpret the classic result as only accounting for bending strains and effectively ignoring the effects of shear.

Example 3.8

Effect of shear on bending deflections

To account for the effects of shear in Example 3.7, one needs to compute the stored energy associated with shear. This contribution can be computed as:

$$\begin{aligned}
\mathcal{W}_{\text{s,shear}} &= \int_V \frac{1}{2} G\gamma^2 \, dV \\
&= \int_V \frac{1}{2} \frac{\tau^2}{G} \, dV \\
&= \int_V \frac{1}{2} \frac{1}{G} \left\{ \frac{P}{2I} \left[\left(\frac{h}{2} \right)^2 - y^2 \right] \right\}^2 \, dV \\
&= \frac{1}{2G} \left(\frac{P}{2I} \right)^2 \int_L \left\{ \int_A \left[\left(\frac{h}{2} \right)^2 - y^2 \right]^2 \, dA \right\} \, dx \\
&= \frac{L}{2G} \left(\frac{P}{2I} \right)^2 \int_A \left[\left(\frac{h}{2} \right)^2 - y^2 \right]^2 \, dA \\
&= \frac{L}{2G} \left(\frac{P}{2I} \right)^2 \frac{bh^5}{30} \\
&= \frac{3}{5} \frac{P^2 L}{GA} \, .
\end{aligned}$$

Here we have used the parabolic shear stress distribution under the assumption of a rectangular cross section with width b and depth h. The derivation of this equation can be found in any good strength of materials/engineering mechanics textbooks. Including this term into the stored energy yields

$$\begin{aligned}
\mathcal{W}_{\text{s}} &= \mathcal{W}_{\text{s,bending}} + \mathcal{W}_{\text{s,shear}} \\
&= \frac{P^2 L^3}{6EI} + \frac{3}{5} \frac{P^2 L}{GA} \, .
\end{aligned}$$

Equating the stored energy with the work done by the load yields

$$\begin{aligned} \mathcal{W}_{\text{in}} &= \mathcal{W}_{\text{s}} \\ \Leftrightarrow \frac{1}{2}P\Delta &= \frac{1}{2}\frac{P^2L^3}{6EI} + \frac{3}{5}\frac{P^2L}{GA} \\ \Leftrightarrow \quad \Delta &= \frac{PL^3}{3EI} + \frac{6}{5}\frac{PL}{GA} \\ \Leftrightarrow \quad \Delta &= \frac{PL^3}{3EI}\left[1 + \frac{3E}{10G}\left(\frac{h}{L}\right)^2\right] . \end{aligned}$$

In the expression for Δ, the term following the 1 is the relative contribution of the shears to the overall deflection. One sees that for slender beams where the aspect ratio of the beam exceeds $1:10$, this shear contribution is of order 0.01 for most metals. This means that neglecting this term only introduces an error of approximately 1%, which is negligible in most structural applications.

Remarks:

(1) This system is statically determinate. We know *a priori*, or have solved for, the expressions for the bending stress $\sigma_{xx} = \sigma$ and shear stress $\tau = \sigma_{xy}$.

(2) Note that in structural elements which carry loads in multiple ways, the stored energy is simply the sum of each contribution. This holds true for linear elastic systems.

3.3 Summary: Work Relations

The work input expression for a linear system is either $P\Delta/2$ or $T\theta/2$. The stored work expressions depend on the manner in which the load is carried. In a general linear elastic setting the strain energy density is given by

$$w = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}) .$$

For the various types of loadings we study, this expression, when integrated over the volume of a load bearing member, reduces to:
Tension-Compression Bars

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\, dV &= \int_L \int_A \frac{1}{2}E\varepsilon^2\, dA\, dx \\ &= \underline{\underline{\int_0^L \frac{1}{2}AE\left(\frac{du}{dx}\right)^2 dx}} \end{aligned}$$

Torsion Rods

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{z\theta}\gamma_{z\theta}\,dV &= \int_L\int_A \frac{1}{2}G\gamma^2\,dA\,dz \\ &= \int_0^L\int_A \frac{1}{2}Gr^2\left(\frac{d\phi}{dz}\right)^2 dA\,dz \\ &= \underline{\underline{\int_0^L \frac{1}{2}GJ\left(\frac{d\phi}{dz}\right)^2 dz}} \end{aligned}$$

Beams in Bending

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dV &= \int_L\int_A \frac{1}{2}E\varepsilon^2\,dA\,dx \\ &= \int_0^L\int_A \frac{1}{2}Ey^2\left(\frac{d^2v}{dx^2}\right)^2 dA\,dx \\ &= \underline{\underline{\int_0^L \frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2 dx}} \end{aligned}$$

These expressions are all given in terms of the relevant kinematic measures of the motion (du/dx, $d\phi/dz$, d^2v/dx^2). Equivalent complementary expressions can also be given in terms of force and moment resultant quantities.

Tension-Compression Bars

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dV &= \int_L\int_A \frac{1}{2E}\sigma^2\,dA\,dx \\ &= \underline{\underline{\int_0^L \frac{1}{2}\frac{R^2}{AE}\,dx}} \end{aligned}$$

Torsion Rods

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{z\theta}\gamma_{z\theta}\,dV &= \int_L\int_A \frac{1}{2G}\tau^2\,dA\,dz \\ &= \underline{\underline{\int_0^L \frac{1}{2}\frac{T^2}{GJ}dz}} \end{aligned}$$

Beams in Bending

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dV &= \int_L\int_A \frac{1}{2E}\sigma^2\,dA\,dx \\ &= \underline{\underline{\int_0^L \frac{1}{2}\frac{M^2}{EI}\,dx}} \end{aligned}$$

Direct Shear in Beams

$$\begin{aligned} W = \int_V \frac{1}{2}\sigma_{xy}\gamma_{xy}\,dV &= \int_L\int_A \frac{1}{2G}\tau^2\,dA\,dx \\ &= \underline{\underline{\alpha\int_0^L \frac{1}{2}\frac{V^2}{GA}\,dx}} \end{aligned}$$

In this last expression, α is a factor that accounts for the shape of the beam's cross-section and the distribution of the shear stresses over it.

Exercises

(3.1) Consider the force field $\boldsymbol{F}(x,y) = fxy\boldsymbol{e}_x + f\boldsymbol{e}_y$, where f is a given constant. Demonstrate that this force field is not conservative.

(3.2) Consider a force field $\boldsymbol{F}(x,y) = F_1xy\boldsymbol{e}_x + F_1y\boldsymbol{e}_y$, where F_1 is a given constant. Is this force field conservative?

(3.3) Consider the force field $\boldsymbol{F}(x,y) = 10xy\boldsymbol{e}_x + 5y\boldsymbol{e}_y$ (N). Does this force field emanate from a potential? i.e., is it conservative?

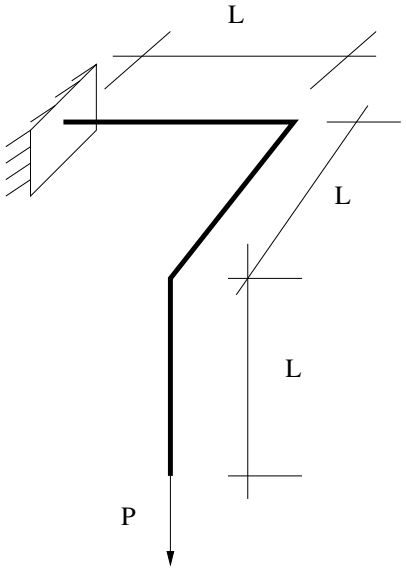
(3.4) Consider a beam of length L with a distributed load $q(x) = q_o + q_1\frac{x}{L}$ acting in the positive y-direction. Assume that the velocity along the loaded face of the beam at a given moment is $\boldsymbol{v}(x) = v_1\boldsymbol{e}_x + v_2\frac{x}{L}\boldsymbol{e}_y$. Find the power of the load at this moment; assume q_o, q_1, v_1, v_2 are given constants.

(3.5) A solid circular bar is bent 90^o at two locations and is built-in at one end. Assume A, I, J, E, and G are constants.

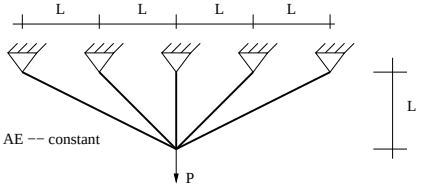
(a) Using conservation of energy, determine a formula for the vertical deflection at the point of load application. [$\alpha = 10/9$ for round bars.]

(b) Let $L = 200$ mm and the diameter of the bar be $d = 30$ mm. What is the percent contribution to the total deflection from axial loading, bending, torsion, and direct shear? Assume $E/G = 2$.

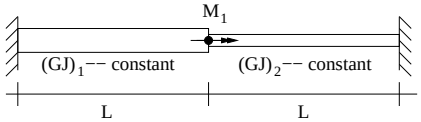
(c) Repeat with $L = 500$ mm and $d = 10$ mm.



(3.6) Use the concept of conservation of energy to determine the relation between the applied force and the vertical deflection at the load point.



(3.7) For the stepped torsion bar below determine the rotation at the point of application of the load. Use the concept of conservation of energy.

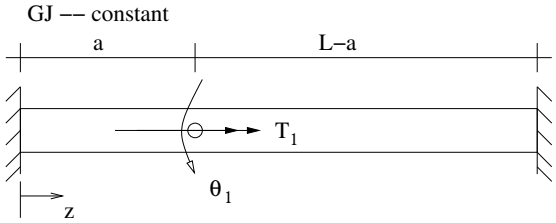


(3.8) Using conservation of energy, find the tip deflection of a linear elastic cantilever beam loaded with a single point force at its end. Account for bending energy as well as shear energy. Assume EI, GA, and α as given constants.

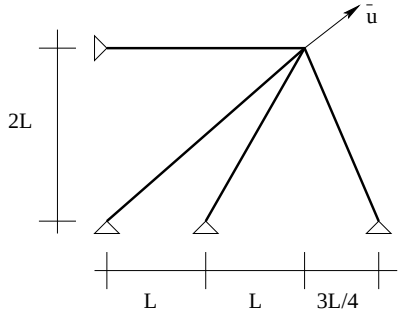
(3.9) Consider a cantilever beam of length L that is loaded with a constant distributed force $q(x) = q_o$. Accounting only for bending energy find an expression for the work done by the load in deflecting the beam. Assume EI is a given constant and express your answer in terms of EI, q_o, and L.

(3.10) Find an expression for the end-rotation of a circular bar loaded with a torque, T_o, at its end. Assume the bar is built-in at the other end, has a length L, and a constant torsional rigidity GJ. Use conservation of energy to solve this problem.

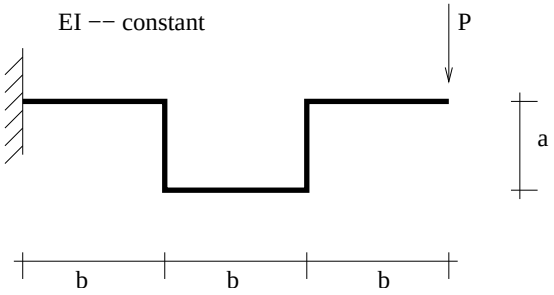
(3.11) For the system shown below, find the relation between the point torque T_1 and the corresponding rotation $\theta_1 = \phi(a)$ using conservation of energy. Use the kinematic expressions for the stored energy.



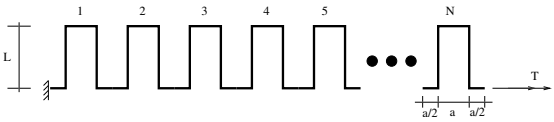
(3.12) For the truss shown below, use conservation of energy to find the load component in the direction of the given displacement at the node where $\bar{\boldsymbol{u}}$ is imposed. Assume $L = 24$ in, $AE = 15 \times 10^6$ lb, and $\bar{\boldsymbol{u}} = 10^{-3}(\boldsymbol{e}_x + \boldsymbol{e}_y)$ in. [Hints: (1) Use the kinematic form of the stored energy. (2) You can exploit the compatibility matrix to help you solve this problem.]



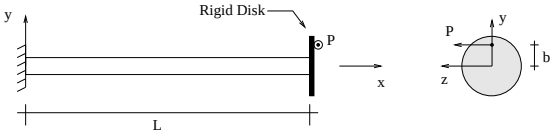
(3.13) Find the vertical deflection at the tip of the structure shown below using a conservation of energy method. Assume all sections are slender. [Hint: use the force expressions for the stored energy.]



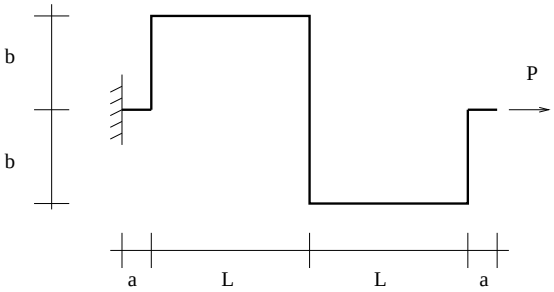
(3.14) Shown is a serpentine spring. The spring is composed of N "hairpin" segments of a round wire with dimensions as shown. Determine an expression for the torsional stiffness of the spring. Express your answer in terms of E, I, J, G, a, L, N. Use conservation of energy to solve; use the force expressions for the stored energy.



(3.15) For the system shown below, use conservation of energy (force version) to determine the deflection in the direction of the load at the point where the load is applied. You may assume that GJ, EI, AE, and $\frac{1}{\alpha}AG$ are all given and constant.



(3.16) A *slender* metal band with constant EI, GJ, and $\frac{1}{\alpha}AG$, as shown, is subjected to a force P. Find the horizontal deflection at the point where the load is applied using conservation of energy.



Potential Energy Methods

4

The energy methods introduced in Chapter 3 are quite powerful and in many instances of great practical utility. Notwithstanding, they have certain limitations. Firstly, the concept of conservation of energy only provides a single scalar equation and thus it only allows for the determination of a single quantity of interest. If a problem of interest, involves more than a single scalar unknown the method does not furnish a sufficient number of equations to allow one to solve for a system's response. It is also noted that the method is not particularly convenient when dealing with non-linear conservative systems. In this chapter, we will take up the topic of *potential energy methods*. This is a class of methods, that are energy based, and avoid the limitations associated with the concept of conservation of energy. One very important advantage of potential energy methods is that they also permit one to devise approximate solution methods in situations where exact solutions are not possible or not needed.

4.1 The Principle of Stationary (Minimum) Potential Energy

Consider the mechanical system consisting of a spring with stiffness k and a mass of weight $W = Mg$ as shown in Fig. 4.1. One can relate the position z of the mass with its displacement Δ by introducing a reference z_0 defined through the relation $z = z_0 - \Delta$. Let us define two potential energies which characterize the forces acting on the mass:

$$\begin{aligned} \Pi_{\text{gravity}} &:= Wz = W(z_0 - \Delta), \\ \Pi_{\text{spring}} &:= \frac{1}{2}k\Delta^2 = \frac{1}{2}k(z_0 - z)^2 . \end{aligned}$$

Π_{gravity} is the potential energy of the gravitational force and Π_{spring} is the potential energy of the spring. The form of theses energies are constructed so that they generate the corresponding forces on the mass after differentiation with respect to position, z. The total potential energy is defined as the sum of these two energies:

$$\Pi_{\text{total}} := \Pi_{\text{gravity}} + \Pi_{\text{spring}} .$$

The *Principle of Stationary Potential Energy* states that:

$$\text{Equilibrium} \quad \Leftrightarrow \quad \Pi_{\text{total}} \text{ is stationary, viz. } -\frac{\partial \Pi_{\text{total}}}{\partial z} = 0.$$

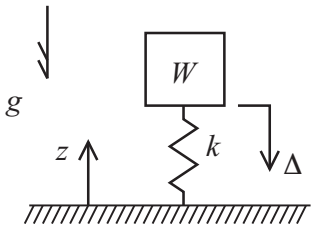


Fig. 4.1 Mass-spring system in gravitational field.

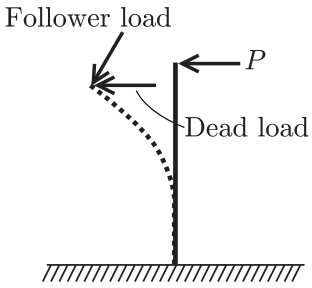


Fig. 4.2 Cantilever beam subjected to a dead load and a follower load.

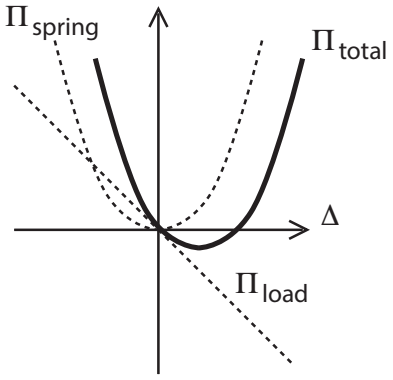


Fig. 4.3 Potential energy for a one degree of freedom variable system.

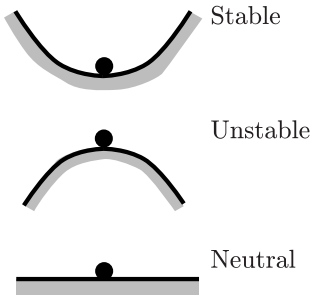


Fig. 4.4 Pictorial aide for understanding the stability condition on the second derivative of the potential energy.

For our example mechanical system we have

$$\begin{aligned} 0 = -\frac{\partial \Pi_{\text{total}}}{\partial z} &= -\left[W - k(z_0 - z)\right] \\ &= -W + k(z_0 - z) \qquad \Rightarrow W = k(z_0 - z) = k\Delta\,. \end{aligned}$$

Remarks:

(1) This principle has nothing to do with conservation of energy. It is a statement of equilibrium.

(2) For dead loads, a type of loading that does not change direction or magnitude with the motion of the structure, one always has the relation:

$$\Pi_{\text{deadload}} = -P \cdot \Delta\,,$$

or in words, Π_{deadload} equals minus the load times the deflection in the direction of the load. See Fig. 4.2 for an example of a dead load and the difference between it and a follower load, which changes direction and depends on the motion of the structure.

(3) For elastic elements in a system, the potential energy expressions happen to coincide with the elastic stored energy expressions:

$$\begin{aligned} \int_L \frac{1}{2} AE(u')^2\, dx &: \text{Tension} - \text{compression bar}, \\ \int_L \frac{1}{2} EI(v'')^2\, dx &: \text{Beam}, \\ \int_L \frac{1}{2} GJ(\phi')^2\, dx &: \text{Torsion bar}\,. \end{aligned}$$

(4) The potential energy is a minimum at equilibrium for stable mechanical systems. As a simple example, consider a mechanical system composed of a spring with spring constant k and a dead load P. The system's total potential energy is given as

$$\Pi_{\text{total}}(\Delta) = \frac{1}{2} k\Delta^2 - P\Delta\,,$$

where Δ is the system motion. Stationarity of Π_{load} implies that $-d\Pi_{\text{load}}/d\Delta = 0 \Rightarrow k\Delta - P = 0$. The second derivative of the total potential energy (evaluated at an equilibrium state) allows one to assess the stability of any equilibrium state. In the present setting, assuming a positive spring constant, one sees that

$$\frac{\partial^2 \Pi_{\text{total}}}{\partial \Delta^2} = k > 0\,,$$

which implies that Π_{total} is a minimum at the equilibrium position $\Delta = \frac{P}{k}$; see Fig. 4.3. The concept of stability can be understood through the analogy of a ball set on a surface; see Fig. 4.4. The shape of the surface corresponds to the shape of the potential

energy function. For a convex Π_{total}, the ball stays in the valley and is in a stable equilibirum state at the bottom of the valley; if one perturbs the position of the ball, it rolls back towards the equilibrium position. For a concave Π_{total}, the ball can roll off the apex and is in an unstable equilibrium state at the top of the hill; if one perturbs the position of the ball, it will roll away from the equilibrium position. For a flat Π_{total}, the ball does not have a preferred position and is in a neutral equilibrium for all possible placements; if one perturbs the position of the ball, there are no forces moving towards or away from its previous equilibrium.

4.1.1 Application to a two degree of freedom problem

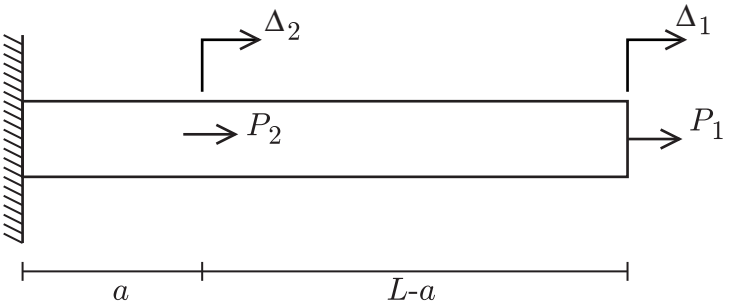


Fig. 4.5 Tension-compression bar with two point loads.

Consider the mechanical system of a tension-compression bar with uniform Young's modulus E and cross-section A shown in Fig. 4.5. A (dead) load of P_1 is applied at $x = L$ and a (dead) load of P_2 at $x = a$. The displacements at these two points are defined as Δ_1 and Δ_2 respectively. The principle of stationary potential energy can be used to determine the relation between applied loads P_1, P_2 and Δ_1, Δ_2. The potential energy due to the load is the sum of the potentials associated with each load:

$$\Pi_{\text{load}} \quad = \quad -P_1\Delta_1 - P_2\Delta_2 \,.$$

To obtain the potential energy of the elastic bar, one must employ some knowledge of the problem. It is known that in the absence of distributed loads and with a constant AE the displacement field is piecewise linear. Applying this fact to the current problem, one has the following expressions for the displacement field $u(x)$ in terms of Δ_1 and Δ_2,

$$u = \begin{cases} \Delta_2 \left(\frac{x}{a}\right) & \text{if } 0 \le x \le a \\ (\Delta_1 - \Delta_2) \left(\frac{x-a}{L-a}\right) + \Delta_2 & \text{if } a \le x \le L \,. \end{cases}$$

Thus, the expression for the potential energy of the elastic bar is

$$\begin{aligned}\Pi_{\text{bar}} &= \int_0^L \frac{1}{2} AE(u')^2 \, dx \\ &= \int_0^a \frac{1}{2} AE \left(\frac{\Delta_2}{a}\right)^2 dx + \int_a^L \frac{1}{2} AE \left(\frac{\Delta_1 - \Delta_2}{L-a}\right)^2 dx \\ &= \frac{1}{2}\frac{AE}{a}(\Delta_2)^2 + \frac{1}{2}\frac{AE}{L-a}(\Delta_1 - \Delta_2)^2 \, .\end{aligned}$$

The total potential energy is given as

$$\begin{aligned}\Pi_{\text{total}} &= \Pi_{\text{bar}} + \Pi_{\text{load}} \\ \Pi_{\text{total}}(\Delta_1, \Delta_2) &= \frac{1}{2}\frac{AE}{a}(\Delta_2)^2 + \frac{1}{2}\frac{AE}{L-a}(\Delta_1 - \Delta_2)^2 - P_1\Delta_1 - P_2\Delta_2 \, .\end{aligned}$$

One observes that Π_{total} is now a function of two variables Δ_1 and Δ_2. For this case, the *Principle of Stationary Potential Energy* states that:

Equilibrium $\quad \Leftrightarrow \quad$ Π_{total} is stationary, $-\dfrac{\partial \Pi_{\text{total}}}{\partial \Delta_i} = 0$ for $i = 1, 2.$

For the given mechanical system one obtains,

$$\begin{aligned}0 = \frac{\partial \Pi_{\text{total}}}{\partial \Delta_1} &= AE\frac{\Delta_1 - \Delta_2}{L-a} - P_1, \\ 0 = \frac{\partial \Pi_{\text{total}}}{\partial \Delta_2} &= AE\frac{\Delta_2}{a} - AE\frac{\Delta_1 - \Delta_2}{L-a} - P_2 \, .\end{aligned}$$

Each of these can be interpreted as the equation representing the equilibrium of forces at each load point. This shows that the statement of stationarity of the potential energy is just a restatement of equilibrium. One has two equations in two unknowns, which can be written in the following matrix form:

$$\begin{bmatrix} \frac{AE}{L-a} & -\frac{AE}{L-a} \\ -\frac{AE}{L-a} & \frac{AE}{L-a} + \frac{AE}{a} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} .$$

It is important to note that the coefficient matrix is symmetric positive definite – a feature of stable conservative mechanical systems.

4.1.2 Application to a single degree of freedom torsion problem

Consider a mechanical system consisting of a torsion bar with uniform shear modulus G and polar moment of inertia J, shown in Fig. 4.6, along with an applied (dead) torque of T_a at $z = a$. The rotation at the point of loading in the positive direction of the load is denoted as θ_a. Let us use the principle of stationary potential energy to determine the relation between the applied torque T_a and the induced motion θ_a.

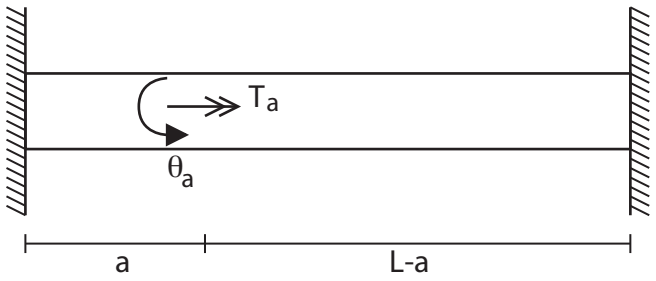


Fig. 4.6 Torsion bar with a point torque and boundary conditions making the problem statically indeterminate.

The potential energy of the torsional load is

$$\Pi_{\text{load}} \quad = \quad -T_a\theta_a \,.$$

To obtain the potential energy of the elastic bar, one must employ some knowledge of the problem. It is known that in the absence of distributed torques, the internal torque is constant. This implies that the twist rate ϕ' is constant and thus ϕ is linear:

$$\begin{aligned} \frac{dT}{dz} + t = 0 \;\text{ and }\; t = 0 \quad &\Rightarrow \quad T(= GJ\phi') \text{ constant} \\ &\Rightarrow \quad \phi' \text{ constant} \\ &\Rightarrow \quad \phi \text{ linear} \,. \end{aligned}$$

Applying this fact to the current problem, one has the following expression for the rotation field $\phi(z)$ in terms of θ_a,

$$\phi = \begin{cases} \frac{\theta_a - 0}{a} x & \text{if } 0 \le x \le a \\ \frac{0-\theta_a}{L-a}(x-a) + \theta_a & \text{if } a \le x \le L \,. \end{cases}$$

The expression for the potential energy of the elastic bar is:

$$\begin{aligned} \Pi_{\text{bar}} \quad &= \quad \int_V \frac{1}{2} G\gamma^2 \; dV \\ &= \quad \int_0^L \frac{1}{2} GJ(\phi')^2 \; dz \\ &= \quad \int_0^a \frac{1}{2} GJ \left(\frac{\theta_a}{a}\right)^2 \; dz + \int_a^L \frac{1}{2} GJ \left(\frac{-\theta_a}{L-a}\right)^2 \; dz \\ &= \quad \frac{1}{2}\frac{GJ}{a} (\theta_a)^2 + \frac{1}{2}\frac{GJ}{L-a} (-\theta_a)^2 \,. \end{aligned}$$

The total potential energy is given as

$$\begin{aligned} \Pi_{\text{total}} \quad &= \quad \Pi_{\text{bar}} + \Pi_{\text{load}} \\ \Pi_{\text{total}}(\theta_a) \quad &= \quad \frac{1}{2}\frac{GJ}{a} (\theta_a)^2 + \frac{1}{2}\frac{GJ}{L-a} (-\theta_a)^2 - T_a\theta_a \,. \end{aligned}$$

One observes that Π_{total} is now a function of θ_a. For this case, the *Principle of Stationary Potential Energy* states that,

$$\boxed{\text{Equilibrium} \quad \Leftrightarrow \quad \Pi_{\text{total}} \text{ is stationary, } -\frac{\partial \Pi_{\text{total}}}{\partial \theta_a} = 0.}$$

For the given mechanical system one obtains:

$$0 = \frac{\partial \Pi_{\text{total}}}{\partial \theta_a} \quad = \quad \frac{GJ}{a}\theta_a + \frac{GJ}{L-a}\theta_a - T_a \,.$$

This equation can be interpreted as the equation representing the equilibrium of torques at the load point. This shows, again, that the statement of stationarity of the potential energy is just a restatement of the governing equilibrium equations. Rearranging gives

$$\boxed{T_a = \left(\frac{GJ}{a} + \frac{GJ}{L-a}\right)\theta_a \,.}$$

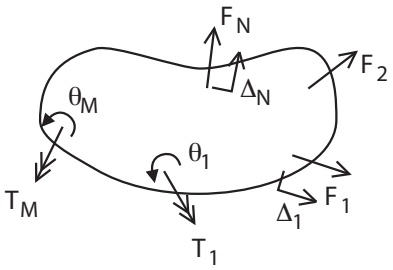


Fig. 4.7 Construction for Castigliano's 1st theorem.

4.1.3 Castigliano's 1st Theorem

The principle of stationary potential energy has a close relationship with *Castigliano's 1st Theorem*. The theorem states the following: Consider an elastic system with N point forces F_i $(i = 1, \ldots, N)$ and M point torques T_j $(j = 1, \ldots, M)$ with load-point displacements Δ_i $(i = 1, \ldots, N)$ and rotations θ_j $(j = 1, \ldots, M)$ defined in the positive directions of the respective loadings; see Fig. 4.7. If Π_{elastic} is the elastic potential energy, then

$$\boxed{\begin{aligned} F_i &= \frac{\partial \Pi_{\text{elastic}}}{\partial \Delta_i} \quad i = 1, \ldots, N \\ T_j &= \frac{\partial \Pi_{\text{elastic}}}{\partial \theta_j} \quad i = 1, \ldots, M \,. \end{aligned}}$$

One can prove this theorem using the principle of stationary potential energy. The total potential energy of the system is

$$\Pi_{\text{total}} \quad = \quad \Pi_{\text{elastic}} - \sum_{i=1}^{N} F_i \Delta_i - \sum_{j=1}^{M} T_j \theta_j \,.$$

At equilibrium, the total potential energy,

$$\Pi_{\text{total}}(\Delta_1, \ldots, \Delta_N, \theta_1, \ldots, \theta_M) \,,$$

must be stationary. This implies the following two conditions,

$$\begin{aligned} 0 &= \frac{\partial \Pi_{\text{total}}}{\partial \Delta_i} = \frac{\partial \Pi_{\text{elastic}}}{\partial \Delta_i} - F_i \quad (i = 1, \ldots, N) \,, \\ 0 &= \frac{\partial \Pi_{\text{total}}}{\partial \theta_j} = \frac{\partial \Pi_{\text{elastic}}}{\partial \theta_j} - T_j \quad (j = 1, \ldots, M) \,. \end{aligned}$$

A simple rearrangement gives Castigliano's 1st Theorem.

4.1.4 Application to an N-node truss

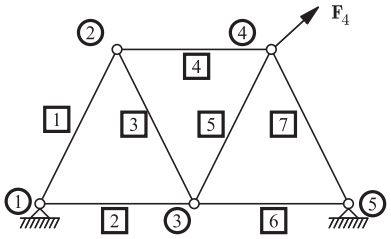


Fig. 4.8 7 bar truss system.

Consider the truss structure shown in Fig. 4.8. One can determine the equilibrium equations for the truss using the principle of stationary potential energy. The potential energy due to the loads is given as

$$\begin{aligned}\Pi_{\text{load}} &= -\boldsymbol{F}_4 \cdot \boldsymbol{u}_4 - \boldsymbol{F}_{1,r} \cdot \boldsymbol{u}_1 - \boldsymbol{F}_{5,r} \cdot \boldsymbol{u}_5 \\ &= -\begin{bmatrix} \boldsymbol{F}_{1,r} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{F}_4 \\ \boldsymbol{F}_{5,r} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \end{bmatrix} = -\boldsymbol{F} \cdot \boldsymbol{u}\,,\end{aligned}$$

where we have defined $\boldsymbol{F}$ and $\boldsymbol{u}$ as in Chapter 2. The subscripted forces $\boldsymbol{F}_{i,r}$ denote the unknown reaction forces at the supports. We include these terms in the same spirit as before even though we know *a priori* that the respective displacements are zero.

To obtain the elastic potential energy of the truss, one can add together the potential energy contribution from each bar. This additive structure is one of the characteristics which makes the use of potential energy a powerful concept in analysis.

$$\begin{aligned}\Pi_{\text{elastic}} &= \sum_{i=1}^{7} \Pi_{\text{bar,i}} \\ &= \sum_{i=1}^{7} \int_0^L \frac{1}{2}(AE)_i(\varepsilon_i)^2 \, dx \\ &= \sum_{i=1}^{7} \frac{1}{2}(AEL)_i(\varepsilon_i)^2 \\ &= \frac{1}{2}\left[\varepsilon_1(AEL)_1\varepsilon_1 + \cdots + \varepsilon_7(AEL)_7\varepsilon_7\right] \\ &= \frac{1}{2}\boldsymbol{\varepsilon}^T \lceil AEL \rfloor \boldsymbol{\varepsilon}\,.\end{aligned}$$

Here we have defined $\boldsymbol{\varepsilon}$ as a vector whose scalar entries are the strains of each bar, and $\lceil AEL \rfloor$ is the diagonal matrix with entries AEL of each bar on the diagonal. Recall the kinematic relation between the vector of nodal displacements $\boldsymbol{u}$ and the strains in the bars $\boldsymbol{\varepsilon}$ from Chapter 2:

$$\boldsymbol{\varepsilon} = \lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{u}\,.$$

If we note that for two matricies $\boldsymbol{C}$ and $\boldsymbol{D}$, that $(\boldsymbol{C}\boldsymbol{D})^T = \boldsymbol{D}^T\boldsymbol{C}^T$, then we have

$$\begin{aligned}\boldsymbol{\varepsilon}^T &= (\lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{u})^T \\ &= \boldsymbol{u}^T\boldsymbol{A}^T \lceil 1/L \rfloor\,.\end{aligned}$$

Substituting this relationship into the expression for the elastic potential

energy yields:

$$\begin{aligned}
\Pi_{\text{elastic}} &= \frac{1}{2}\boldsymbol{\varepsilon}^T \lceil AEL \rfloor \boldsymbol{\varepsilon} \\
&= \frac{1}{2}\left(\boldsymbol{u}^T \boldsymbol{A}^T \lceil 1/L \rfloor\right) \lceil AEL \rfloor \left(\lceil 1/L \rfloor \boldsymbol{A}\boldsymbol{u}\right) \\
&= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}\boldsymbol{u} \\
&= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u}\,.
\end{aligned}$$

Here, the definition of the stiffness matrix for the truss, $\boldsymbol{K} := \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}$, has been used. The total potential energy is now given as

$$\begin{aligned}
\Pi_{\text{total}} &= \Pi_{\text{elastic}} + \Pi_{\text{load}} \\
\Pi_{\text{total}}(\boldsymbol{u}) &= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} - \boldsymbol{F}_4 \cdot \boldsymbol{u}_4 - \boldsymbol{F}_{1,r} \cdot \boldsymbol{u}_1 - \boldsymbol{F}_{5,r} \cdot \boldsymbol{u}_5 \\
&= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} - \boldsymbol{F} \cdot \boldsymbol{u}\,.
\end{aligned}$$

One observes that Π_{total} is now a function of the vector of nodal displacements $\boldsymbol{u}$.

For this case, the *Principle of Stationary Potential Energy* states that:

$$\boxed{\begin{array}{ccc} \text{Equilibrium} & \Leftrightarrow & \Pi_{\text{total}} \text{ is stationary,} \\ -\dfrac{\partial \Pi_{\text{total}}}{\partial u_I} = 0 \text{ for } I = 1, \ldots, 10. & & \end{array}}$$

Here we have used the fact that the displacement vector $\boldsymbol{u}$ has a total of 10 degrees of freedom for our particular 2-dimensional truss structure with 5 nodes. Depending on the context the vector of nodal displacements will be expressed in various ways:

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{10} \end{bmatrix}\,.$$

The second expression above emphasizes the grouping into the displacements at each node, $\boldsymbol{u}_i$, where

$$\boldsymbol{u}_i := \begin{bmatrix} u_{i,x} \\ u_{i,y} \end{bmatrix}\,.$$

The third expression emphasizes that there are actually a total of 10 scalar degrees of freedom. Similar notation is employed for the vector of applied forces $\boldsymbol{F}$ such that one has

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \vdots \\ \boldsymbol{F}_5 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{10} \end{bmatrix}\,.$$

The second expression above emphasizes the grouping of the forces at each node:

$$\boldsymbol{F}_i \;:=\; \begin{bmatrix} F_{i,x} \\ F_{i,y} \end{bmatrix} .$$

The third expression emphasizes that there are actually a total of 10 scalar forces corresponding to the degrees-of-freedom.

Since

$$\frac{\partial \Pi_{\text{total}}}{\partial u_I} = \frac{\partial \Pi_{\text{elastic}}}{\partial u_I} + \frac{\partial \Pi_{\text{load}}}{\partial u_I} ,$$

we can separately compute the two contributions to the equilibrium equations. Let us first compute $\partial \Pi_{\text{load}}/\partial u_I$. The expression for Π_{load} can be written as:

$$\begin{aligned} \Pi_{\text{load}} &= -\boldsymbol{F} \cdot \boldsymbol{u} \\ &= -\sum_{I=1}^{10} F_I u_I . \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial \Pi_{\text{load}}}{\partial u_L} &= -\frac{\partial}{\partial u_L} \left(\sum_{I=1}^{10} F_I u_I \right) \\ &= -\sum_{I=1}^{10} F_I \frac{\partial u_I}{\partial u_L} . \end{aligned}$$

The expression $\partial u_I/\partial u_L$ is nonzero only when $I = L$. One can use the Kronecker-delta δ_{IJ} to express this relationship, since

$$\delta_{IJ} \;=\; \begin{cases} 1 & I = J \\ 0 & I \neq J . \end{cases}$$

Using the Kronecker-delta,

$$\begin{aligned} \frac{\partial \Pi_{\text{load}}}{\partial u_L} &= -\sum_{I=1}^{10} F_I \delta_{IL} \\ &= -F_L . \end{aligned}$$

The summation above has been removed since only the term where $I = L$ remains. The resulting expression can be written as:

$$\begin{bmatrix} \dfrac{\partial \Pi_{\text{load}}}{\partial u_1} \\ \vdots \\ \dfrac{\partial \Pi_{\text{load}}}{\partial u_{10}} \end{bmatrix} = \begin{bmatrix} -F_1 \\ \vdots \\ -F_{10} \end{bmatrix} .$$

This motivates the use of the compact notation

$$\frac{\partial \Pi_{\text{load}}}{\partial \boldsymbol{u}} = -\boldsymbol{F}\,.$$

Since one also has the relation

$$\begin{bmatrix} \dfrac{\partial \Pi_{\text{load}}}{\partial u_{i,x}} \\ \dfrac{\partial \Pi_{\text{load}}}{\partial u_{i,y}} \end{bmatrix} = \begin{bmatrix} -F_{i,x} \\ -F_{i,y} \end{bmatrix},$$

one can equivalently write

$$\frac{\partial \Pi_{\text{load}}}{\partial \boldsymbol{u}_i} = -\boldsymbol{F}_i\,.$$

For our example truss, we have more specifically that

$$\frac{\partial \Pi_{\text{load}}}{\partial \boldsymbol{u}} = -\begin{bmatrix} \boldsymbol{F}_{1,r} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{F}_4 \\ \boldsymbol{F}_{5,r} \end{bmatrix}, \qquad \frac{\partial \Pi_{\text{load}}}{\partial \boldsymbol{u}_i} = \begin{cases} -\boldsymbol{F}_{1,r} & i = 1 \\ \boldsymbol{0} & i = 2,3 \\ -\boldsymbol{F}_4 & i = 4 \\ -\boldsymbol{F}_{5,r} & i = 5\,. \end{cases}$$

Let us next compute $\partial \Pi_{\text{elastic}}/\partial u_I$. The expression for Π_{elastic} can be written as:

$$\begin{aligned} \Pi_{\text{elastic}} &= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} \\ &= \frac{1}{2}\sum_{I=1}^{10}\sum_{J=1}^{10} u_I K_{IJ} u_J\,. \end{aligned}$$

Thus,

$$\begin{aligned} 2\frac{\partial \Pi_{\text{elastic}}}{\partial u_L} &= \frac{\partial}{\partial u_L}\left(\sum_{I=1}^{10}\sum_{J=1}^{10} u_I K_{IJ} u_J\right) \\ &= \sum_{I=1}^{10}\sum_{J=1}^{10} \frac{\partial}{\partial u_L}\left(u_I K_{IJ} u_J\right) \\ &= \sum_{I=1}^{10}\sum_{J=1}^{10} \left(\frac{\partial u_I}{\partial u_L} K_{IJ} u_J + u_I K_{IJ} \frac{\partial u_J}{\partial u_L}\right) \\ &= \sum_{I=1}^{10}\sum_{J=1}^{10} \left(\delta_{IL} K_{IJ} u_J + u_I K_{IJ} \delta_{JL}\right) \\ &= \sum_{J=1}^{10}\left(\sum_{I=1}^{10} \delta_{IL} K_{IJ}\right) u_J + \sum_{I=1}^{10} u_I \left(\sum_{J=1}^{10} K_{IJ}\delta_{JL}\right) \\ &= \sum_{J=1}^{10} K_{LJ} u_J + \sum_{I=1}^{10} u_I K_{IL}\,. \end{aligned}$$

Here the Kronecker-delta has again been employed. Next we use the fact that $\boldsymbol{K}$ is symmetric, i.e. $\boldsymbol{K}^T = \boldsymbol{K}$ or equivalently $K_{IJ} = K_{JI}$:

$$\begin{aligned} 2\frac{\partial \Pi_{\text{elastic}}}{\partial u_L} &= \sum_{J=1}^{10} K_{LJ} u_J + \sum_{I=1}^{10} u_I K_{LI} \\ &= 2\sum_{J=1}^{10} K_{LJ} u_J \\ &= 2\left(\boldsymbol{K}\boldsymbol{u}\right)_L . \end{aligned}$$

The expression $(\boldsymbol{K}\boldsymbol{u})_L$ denotes the Lth component of the vector $\boldsymbol{K}\boldsymbol{u}$. The resulting expression can be written as:

$$\begin{bmatrix} \dfrac{\partial \Pi_{\text{elastic}}}{\partial u_1} \\ \vdots \\ \dfrac{\partial \Pi_{\text{elastic}}}{\partial u_{10}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{K}\boldsymbol{u})_1 \\ \vdots \\ (\boldsymbol{K}\boldsymbol{u})_{10} \end{bmatrix} .$$

This motivates the use of the compact notation

$$\frac{\partial \Pi_{\text{elastic}}}{\partial \boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u}$$

and so we see that the principle of stationary potential energy yields the expected equilibrium equations: $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$.

4.2 The Method of Ritz

In the developments so far, the critical step in exploiting the principle of stationary potential has been to obtain a finite set of parameters that is able to fully characterize the motion of our system and thus the potential energy. Consider the tension-compression bar example with two applied point loads. Though there are displacements defined at any point along the bar, in other words the displacements are a field $u(x)$, the entire field can be fully characterized by two parameters, the displacement at the locations of the applied loads; see Fig. 4.9.

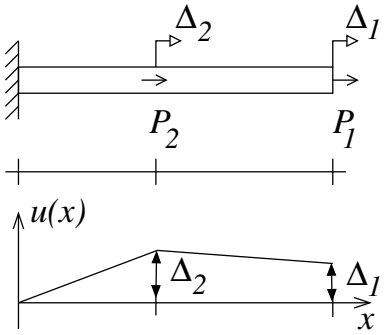


Fig. 4.9 Tension-compression bar with 2 point loads.

It is not generally true that this kind of fortunate case will occur. Also, even in the cases that one can find such a characterization, the procedure to obtain the characterization may be tedious and require as much effort as solving the problem directly from the governing differential equations. Consider, for example, the cantilever beam shown in Fig. 4.10 where one would like to find the tip displacement Δ and rotation θ in terms of the load P. The elastic potential energy and the energy due to the load are defined as:

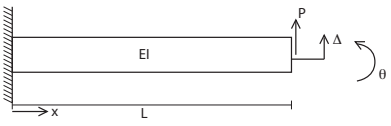


Fig. 4.10 Cantilever beam with and end-load.

$$\begin{aligned} \Pi_{\text{load}} &= -P\Delta, \\ \Pi_{\text{elastic}} &= \int_0^L \frac{1}{2} EI\kappa^2 \, dx \, , \end{aligned}$$

with the total potential energy Π_{total} being their sum. Through the use of the principal of stationary potential energy, one would like to obtain the desired relationships via the equilibrium relations

$$\begin{aligned} 0 &= \frac{\partial \Pi_{\text{total}}}{\partial \Delta} = \frac{\partial \Pi_{\text{elastic}}}{\partial \Delta} - P, \\ 0 &= \frac{\partial \Pi_{\text{total}}}{\partial \theta} = \frac{\partial \Pi_{\text{elastic}}}{\partial \theta}\,. \end{aligned}$$

To compute the derivatives of the potential energy with respect to Δ and θ, one requires a representation of the curvature $\kappa(x)$ in terms of Δ and θ. This relationship is non-trivially given as

$$\kappa(x) = 6\left(\frac{\theta}{L^2} - \frac{2\Delta}{L^2}\right)x + \left(\frac{3\Delta}{L^2} - \frac{\theta}{L}\right), \tag{4.1}$$

which cannot be simply determined without more than a basic knowledge of the physics at hand.

If we still want to use the principle of stationary potential energy but do not know eqn (4.1), there is still the option of finding an approximate solution. This approximate methodology exploits the special character that we have for the total potential energy of the mechanical systems that we investigate:

> For stable systems, the potential energy,
>
> $$\Pi_{\text{total}} = \Pi_{\text{elastic}} + \Pi_{\text{load}}\,,$$
>
> is not only stationary for equilibrium states, but also a minimum.

Exploiting this fact for the beam, we look for the equilibrium deflection $v(x)$, such that

$$\Pi_{\text{total}}(v(x)) = \int_0^L \frac{1}{2} EI(v'')^2 \, dx - Pv(L) \tag{4.2}$$

is a minimum. Here we observe that the scalar value of the total potential energy depends on the function $v(x)$ defining the shape of the deflected beam. To compute its minimum one is tempted to try to take the derivative of Π_{total} with respect to $v(x)$. However as defined in eqn (4.2), Π_{total} is a function of a function and it is not obvious how to take its derivative with respect to its argument $v(x)$ which itself is a function. Notwithstanding, we still have the knowledge that the minimizing deflection of the beam $v(x)$ is the function, out of all functions, that renders the total potential energy a minimum. So another method we could follow is to search amongst the set of all functions for the one that minimizes the total potential energy. This unfortunately is a rather daunting task since there are an infinite number of functions we could choose from. The search is in an infinite dimensional space.

The key to rendering this problem tractable is the method of Ritz. The method of Ritz, tells us to forego looking for an exact solution and to look instead for an approximate solution from within a (well selected) finite set of functions.

The Method of Ritz:
Search only over a finite collection of functions and find the one that minimizes Π_{total}. This minimizer will yield an approximate solution.

Formally one must guess a general form of the solution:

$$\boxed{v(x) = \sum_{i=1}^{N} c_i f_i(x)\,.}$$

Here:

- The functions $f_i(s)$ $(i = 1, \ldots, N)$ are *known* (Ritz) functions that satisfy the kinematic boundary conditions. In beam problems, for example, specified values on the deflection v and rotation θ are kinematic boundary conditions. If one does not employ this requirement, the expression for the potential energy needs to be modified for the potential energy associated with the support forces. For simple hand computations, this leads to a more awkward set-up of the problem. Thus we opt for the kinematic requirements on the Ritz functions for hand solutions. In this case of automated solutions, it often is more convenient to forego the kinematic requirements and to include the extra terms associated with the boundary forces.
- The coefficients c_i $(i = 1, \ldots, N)$ are *unknown* parameters that are to be determined.

In this setting, we reduce the potential energy from a function of a function to one in terms of the finite number of parameters c_i:

$$\Pi_{\text{total}}(v(x)) \quad \Rightarrow \quad \Pi_{\text{total}}(c_1, \ldots, c_N)\,.$$

Let us look at the simple cantilever shown in Fig. 4.10 again. To find the equilibrium shape of the beam, we need to minimize

$$\Pi_{\text{total}}(v(x)) \quad = \quad \int_0^L \frac{1}{2} EI(v'')^2 \, dx - Pv(L)$$

over all functions in the (trial solution space) $\mathcal{S} = \{v(x) \mid v(0) = v'(0) = 0\}$. As an approximation let us consider minimizing Π_{total} over the subset

$$\tilde{\mathcal{S}} = \left\{ f(x) \;\middle|\; f(x) = Cx^2\,; \quad C \in \mathbb{R} \right\} \subset \mathcal{S}\,.$$

For all functions $v \in \tilde{\mathcal{S}}$, we have $v'' = 2C$ and thus

$$\Pi_{\text{total}}(v(x)) \quad \Rightarrow \quad \Pi_{\text{total}}(C) = \int_0^L \frac{1}{2} EI(2C)^2 \, dx - PCL^2\,.$$

Looking for the stationary point with respect to C, the only degree of freedom, yields

$$\begin{aligned} 0 = \frac{\partial \Pi_{\text{total}}}{\partial C} &= \int_0^L 4EIC\, dx - PL^2 \\ &= 4EICL - PL^2\,, \\ \Rightarrow \quad C &= \frac{PL}{4EI}\,. \end{aligned}$$

Thus the approximation we obtain for the solution is:

$$v_{\text{approx}}(x) = \frac{PL}{4EI}x^2 \in \tilde{\mathcal{S}}\,.$$

To assess how good the solution is, let us compare the tip deflection to the exact answer:

$$\begin{aligned} v_{\text{approx}}(L) &= \frac{PL^3}{4EI}\,, \\ v_{\text{exact}}(L) &= \frac{PL^3}{3EI}\,. \end{aligned}$$

The relative error of the approximation is defined as

$$\begin{aligned} \text{relative error} &= \frac{|v_{\text{approx}}(L) - v_{\text{exact}}(L)|}{|v_{\text{exact}}(L)|} \\ &= 0.25\,. \end{aligned}$$

Thus there is a relative error of 25% at the tip. If we had selected to minimize the potential energy over all the functions $v(x)$ in the set,

$$\tilde{\mathcal{S}} = \left\{ f(x) \;\middle|\; f(x) = Cx^2 + Dx^3\,; \quad C, D \in \mathbb{R} \right\} \subset \mathcal{S}\,,$$

the potential energy would have the form

$$\Pi_{\text{total}}(v(x)) \quad \Rightarrow \quad \Pi_{\text{total}}(C, D)\,.$$

The stationary point for this case yields the exact solution since the exact solution has the form:

$$v_{\text{exact}}(x) = -\frac{P}{6EI}x^3 + \frac{PL}{2EI}x^2\,,$$

which is included in the set $\tilde{\mathcal{S}}$. Graphically, the situation is as depicted in Fig. 4.11. The set of all possible functions that satisfy the kinematic boundary conditions, $\mathcal{S}$, is infinite dimensional. Our first selection for $\tilde{\mathcal{S}}$ was a one-dimensional subset of $\mathcal{S}$ and it did not contain the exact solution. Our second selection for $\tilde{\mathcal{S}}$ was a two-dimensional subset of the infinite set $\mathcal{S}$ and it contained our first set as well as the exact solution. The basic property of the method of Ritz is that it gives you the best answer within the set of functions you have picked. Thus one sees one will recover the exact solution if it is contained within one's guess.

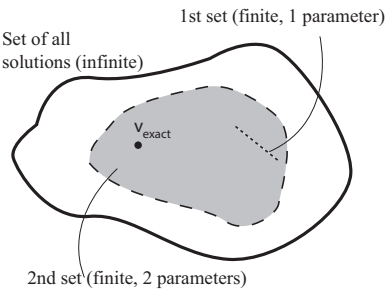


Fig. 4.11 Topology of the method of Ritz for different selections for the set of solutions (approximations).

Example 4.1

Cantilever with general loading

Consider the case of a multiply loaded cantilever as shown in Fig. 4.12. The cantilever is loaded with N_p point loads P_i $(i = 1, \ldots, N_p)$ at points x_i^p $(i = 1, \ldots, N_p)$ and N_m point moments M_k $(k = 1, \ldots, N_m)$ at points x_k^m $(k = 1, \ldots, N_m)$. Find a general approximate solution for the beam's deflection.

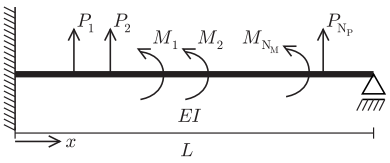


Fig. 4.12 A pin-clamped beam with multiple point moments and point forces.

Solution: The total potential energy is

$$\Pi_{\text{total}}(v(x)) \quad = \quad \int_0^L \frac{1}{2} EI(v'')^2 \, dx - \sum_{j=1}^{N_p} P_j v(x_j^p) - \sum_{k=1}^{N_m} M_k v'(x_k^m) \, .$$

We will minimize the energy over all functions $v(x)$ in the set

$$\tilde{\mathcal{S}} = \left\{ f(x) \;\middle|\; f(x) = \sum_{i=1}^{N} c_i f_i(x) \, ; \quad \{c_i\}_{i=1}^N \in \mathbb{R} \right\} ,$$

where the functions $f_i(x)$ have the property that $f_i(0) = f_i'(0) = f_i(L) = 0$. Thus $\tilde{\mathcal{S}} \subset \mathcal{S} = \{v(x) \mid v(0) = v'(0) = v(L) = 0\}$. Then,

$$\begin{aligned} \Pi_{\text{total}}(v(x)) \Rightarrow \Pi_{\text{total}}(c_1, \ldots, c_N) \quad = \quad & \int_0^L \frac{1}{2} EI \left(\sum_{i=1}^{N} c_i f_i''(x) \right)^2 dx \\ & - \sum_{j=1}^{N_p} P_j \left(\sum_{i=1}^{N} c_i f_i(x_j^p) \right) \\ & - \sum_{k=1}^{N_m} M_k \left(\sum_{i=1}^{N} c_i f_i'(x_k^m) \right) . \end{aligned}$$

Looking for the stationary point with respect to the parameters $\{c_i\}_{i=1}^N$ requires

$$0 = \frac{\partial \Pi_{\text{total}}}{\partial c_l}, \qquad (l = 1, \ldots, N) \, .$$

This yields

$$\begin{aligned} & 0 = \int_0^L \frac{1}{2} EI 2 \left(\sum_{i=1}^{N} c_i f_i''(x) \right) \cdot f_l''(x) \, dx - \sum_{j=1}^{N_p} P_j f_l(x_j^p) - \sum_{k=1}^{N_m} M_k f_l'(x_k^m) \\ \Rightarrow \quad & \sum_{i=1}^{N} \left(\int_0^L EI f_i''(x) \cdot f_l''(x) \, dx \right) c_i = \sum_{j=1}^{N_p} P_j f_l(x_j^p) + \sum_{k=1}^{N_m} M_k f_l'(x_k^m) \\ \Rightarrow \quad & \sum_{i=1}^{N} K_{li} c_i = F_l \\ \Rightarrow \quad & \boldsymbol{K c} = \boldsymbol{F} \, . \end{aligned}$$

Here we have defined

$$K_{li} \quad := \quad \int_0^L EI f_i''(x) \cdot f_l''(x)\, dx$$

and

$$F_l \quad := \quad \sum_{j=1}^{N_p} P_j f_l(x_j^p) + \sum_{k=1}^{N_m} M_k f_l'(x_k^m)\,,$$

where the matrix $\boldsymbol{K}$ has components K_{li} and the vector $\boldsymbol{F}$ has components F_l. By solving the linear system of equations $\boldsymbol{Kc} = \boldsymbol{F}$ for the vector of *generalized displacements* $\boldsymbol{c}$, one obtains the approximate solution

$$v(x) = \sum_{i=1}^{N} c_i f_i(x)\,.$$

By defining the vector of functions

$$\boldsymbol{f}(x) \quad := \quad \begin{bmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{bmatrix}\,,$$

one can also write the final result as

$$v(x) = \boldsymbol{c}^T \boldsymbol{f}(x)\,.$$

Remarks:

(1) The final result is an N-dimensional approximation to the true solution.

(2) A judicious choice for the Ritz functions $f_i(x)$ can lead to a very good answer with only a few functions.

(3) The accuracy of the method always increases with increasing N.

Example 4.2

Tension-compression bar with general loading

Consider a multiply loaded tension-compression bar that is builit-in at $x = 0$ and $x = L$. The bar is loaded with N_p point loads P_i ($i = 1, \ldots, N_p$) at points x_i^p ($i = 1, \ldots, N_p$). Find a general approximate solution for the bar's motion using the principle of stationary potential energy.

Solution: The total potential energy is

$$\Pi_{\text{total}}(u(x)) \quad = \quad \int_0^L \frac{1}{2} AE(u')^2 \, dx - \sum_{j=1}^{N_p} P_j u(x_j^p) \, .$$

We will minimize the energy over all functions $u(x)$ in the set

$$\tilde{\mathcal{S}} = \left\{ f(x) \;\middle|\; f(x) = \sum_{i=1}^{N} c_i f_i(x) \, ; \quad \{c_i\}_{i=1}^N \in \mathbb{R} \right\} ,$$

where $f_i(0) = f_i(L) = 0$. Thus, $\tilde{\mathcal{S}} \subset \mathcal{S} = \{u(x) \mid u(0) = u(L) = 0\}$. This implies that

$$\begin{aligned} \Pi_{\text{total}}(u(x)) \Rightarrow \Pi_{\text{total}}(c_1, \ldots, c_N) \quad = \quad & \int_0^L \frac{1}{2} AE \left(\sum_{i=1}^{N} c_i f_i'(x) \right)^2 dx \\ & - \sum_{j=1}^{N_p} P_j \left(\sum_{i=1}^{N} c_i f_i(x_j^p) \right) . \end{aligned}$$

Looking for the stationary point with respect to $\{c_i\}_{i=1}^N$ requires

$$0 = \frac{\partial \Pi_{\text{total}}}{\partial c_l}, \qquad (l = 1, \ldots, N) \, .$$

This yields

$$\begin{aligned} & 0 = \int_0^L \frac{1}{2} AE2 \left(\sum_{i=1}^{N} c_i f_i'(x) \right) \cdot f_l'(x) \, dx - \sum_{j=1}^{N_p} P_j f_l(x_j^p) \\ \Rightarrow \quad & \sum_{i=1}^{N} \left(\int_0^L AE f_i'(x) \cdot f_l'(x) \, dx \right) c_i = \sum_{j=1}^{N_p} P_j f_l(x_j^p) \\ \Rightarrow \quad & \sum_{i=1}^{N} K_{li} c_i = F_l \\ \Rightarrow \quad & \boldsymbol{K}\boldsymbol{c} = \boldsymbol{F} \, . \end{aligned}$$

The components of the matrix $\boldsymbol{K}$ and the vector $\boldsymbol{F}$ are given as

$$K_{li} \quad := \quad \int_0^L AE f_i'(x) \cdot f_l'(x) \, dx \, , \tag{4.3}$$

$$F_l \quad := \quad \sum_{j=1}^{N_P} P_j f_l(x_j^P) \, . \tag{4.4}$$

By solving the linear system of equations $\boldsymbol{K}\boldsymbol{c} = \boldsymbol{F}$ for the vector of *generalized displacements* $\boldsymbol{c}$, one obtains the approximate Ritz solution

$$u(x) = \sum_{i=1}^{N} c_i f_i(x) \, .$$

By defining the vector of functions

$$\boldsymbol{f}(x) \quad := \quad \begin{bmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{bmatrix},$$

one can also write this as

$$u(x) = \boldsymbol{c}^T \boldsymbol{f}(x)\,.$$

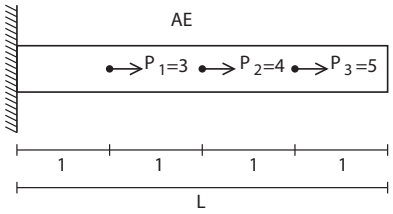


Fig. 4.13 Tension-compression bar with three point loads.

Example 4.3

Tension-compression bar with 3 point loads

Consider the bar shown in Fig. 4.13. The bar is loaded with 3 point forces at locations x_1, x_2, and x_3:

$$\begin{aligned} P_1 &= 3, \quad x_1 = 1 \\ P_2 &= 4, \quad x_2 = 2 \\ P_3 &= 5, \quad x_3 = 3\,. \end{aligned}$$

Find an approximate solution for the bar's motion.

Solution: We are interested in an approximate solution of the form

$$u(x) \quad = \quad \sum_{i=1}^{N} c_i f_i(x) \in \tilde{\mathcal{S}} \subset \mathcal{S}\,,$$

where the unknown coefficients, c_i, are to be determined by minimizing the potential energy for the system:

$$\Pi_{\text{total}}(u(x)) \quad = \quad \int_0^L \frac{1}{2} AE(u')^2 \, dx - \sum_{j=1}^{3} P_j u(x_j)\,.$$

To do so, one must select the functions $f_i(x)$ so that they satisfy the kinematic (displacement) boundary condition, viz.,

$$f_i(0) = 0\,.$$

This ensures that $\tilde{\mathcal{S}} \subset \mathcal{S} = \{u(x) \mid u(0) = 0\}$. Some possible choices for these functions are the polynomial and sine functions:

$$\begin{aligned} f_i(x) &= \left(\frac{x}{L}\right)^i \qquad (i > 0)\,, \\ f_i(x) &= \sin\left(i\pi \frac{x}{L}\right)\,. \end{aligned}$$

Here, let us select the polynomials for the approximation and use the results of Example 4.2. The forcing vector $\boldsymbol{F}$, according to eqn (4.4), has components

$$\begin{aligned} F_i &= \sum_{i=1}^{3} P_j f_i(x_j) \\ &= 3f_i(1) + 4f_i(2) + 5f_i(3) \\ &= 3\left(\frac{1}{4}\right)^i + 4\left(\frac{2}{4}\right)^i + 5\left(\frac{3}{4}\right)^i . \end{aligned}$$

The stiffness matrix $\boldsymbol{K}$, according to eqn (4.3), has components

$$\begin{aligned} K_{ij} &= \int_0^4 AE f_i' f_j' \, dx \\ &= \int_0^4 AEi \left(\frac{x}{L}\right)^{i-1} \left(\frac{1}{L}\right) j \left(\frac{x}{L}\right)^{j-1} \left(\frac{1}{L}\right) \, dx \\ &= \int_0^4 AE \frac{ij}{L^{i+j}} x^{i+j-2} \, dx \\ &= \frac{ij}{L^{i+j}} \frac{AE}{i+j-1} \left[x^{i+j-1}\right]_0^L \\ &= \frac{ij}{i+j-1} \frac{AE}{L} . \end{aligned}$$

Remarks:

(1) By solving the equations

$$\boldsymbol{K}\boldsymbol{c} = \boldsymbol{F},$$

one obtains the approximate solution.

(2) For small values for N this is a tractable hand solution. For larger values, this leads to a reasonable numerical methodology which is the basis of much of modern computational engineering and science.

(3) The exact solution for the displacement field is shown in Fig. 4.14. Figure 4.14 also shows the approximate solution for the case of $N = 10$. The agreement is seen to be quite good.

(4) If we had chosen not to respect the kinematic boundary conditions at $x = 0$ in our formulation, then we would need to add the potential energy of the reaction forces $R(0)$ into our total potential energy:

$$\Pi_{\text{total}}(u(x)) = \int_0^L \frac{1}{2} AE(u')^2 \, dx - \sum_{j=1}^{3} P_j u(x_j) - R_0 u(0) . \quad (4.5)$$

Here, R_0 is defined to be the force the support applies to the bar taken as positive in the positive x-direction. If one chooses to use eqn (4.5), then one can utilize the fact that $u(0) = 0$ to help solve the equations.

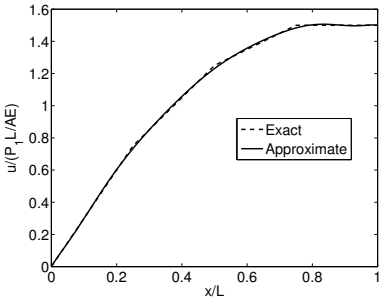


Fig. 4.14 Exact and approximate solution to Example 4.3 using $N = 10$, polynomial terms.

In particular, consider

$$u(x) = \sum_{i=0}^{2} c_i f_i(x) \in \tilde{\mathcal{S}}\,,$$

where $f_i(x) = (x/L)^i$. In this case we will have 3 equilibrium equations $\partial\Pi_{\text{total}}/\partial c_i = 0$ with a total of 4 unknowns c_0, c_1, c_2, and R_0. As 4th equation we can use the fact that we must have

$$u(0) = c_0 f_0(0) + c_1 f_1(0) + c_2 f_2(0) = 0\,.$$

With some rearrangement, the full set of linear equations to be solved can be written as

$$\begin{bmatrix} 0 & 0 & 0 & -f_0(0) \\ 0 & \frac{AE}{L} & \frac{AE}{L} & -f_1(0) \\ 0 & \frac{AE}{L} & \frac{4AE}{3L} & -f_2(0) \\ -f_0(0) & -f_1(0) & -f_2(0) & 0 \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ R_0 \end{pmatrix} = \begin{pmatrix} 12 \\ 26/4 \\ 4 \\ 0 \end{pmatrix}.$$

Given the properties of our chosen $f_i(x)$, it is easy to see that $R_0 = -12$ (as it should) and $c_0 = 0$ (as it should).

4.2.1 Analogy between vectors and functions

In the process of selecting functions for approximating solutions in the Ritz method, one desires to select them as linearly independent as possible. This notion can be motivated through an example from linear algebra. Consider a vector $\boldsymbol{a}$ that we would like to write as a linear combination of two other vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$; i.e., we would like to determine the coefficients c_1 and c_2 in the expression

$$\boldsymbol{a} \;=\; c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2\,.$$

These coefficients can be determined by taking the dot product of $\boldsymbol{a}$ with the two vectors resulting in two conditions or equations that can be solved for c_1 and c_2. Let us define the notation for the dot product as

$$(\boldsymbol{v},\boldsymbol{w}) \;:=\; \boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{v}^T\boldsymbol{w}\,.$$

With this notation, the norm or size of a vector can be expressed as

$$\|\boldsymbol{v}\| \;=\; \sqrt{(\boldsymbol{v},\boldsymbol{v})} = \sqrt{\boldsymbol{v}^T\boldsymbol{v}}\,.$$

The angle θ between two vectors can be defined via

$$(\boldsymbol{v},\boldsymbol{w}) \;:=\; \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta\,.$$

This is graphically depicted in Fig. 4.15. Using this notation,

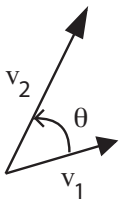


Fig. 4.15 Angle between two vectors.

$$\begin{aligned}(\boldsymbol{v}_1, \boldsymbol{a}) &= c_1(\boldsymbol{v}_1, \boldsymbol{v}_1) + c_2(\boldsymbol{v}_1, \boldsymbol{v}_2)\,,\\ (\boldsymbol{v}_2, \boldsymbol{a}) &= c_1(\boldsymbol{v}_2, \boldsymbol{v}_1) + c_2(\boldsymbol{v}_2, \boldsymbol{v}_2)\,.\end{aligned}$$

In matrix form this becomes:

$$\begin{bmatrix}(\boldsymbol{v}_1, \boldsymbol{a})\\ (\boldsymbol{v}_2, \boldsymbol{a})\end{bmatrix} = \begin{bmatrix}(\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2)\\ (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}\,.$$

This linear system of equations will have a unique solution as long as the determinant of the coefficient matrix is non-zero.

$$\begin{aligned}\det\left(\begin{bmatrix}(\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2)\\ (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2)\end{bmatrix}\right) &= (\boldsymbol{v}_1, \boldsymbol{v}_1)(\boldsymbol{v}_2, \boldsymbol{v}_2) - (\boldsymbol{v}_1, \boldsymbol{v}_2)^2\\ &= \|\boldsymbol{v}_1\|^2\|\boldsymbol{v}_2\|^2 - \|\boldsymbol{v}_1\|^2\|\boldsymbol{v}_2\|^2\cos^2\theta\\ &= \|\boldsymbol{v}_1\|^2\|\boldsymbol{v}_2\|^2(1 - \cos^2\theta)\\ &= \|\boldsymbol{v}_1\|^2\|\boldsymbol{v}_2\|^2\sin^2\theta\,.\end{aligned}$$

The value of the determinant will be zero when $\theta = 0$, i.e. when the two vectors are parallel to each other. In the case where θ is nearly zero, the equations can technically be solved; however, in finite precision arithmetic, the situation (numerically) is not well behaved. The farthest one can get from this problematic case is when the two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal to each other. This is the most desired situation.

One can make similar arguments about functions (in place of vectors). For example, in the Ritz method, consider the case where one is trying to express a function $g(x)$ as a linear combination of two functions $f_1(x)$ and $f_2(x)$ on the interval $[a, b]$:

$$g(x) = c_1 f_1(x) + c_2 f_2(x)\,.$$

To be able to determine the coefficients c_1 and c_2, one wants the functions $f_1(x)$ and $f_2(x)$ to be as orthogonal as possible on the interval $[a, b]$. To define orthogonality one must define the notion of a dot product for functions. The standard dot product (or inner product) for functions is defined as

$$(f, g) := \int_a^b f(x)g(x)\;dx\,.$$

This is sometimes called the L^2 inner product. Using this notation one can also define the size (or norm) of a function in a manner similar to norms of vectors in $\mathbb{R}^n$:

$$\|f\| := \sqrt{(f, f)} = \sqrt{\int_a^b f(x)f(x)\;dx}\,.$$

This is known as the L^2 norm of a function. The angle θ between two functions $f(x)$ and $g(x)$ on the interval $[a, b]$ can also be defined via the relation

$$(f, g) = \|f\|\,\|g\|\cos\theta\,.$$

Thus to check whether two functions are orthogonal, one must check the quantity

$$\cos\theta \;=\; \frac{(f,g)}{\|f\|\,\|g\|}\,.$$

It is important to note that one cannot just check (f,g).

To make this notion more concrete, let us check whether the polynomial functions

$$f_i(x) \;=\; \left(\frac{x}{L}\right)^i$$

are orthogonal on the interval $[0,L]$ for $i>1$. The quantities needed to compute the angle θ between functions f_i and f_j are as follows:

$$\begin{aligned}\|f_i\| &= \sqrt{\frac{1}{2i+1}L}\,,\\ (f_i,f_j) &= \frac{1}{i+j+1}L\,.\end{aligned}$$

Thus,

$$\cos\theta_{ij} \;=\; \frac{\sqrt{(2i+1)(2j+1)}}{(i+j+1)}\,.$$

The functions are seen to not be parallel since the angle between any two is never zero – the cosine of the angle is never equal to unity for $i\neq j$. The functions are also seen to not be orthogonal as $\cos\theta_{ij}\neq 0$ for any $i\neq j$. However, as the number of functions N increases, the number of combinations of functions which have $\cos\theta_{ij}\approx 1$ increases, implying that more and more functions are becoming nearly linearly dependent (parallel). Thus increasing N while using these simple polynomials will not lead to a good situation. Small values of N are fine and useful for hand computation but large values are ill-advised.

On the other hand, the functions

$$f_i(x) \;=\; \sin\left(i\pi\frac{x}{L}\right)$$

are orthogonal on the interval $[0,L]$, since

$$(f_i,f_j) \;=\; 0\,,$$

for any combination of $i\neq j$. Using this concept of orthogonality of functions, one can also create an orthogonal set of functions from a non-orthogonal set of functions. This method is based on Gram-Schmidt orthogonalization. For example, orthogonalizing the polynomials on the interval $[-1,1]$ produces the Lagrange polynomials. The first three of which are

$$\begin{aligned}P_0(x) &= 1,\\ P_1(x) &= x,\\ P_2(x) &= \frac{1}{2}(3x^2-1)\,.\end{aligned}$$

One can easily confirm that these are orthogonal by computing

$$(P_i, P_j) \quad = \quad \int_{-1}^{1} P_i(x) P_j(x)\, dx$$

which is equal to zero for $i \neq j$.

4.2.2 Evaluating the quality of a given approximation

Once an approximation $u_a(x)$ is obtained through the Ritz method, one would like to estimate the quality of the approximation. In the case that one has the exact solution $u_e(x)$ to the problem, one can check:

- Pointwise error: the error at each point.
- L^2 error : the error over the whole domain.

Each type of error comes in two flavors:

- Absolute error.
- Relative error.

These are computed as presented in Table 4.1. Note that the pointwise error is a function of x as opposed to the L^2 error which is a scalar valued quantity. The L^2 error is often preferred due to its scalar nature and its ability to represent the average error over the whole domain in a single number. The relative error is also often preferred since it gives information on how many digits of accuracy the solution has been able to attain. However, care must be exercised when looking at relative pointwise error since the denominator may be zero at certain locations.

Table 4.1 Various error types and their definitions.

Type	Pointwise	L^2
Abs.	$\lvert u_a(x) - u_e(x)\rvert$	$\lVert u_a - u_e \rVert$
Rel.	$\frac{\lvert u_a(x) - u_e(x)\rvert}{\lvert u_e(x)\rvert}$	$\frac{\lVert u_a - u_e \rVert}{\lVert u_e \rVert}$

4.2.3 Estimating error in Ritz's method

The objective of Ritz's method is to reduce the error in the approximation. With respect to our definitions, this means that what we would like to happen is that error goes to zero as (the number of approximating functions) $N \to \infty$. Knowing how the error depends upon N (once the approximation functions have been chosen) is crucial to being able to estimate the error that one has made. This is an advanced topic which we will not treat in extensive detail. But it should be observed that the general characteristic of Ritz's method applied to elastic systems is that convergence to zero error occurs starting with the most significant digit in the approximation and continuing to the least significant digit. Thus one can perform computations with increasing values of N and compare solutions from one value to another. As the digits of the approximate solution take on the values of the exact solution they will stop changing as N increases. In this way one can estimate *how may digits of accuracy* one has in the approximation and thus one can also estimate the percent error one has made. This can be done on a pointwise basis or on an average basis.

Example 4.4

Convergence of the method of Ritz for a double built-in beam with a point load.

Consider a beam that is built in at both ends and subjected to a point force P at mid-span. Approximately solve for the beam's deflection and estimate the number of correct digits.

Solution: As an approximate space of solutions, we will use the set

$$\widetilde{\mathcal{S}} = \left\{ v(x) \;\middle|\; v(x) = \sum_{i \in \text{odd}} C_i [\cos(2\pi i x/L) - 1] \right\} .$$

Note that we restrict our sum to functions which are symmetric about the center of the span and each function individually satisfies the kinematic boundary conditions. Employing the principle of stationary potential energy one can solve for the C_i by hand to give

$$C_i = -\frac{4PL^3}{EI(2\pi i)^4} .$$

In order to estimate the error in the approximate solution, we compute the L^2 norm of the approximate deflection as a function of the number of terms in the approximation. Doing so gives

$$\|v(x)\| = \frac{|P|L^{7/2}}{4EI\pi^4} \sqrt{\frac{1}{2} \sum_{i \in \text{odd}} \frac{1}{i^8} + \left(\sum_{i \in \text{odd}} \frac{1}{i^4} \right)^2} .$$

Table 4.2 shows the result of this computation, where the first column indicates how many terms we have used in our approximation and the second column gives values of the L^2 norm divided by $|P|L^{7/2}/4EI\pi^4$. The underline indicates the digits which remain unchanged as we added new terms to our approximation. It is reasonable to assume these digits are converged.

Table 4.2 Convergence behavior of approximate solution.

No. Terms	Norm
1	1.22474487139
2	1.23487650463
3	1.23618903404
4	1.23653070762
5	1.23665574650
6	1.23671178040
7	1.23674050485
8	1.23675671035
9	1.23676653310
10	1.23677282839
Exact	1.23678983555

Remarks:

(1) From the result we can conclude with some confidence, for example, that with only three terms our result is accurate to three digits or has error less than 0.1%. Also shown in Table 4.2 is the exact solution computed by solving the governing differential equation. This verifies our conclusion.

(2) This type analysis is not an exact error analysis but suffices in most situations.

(3) If we had not restricted i to be odd, we would have had essentially the same result. The main difference would have been that half of the generalized displacements would have been zero. Exploiting symmetry merely simplified the details of the analysis.

4.2.4 Summary of the Method of Ritz

The method of Ritz can be summarized in the following three steps.

(1) Form the total potential energy Π_{total} which is a sum of the potential energy of the elastic elements of the system and the potential energy due to the load:

$$\Pi_{\text{total}} = \Pi_{\text{elastic}} + \Pi_{\text{load}} .$$

The total potential energy is:

- stationary at equilibrium and
- minimum at a stable equilibrium.

(2) Select an approximate form of the solution

$$u(x) \approx \sum_{i=1}^{N} c_i f_i(x),$$

where c_i are the unknown coefficients which must be determined and $f_i(x)$ are known functions which generally satisfy the kinematic boundary conditions.

(3) Find the stationary points of Π_{total}:

$$\frac{\partial \Pi_{\text{total}}}{\partial c_i} = 0, \quad \text{for } i = 1, \ldots, N .$$

This yields a linear system of N equations in N unknowns. Its solution provides the values for the parameters c_i.

Remarks:

(1) Each function $f_i(x)$ should satisfy the kinematic boundary conditions. If they do not satisfy the kinematic boundary conditions, then one must include extra terms in Π_{load} to account for the unknown reactions.

(2) One desires that the functions $f_i(x)$ be as orthogonal as possible:

$$\int_0^L f_i f_j \, dx = \delta_{ij} .$$

(a) For polynomials, the Legendre polynomials are orthogonal on the interval $[-1, 1]$. One can scale and translate these functions, so they are orthogonal on any desired interval $[a, b]$. Given the nth-order Legendre polynomial $P_n(\xi)$ defined on $[-1, 1]$ it can be translated to be defined on the interval $[x_l, x_u]$ by the mapping

$$\xi(x) = \frac{2}{x_u - x_l}(x - x_l) - 1 .$$

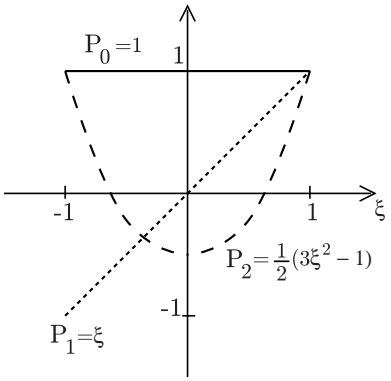


Fig. 4.16 The first three Legendre Polynomials.

Note that $\xi|_{x=x_l} = -1$ and $\xi|_{x=x_u} = 1$. The translated version of $P_n(\xi)$ is

$$f_n(x) \quad = \quad P_n(\xi(x))\,.$$

The first 3 Legendre polynomials are as shown in Fig. 4.16:

$$\begin{aligned} P_0(\xi) &= 1, \\ P_1(\xi) &= \xi, \\ P_2(\xi) &= \frac{1}{2}(3\xi^2 - 1)\,. \end{aligned}$$

(b) One can also pick sine and cosine functions such as:

$$f_n(\xi) \quad = \quad \sin(n\pi\xi)\,, \quad \xi \in [0,1]\,.$$

(3) If

$$u_{\text{exact}} \quad = \quad \sum_{i=1}^{N} c_i f_i(x)$$

for some c_is, then the Ritz method will give you the exact solution.

(4) For distributed (dead) loads, the potential energy of the load is derived as follows. Consider, for example, the tension-compression bar with a distributed load of $b(x)$. The force acting on an element of thickness dx is $b(x)dx$, and the potential energy of this load is $-u(x)b(x)dx$. One must sum this along the length of the whole bar, resulting in the expression:

$$\Pi_{\text{load}} \quad = \quad -\int_0^L u(x)b(x)dx\,.$$

One has the following expressions for torsional distributed loads $t(z)$ and transverse distributed loads $q(x)$:

$$\Pi_{\text{load}} \quad = \quad \begin{cases} -\int_0^L t(z)\phi(z)\ dz & \text{Torsional distributed load} \\ -\int_0^L q(x)v(x)\ dx & \text{Transverse distributed load}\,. \end{cases}$$

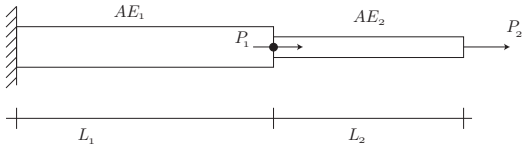
Exercises

(4.1) Rework Exercise 3.6 using the concept of stationary potential energy. You should get the same answer.

(4.2) Rework Exercise 3.7 using the concept of stationary potential energy. You should get the same answer.

(4.3) Consider a stepped bar subjected to two axial forces P_1 and P_2. Find the *exact* displacement field

for this system using a potential energy method.



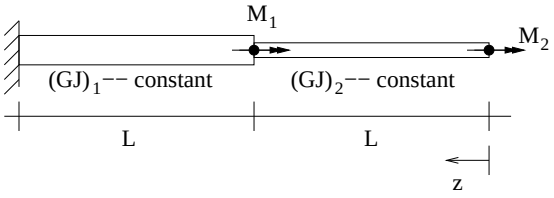
(4.4) Consider a stepped torsion rod with torsional rigidity $(GJ)_1$ for $0 \le z \le L$ and $(GJ)_2$ for $L \le z \le 2L$. The rod is built-in at $z = 0$ and $z = 2L$; it is subject to a point torque of magnitude T_a at $z = L$.

(a) Find the rotation at $z = L$ using conservation of energy.

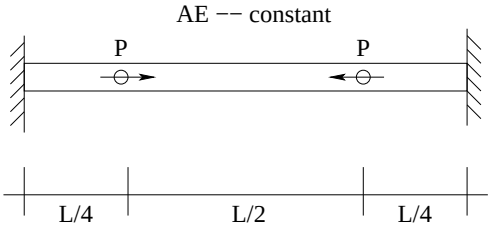
(b) Find the rotation at $z = L$ using the principle of stationary potential energy. Assume that the rotation field is piecewise linear.

You should get the same answer for both parts!

(4.5) Use the principle of stationary potential energy to find the rotations at the two load points for the stepped rod shown below.



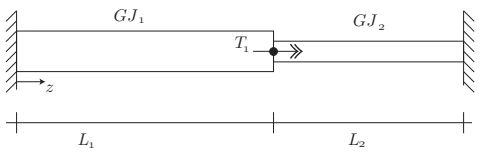
(4.6) Consider the elastic rod shown below.



(a) Using an approximate potential energy method find the displacement field of the rod; use the space of approximate solutions $\tilde{\mathcal{S}} = \{u(x) \mid u(x) = C\sin(2\pi x/L)\}$.

(b) Determine the strain at the center of the rod.

(4.7) Consider a stepped to torsion bar of length $L = L_1 + L_2$ subjected to a single torques T_1.



(a) Write an expression for the space of trial solutions, $\mathcal{S}$, for this problem.

(b) Approximate the space of solutions as

$$\tilde{\mathcal{S}} = \{\phi(z) \mid \phi(z) = Az(z - L), \quad A \in \mathbb{R}\}$$

and find an approximation to the rotation field. *In your final answer you may define symbols to represent integrals of polynomials. For example, if you find the integral* $\int_0^L (3z^2 + 4)^2\, dz$ *in your answer, you do not need to compute it. Merely define* $I = \int_0^L (3z^2+4)^2\, dz$ *and use* I *in your answer.*

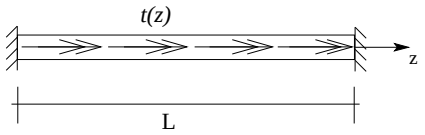
(4.8) Consider a cantilever beam with a uniform distributed load $q(x) = q_o$. Assume a deflection solution of the form $v(x) = Cx^2$ and determine an *approximate* solution by minimizing the potential energy. Compare the tip deflection to the exact solution.

(4.9) Consider a cantilever beam with a uniform distributed load $q(x) = q_o$. Assume a deflection solution of the form $v(x) = C(x/L)^2 + D(x/L)^3$ and determine an *approximate* solution by minimizing the potential energy. Compare the tip deflection to the exact solution.

(4.10) Consider a round elastic bar of length L with constant shear modulus, G, and polar moment of inertia, J. The bar is built-in at both ends and subject to a spatially varying distributed torsional load

$$t(z) = p\sin(\frac{2\pi}{L}z)\,,$$

where p is a constant with units of torque per unit length. Find an *approximate* expression for the rotation field using the principle of stationary potential energy using a one term approximation. Compare your result to the exact solution.



(4.11) Consider a pin-pin beam of length L with equal transverse loads of magnitude P in the positive and negative directions at $x = L/4$ and $x = 3L/4$, respectively. By approximately minimizing the potential energy of the system find the displacement field for the beam. Use a one term approximation. Compare your approximation to the exact answer.

(4.12) Carefully derive the matrix equations that would result from using the method of Ritz on an elastic tension-compression bar that is built-in at both ends in the presence of point forces and distributed loads.

(4.13) Carefully derive the matrix equations that would result from using the method of Ritz on an elastic tension-compression bar problem fixed at its right end and subject to a distributed force $b(x)$.

(4.14) Consider a round elastic bar of length L with constant shear modulus, G, and polar moment of inertia, J. The bar is built-in at both ends and subject to a spatially varying distributed torsional load

$$t(z) = p\frac{z}{L}\,,$$

where p is a constant with units of torque per unit length. Find an *approximate* expression for the the rotation field using the principle of stationary potential energy. Compare your result to the exact solution.

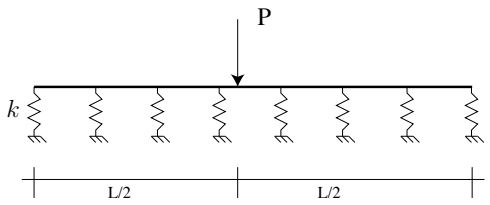
(4.15) Consider a doubly built-in beam of length L with a transverse load of magnitude P in the positive direction at $x = L/2$.

(a) By approximately minimizing the potential energy of the system find the displacement field for the beam: use a subspace with one degree of freedom.

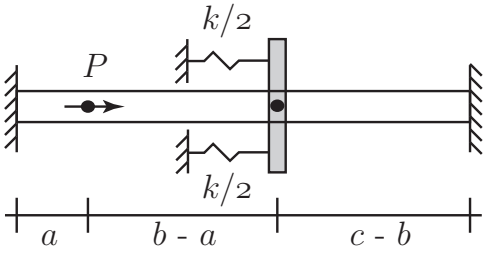
(b) Compare your approximation to the exact answer with an accurate plot of normalized non-dimensional deflection versus non-dimensional position. Make sure that you clearly label your axes, have a proper aspect ratio, legend, etc.

(c) Further assess the accuracy of your approximate solution by computing the relative displacement error at the middle of the beam (as a percentage).

(4.16) The system shown consists of a beam with constant bending rigidity EI and a Winkler foundation with foundation stiffness k whose dimensions are force per unit length per unit length (equivalently force per unit area). The beam is subjected to a dead load at mid-span. Write an expression for the total potential energy of the system in terms of the beam deflection $v(x)$ and any other needed system parameters.



(4.17) Consider a tension-compression bar with constant AE that is built in at both ends, subjected to a point force as shown, and connected to two linear springs via a rigid bar. Using a potential energy method, (1) determine the motion at the location of the load and at the location where the rigid bar is attached and (2) determine the strain in the bar near the right-hand support.



MATLAB Exercises

(4.18) Consider a linear elastic bar with cross-sectional properties $AE = 450\times10^6$ lbf and length 5 ft which is built-in at both ends. The bar is loaded with an axial point-force in its center of magnitude 450 kips. Solve for the displacement and strain fields in the bar using the method of Ritz and the basis functions $f_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$. How many terms in the expansion are required to reduce the relative L^2 error in the displacements to 1%? Use the exact answer for the error computation. Numerical quadrature is acceptable. [Extra: How many terms are needed to do the same for the strains? Why is it so hard to converge the strains?]

(4.19) Consider a cantilever beam (built-in at $x = 0$) of length $L = 2$ m carrying a uniform distributed load of 5000 N/m. The flexural rigidity of the beam is $EI(x) = (EI)_1 + (EI)_2(1 - \frac{x}{L})$, where $(EI)_1 = 30\times 10^6$ MN $\cdot$ mm^2 and $(EI)_2 = 60 \times 10^6$ MN $\cdot$ mm^2. Solve for the displacement field in the beam using

the method of Ritz and the basis functions

$$f_n(x) = (x/L)^{n+1} \text{ for } n = 1, 2, 3, \ldots$$

Make a plot of the converged shape of the beam. Make sure to properly label your graph.

(4.20) Consider a linear elastic bar with cross-sectional properties $AE = 330 \times 10^6$ lbf and length 4 ft which is built-in at both ends. The bar is loaded with a distributed force $b(x) = 100$ kips/ft. Solve for the displacement and strain fields in the bar using the method of Ritz and the basis functions $f_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$. How many terms in the expansion are required to reduce the relative L^2 error in the displacements to 1%? To compute the errors you can use an approximate quadrature, say something simple like Riemann sums. How many terms are needed to achieve the same with respect to the strains?

Buckling

5

In this chapter notes we will consider the problem of buckling instability. In particular, we will apply the concept of potential energy to examine this class of problems. We will start with some simple systems composed of rigid bars and springs and then will turn our attention to continuously deformable systems and approximate solutions.[1] It is of importance to recall:

(1) Both buckled and unbuckled solutions correspond to equilibrium states of a system.

(2) As the load is increased one approaches the critical point. At the critical point one has the emergence of multiple equilibrium solutions.

(3) An equilibrium solution that is stable has a positive second derivative of the potential energy.

(4) If one only wishes to determine the buckling load and perhaps some general information about the buckled shape, then one can simplify the analysis by using small deflection and rotation assumptions. This results in an eigenvalue type problem, where the eigenvalues correspond to the buckling loads and the eigenvectors to the buckled shapes.

[1] For a refresher on basic concepts of buckling see Sections 12.1 – 12.3 in S. Govindjee *Engineering Mechanics of Deformable Solids*, Oxford University Press, Oxford (2013).

5.1 Instability of discrete systems

Let us consider a simple mechanical system consisting of a rigid bar and a rotational spring as shown in Fig. 5.1. The bar is pinned at its base and further supported by a rotational spring with spring constant k. At the other end of the bar, it is subject to an axial (compressive) load P. Here we are interested in the equilibrium solutions of the system. Since these correspond to stationary points of the total potential energy, let us take the approach which employs the principle of stationary potential energy. The total potential energy for this system consists of the elastic potential energy and the potential energy of the load:

$$\begin{aligned} \Pi_{\text{elastic}} &= \frac{1}{2}k\theta^2, \\ \Pi_{\text{load}} &= -PL(1-\cos\theta)\,. \end{aligned}$$

Thus the total potential energy

$$\Pi_{\text{total}}(\theta) = \frac{1}{2}k\theta^2 - PL(1-\cos\theta)\,.$$

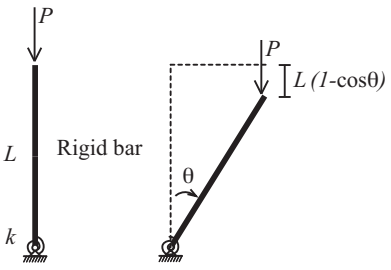


Fig. 5.1 Rigid bar supported by a tosionsal spring.

The stationary points of the potential energy are obtained from

$$\frac{d\Pi_{\text{total}}}{d\theta} \quad = \quad k\theta - PL\sin\theta = 0\,.$$

The θ's which satisfy this equation for a given P are the equilibrium solutions. Note the use of the plural *solutions*, as here as in opposition to all other problems we have treated up to now, the problem does not necessarily have a unique solution. In fact, there are two possible solutions:

(1) The trivial solution (where the bar remains straight-up-and-down)

$$\theta = 0\,, \tag{5.1}$$

true for any given load P.

(2) The non-trivial solution

$$P = \frac{k}{L}\frac{\theta}{\sin\theta}\,. \tag{5.2}$$

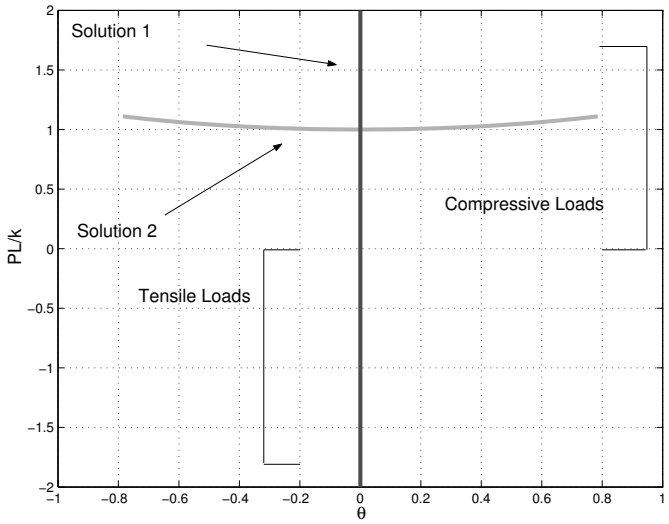


Fig. 5.2 Solutions θ for varying loads P. For loads $P > k/L$ there are multiple solutions.

A plot of the loci of points (θ, P) which satisfy either equilibrium solution is given in Fig. 5.2. The plot exhibits an interesting transition in behavior as the load approaches

$$P_{\text{cr}} \quad = \quad \frac{k}{L}$$

from below. This load is called the *critical load* and denotes the load past which the trivial solution $\theta = 0$ is no longer stable and no longer unique. For loads below P_{cr}, there is only the trivial solution. Above

this load there are three solutions: $\theta = 0$ and two solutions implicitly given by eqn (5.2).

To actually ascertain the stability of the different equilibrium solutions, we need to look at the second derivatives of the potential energy as these tell us if the forces acting on the system serve to restore equilibrium after a perturbation or serve to move the system even further from the equilibrium state. For our example:

$$\frac{d^2\Pi_{\text{total}}}{d\theta^2} = k - PL\cos\theta\,.$$

The stability of the solution is:

- On solution path 1:

$$\frac{d^2\Pi_{\text{total}}}{d\theta^2} = \begin{cases} k - PL > 0 & \text{for } P < \frac{k}{L} \\ k - PL < 0 & \text{for } P > \frac{k}{L} \end{cases}$$

- On solution path 2:

$$\begin{aligned} \frac{d\Pi_{\text{total}}}{d\theta^2} &= k - \left(\frac{k}{L}\frac{\theta}{\sin\theta}\right) L\cos\theta \\ &= k\,(1 - \theta\cot\theta) > 0 \qquad \forall\theta. \end{aligned}$$

Thus we see that solution (5.1) is the stable solution for all loads $P < P_{\text{cr}}$ but for loads $P > P_{\text{cr}}$ it is unstable (negative second derivative of the potential energy). We also see that for this case that solution (5.2) is stable.

Remarks:

(1) If one is only interested in determining the critical load and perhaps linearized information about the buckled state of the system, then one can make a small angle approximation. If we assume small values of θ, $|\theta| \ll 1$, then

$$1 - \cos\theta \approx 1 - \left[1 - \frac{\theta^2}{2}\right] = \frac{\theta^2}{2}\,,$$

such that,

$$\Pi_{\text{load}} = -PL\frac{\theta^2}{2}\,.$$

Therefore,

$$\Pi_{\text{total}} \approx \frac{1}{2}k\theta^2 - PL\frac{\theta^2}{2}\,.$$

Using this expression to compute the equilibrium equations gives:

$$\begin{aligned} & \frac{d\Pi_{\text{total}}}{d\theta} \approx k\theta - PL\theta = 0 \\ \Rightarrow\quad & \theta(k - PL) = 0 \\ \Rightarrow\quad & \theta = 0 \;\text{ or }\; P = \frac{k}{L} \end{aligned}$$

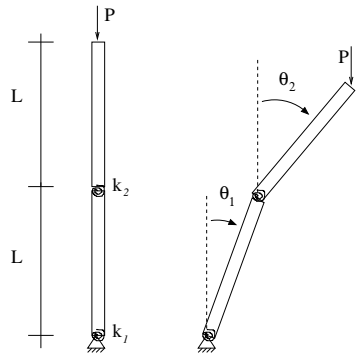


Fig. 5.3 2 rigid element and 2 rotational spring (2 DOF) system

The analysis reveals the critical load of the system $P_{\rm cr} = k/L$ but does not reveal too much about the buckled state. It says $\theta = 0$ and P arbitrary is an equilibrium solution and that $P = k/L$ and θ arbitrary is another solution. If we take a second derivative, we find

$$\frac{d^2\Pi_{\rm total}}{d\theta^2} \approx k - PL\,.$$

This shows that the trivial (unbuckled) configuration $\theta = 0$ is stable for $P < P_{\rm cr}$ and unstable above this load. It tells us that the second solution is neutrally stable near $\theta \approx 0$.

Example 5.1

Buckling of two rigid links connected by torsional springs

As an example of this type of analysis for a multi-degree of freedom system, consider the two degree-of-freedom system in Fig. 5.3. The system consists of two rigid bars connected by torsional springs. Each bar has length L and the springs have stiffness k_1 and k_2. We will use the angle from the vertical to parameterize the possible configurations of the system; see Fig. 5.3. The total potential energy of the system can be expressed as:

$$\Pi(\theta_1,\theta_2) \;=\; \frac{1}{2}k_1\theta_1^2 + \frac{1}{2}k_2(\theta_2-\theta_1)^2 - P\left[L\frac{\theta_1^2}{2} + L\frac{\theta_2^2}{2}\right].$$

Here we have already employed the assumption of $|\theta_1| \ll 1$ and $|\theta_2| \ll 1$ to simplify the expression for the potential energy of the load. Note that this will restrict the information which we can extract from the analysis. Notwithstanding, we will be able to determine the critical load and we will be able to make some assessment of the buckled state of the system.

For equilibrium, the potential energy must be stationary and this implies

$$\begin{aligned}\frac{\partial\Pi}{\partial\theta_1} &= 0,\\ \frac{\partial\Pi}{\partial\theta_2} &= 0\,.\end{aligned}$$

This leads to a linear system of equations consisting of two equations in two unknowns:

$$\begin{bmatrix} k_1+k_2-PL & -k_2 \\ -k_2 & k_2-PL \end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

This equation is similar to the equation $(k-PL)\theta = 0$ in the one degree of freedom case. The combinations of (θ_1,θ_2) which satisfy this (homogeneous) linear system of equations for a given load P are the solutions which satisfy equilibrium. One has two different cases of solutions:

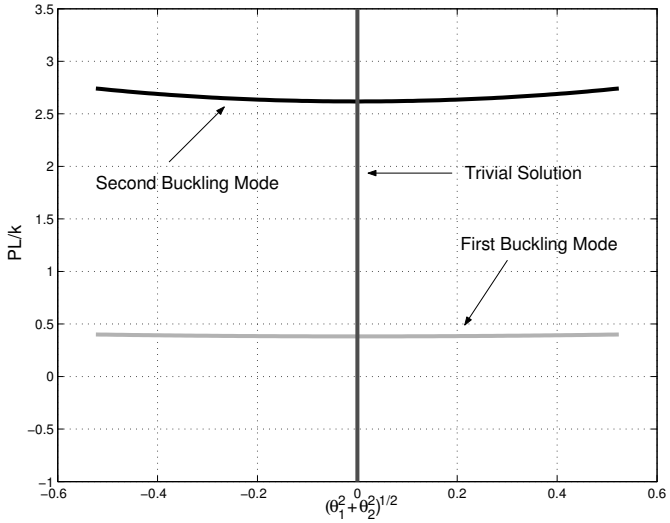


Fig. 5.4 Solution branches for varying loads P in a two degree of freedom system. Numerical values are for the full non-linear case with $k_1 = k_2$. The trivial solution is stable below the first critical load.

(1) $\boldsymbol{\theta} = \mathbf{0}$, which gives the trivial (straight-up-and-down) solution. This solution is valid for all loads P – even if it is possibly an unstable equilibrium.

(2) A non-trivial solution, $\boldsymbol{\theta}$, is only possible when $\det[\square] = 0$. This will only occur for specific values of P assuming that the bar lengths and the springs are fixed. These values are the eigenvalues associated with the generalized eigenvalue problem:

$$\left(\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - P \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \right) \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

For our two degree of freedom problem there are two such load values $P = P_1, P_2$ for which non-trivial solutions exist, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. These loads are called the *buckling loads* and the smallest buckling load is called the *critical load*:

$$P_{\text{cr}} := \min(P_1, P_2) .$$

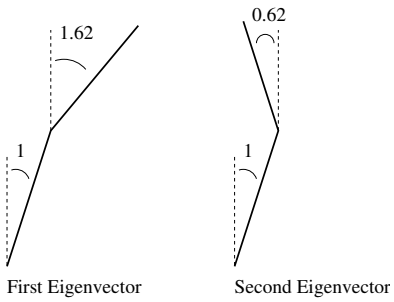


Fig. 5.5 Mode shapes (eigenvectors) for the two degree of freedom system. Numerical vaules are for the case of $k_1 = k_2$.

Figure 5.4 displays the (full non-linear) solutions using the norm of $\boldsymbol{\theta}$ as the abscissa. For loads $P < P_{\rm cr}$, the trivial (straight) solution is stable. Above this load the trivial solution will be unstable. The two buckling solutions are depicted by the two sets of pitchfork arms, the lowest of which is typically a stable equilibrium branch of the solution. Note that our linearized solution does not provide equations for the solution away from the small angle approximation. To obtain the details of the curves shown, one needs to perform a full non-linear analysis. What one can determine from the linear solutions is the general configuration into which the system buckles on each branch. This information is contained in the eigenvectors corresponding to the two buckling loads P_1, P_2. Figure 5.5 shows the eigenvectors and the ratios of the eigenvector components.

5.2 Instability of continuous systems

Before investigating the stability of continuous mechanical systems via potential energy let us review some classical results for beam buckling that are typically derived using ordinary differential equation methods. The classic results are associated with four basic sets of beam boundary conditions. These are:

- Clamped-Free,
- Pin-Pin,
- Clamped-Pin, and
- Clamped-Clamped

under the action of a compressive axial load as shown in Fig. 5.6.

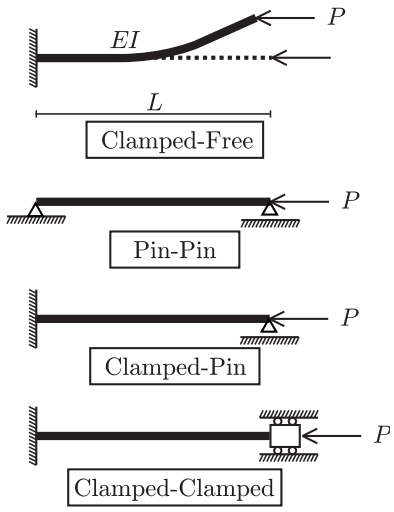


Fig. 5.6 The four classical buckling configurations. The pin-pin case gives rise to the so-called Euler buckling load.

For all cases, the governing differential equation of equilibrium is:

$$EIv'''' + Pv'' = 0\,. \tag{5.3}$$

This equation has two classes of solutions:

(1) $v(x) = 0$, which is the trivial (unbuckled) solution which exists for any load P.

(2) $v(x) \neq 0$, which is the non-trivial solution which exists only for certain loads P.

One is faced with following questions:

- When can one have a non-trivial solution $v(x)$?
- What are the loads P for a non-trivial solution?

The loads P corresponding to non-trivial solutions can be found by solving eqn (5.3) subject to the given boundary conditions of the problem. An outline of the solution method is as follows:

(1) Assume $v(x) = Ce^{sx}$, and insert this into eqn (5.3) to yield the characteristic equation

$$C(EIs^2 + P)s^2 = 0\,.$$

The four roots of this equation are $s = 0, 0, \pm i\sqrt{\dfrac{P}{EI}}$, where $i = \sqrt{-1}$. This yields the following (homogeneous) solution:

$$v(x) = c_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + c_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) + c_3 x + c_4\,.$$

(2) Applying the boundary conditions (there are four) generates four equations for the four unknowns c_1, c_2, c_3, and c_4. These equations have the form

$$\begin{aligned} \boldsymbol{M}(P)\boldsymbol{c} &= \boldsymbol{0}\,, \\ \boldsymbol{c} &:= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}\,, \end{aligned}$$

where $\boldsymbol{M}(P)$ is a 4-by-4 matrix depending on P.

(3) A non-trivial solution, $\boldsymbol{c} \neq \boldsymbol{0}$, is only possible when $\det[\boldsymbol{M}(P)] = 0$. This condition yields a transcendental equation in P. By solving for the roots of this relation, one obtains the buckling loads. In this setting there are an infinite number of buckling loads; the smallest of these is the critical load for the system.

For the four representative cases the results are:

- Pin-Pin (Euler load):

$$P_{\text{cr}} = \pi^2 \frac{EI}{L^2}$$

- Clamped-Pin:

$$P_{\text{cr}} = (4.497)^2 \frac{EI}{L^2}$$

- Clamped-Clamped:

$$P_{\text{cr}} = (2\pi)^2 \frac{EI}{L^2}$$

- Clamped-Free:

$$P_{\text{cr}} = (\pi/2)^2 \frac{EI}{L^2}$$

5.2.1 Beam buckling using potential energy

The instability of continuous mechanical systems, e.g., beams, can be treated in a manner similar to the discrete mechanical systems consisting of rigid bars and springs. One must determine the elastic potential energy and the potential energy of the load. The elastic potential energy is as before:

$$\Pi_{\text{elastic}} := \int_0^L \frac{1}{2} EI(v'')^2 \, dx \, .$$

For the potential of the load, we have

$$\begin{aligned} \Pi_{\text{load}} &:= -P \cdot \text{ displacement at load point} \\ &= -P \int_0^L \frac{1}{2}\theta^2 \, dx, \\ &= -P \int_0^L \frac{1}{2}(v')^2 \, dx \, . \end{aligned}$$

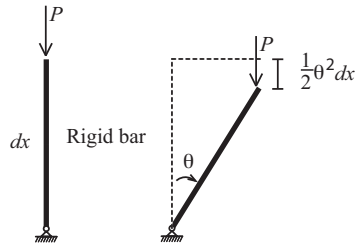


Fig. 5.7 Displacement drop of an infinitesimal element.

The way that the displacement at the load point is derived is through the following argument. Consider an infinitesimal segment of the beam of length dx. This segment rotates (during buckling) by an angle $\theta = v'$. The resulting vertical drop associated with the segment is equal to $-\frac{1}{2}\theta^2 \, dx$; see Fig. 5.7. For the total contribution from each segment of the beam, one must add up the contributions from each segment; i.e., one must integrate the expression over the interval $[0, L]$.

The total potential energy is defined as the sum of these two contributions:

$$\begin{aligned} \Pi_{\text{total}}(v(x)) &= \Pi_{\text{elastic}} + \Pi_{\text{load}} \\ &= \int_0^L \frac{1}{2} EI(v'')^2 \, dx - P \int_0^L \frac{1}{2}(v')^2 \, dx \, . \end{aligned}$$

To approximately find the loads at which instability occur (buckling loads) and to estimate the buckling modes, one can apply the method of Ritz. As was done previously, we will assume an approximate form for the solution $v(x)$ which satisfies the kinematic boundary conditions and contains free parameters. These will then be determined by finding the stationary conditions for the potential energy.

Example 5.2

One parameter approximation for clamped-free beam buckling

Consider the clamped-free beam subjected to an axial compression as shown in Fig. 5.6. Find an approximation to the buckling load.

Solution: Consider as a simple approximation for $v(x)$ the form

$$v(x) \approx Cx^2,$$

where C is an undetermined coefficient. To insert into the potential energy expression, we need the derivatives

$$\begin{aligned} v'(x) &= 2Cx, \\ v''(x) &= 2C\,. \end{aligned}$$

Inserting these into the expression for the total potential energy gives

$$\begin{aligned} \Pi_{\text{total}}(v(x)) \xrightarrow{v(x)=Cx^2} \Pi_{\text{total}}(C) &= \int_0^L \frac{1}{2} EI 4C^2 \, dx - P \int_0^L \frac{1}{2} 4C^2 x^2 \, dx \\ \frac{d\Pi_{\text{total}}}{dC} &= 4EICL - P4C \cdot \frac{1}{3} L^3 \\ &= C\left[4EIL - P\frac{4}{3}L^3\right] = 0\,. \end{aligned}$$

This yields two solutions:

(1) $C = 0$, which is the trivial solution with $v(x) = 0$ corresponding to any load P.

(2) $C \neq 0$, which is the non-trivial solution corresponding to a load of $P = \frac{3EI}{L^2}$. This value for the load is the critical buckling load P_{cr} and is the load at which instability occurs.

The exact critical buckling load for this configuration is, $P_{\text{cr}}^{\text{exact}} = \dfrac{\pi^2}{4}\dfrac{EI}{L^2}$. The relative error in the approximation is:

$$\text{relerr} = \left|\frac{\pi^2/4 - 3}{\pi^2/4}\right| = \left|1 - \frac{12}{\pi^2}\right| = 21\%\,.$$

Remarks:

(1) The approximation here is not too good. This is a reflection of an overly simplistic guess to the buckling shape. Notwithstanding, the analysis certainly gives a reasonable order of magnitude approximation to the buckling load.

Example 5.3

Two parameter approximation for clamped-free buckling

An advantage of the Ritz method is the ability to improve the approximate solution by expanding the possible solutions in a systematic fashion. Consider the two parameter approximation

$$v(x) \approx C\left(\frac{x}{L}\right)^2 + D\left(\frac{x}{L}\right)^3\,.$$

It is advantageous numerically and in terms of understanding the phenomenon to consider non-dimensionalized forms for the functions in the Ritz method, e.g., in this case the selection of $\left(\frac{x}{L}\right)^2$ instead of x^2. Doing this, the coefficients C, D have units of length and are often called the generalized displacements. The needed derivatives of the approximation are

$$\begin{aligned} v'(x) &= \frac{2C}{L}\left(\frac{x}{L}\right) + \frac{3D}{L}\left(\frac{x}{L}\right)^2, \\ v''(x) &= \frac{2C}{L^2} + \frac{6D}{L^2}\left(\frac{x}{L}\right). \end{aligned}$$

The total potential energy yields:

$$\begin{aligned} \Pi_{\text{total}}(v(x)) \to \Pi_{\text{total}}(C,D) &= \int_0^L \frac{1}{2} EI \left[\frac{2C}{L^2} + \frac{6D}{L^2}\left(\frac{x}{L}\right)\right]^2 dx \\ &- P\int_0^L \frac{1}{2}\left[\frac{2C}{L}\left(\frac{x}{L}\right) + \frac{3D}{L}\left(\frac{x}{L}\right)^2\right]^2 dx\,. \end{aligned}$$

The stationary conditions for Π_{total} give the equilibrium equations for the system:

$$\left.\begin{array}{l} \dfrac{\partial \Pi_{\text{total}}}{\partial C} = 0 \\ \dfrac{\partial \Pi_{\text{total}}}{\partial D} = 0 \end{array}\right\} \Rightarrow \begin{bmatrix} \frac{4EI}{L^3} - \frac{P}{L}\frac{4}{3} & \frac{6EI}{L^3} - \frac{P}{L}\frac{3}{2} \\ \frac{6EI}{L^3} - \frac{P}{L}\frac{3}{2} & \frac{12EI}{L^3} - \frac{P}{L}\frac{9}{5} \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5.4)$$

There are two possible ways to satisfy these homogeneous equations:

(1) Take $C = D = 0$, which is the straight trivial solution with $v(x) = 0$ and is valid for any load P.

(2) If however $\det[\square] = 0$, then C, D can take on non-trivial values and still satisfy eqn (5.4). This condition will hold true only for certain values of P. The minimum of these values is the critical load.

The buckling loads and the corresponding non-trivial solutions $[C, D]^T$ can be found by rewriting eqn (5.4) in the form:

$$\left(\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} - \frac{PL^2}{EI}\begin{bmatrix} 4/3 & 3/2 \\ 3/2 & 9/5 \end{bmatrix}\right)\begin{bmatrix} C \\ D \end{bmatrix} = \mathbf{0}\,.$$

By defining

$$\begin{aligned} \boldsymbol{A} &:= \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}, \\ \boldsymbol{B} &:= \begin{bmatrix} 4/3 & 3/2 \\ 3/2 & 9/5 \end{bmatrix}, \\ \lambda &:= \frac{PL^2}{EI}, \\ \boldsymbol{x} &:= \begin{bmatrix} C \\ D \end{bmatrix}, \end{aligned}$$

these equations have the form:

$$(\boldsymbol{A} - \lambda \boldsymbol{B})\, \boldsymbol{x} = \boldsymbol{0},$$

which is known as a *Generalized Eigenvalue Problem*. Note that in the case that $\boldsymbol{B}$ is the identity matrix, one has $\boldsymbol{Ax} = \lambda \boldsymbol{x}$ which is the standard eigenvalue problem.[2] The eigenvalues are computed by setting the determinant of $\boldsymbol{A} - \lambda \boldsymbol{B}$ zero which results in a quadratic polynomial for λ. The two solutions are:

$$\lambda = \frac{PL^2}{EI} = 2.486 \quad \text{and} \quad 32.1807\,.$$

Since the smallest buckling load is the critical load:

$$P_{\text{cr}} \;=\; 2.486 \frac{EI}{L^2}\,.$$

Remarks:

(1) With respect to the exact solution, this has a relative error of 0.75%.

(2) If one computes the associated eigenvector $[C, D]^T$, then one can plot the basic shape of the buckling mode as $v(x) = C(x/L)^2 + D(x/L)^3$. Doing so results in $[C, D]^T = [-1.3, 0.4]^T$, and the corresponding buckling mode(shape) is shown in Fig. 5.8.

[2] This generalized eigenvalue problem can be efficiently solved in MATLAB by the function call `eig(A,B)`.

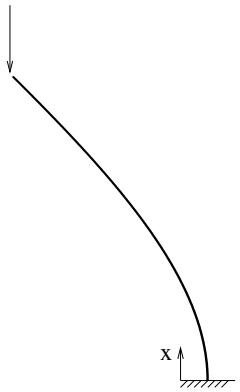


Fig. 5.8 Approximate two parameter buckling mode solution to the clamped-free bucking problem.

Example 5.4

Buckling of a cantilever supported at mid-span by a spring support

Consider computing the buckling load of a cantilever with a lateral spring support placed at mid-span as shown in Fig. 5.9. The calculation involves only a slight modification to the solution of Example 5.3. One only has to account for the additional contribution of the elastic potential energy associated with the spring. This is simply added to the total potential energy as shown in the following formulation:

$$\begin{aligned} \Pi_{\text{total}} &= \Pi_{\text{elastic,beam}} + \Pi_{\text{elastic,spring}} + \Pi_{\text{load}}\,, \\ \Pi_{\text{elastic,spring}} &:= \frac{1}{2} k \left[v\left(\frac{L}{2}\right) \right]^2 . \end{aligned}$$

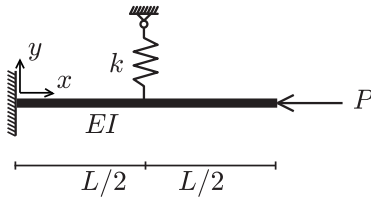


Fig. 5.9 Cantilever supported by a spring at mid-span.

Using the approximation

$$v(x) = C\left(\frac{x}{L}\right)^2 + D\left(\frac{x}{L}\right)^3 ,$$

one has

$$\Pi_{\text{elastic,spring}} \;=\; \frac{1}{2} k \left[C\left(\frac{1}{2}\right)^2 + D\left(\frac{1}{2}\right)^3 \right]^2 .$$

Partial derivatives with respect to C and D of this added term yield new contributions to the equilibrium equations in terms of k.

$$\left(\underline{\underline{\frac{kL^3}{64EI} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}}} + \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} - \frac{PL^2}{EI} \begin{bmatrix} 4/3 & 3/2 \\ 3/2 & 9/5 \end{bmatrix} \right) \begin{bmatrix} C \\ D \end{bmatrix} = \mathbf{0} \,.$$

Taking the determinant of the coefficient matrix and setting it equal to zero yields a polynomial for the eigenvalues – the smallest of which is the critical load. The corresponding eigenvector then defines the general buckled shape of the system.

5.2.2 Buckling with distributed loads

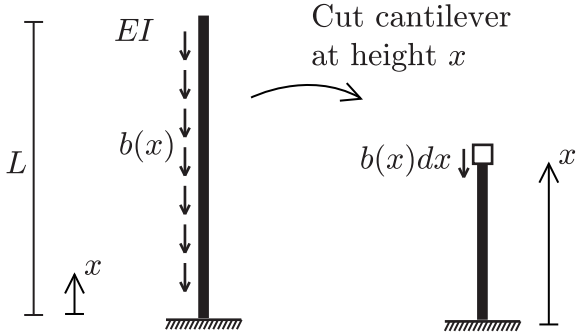


Fig. 5.10 Cantilever beam with a distributed axial load.

Consider a cantilever beam subjected to a distributed load as shown in Fig. 5.10. The total potential energy for this system is given as,

$$\Pi_{\text{total}} := \int_0^L \frac{1}{2} EI(v'')^2 \, dx - \int_0^L \left[\{b(x)dx\} \left\{ \int_0^x \frac{1}{2}(v'(\bar{x}))^2 \, d\bar{x} \right\} \right] ,$$

where the first integral is the elastic potential energy and the second integral is the potential of the load. The expression for the potential of the load appears rather complex but can be justified if examined closely. The first component under the outer integral represents the load for a small segment of material near x and the second component represents the displacement drop at point x. The expression for this displacement drop is necessarily an integral as was required in the earlier examples. Its limits are 0 to x to account for all rotation effects below the point in question. Recall that for a cantilever of length L with end load P, the contribution to the potential energy due to the load is

$$-P \int_0^L \frac{1}{2}(v')^2 \, dx \,.$$

What has been done is a simple replacement of L with x and P with $b(x)dx$ and the summation of this contribution from 0 to L.

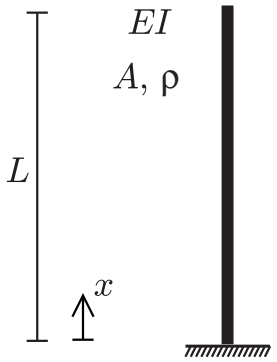


Fig. 5.11 Collapse of a column under self-weight – the tree problem.

Example 5.5

Buckling behavior under a uniform distributed load

Let us consider the problem of a tall tree and determine how tall it can grow before it will collapse due to buckling. For this situation the loading is a distributed load which is constant $b(x) = A\rho g = \gamma$, where A is the cross-sectional area of the tree (assumed constant), ρ is the mass density, and g is the gravitational constant. Thus γ is simply the weight per unit length of the tree. To solve, let us determine the buckling condition for the system in terms of γ. We will then invert this relation at the end to find a restriction of the height of the tree in terms of γ.

For this case the potential energy takes the form,

$$\Pi_{\text{total}} = \int_0^L \frac{1}{2} EI(v'')^2 \, dx - \gamma \int_0^L \left\{ \int_0^x \frac{1}{2}(v'(\bar{x}))^2 \, d\bar{x} \right\} \, dx.$$

To approximately solve this problem, consider a one-parameter approximation for the solution:

$$v(x) \approx Cf(x)\,,$$

where $f(x)$ is a known function (we will specify it later) and C is the undetermined coefficient. Under this assumed form,

$$\Pi_{\text{total}}(v) \Rightarrow \Pi_{\text{total}}(C) = \int_0^L \frac{1}{2} EI(Cf'')^2 \, dx - \gamma \int_0^L \left\{ \int_0^x \frac{1}{2}(Cf'(\bar{x}))^2 \, d\bar{x} \right\} \, dx\,.$$

The stationary condition implies

$$\frac{d\Pi_{\text{total}}(C)}{dC} = C \left[\int_0^L EI(f'')^2 \, dx - \gamma \int_0^L \left\{ \int_0^x (f'(\bar{x}))^2 \, d\bar{x} \right\} \, dx \right] = 0.$$

For a non-trivial solution $C \neq 0$ one must have

$$\gamma = \frac{\int_0^L EI(f'')^2 \, dx}{\int_0^L \left\{ \int_0^x (f'(\bar{x}))^2 \, d\bar{x} \right\} \, dx}\,.$$

Let us now select the specific Ritz function $f(x) = x^2$. With this approximation,

$$\begin{aligned} \gamma &= \frac{\int_0^L EI(2)^2 \, dx}{\int_0^L \left\{ \int_0^x (2\bar{x})^2 \, d\bar{x} \right\} \, dx} \\ &= \frac{4EIL}{\int_0^L \left\{ \frac{4}{3}x^3 \right\} \, dx} \\ &= \frac{4EIL}{\frac{1}{3}L^4} \\ &= \frac{12EI}{L^3}\,. \end{aligned}$$

For $\gamma < 12EI/L^3$ there will be no buckling and thus the height restriction is

$$L^3 < \frac{12EI}{\gamma} = \frac{12EI}{\rho g A} = \frac{12Er^2}{\rho g}\,,$$

where for the last equality we have introduced the radius of gyration $r^2 = I/A$ of the cross-section.

Remarks:

(1) From the result, we see that denser trees are necessarily shorter. Likewise a tree can grow taller, if it increases its radius of gyration. The dependency of height to radius is a power law relation with exponent 2/3.

(2) We can assess the accuracy of our computation since there is a known reference solution. Up to 4 digits this is[3]

$$\gamma_{\text{exact}} = 7.837\frac{EI}{L^3}\,.$$

Comparing, we find that the relative error for our simple approximation is $\left|\dfrac{12 - 7.837}{7.837}\right| \approx 53\%$, which is not too good. Adding additional polynomial terms would greatly improve the accuracy.

(3) As an alternative choice one can pick $f(x) = 1 - \cos\left(\frac{\pi}{2}\frac{x}{L}\right)$. This yields

$$\gamma = 8.2979\frac{EI}{L^3},$$

which gives a relative error of approximately only 6%.

(4) Every tree you see must know this relation!

[3]This solution can be computed from the governing ordinary differential equation. However, it requires knowledge of Bessel functions of fractional order.

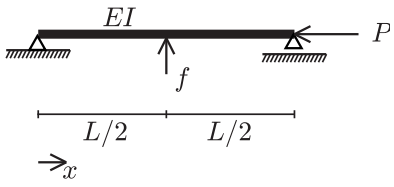


Fig. 5.12 Simply supported beam with an intermediate transverse load and a compressive axial force.

5.2.3 Deflection behavior for beams with compressive axial loads and transverse loads

Consider a simply supported beam with a fixed load f applied at midspan as shown in Fig. 5.12. Additionally, the beam is subjected to a compressive axial load P. In such situations, the compressive load sets up a situation where the beam deflection has a buckling-like behavior. We can gain an appreciation for the situation by looking at an approximate solution to the system's response. The total potential energy for this mechanical system is

$$\Pi_{\text{total}} = \int_0^L \frac{1}{2}EI(v'')^2\,dx - P\int_0^L \frac{1}{2}(v')^2\,dx - fv\left(\frac{L}{2}\right)\,.$$

If $f = 0$, we are looking at a classical buckling problem; viz., the beam remains straight until a critical load is reached after which the beam

bends suddenly. The critical load for the configuration shown is $P_{\rm cr} = \pi^2 EI/L^2$. Let us investigate the behavior for $f \neq 0$.

As we have had previously, the stationary conditions of the potential energy produce the equilibrium equations for the system. Let us compute an approximate solution using the form

$$v(x) \approx C \sin\left(\pi \frac{x}{L}\right) .$$

The derivatives of this function are

$$\begin{aligned} v'(x) &= C\frac{\pi}{L}\cos\left(\pi\frac{x}{L}\right) , \\ v''(x) &= -C\left(\frac{\pi}{L}\right)^2 \sin\left(\pi\frac{x}{L}\right) . \end{aligned}$$

Inserting these into the potential energy yields

$$\begin{aligned} \Pi_{\rm total} &= \int_0^L \frac{1}{2}EI\left(\frac{\pi}{L}\right)^4 C^2 \sin^2\left(\pi\frac{x}{L}\right)\, dx \\ &\quad -P\int_0^L \frac{1}{2}\left(\frac{\pi}{L}\right)^2 C^2\cos^2\left(\pi\frac{x}{L}\right)\, dx - fC\sin\left(\pi\frac{L/2}{L}\right) \\ &= \int_0^L \frac{1}{2}EI\left(\frac{\pi}{L}\right)^4 C^2\left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right\}\, dx \\ &\quad -P\int_0^L \frac{1}{2}\left(\frac{\pi}{L}\right)^2 C^2\left\{\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right\}\, dx - fC \\ &= \frac{1}{4}EI\left(\frac{\pi}{L}\right)^4 C^2 L - P\frac{1}{4}\left(\frac{\pi}{L}\right)^2 C^2 L - fC\,. \end{aligned}$$

The stationary condition yields

$$\begin{aligned} 0 = \frac{d\Pi_{\rm total}}{dC} &= \frac{1}{2}EI\left(\frac{\pi}{L}\right)^4 CL - P\frac{1}{2}\left(\frac{\pi}{L}\right)^2 CL - f \\ &= C\left\{\frac{1}{2}EI\left(\frac{\pi}{L}\right)^4 L - P\frac{1}{2}\left(\frac{\pi}{L}\right)^2 L\right\} - f = 0 \end{aligned}$$

and thus

$$\begin{aligned} C &= \frac{f}{\dfrac{EI\pi^4}{2L^3} - P\dfrac{\pi^2}{2L}} \\ &= \frac{f\ 2L/\pi^2}{\dfrac{EI\pi^2}{L^2} - P} \\ &= \frac{2L}{\pi^2}\frac{f}{P_{\rm cr} - P}\,. \end{aligned}$$

The approximate solution has the form

$$v(x) \approx \frac{2L}{\pi^2}\frac{f}{P_{\rm cr}-P}\sin\left(\pi\frac{x}{L}\right) .$$

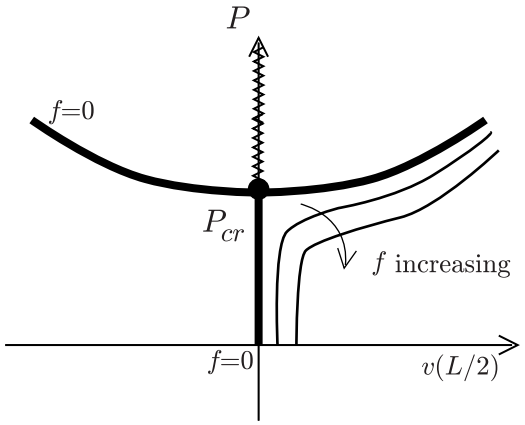


Fig. 5.13 Relationship between mid-span displacement $v(L/2)$ and the compressive load P.

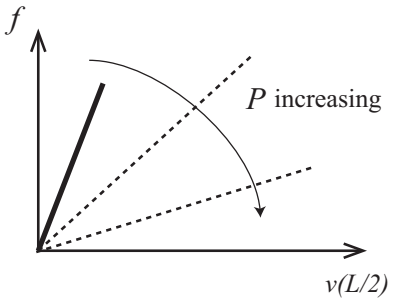


Fig. 5.14 Relationship between mid-span displacement $v(L/2)$ and the transverse load f. The slope of the response curve $\frac{\pi^2}{2L}(P_{\rm cr}-P) \to 0$ (the system becomes softer) as $P \to P_{\rm cr}$.

The relationship between P and $v(L/2)$ for a fixed f is shown in Fig. 5.13. The relationship between f and $v(L/2)$ for a fixed P is shown in Fig. 5.14.

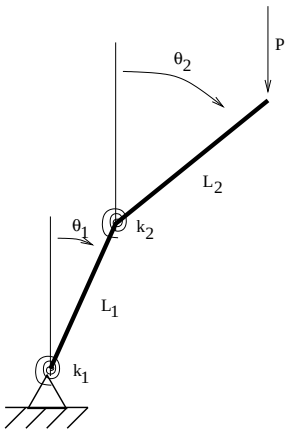
Remarks:

(1) Observe how the response curve in the $v(L/2)-P$ plane converges to the case for $f = 0$ as f decreases; see Fig. 5.13. The behavior is buckling-like for this system.

(2) Note that for small axial loads, the system behaves like a standard beam in bending. However, as the axial load approaches the Euler load, the deflection is greatly amplified by the factor $1/(P_{\rm cr} - P)$; see Fig. 5.14.

Exercises

(5.1) For the two-degree-of-freedom system made up of rigid bars and torsional springs, as shown,



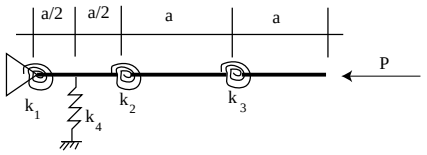
(a) Find the total potential energy Π without making a small angle approximation.

(b) Reduce your expression in part (a) to one appropriate for small angles.

(5.2) The linkage shown below is made of three rigid bars, three torsional springs, and one extensional spring.

(a) Set up the potential energy expression for the system assuming small motions.

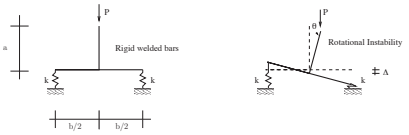
(b) Find the governing system of equilibrium equations that one would have to solve in order to determine the critical load; write your answer in matrix form and indicate in words the remaining steps that would be needed to solve the problem. **Do not solve the equations.**



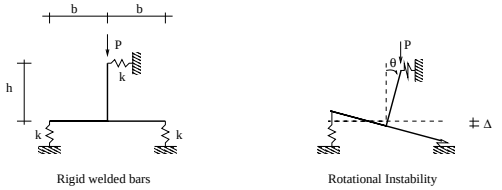
(5.3) As the load P is increased on the structure shown, the rigid inverted-T will displace uniformly downwards. At a certain load P the structure will experience a rotational instability.

(a) Assuming small motions, write an expressions for the system's potential energy. (Assume that horizontal translation of the point where the bars meet is not permitted.)

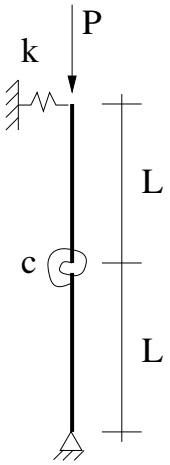
(b) Find the equilibrium equations and determine the critical load.



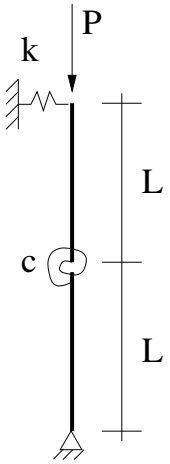
(5.4) Shown is an inverted-T made of rigid welded bars. It is supported by two vertical springs (which can only provide vertical forces) and a horizontal spring (which can only provide a horizontal force). When a load P is applied to the inverted-T, it displaces uniformly downwards without rotation. However, as the load in increased, a critical value is reached and the inverted-T displays a sudden rotational instability (as shown). Determine this critical load.



(5.5) Shown below is a structure that may be idealized as being composed of two rigid links of length $L = 2$ (m) that are joined by a torsional spring with spring constant $c = 2$ (kN-m/rad). The top of the structure is supported by a flexible support with spring constant $k = 1$ (kN/m). Determine the critical load of the structure and sketch the deflected shape just after collapse.



(5.6) Shown below is a structure that may be idealized as being composed of two rigid links of length L that are joined by a torsional spring with spring constant c. The top of the structure is supported by a flexible support with spring constant k.



(a) Find the total potential energy Π without making a small angle approximation.

(b) Reduce your expression in part (a) to one appropriate for small angles.

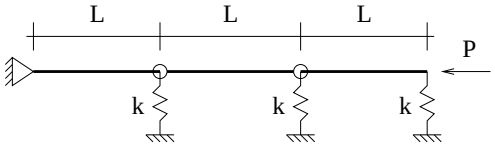
(c) Determine the critical load using your expression from part (b) and *accurately sketch/plot* the deflected shape just after collapse. As degrees of freedom, choose the rotations of the bars with respect to the vertical.

(5.7) Consider the three (rigid) bar system shown below where $k = 100$ kN/m and $L = 0.3$ m.

(a) Find the three buckling loads and their associated buckling modes/shapes. *Accurately sketch/plot* the buckling modes. [Hint: Do not do this entirely by hand.]

(b) Which of the three is the critical mode shape?

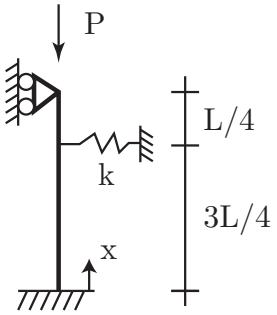
(c) If the spring constant nearest the support is quadrupled in value, what is the new critical load and mode shape? *Accurately sketch/plot* the critical mode.



(5.8) Shown below is a linear elastic beam with constant bending rigidity EI. Assume an approximate solution space of the form

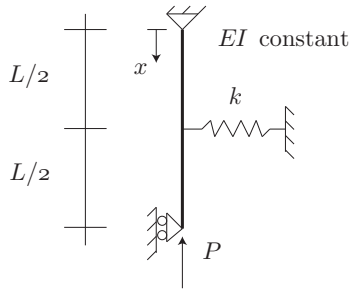
$$\tilde{\mathcal{S}} = \{v(x) \mid v(x) = A(x^3 - x^2 L)\},$$

where A is a parameter and find an expression for the buckling load.



(5.9) The determine the buckling load for the pin-pin beam shown below with lateral spring support of stiffness k. Approximate the space of solutions as

$$\tilde{\mathcal{S}} = \{v(x) \mid v(x) = A\sin(\pi x/L), \quad A \in \mathbb{R}\}$$



(5.10) Consider a column with length $L = 1$ m and a 1×1 cm^2 square cross-section. The column has pin and pin-roller supports at $x = 0$ and $x = L$, respectively. Further, it is supported at its midspan by a linear spring with spring constant $k = 0.5$ N/mm. The column is subjected to an axial compressive force P at the pin-roller support. Find the critical load using an approximate potential energy method with a single parameter. Assume $E = 200$ kN/mm^2.

(5.11) Consider the system in Exercise 5.10 except that the axial compressive load is now applied at $x = 0.75$ m instead of at $x = 1$ m. Find the critical load using an approximate potential energy method with a single parameter.

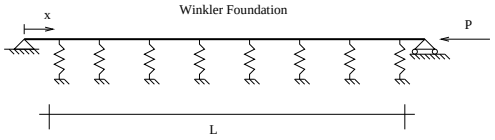
(5.12) Consider a column with length $L = 1.5$ m and a 1.2×1.2 cm^2 square cross-section. The column is pinned at $x = 0$ and is supported by a pin-roller at $x = L$ – i.e. it is simply supported. Further, it is supported at $x = L/4$ by a linear spring with spring constant $k = 0.5$ N/mm. The column is subjected to an axial compressive force P at the pin-roller support. Estimate the critical load using an approximate potential energy method with a single parameter. Assume $E = 200$ kN/mm^2.

(5.13) Consider the system in Exercise 5.12 except that the axial compressive load is now applied at $x = 3L/4$ instead of at $x = L$. Find the critical load using an approximate potential energy method with a single parameter.

(5.14) Find the paper `http://dx.doi.org/10.1098/rspa.2010.0505` and go to the 'Data Supplement' under the 'Figures & Data' tab. Watch the video provided. Then, set up the equations for the tensile buckling problem with rigid bars that is described and solve for the critical 'buckling' load.

(5.15) Consider a beam supported by a Winkler foundation. The beam is 100 ft long with a Young's modulus of $E = 30 \times 10^6$ psi and a cross-sectional area moment of inertia $I = 77.4$ in^4. Assume

a (continuously distributed) foundation stiffness $k = 100$ lb/in^2 and find the axial buckling load (with small deformation assumptions) and buckling mode. To solve this problem assume an approximation of the form $v(x) = \sum_{i=1}^{n} c_i f_i(x)$ where $f_i = \sin(\frac{\pi i}{L} x)$.



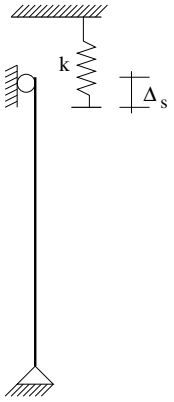
(5.16) The spring shown below is Δ_s longer that the gap above the beam. It is compressed and then slid into place above the beam.

(a) Explain why the potential energy of the spring is given by

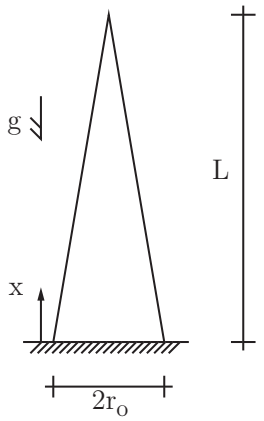
$$\Pi_{\text{spring}} = +\frac{1}{2}k\left[\Delta_s - \int_0^L \frac{1}{2}(v')^2\,dx\right]^2 .$$

(b) Assume an approximate beam deflection of the form $v(x) \approx A\sin(\pi x/L)$ and determine the response of the system. In particular, find the combination of system parameters that will lead to buckling of the beam.

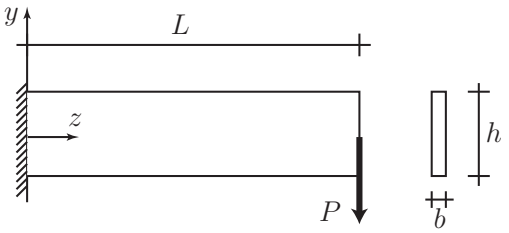
Assume the beam has bending stiffness EI and length L.



(5.17) Consider an Aluminum, vertical, tapered circular column of height L, radius $r(x) = r_o - r_o x/L$, density ρ, clamped at the base and free at the top. How tall can the column be before it buckles due to self-weight? For properties, let $E = 11$ GPa, $r_o = 0.3$ m, $\rho = 2700$ kg/m^3, and $g = 9.81$ m/s^2. Assume $\tilde{\mathcal{S}} = \{v(x) \mid v(x) = c_1 x^2 + c_2 x^3\,, \quad c_1, c_2 \in \mathbb{R}\}$. [Hint: Compute the necessary integrals numerically or with a symbolic manipulation program. The determinant equation will result in a high degree polynomial; solve for its roots numerically and then select the physically meaningful root.]



(5.18) Consider a deep cantilever beam $h \gg b$ of length L. If such a beam is loaded with a tip-load P, it will bend (very slightly) in the y, z-plane, giving bending about the x-axis. However, if the load P exceeds a critical value P_{cr}, the beam will undergo a so-called lateral-torsional instability – a form of buckling in which the beam twists about the z-axis while it simultaneously bends about the y-axis.



The potential energy for the system (with respect to twisting) is given by the relation:

$$\Pi[\phi(z)] = \int_0^L \frac{1}{2}(GJ)_{\text{eff}}(\phi'(z))^2\,dz - \int_0^L \frac{1}{2}\frac{P^2}{EI_y}(L-z)^2\phi^2(z)\,dz\,,$$

where $(GJ)_{\text{eff}} = \frac{1}{3}hb^3G$ is the torsional rigidity and $I_y = hb^3/12$ is the area moment of inertia about the y-axis. Assume $\tilde{\mathcal{S}} = \{\phi(z) \mid \phi(z) = C(z^2 - 2Lz), \quad C \in \mathbb{R}\}$ and find an approximation for P_{cr}.

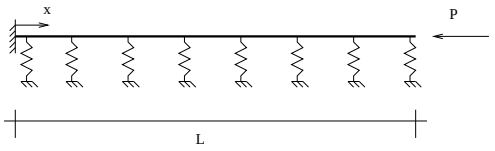
[Remark: The exact buckling mode is given by $\phi(z) = \sqrt{L-z}J_{-1/4}\left(2.0063(L-z)^2/L^2\right)$ with $P_{cr} = \frac{4.013}{L^2}\sqrt{EI_y(GJ)_{\text{eff}}}$ – see e.g. §6.3 in S.P. Timoschenko and J.M Gere, *Theory of Elastic Stabil-*

ity, McGraw-Hill (1961). The function $J_{-1/4}$ is the Bessel function of the first kind of order $-\frac{1}{4}$.]

MATLAB Exercises

(5.19) Consider, as shown, a beam supported by a distributed spring foundation. Assume the beam is 100 ft long with a Young's modulus of $E = 30 \times 10^6$ psi and a cross-sectional area moment of inertia $I = 77.4$ in^4. Assume a foundation stiffness of $k = 100$ lb/in^2 and find the axial buckling load (with small deformation assumptions). To solve this problem you should write a program that implements the following general approximation $v(x) = \sum_{i=1}^{n} c_i f_i(x)$ where $f_i = (x/L)^{i+1}$.

[Warning: The $f_i(x)$ are not orthogonal. At some point, increasing n will actually lead to a worse result due to round-off errors.]



Virtual Work

6

6.1 Principle of Virtual Work

When using the Principle of Stationary Potential Energy to find the equilibrium states for a mechanical system we are forced to make the assumption that the system involved is conservative. This restricts the type of problems that can be analyzed. In this section we will present a special case of the Principle of Virtual Work. The Principle of Virtual Work is a more general way of stating equilibrium and is not restricted to the conservative setting. The special case we will present is also known as the Method of Virtual Displacements and we will use these terms interchangeably. For the most part, we will discuss systems that happen to be conservative. This is done to keep the presentation familiar. However, it should not be forgotten that the methodology developed also applies to non-conservative systems.

6.1.1 Motivating example: Tension-compression bar

Consider the tension-compression bar shown in Fig. 6.1. The bar is subjected to an end-force F and a distributed load $b(x)$; the loads may or may not be conservative. The governing equilibrium equation for the problem is:

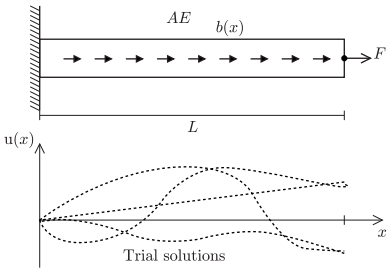


Fig. 6.1 Tension-compression bar (top) and examples of trial solutions (bottom).

Differential equation of equilibrium (**strong form**):

$$\frac{dR}{dx} + b = 0\,.$$

The kinematic boundary conditions for the problem tell us that the solution $u(x)$ must satisfy the condition $u(0) = 0$. Let us define the set (space) of admissible displacements $\mathcal{S}$ as the set of all functions which satisfy the kinematic boundary conditions. The true displacement solution must be in this set of functions:

Set (space) of admissible displacements:

$$\mathcal{S} = \{u(x) \mid \ u(0) = 0\}$$

Every functions $u \in \mathcal{S}$ satisfies the displacement (kinematic) boundary conditions.

Let us also define a second set of functions called the set (space) of test functions (virtual displacements) $\mathcal{V}$.

Set (space) of test solutions (virtual displacements):

$$\mathcal{V} = \{\delta u(x) \mid \delta u = 0 \text{ at all points where } u \text{ is known.}\}$$

The requirements on the functions $\delta u \in \mathcal{V}$ turn out to be a very convenient choice.

To proceed further, we will now convert the strong form of the equilibrium equation to the so-called weak form. The weak form of the equilibrium equation can be derived from the strong form strictly by algebraic and calculus manipulations – i.e. without the use of any further physical information about the problem.

(1) Start with the differential equation of equilibrium

$$\frac{dR}{dx} + b = 0 \ .$$

(2) Multiply the differential equation by an arbitrary function $\delta u \in \mathcal{V}$

$$\delta u \, \frac{dR}{dx} + \delta u \, b = 0 \ .$$

(3) Integrate the over the length of the bar (i.e., $[0, L]$)

$$\int_0^L \delta u \, \frac{dR}{dx} \, dx + \int_0^L \delta u \, b \, dx = 0 \ .$$

(4) Exploit the product rule of differentiation

$$\int_0^L \frac{d}{dx} \left[\delta u R\right] - \frac{d\delta u}{dx} R \, dx + \int_0^L \delta u \, b \, dx = 0 \ .$$

(5) Integrate out the exact differential $\frac{d}{dx}\left[\delta u R\right]$.

$$\begin{aligned} & \delta u R|_0^L - \int_0^L \frac{d\delta u}{dx} R \, dx + \int_0^L \delta u b \, dx = 0 \\ \Rightarrow \quad & \delta u(L) R(L) - \delta u(0) R(0) - \int_0^L \frac{d\delta u}{dx} R \, dx + \int_0^L \delta u \, b \, dx = 0 \ . \end{aligned}$$

(6) Note that $\delta u(0) = 0$ (since $\delta u \in \mathcal{V}$) and that $R(L) = F$ (according to the force boundary condition)

$$\delta u(L)F - \int_0^L \frac{d\delta u}{dx} R \, dx + \int_0^L \delta u \, b \, dx = 0 \, .$$

(7) Group terms and define $\delta\varepsilon := \frac{d\delta u}{dx}$

$$\int_0^L \delta\varepsilon R \, dx = \int_0^L \delta u \, b \, dx + \delta u(L)F \, .$$

This last equation is known as the weak form, the weak statement of equilibrium, or the virtual work equation:

Weak statement of equilibrium (**weak form**):

$$\int_0^L \delta\varepsilon R \, dx = \int_0^L \delta u \, b \, dx + \delta u(L)F \, .$$

Remarks:

(1) The *Principle of Virtual Work* states that a system is in equilibrium if and only if the weak equilibrium statement holds $\forall\ \delta u \in \mathcal{V}$. As presented here, we have derived this result. Thus in this context it is sometimes also known as the virtual work theorem.

(2) Note that we can follow the steps outlined above in the reverse order (going from Step 7 to Step 1). This implies that the differential equation of equilibrium and weak statement of equilibrium are equivalent, stating exactly the same condition:

Differential equation of equilibrium (**strong form**)

$\Leftrightarrow$

Principle of Virtual Work (**weak form**)

Thus the Principle of Virtual Work is a restatement of the differential equation of equilibrium, such that *they imply one and the same thing.*

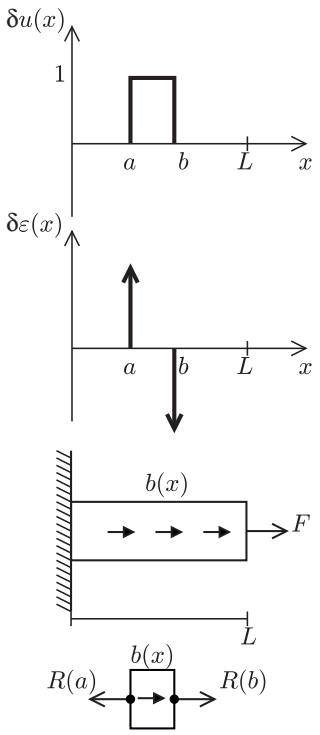


Fig. 6.2 Specific choice of a test function δu and the resulting equilibrium condition.

Example 6.1

Weak form content

To partially illustrate the equivalence of the weak and strong forms let us select a test function $\delta u \in \mathcal{V}$ of the form

$$\delta u(x) = H(x - a) - H(x - b) \, .$$

This function and its derivative ($\delta\varepsilon = (d/dx)\delta u$)

$$\delta\varepsilon = \delta(x-a) - \delta(x-b)$$

are depicted in Fig. 6.2. Clearly this function satisfies the conditions required for δu to be a member of $\mathcal{V}$. Substituting this into the weak form yields

$$\begin{aligned} &\int_0^L \left[\delta(x-a) - \delta(x-b)\right] R(x)\ dx \\ &\qquad = \int_0^L \left[H(x-a) - H(x-b)\right] b\ dx + F\delta u(L) \\ \Rightarrow \quad & R(a) - R(b) = \int_a^b b\ dx\ , \end{aligned}$$

which is just a statement of equilibrium for the segment $[a, b]$ of the bar. The difference between the internal forces $R(a)$ and $R(b)$ is equal to the integral of the distributed load b over $[a, b]$; see Fig. 6.2. Thus, our chosen *test function* tests global force equilibrium over the segment $[a, b]$.

Remarks:

(1) Hueristically one can think for choosing many similar test functions δu such that one finds that every segment of the bar must be in equilibrium and hence the whole bar must be in equilibrium.

Remarks:

(1) The space of admissible displacements $\mathcal{S}$ is sometimes called the space of trial solutions.

(2) The quantity $\delta\varepsilon = (d/dx)\delta u$ is often called the virtual strain in analogy to real strains $\varepsilon = (d/dx)u$.

(3) The quantity

$$\int_0^L \delta\varepsilon\, R\ dx\ = \int_0^L \int_A \delta\varepsilon\, \sigma\ dA\ dx$$

is known as the *internal virtual work*. Observe that if we assume δu has dimensions of length, then this term has dimensions of work. In general, the internal virtual work is the integral over the body of the virtual strains times the real stresses.

(4) The quantity

$$\int_0^L \delta u\, b\ dx + \delta u(L)F$$

is called the *external virtual work*. In general the external virtual work is equal to the product of the virtual displacements times the real forces (integrated if given as distributions).

(5) The Principle of Virtual Work can be stated as:

Internal Virtual Work = External Virtual Work

(6) Applying the Principle of Virtual Work, with our particular form of the virtual work equation, is often called the method of virtual displacements.

(7) In deriving the weak statement of equilibrium, note that the force boundary conditions were incorporated (Step 6.). Thus one has the following more precise equivalence:

Differential equation of equilibrium $\oplus$ Force B.C.s
$\Leftrightarrow$
Weak statement of equilibrium

(8) The displacement (kinematic) boundary conditions are also called the *essential boundary conditions*.

(9) The force boundary conditions are sometimes referred to as the *natural boundary conditions*, because they are "naturally" pre-embedded in the weak statement of equilibrium.

(10) Every choice of a test function in the virtual work equation yields a generalized equilibrium equation for the system.

6.1.2 Summary of the steps leading to the virtual work equations

In order to derive a weak statement of equilibrium (in the context of virtual displacements), one has to execute the following basic steps:

(1) Specify the governing differential equation of equilibrium.

(2) Determine the trial solution space $\mathcal{S}$ and test function space $\mathcal{V}$ for the given problem.

(3) Multiply the differential equation by a test function $\delta u \in \mathcal{V}$ and integrate over the whole domain.

(4) Integrate by parts (product rule of differentiation followed by integration of an exact differential) to obtain the weak statement of equilibrium.

6.1.3 Solutions

The Principle of Virtual Work (Method of Virtual Displacements) can be used as a tool for finding exact as well as approximate solutions to the equations of equilibrium. The methodology is summarized as:

Find $u \in \mathcal{S}$ such that

$$\int_0^L \delta\varepsilon\, R\, dx = \int_0^L \delta u\, b\, dx + F\delta u(L)$$

for all $\delta u \in \mathcal{V}$ where $\delta\varepsilon = \delta u'$ and $R = R(u')$.

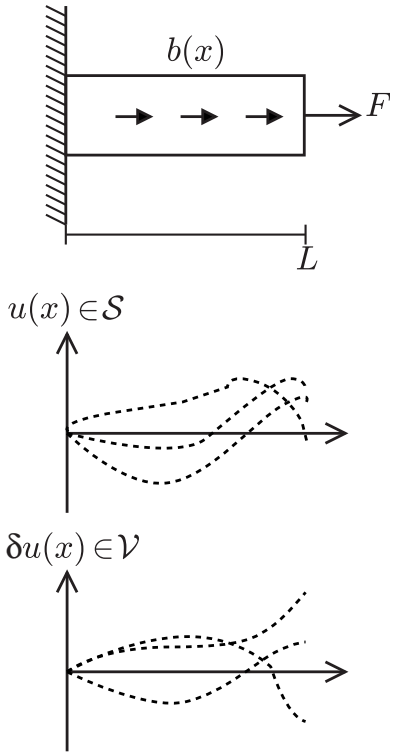


Fig. 6.3 The search problem for a tension-compression bar; possible trial solutions u and test functions δu.

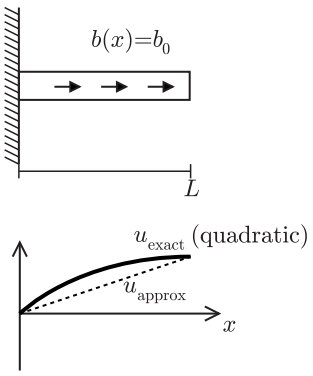


Fig. 6.4 Tension-compression bar with distributed load, the exact solution and approximate solution obtained from a 1 parameter approximation.

For the special case of a linear elastic material, $R(u') = AEu'$, one has

$$\int_0^L \delta u' AEu' \, dx \; = \int_0^L \delta u \, b \, dx + F\delta u(L).$$

This can be considered an infinite dimensional search problem where one iterates through the following steps:

(1) Pick a function $u \in \mathcal{S}$.

(2) Test this function against all possible $\delta u \in \mathcal{V}$; i.e., make sure the virtual work equation holds true for all $\delta u \in \mathcal{V}$.

(3) If the equation does not hold for some δu, then go back to Step 1 and repeat the process.

For the case of a tension-compression bar fixed at one end, some possible trial solutions and test functions are shown in Fig. 6.3. This is the case, if one desires to find the "exact" solution. If one desires only an approximate solution, the method becomes simpler and more tractable. The steps are quite similar to the method of Ritz applied to the Principle of Stationary Potential Energy, in that one selects an approximate form for the functions involved and then solves a finite dimensional problem. In the case of the Principle of Virtual Work, one selects, an approximation for the trial solution space $\mathcal{S}$ and test function space $\mathcal{V}$. To make this more concrete, let us consider an example.

Example 6.2

Bar with a constant distributed load

Consider the linear elastic tension-compression bar with one end fixed as shown in Fig. 6.4. The bar is subjected to a constant distributed load. The exact solution for this problem is a quadratic function. Find a one term approximate linear solution.

Solution: The virtual work equation for this problem is:

$$\int_0^L \delta\varepsilon R \, dx = \int_0^L \delta u \, b_0 \, dx \; .$$

Let us select the following one parameter approximations for $\mathcal{S}$ and $\mathcal{V}$,

$$\tilde{\mathcal{S}} = \{u(x) \mid u(x) = Cx\} \qquad \subset \mathcal{S},$$
$$\tilde{\mathcal{V}} = \{\delta u(x) \mid \delta u(x) = \delta Cx\} \qquad \subset \mathcal{V}.$$

Note that we have chosen the functions in $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{V}}$ to have the same form. This is called a Bubnov-Galerkin approximation. Inserting the selected approximate forms for u and δu into the virtual work equation yields:

$$\int_0^L (\delta C) AEC \; dx = \int_0^L (\delta C x) b_0 \; dx$$
$$\Rightarrow \quad \delta C \left[\int_0^L AEC \; dx - \int_0^L x b_0 \; dx \right] = 0 \; ,$$

where we have used the fact that the bar is linear elastic and thus $R(u') = AEu' = AEC$. Since the last statement must hold for any δC (the Principle of Virtual Work), one must have:

$$\int_0^L AEC \; dx - \int_0^L x b_0 \; dx = 0 \, ,$$

which represents a single scalar equation for the one unknown, C. Solving for C gives:

$$C = \frac{\frac{b_0 L^2}{2}}{AEL} = \frac{b_0 L}{2AE} \; .$$

Thus the approximate solution is,

$$u_{\text{approx}}(x) = \frac{b_0 L}{2AE} x \; .$$

Remarks:

(1) Figure 6.4 compares the approximate solution with the exact solution. The approximation is seen to be reasonable for a single parameter approximation.

(2) It should be emphasized that should one have treated this problem with the Principle of Stationary Potential Energy assuming an approximation $u = Cx$, one would have obtained the same equation as above for C. (This of course is only true when one assumes an elastic material.)

The key to a good approximation using the Principle of Virtual Work, lies in selecting good approximations for the solution space $\mathcal{S}$ and test function space $\mathcal{V}$. For the example problem, we had

$$\begin{aligned} \mathcal{S} &= \{u(x) \mid u(0) = 0\} \, , \\ \mathcal{V} &= \{\delta u(x) \mid \delta u(0) = 0\} \, . \end{aligned}$$

One must select the approximate trial solution space $\tilde{\mathcal{S}}$ and approximate test function space $\tilde{\mathcal{V}}$ so that they are subspaces of $\mathcal{S}$ and $\mathcal{V}$,

$$\begin{aligned} &\tilde{\mathcal{S}} \subset \mathcal{S}, \\ &\tilde{\mathcal{V}} \subset \mathcal{V} \; . \end{aligned}$$

Observe how our selection in the example,

$$\tilde{\mathcal{S}} = \tilde{\mathcal{V}} = \{f(x) \mid f(x) = Cx\},$$

satisfied this property. To extend the method to obtain better approximations, just as in the Method of Ritz, one can consider an approximation constructed from a linear combination of many functions. $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{V}}$ can be extended to the form:

$$\begin{aligned}
\hat{\mathcal{S}} &= \left\{ u(x) \;\middle|\; u = \sum_{n=1}^{N_u} u_n f_n(x);\ \text{Chosen to satisfy kinematic BCs} \right\}, \\
\hat{\mathcal{V}} &= \left\{ \delta u(x) \;\middle|\; \delta u = \sum_{m=1}^{N_{\delta u}} \delta u_m g_m(x);\ \text{Chosen to be zero where} \right. \\
&\qquad\qquad \left. \text{the kinematic BCs are specified} \right\},
\end{aligned}$$

where we have used the notation of u_n and δu_m for the parameters. Inserting these forms into the virtual work equation for the example gives:

$$\int_0^L \sum_{m=1}^{N_{\delta u}} \delta u_m g'_m(x) R \left(\sum_{n=1}^{N_u} u_n f'_n(x) \right) \, dx = \int_0^L b_0 \sum_{m=1}^{N_{\delta u}} \delta u_m g_m(x) \, dx.$$

For the linear elastic material case, $R = AEu'$, one has:

$$\begin{aligned}
& \int_0^L \sum_{m=1}^{N_{\delta u}} \delta u_m g'_m(x) AE \sum_{n=1}^{N_u} u_n f'_n(x) \, dx = \int_0^L b_0 \sum_{m=1}^{N_{\delta u}} \delta u_m g_m(x) \, dx \\
\Rightarrow \quad & \sum_{m=1}^{N_{\delta u}} \delta u_m \underbrace{\left[\int_0^L g'_m(x) AE \sum_{n=1}^{N_u} u_n f'_n(x) \, dx - \int_0^L b_0 g_m(x) \, dx \right]}_{H_m} = 0 \\
\Rightarrow \quad & \sum_{m=1}^{N_{\delta u}} \delta u_m H_m \ .
\end{aligned}$$

This last equation must hold $\forall \delta u_m$ which implies that $H_m = 0$ for all m. Thus

$$\begin{aligned}
& \sum_{n=1}^{N_u} \int_0^L g'_m(x) AE f'_n(x) \, dx \; u_n = \int_0^L b_0 g_m(x) \, dx \\
\Rightarrow \quad & \sum_{n=1}^{N_u} K_{mn} u_n = F_m \\
\Rightarrow \quad & \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F} \ .
\end{aligned}$$

Here $\boldsymbol{K}$ is a $N_{\delta u}$-by-N_u matrix whose mn-th entry is defined by

$$K_{mn} = \int_0^L g'_m(x) AE f'_n(x) \, dx \ ,$$

$\boldsymbol{F}$ is a $N_{\delta u}$-by-1 vector whose m-th entry is defined by

$$F_m = \int_0^L g_m(x)\ b_0\ dx\ ,$$

and $\boldsymbol{u}$ is a N_u-by-1 vector whose n-th entry is u_n.

Remarks:

(1) If $g_m = f_m$ and $N_u = N_{\delta u}$ (the usual choice), then $\boldsymbol{K}$ is square and symmetric. This is the (Bubnov)-Galerkin method.

(2) The choices for f_n and g_m are essentially arbitrary after the basic conditions have been satisfied (viz., the final expression for $u(x)$ satisfies the essential boundary conditions and $\delta u(x)$ is equal to zero at these points).

6.1.4 Equivalence between the Principle of Virtual Work and the Principle of Stationary Potential Energy

Through the previous examples, one may have observed a notable similarity between the Principle of Virtual Work and the Principle of Stationary Potential Energy. This is not merely a coincidence. One can show that they are formally equivalent for conservative systems. Here, let us derive their equivalence for the simple case of an elastic tension-compression bar subject to a conservative constant distributed load as shown in Fig. 6.5.

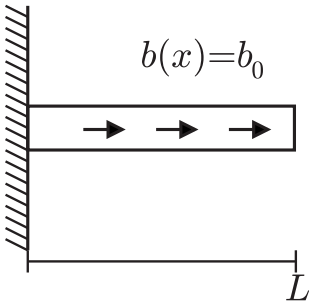


Fig. 6.5 Tension-compression bar with distributed load.

The total potential energy for this system is

$$\Pi(u) \quad = \quad \int_0^L \frac{1}{2} AE(u')^2\ dx - \int_0^L b_0 u\ dx\,.$$

Let us search for the stationary point of Π over,

$$\mathcal{S} \quad = \quad \{u \mid u(0) = 0\}\ .$$

Assume that $u^*(x)$ is the function(point) which gives the minimum value of Π. Then if one adds any multiple $\alpha \in \mathbb{R}$ of any function δu to u^*, the value of Π should increase:

$$\Rightarrow \Pi(u^*) \leq \Pi(u^* + \alpha\delta u)\,.$$

Here δu is selected such that

$$\delta u \in \mathcal{V} = \{\delta u \mid \delta u(0) = 0\}\ .$$

Define now

$$\begin{aligned} h(\alpha) \quad &:= \quad \Pi(u^* + \alpha\delta u) \\ &= \quad \int_0^L \frac{1}{2} AE\left[(u^* + \alpha\delta u)'\right]^2\ dx - \int_0^L b_0\,(u^* + \alpha\delta u)\ dx\ . \end{aligned}$$

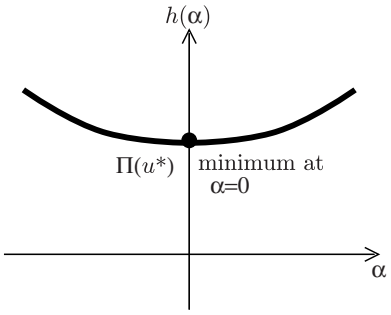


Fig. 6.6 The $h(\alpha)$ function and its minimum (stationary) value at $\alpha = 0$.

By construction, $h(\alpha)$ has a minimum (stationary) point at $\alpha = 0$; thus,

$$\left.\frac{dh}{d\alpha}\right|_{\alpha=0} = 0\,.$$

This situation is depicted in Fig. 6.6. The derivative can be evaluated as:

$$\begin{aligned}\left.\frac{dh}{d\alpha}\right|_{\alpha=0} &= \int_0^L \frac{1}{2}AE2\left[u^{*\prime} + \alpha\delta u'\right]\delta u'\;dx - \left.\int_0^L b_0\delta u\;dx\right|_{\alpha=0} \\ &= \int_0^L AE(u^*)'\delta u'\;dx - \int_0^L b_0\delta u\;dx \\ &= \int_0^L R\delta u'\;dx - \int_0^L b_0\delta u\;dx\;\;.\end{aligned}$$

In the last line we have made use of the linear elastic relation $R = AEu^{*\prime}$. Noting that the derivative of $h(\alpha)$ with respect to α at $\alpha = 0$ is zero by construction, we can rewrite this as

$$\int_0^L R\delta u'\;dx = \int_0^L b_0\delta u\;dx\,,$$

which must hold $\forall\delta u \in \mathcal{V}$. Note that this is exactly the statement of the Principle of Virtual Work.

Remarks:

(1) It should be emphasized that the method used to show the equivalence is modestly general and can be applied to other mechanical systems as well.

(2) In this type of analysis, one often defines the symbol $\delta\Pi$ which is known as the variation (or variational derivative) of Π. It is defined as

$$\delta\Pi \;:=\; \left.\frac{dh}{d\alpha}\right|_{\alpha=0}\;.$$

For the given example problem, one has

$$\delta\Pi = \int_0^L R\delta u'\;dx - \int_0^L b_0\delta u\;dx\;.$$

(3) The variational derivative is equivalent to the directional derivative introduced in vector calculus. Given a scalar-valued function $f(\boldsymbol{x})$ which depends on the position $\boldsymbol{x} \in \mathbb{R}^n$ in n-dimensions, the directional derivative of f is defined as:

$$\left.\frac{d}{d\alpha}f\left(\boldsymbol{x} + \alpha\boldsymbol{v}\right)\right|_{\alpha=0}\;.$$

This quantity is also denoted $Df(\boldsymbol{x})[\boldsymbol{v}]$. In the case of the potential energy Π, replace f by Π, $\boldsymbol{x}$ by u, and $\boldsymbol{v}$ by δu. Thus we have the equivalence

$$Df(\boldsymbol{x})[\boldsymbol{v}] \backsim \delta\Pi(u)[\delta u] = \delta\Pi(u;\delta u)\;.$$

6.2 Torsion

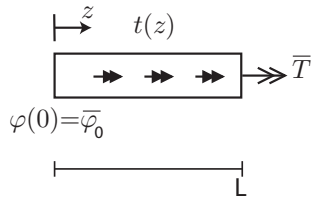


Fig. 6.7 Torsion bar with distributed torque $t(z)$, and end-torque $\bar{T}$ and end-rotation $\bar{\phi}_o$.

Let us now consider the Principle of Virtual work for the problem of torsion of a bar. Consider a bar where the left end has an imposed rotation, an applied distributed torque, and an end-torque as shown in Fig. 6.7. The kinematic boundary condition for the problem is

$$\phi(0) = \bar{\phi}_0$$

and the torque boundary condition is

$$T(L) = \bar{T}\,.$$

The differential equation governing equilibrium (strong form) is given by

$$\frac{dT}{dz} + t = 0\,.$$

The kinematic boundary conditions of the problem define the form for the trial solution space

$$\mathcal{S} \;=\; \left\{\phi(z) \mid \phi(0) = \bar{\phi}_0\right\}$$

and the test function space

$$\mathcal{V} \;=\; \left\{\delta\phi(z) \mid \delta\phi(0) = 0\right\}\,.$$

To derive the virtual work equation (weak statement of equilibrium), one takes the strong form of equilibrium and multiplies it by a test function $\delta\phi \in \mathcal{V}$ and integrates over the whole domain $[0, L]$:

$$\begin{aligned}
&\int_0^L \delta\phi \frac{dT}{dz} + \delta\phi t \; dz = 0 \\
\Rightarrow\quad &\int_0^L (\delta\phi T)' - \delta\phi' T + \delta\phi t \; dz = 0 \\
\Rightarrow\quad &[\delta\phi T]_0^L - \int_0^L \delta\phi' T \; dz + \int_0^L \delta\phi t \; dz = 0 \\
\Rightarrow\quad &\int_0^L \delta\phi' T \; dz = [\delta\phi T]_0^L + \int_0^L \delta\phi t \; dz \\
\Rightarrow\quad &\int_0^L \delta\phi' T \; dz = \delta\phi(L)T(L) - \delta\phi(0)T(0) + \int_0^L \delta\phi t \; dz \\
\Rightarrow\quad &\int_0^L \delta\phi' T \; dz = \int_0^L \delta\phi t \; dz + \delta\phi(L)\bar{T}\,.
\end{aligned}$$

By defining the virtual twist rate $\delta\alpha := \delta\phi'$ and twist rate $\alpha := \phi'$, one obtains the virtual work equation for this system:

$$\underbrace{\int_0^L \delta\alpha T(\alpha) \; dz}_{\text{Internal V.W.}} = \underbrace{\int_0^L \delta\phi t \; dz + \delta\phi(L)\bar{T}}_{\begin{array}{c}\text{External V.W. from distributed torque} \\ \oplus \\ \text{External V.W. from point torque at } z = L\end{array}}$$

The problem statement for finding the solution ϕ to the problem becomes:

Find $\phi \in \mathcal{S}$ such that,

$$\int_0^L \delta\alpha T \; dz = \int_0^L \delta\phi \, t \; dz + \bar{T}\delta\phi(L),$$

for all $\delta\phi \in \mathcal{V}$ where $\delta\alpha = \delta\phi'$ and $T = T(\alpha)$.

For the special case of a linear elastic material, $T(\alpha) = GJ\alpha = GJ\phi'$, one has

$$\int_0^L \delta\phi' GJ\phi' \; dz \; = \int_0^L \delta\phi \, t \; dz + \bar{T}\delta\phi(L).$$

It should be emphasized that $\mathcal{S}$ and $\mathcal{V}$ change with changing kinematic boundary conditions.

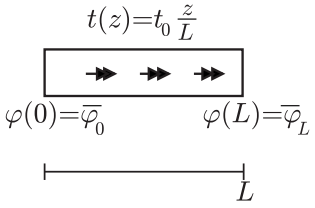


Fig. 6.8 Torsion bar with rotation specifed at both ends.

Example 6.3

Fixed end-rotations

Consider a torsion rod with a linear distributed load where rotations are also imposed at both ends, $\phi(0) = \bar{\phi}_0$ and $\phi(L) = \bar{\phi}_L$; see Fig. 6.8. State the virtual work equation along with the required function spaces.

Solution: From the given boundary conditions, we have that $\mathcal{S}$ and $\mathcal{V}$ are:

$$\begin{aligned} \mathcal{S} &= \left\{\phi(z) \mid \phi(0) = \bar{\phi}_0 \text{ and } \phi(L) = \bar{\phi}_L\right\}, \\ \mathcal{V} &= \left\{\delta\phi(z) \mid \delta\phi(0) = 0 \text{ and } \delta\phi(L) = 0\right\}. \end{aligned}$$

In this case the only contribution to the external virtual work is from the distributed load. Thus, the expression for the weak statement of equilibrium (the virtual work equation) is

$$\int_0^L \delta\alpha T \; dz = \int_0^L \delta\phi \, t_0 \frac{z}{L} \; dz \; .$$

Example 6.4

No kinematic boundary conditions

Consider a torsion rod where torques are imposed at both ends, $T(0) = \bar{T}_0$ and $T(L) = \bar{T}_L$; see Fig. 6.9. State the virtual work equation along with the required function spaces.

Solution: From the given boundary conditions, we have that $\mathcal{S}$ and $\mathcal{V}$ are

$$\begin{aligned} \mathcal{S} &= \{\phi(z) \mid \text{no conditions}\} \ , \\ \mathcal{V} &= \{\delta\phi(z) \mid \text{no conditions}\} \ . \end{aligned}$$

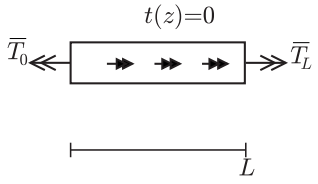


Fig. 6.9 Torsion bar with torques specifed at both ends.

In this case there is no contribution to the external virtual work from a distributed load but there are contribution from the two applied loads. Thus, the expression for the virtual work equation reads:

$$\int_0^L \delta\alpha T \ dz = \delta\phi(L)\bar{T}_L - \delta\phi(0)\bar{T}_0 \ .$$

Remarks:

(1) As an example of how the virtual work equation embeds different concepts of equilibrium, consider the test function $\delta\phi = \delta C$, any constant. Then $\delta\alpha = 0$, which implies,

$$\begin{aligned} & 0 = \delta C\bar{T}_L - \delta C\bar{T}_0 \qquad \forall \delta C \\ \Rightarrow \quad & \delta C\left[\bar{T}_L - \bar{T}_0\right] = 0 \qquad \forall \delta C \\ \Rightarrow \quad & \bar{T}_L - \bar{T}_0 = 0 \\ \Rightarrow \quad & \bar{T}_L = \bar{T}_0 \ . \end{aligned}$$

Thus one sees that choosing an arbitrary constant test function implies global (moment) equilibrium of the bar about the z-axis.

Remarks:

(1) As with the tension-compression bar, the Principle of Virtual Work applied to torsion problems can be used as a tool for computing approximate solutions. One simply selects a set of finitely parameterized trial solutions *and* test functions; see Fig. 6.10. These are then substituted into the virtual work equation which provides the necessary relations for determining the parameters in the approximation. Selections similar to what we used with the Principle of Stationary Potential Energy are appropriate here.

6.3 Bending

The use of virtual work to express the equilibrium equations for beam bending follows the same pattern that we have developed for tension-compression bars and torsion rods. Consider a beam fixed at one end

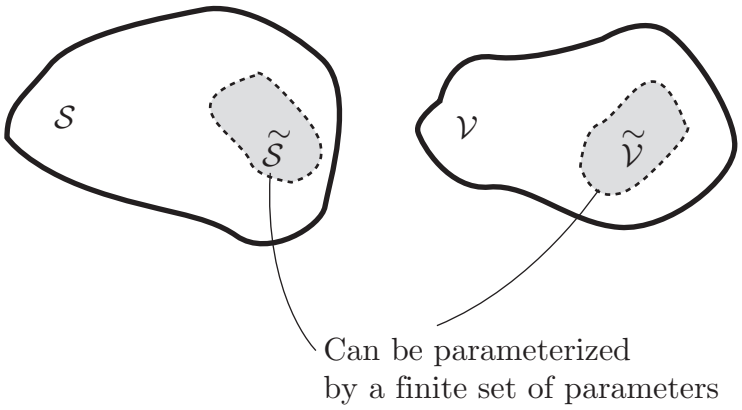


Fig. 6.10 Trial solution space $\mathcal{S}$ and test function space $\mathcal{V}$ approximated by subspaces $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{V}}$ parameterized by a finite set of parameters.

and subjected to a distributed load, an end-shear, and an end-moment as is shown in Fig. 6.11. The kinematic boundary conditions for this problem are:

$$v(0) = 0 \ , \qquad \theta(0) = v'(0) = 0 \ ,$$

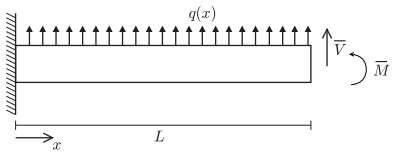


Fig. 6.11 Cantilever beam with distributed load $q(x)$, end-shear $\bar{V}$, and end-moment $\bar{M}$.

and the force boundary conditions are:

$$V(L) = -EIv'''(L) = \bar{V} \ , \qquad M(L) = EIv''(L) = \bar{M} \ .$$

The differential equations governing equilibrium are:

$$\begin{aligned} \frac{dM}{dx} + V &= 0 \ , \\ \frac{dV}{dx} + q &= 0 \ . \end{aligned}$$

The kinematic boundary conditions for the problem define the form for the trial solution space

$$\mathcal{S} \ = \ \{v(x) \mid v(0) = 0 \text{ and } v'(0) = 0\}$$

and the test function space

$$\mathcal{V} \ = \ \{\delta v(x) \mid \delta v(0) = 0 \text{ and } \delta v'(0) = 0\} \ .$$

Before formally deriving the virtual work expression, let us reason out its form. The internal virtual work is in general the real stresses times the virtual strains integrated over the volume. For a beam in bending this gives

$$\int_V \sigma\delta\varepsilon + \tau\delta\gamma \, dV \, ,$$

where σ is the bending stress, τ the shear stress, and $\delta\varepsilon$ and $\delta\gamma$ the corresponding virtual normal and shear strains. In Bernoulli-Euler beams, the kinematic assumptions require the shear strains to be zero and thus we also have $\delta\gamma = 0$. The virtual normal strains are also restricted by the kinematic assumptions, which tells us that $\varepsilon = -y\kappa = -yv''$.

In particular, this restricts the virtual normal strains to the form $\delta\varepsilon = -y\delta\kappa = -y\delta v''$. Substituting these observations into the general virtual work expression yields for the internal virtual work:

$$\int_V \sigma\delta\varepsilon = \int_L \int_A -y\delta\kappa\sigma \, dA dx = \int_0^L \delta\kappa M \, dx$$

The external virtual work is the product of the real external load with the virtual motions. In our case there is a distributed load and two point loads. Multiplying each by the corresponding motion where it acts gives for the external virtual work:

$$\int_0^L \delta v \; q \; dx + \delta v(L)\bar{V} + \delta\theta(L)\bar{M} \, ,$$

where we have defined $\delta\theta = \delta v'$. The final result is then:

$$\underbrace{\int_0^L \delta\kappa M(\kappa) \; dx}_{\text{Internal V.W.}} = \underbrace{\int_0^L \delta v \; q \; dx + \delta v(L)\bar{V} + \delta\theta(L)\bar{M} \, .}_{\begin{array}{c}\text{External V.W. from distributed load} \\ \bigoplus \\ \text{External V.W. from point force at } x = L \\ \bigoplus \\ \text{External V.W. from point moment at } x = L\end{array}}$$

To formally derive the weak statement of equilibrium from the governing differential equations, let us first combine the two equations defining equilibrium into one by eliminating the shear force between them. This yields:

$$\frac{d^2 M}{dx^2} - q = 0 \; .$$

Now multiply by a test function $\delta v \in \mathcal{V}$ and integrate over the whole

domain $[0, L]$:

$$\begin{aligned}
&\int_0^L \delta v M'' - \delta v\, q\ dx = 0 \\
\Rightarrow\quad &\int_0^L [(\delta v M')' - \delta v' M'] - \delta v\, q\ dx = 0 \\
\Rightarrow\quad &\int_0^L [(\delta v M')' - \{(\delta v' M)' - \delta v'' M\}] - \delta v\, q\ dx = 0 \\
\Rightarrow\quad &\int_0^L (-\delta v V)' - (\delta v' M)' + \delta v'' M - \delta v\, q\ dx = 0 \\
\Rightarrow\quad &\int_0^L \delta v'' M\ dx = \int_0^L \delta v\, q\ dx + \int_0^L (\delta v' M)'\ dx + \int_0^L (\delta v V)'\ dx \\
\Rightarrow\quad &\int_0^L \delta v'' M\ dx = \int_0^L \delta v\, q\ dx + [\delta v' M]_0^L + [\delta v V]_0^L \\
\Rightarrow\quad &\int_0^L \delta v'' M\ dx = \int_0^L \delta v\, q\ dx + \delta v'(L) M(L) - \delta v'(0) M(0) \\
&\qquad\qquad\qquad\qquad + \delta v(L) V(L) - \delta v(0) V(0) \\
\Rightarrow\quad &\int_0^L \delta v'' M\ dx = \int_0^L \delta v\, q\ dx + \delta v'(L) \bar{M} + \delta v(L) \bar{V}\ .
\end{aligned}$$

Using the notation for virtual rotations and curvature one alternatively has

$$\int_0^L \delta\kappa M\ dx = \int_0^L \delta v\, q\ dx + \delta\theta(L) \bar{M} + \delta v(L) \bar{V}\ .$$

One observes that this is identical to the form which we reasoned out above.

The problem statement for finding the solution $v(x)$ to the problem becomes:

Find $v \in \mathcal{S}$ such that,

$$\int_0^L \delta\kappa M\ dx = \int_0^L \delta v\, q\ dx + \bar{M}\delta\theta(L) + \bar{V}\delta v(L),$$

for all $\delta v \in \mathcal{V}$ where $\delta\theta = \delta v'$, $\delta\kappa = \delta v''$ and $M = M(v'')$.

For the special case of a linear elastic material, $M = EIv''$, one has

$$\int_0^L \delta v'' EI v''\ dx = \int_0^L \delta v\, q\ dx + \bar{M}\delta\theta(L) + \bar{V}\delta v(L)\ .$$

Remarks:

(1) As with the tension-compression bar and the torsion rod, the Principle of Virtual Work can be used as a tool for computing exact as well as approximate solutions to beam problems.

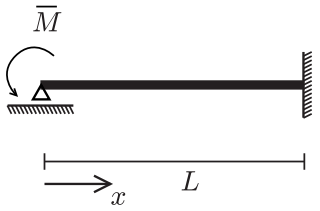


Fig. 6.12 Beam with end moment.

Example 6.5

Indeterminate beam with end-moment

Consider the beam shown in Fig. 6.12, which is simply supported at one end and fixed at the other. A moment $\bar{M}$ is applied at the left end. Define the function spaces $\mathcal{S}$ and $\mathcal{V}$ and use a subset of them to find an approximate solution to the given problem.

Solution: The kinematic boundary conditions provide the form for the solution space and the test space as:

$$\begin{aligned} \mathcal{S} &= \{v(x) \mid v(0) = v(L) = v'(L) = 0\} \ , \\ \mathcal{V} &= \{\delta v(x) \mid \delta v(0) = \delta v(L) = \delta v'(L) = 0\} \ . \end{aligned}$$

Let us consider the following one parameter approximations for $\mathcal{S}$ and $\mathcal{V}$

$$\begin{aligned} \tilde{\mathcal{S}} &= \left\{v(x) \mid v(x) = Ax(x-L)^2\right\} \ , \\ \tilde{\mathcal{V}} &= \left\{\delta v(x) \mid \delta v(x) = \delta A x(x-L)^2\right\} \ . \end{aligned}$$

The set $\tilde{\mathcal{S}}$ is parameterized by A and the set $\tilde{\mathcal{V}}$ is parameterized by δA. Both approximation spaces are proper subsets of their respective full spaces.

For the given problem the relevant virtual work equation is:

$$\int_0^L \delta\kappa M \ dx = \bar{M}\delta\theta(0) \ .$$

From our approximation spaces, we have

$$\begin{aligned} \kappa &= A\left[6x - 4L\right], \\ \delta\theta &= \delta A\left[3x^2 - 4xL + L^2\right], \\ \delta\kappa &= \delta A\left[6x - 4L\right]. \end{aligned}$$

If we assume the beam to be linear elastic, then the Principle of Virtual Work states:

$$\begin{aligned} &\int_0^L \delta A\left[6x - 4L\right] \ EI \ A\left[6x - 4L\right] \ dx = \bar{M}\delta AL^2 \qquad \forall \delta A \\ \Rightarrow \quad A &= \frac{\bar{M}L^2}{\int_0^L EI\left[6x - 4L\right]^2 \ dx} = \frac{\bar{M}}{4EIL} \, . \end{aligned}$$

Thus the final approximate solution is given by:

$$v(x) = \frac{\bar{M}}{4EIL} x(x-L)^2 \, .$$

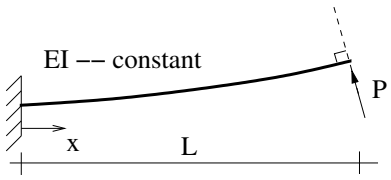


Fig. 6.13 Cantilever beam subjected to a follower force.

Example 6.6

Non-conservative load. The beam shown in Fig. 6.13 is subjected to a force that remains perpendicular to the beam even as it rotates. Such a force is known as a follower force and it is a type of non-conservative load. Use the principle of virtual displacements to find an approximate expression for the beam's deflection.

Solution: The definitions for the internal virtual work do not change due to the presence of the follower load. The only issue that requires attention is the fact that the vertical component of the force will be $P\cos(v'(L))$. Thus the virtual work theorem will read

$$P\cos\left(v'(L)\right)\delta v(L) = \int_0^L v''EI\delta v''\,dx\,.$$

In what follows we will approximate $\cos((v'(L))$ as $1-(v'(L))^2/2$ and for approximate function spaces we will pick the forms $v = Cx^2$ and $\delta v = \delta C x^2$. Inserting the approximations into the virtual work theorem and requiring it to hold true for all δC, yields

$$(2PL^4)C^2 + (4EIL)C - PL^2 = 0\,.$$

Solving for C then gives:

$$C = \frac{-4EIL \pm \sqrt{(4EIL)^2 + 8P^2L^6}}{4PL^4}\,.$$

There are two solutions. However, the solution associated with the negative sign in front of the radical leads to the non-physical result that the beam moves in the opposite direction of the load. Thus we discard that possibility and find that

$$\begin{aligned} C &= \frac{-4EIL + \sqrt{(4EIL)^2 + 8P^2L^6}}{4PL^4} \\ &= \frac{EI}{PL^3}\left[-1 + \sqrt{1 + \frac{1}{2}\left(\frac{PL^2}{EI}\right)^2}\right]\,. \end{aligned}$$

Remarks:

(1) This problem illustrates two important features of the principle of virtual work: (a) it applies to non-linear problems (as does stationary potential enegy) and (b) that it applies to non-conservative problems (to which stationary potential energy does not). The load in this case was not conservative and thus the use of Ritz's method is precluded.

(2) Note that if the load is very small, then the radical can be expanded in a Taylor series to show that $C \approx PL/4EI$ which is the result we had for this level of approximation when we treated the case of a dead-loaded cantilever. So for small forces, follower loads and dead-loads give the same response as one would intuitively expect.

6.3.1 Generalizations

So far we have used the Principle of Virtual Work to look at simple problems so that we can better understand what it means and what its properties are. The principle can also be applied far more generally without much added effort. One needs to observe that the internal virtual work is always given by the real stresses times the virtual strains integrated over the volume of the system. In particular, if there are multiple types of stress (say bending and torsional), then we need to account for both of them. This occurs by simply adding their contributions together. Likewise, since internal virtual work is an integral, if the system has complex shape, we can simply break down the computation of the internal virtual work into the sum of the integrals over sub-parts of the system. The same holds for external virtual work. It too is an additive quantity. Each type of load, additively contributes to the total external virtual work.

Example 6.7

Bending and torsion Consider a cantilevered rod that is subjected to both an end-shear and an end-torque. Write down the appropriate virtual work expression.

Solution: For the internal virtual work we need to sum the contributions from bending and torsion this gives:

$$\int_0^L \delta v'' M \; dx + \int_0^L \delta \phi' T \; dx \, .$$

For the external virtual work we need to sum the contributions from each load acting on the system. Here this gives

$$\bar{V}\delta v(L) + \bar{T}\delta\phi(L).$$

The virtual work equation is thus

$$\int_0^L \delta v'' M \; dx + \int_0^L \delta \phi' T \; dx = \bar{V}\delta v(L) + \bar{T}\delta\phi(L)$$

and the Principle of Virtual Work tells us that for equilibrium this equation must hold for all virtual deflections and virtual rotations (in the appropriate test spaces).

Example 6.8

Angle frame

Consider the angle frame in Fig. 6.14. Element 2 is in a state of axial stress as well as bending and Element 1 is in a state of bending. Find expressions for the internal virtual work and the external virtual work.

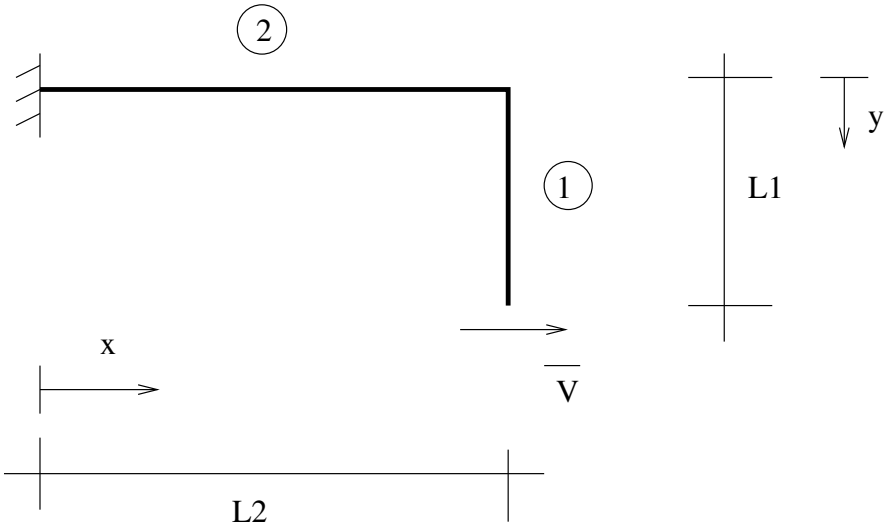


Fig. 6.14 Angle frame.

Solution: Let us define the following internal resultants and virtual fields:

$\delta v_1(y)$: Virtual transverse deflection of element 1
$\delta u_2(x)$: Virtual axial displacement of element 2
$\delta v_2(x)$: Virtual transverse deflection of element 2
$M_1(y)$: Bending moment in element 1
$M_2(x)$: Bending moment in element 2
$R_2(x)$: Axial force in element 2 .

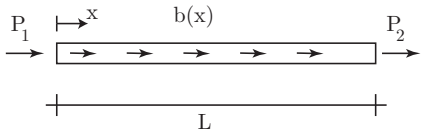
It should be noted that there are no kinematic conditions on $\delta v_1(y)$ but $\delta u_2(x)$ and $\delta v_2(x)$ need to respect the kinematic conditions provided by the built-in support. With these definitions, we can construct the relevant expressions

$$\begin{aligned} \text{External V.W.} &= \bar{V}\delta v_1(L) \\ \text{Internal V.W.} &= \int_0^{L_1} \delta v_1'' M_1 \, dy + \int_0^{L_2} \delta v_2'' M_2 \, dx + \int_0^{L_2} \delta u_2' R_2 \, dx. \end{aligned}$$

Exercises

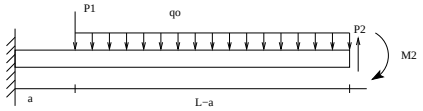
(6.1) Write the virtual work statement for the following system. Make sure to define the solution space $\mathcal{S}$; use no restrictions on the test function space $\mathcal{V}$.

(6.2) Write the virtual work statement for the linear elastic bar shown below; assume $b(x)$ is given. Make sure to define a suitable solution space $\mathcal{S}$ and a suitable space of test functions $\mathcal{V}$.

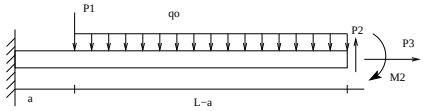


(6.3) Write the virtual work statement for the following system. Make sure to define the solution space $\mathcal{S}$; use no restrictions on the test function space $\mathcal{V}$.

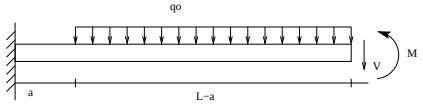
(6.4) Write the virtual work statement for the following system. Make sure to define the solution space $\mathcal{S}$; use no restrictions on the test function space $\mathcal{V}$.



(6.5) Write the virtual work statement for the following system. Make sure to define the solution space $\mathcal{S}$; use a test function space $\mathcal{V}$ that eliminates support reactions. (Hint: virtual work expressions are additive like real work.)

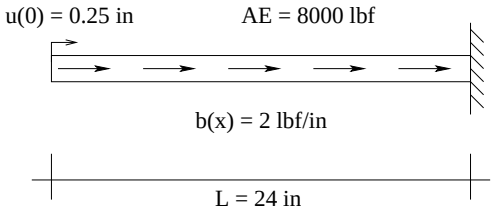


(6.6) For the configuration shown, derive the virtual work equation starting from $(d^2/dx^2)M = q$.



(6.7) Derive, starting from $dT/dz + t = 0$, the weak equilibrium equation for a bar of length L in torsion that is subjected to a constant distributed torque $t(z) = t_o$, an applied end-rotation $\phi(0) = \bar{\phi}$, and an applied torque T_L at $z = L$.

(6.8) Consider the elastic rod shown below. Starting from the relevant governing ordinary differential equation, derive the relevant statement of virtual work (weak form expression). Be sure to explicitly define your space of trial solutions and test functions.



(6.9) Consider a tension-compression bar with N applied forces F_i at $x = iL/N$ for $i = 1, 2, \ldots, N$. Formulate the principle of virtual work for this system assuming that $R(0) = 0$.

(6.10) Consider a tension-compression bar where $u(0) = 0$. The bar is subject to 3 forces $F_i = i\bar{F}$ $(i = 1, 2, 3)$ for $\bar{F} = 1$ N at $x_i = iL/3$, where $L = 3$ mm. Find an approximate solution to the weak form by using a sub-space of functions that includes linear and quadratic terms. Assume that $A = 1$ mm^2 and $E = 10$ MPa.

(6.11) Consider the potential energy for the system described in Exercise 6.7. Compute the variational derivative of the energy. Your answer should match the result you got in Exercise 6.7.

(6.12) Consider a (Bernoulli-Euler) beam of length L and bending stiffness EI subject to a transverse loading $q(x)$ and boundary conditions $\theta(0) = \theta_o$, $V(0) = V_o$, $V(L) = V_L$, and $\theta(L) = \theta_L$. Starting from the strong form, derive the weak equilibrium equation for the beam. Use a space of test functions that eliminates any end reactions.

(6.13) Consider a problem with unknown $w(x)$, where $w(x)$ can be found by minimizing the following po-

tential energy expression:

$$\Pi[w(x)] = \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L b(x)w(x)\,dx\,,$$

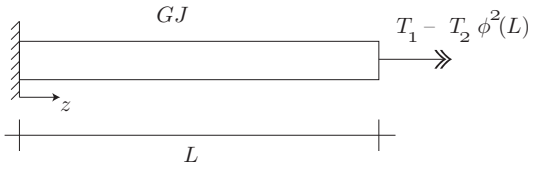
where $b(x)$ is a given loading and the domain of interest is $[0, L]$. Assume that $w(x)$ and its first derivative are known at $x = 0$ and $x = L$, such that the space of solutions can be expressed as

$$\mathcal{S} = \left\{w(x) \mid w(0),\, w'(0),\, w(L),\, w'(L) \text{ known}\right\}\,.$$

(a) Define an appropriate space of test functions, $\mathcal{V}$.

(b) Find the weak form expression (virtual work equation) for this problem by taking the variational derivative of Π.

(6.14) The torsion bar shown is subjected to a positive end-torque of the form $T_1 - T_2\phi^2(L)$, where $T_1, T_2 \in \mathbb{R}_+$ (positive scalars).



(a) Write the virtual work expression for this problem.

(b) Express the space of solutions as

$$\mathcal{S} = \{\phi(z) \mid \phi(z) = Az\}$$

and the space of test functions as

$$\mathcal{V} = \{\delta\phi(z) \mid \delta\phi(z) = \delta Az\}$$

and find the rotation field for this problem.

Two-dimensional problems

7

7.1 The Method of Ritz: Application to two dimensional problems

The systems to which one can apply the Method of Ritz and/or the Virtual Work Theorem is in no way restricted to the one-dimensional systems we have considered up to this point (tension-compression bars, torsion rods, beams). In fact, the application of these methods to multi-dimensional systems is quite similar to what we have already learned. This holds for multi-dimensional problems with scalar-valued unknowns as well as vector-valued unknowns. These multi-dimensional applications are in fact one of the main reasons that these concepts are so important. In this chapter, we will look at the extension of our methods to a few model two-dimensional problems where the primary unknowns are scalar-valued as well as vector-valued. Examples will be both structural as well as from other areas of engineering. In what follows, we will describe some important multi-dimensional problems and then examine the application of our methods to them.

7.1.1 Anti-plane shear

Anti-plane shear is a special case of three-dimensional elasticity. In three-dimensional elasticity there are 3 primary unknowns, the displacements $u(x,y,z),\ v(x,y,z),\ w(x,y,z)$ in the three coordinate directions x,y,z. In problems where anti-plane shear applies, $u \approx v \approx 0$ and $w = w(x,y)$ is not a function of z. This kinematic assumption reduces a three-dimensional elasticity problem to a single scalar-valued problem in two dimensions. For these assumptions to make reasonable sense the only allowed loadings must occur in the z-direction; see Fig. 7.1.

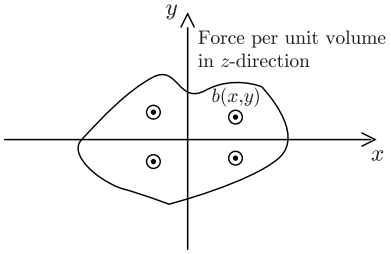


Fig. 7.1 Anti-plane strain body with distributed load.

The kinematic assumption for anti-plane shear gives rise to only two

non-zero strains:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0, \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0, \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0, \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\frac{\partial w}{\partial y} = \frac{1}{2}\gamma_{yz} \\
\varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\frac{\partial w}{\partial x} = \frac{1}{2}\gamma_{zx} \ .
\end{aligned}$$

The components of the strain can be arranged in a matrix as

$$\text{Strains}: \ \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial w}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{1}{2}\frac{\partial w}{\partial y} & 0 \end{bmatrix} \ .$$

If we wish to apply the Principle of Stationary Potential Energy, we need an expression for the stored elastic energy in a state of anti-plane shear. In anti-plane shear there are only two non-zero shear strains and the strain energy (density) associated with these two strains is given by

$$\frac{1}{2}G\gamma_{xz}^2 + \frac{1}{2}G\gamma_{yz}^2 \quad \text{[Energy/Volume]} \ .$$

The elastic energy over the whole domain is the integral of this quantity:

$$\Pi_{\text{elastic}} = \int_{\text{volume}} \frac{1}{2}G\left[\gamma_{xz}^2 + \gamma_{yz}^2\right] \, dV \ .$$

In order to form the total potential energy, one also needs an expression for the potential of the load. If we consider a distributed load per unit volume in the z-direction, $b(x, y)$, then

$$\Pi_{\text{load}} = -\int_{\text{volume}} b \ w \ dV \ .$$

The sum of the two quantities yields the total potential energy:

$$\begin{aligned}
\Pi_{\text{total}} &= \Pi_{\text{elastic}} + \Pi_{\text{load}} \\
&= \int_{\text{A}} \frac{1}{2}Gt\left[\gamma_{xz}^2 + \gamma_{yz}^2\right] \, dA - \int_{\text{A}} bt \ w \ dA \ .
\end{aligned}$$

In the above we have assumed that the thickness t of the body is constant. This has allowed us to to explicitly integrate out the z direction, since all the other quantities are assumed to be independent of z.

Note that the expression for $\Pi_{\text{total}}(w)$ depends on the function $w(x, y)$ which in turn depends on the coordinates x, y. For w to satisfy equilibrium, it must be a stationary point of Π_{total} – the Principle of Stationary

Potential Energy. Compared with the 1-dimensional case which depends only on x, w here depends on two variables x, y. As before, let us consider the determination of approximate solutions. In that case we need to construct an approximate form for $w(x, y)$ and like before we will require the approximations to satisfy the the kinematic boundary conditions. In fact we can proceed in an identical manner as we did with one-dimensional problems and write

$$w(x, y) \approx \sum_{A=1}^{N} w_A f_A(x, y),$$

where $f_A(x, y)$ are known functions and w_A are coefficients which must be determined. The only difference with the 1-dimensional case is that the functions f_A depend not just on the x coordinate but also on the y coordinate.

For compactness, we use the following notation:

$$\begin{aligned} \nabla w &= \begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} \\ \nabla w \cdot \nabla w &= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 . \end{aligned}$$

Substituting the approximation into the potential energy yields,

$$\begin{aligned} \Pi_{\text{total}}(w) \Rightarrow \Pi_{\text{total}}(\boldsymbol{w}) &= \int_A \frac{1}{2} Gt \nabla w \cdot \nabla w \; dA - \int_A btw \; dA \\ &= \int_A \frac{1}{2} Gt \left(\sum_{A=1}^{N} w_A \nabla f_A\right) \cdot \left(\sum_{A=1}^{N} w_A \nabla f_A\right) \; dA \\ &- \int_A bt \left(\sum_{A=1}^{N} w_A f_A(x, y)\right) \; dA \; . \end{aligned}$$

The stationary condition on Π_{total} implies

$$\begin{aligned} 0 = \frac{\partial \Pi_{\text{total}}}{\partial w_D} &= \int_A \frac{1}{2} Gt \left[\nabla f_D \cdot \left(\sum_{A=1}^{N} w_A \nabla f_A\right) + \left(\sum_{A=1}^{N} w_A \nabla f_A\right) \cdot \nabla f_D\right] \; dA \\ &\quad - \int_A btf_D \; dA \\ &= \sum_{A=1}^{N} \left[\int_A Gt \nabla f_D \cdot \nabla f_A \; dA\right] w_A - \int_A btf_D \; dA \\ &= \sum_{A=1}^{N} K_{DA} w_A - F_D \\ &\Rightarrow \quad \boldsymbol{Kw} - \boldsymbol{F} \; . \end{aligned}$$

Here $\boldsymbol{w}$ denotes a vector with entries w_A $(A = 1, \ldots, N)$. We have also defined the matrix $\boldsymbol{K}$ (N-by-N) and vector $\boldsymbol{F}$ (N-by-1) which have

entries

$$\begin{aligned} K_{DA} &:= \int_A Gt\nabla f_D \cdot \nabla f_A \; dA, \\ F_D &:= \int_A bt f_D \; dA \; . \end{aligned}$$

By solving the linear system $\boldsymbol{K}\boldsymbol{w} = \boldsymbol{F}$ for the undetermined coefficients $\boldsymbol{w}$, one obtains the approximate solution as

$$w_{\text{approx}} = \sum_{A=1}^{N} w_A f_A(x, y) \; .$$

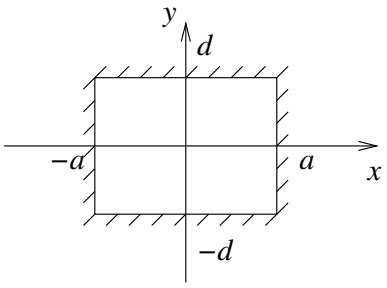


Fig. 7.2 Anti-plane shear problem setup: loading $b(x, y) = b_o$, through thickness dimension t, and shear modulus G. Edges are all restrained.

Example 7.1

Anti-plane shear rectangular domain Consider an anti-plane shear problem defined on a rectangular domain of side lengths $2a$ and $2d$ with fixed boundary conditions on the perimeter. The coordinate system is set up as is shown in Fig. 7.2. Assume further that the body force is constant; i.e., $b(x, y) = b_0$. Find an approximate solution to this problem.

Solution: In order to apply the Method of Ritz, one must select an approximate form for the solution which satisfies the kinematic boundary condition. This approximate form can be constructed by multiplying two functions which satisfy the boundary conditions in x and y separately. Possible choices for a function which is equal to zero at $x = -a, a$ are,

$$\cos\left(\frac{\pi}{2}\frac{x}{a}\right) \text{ or } \left(1 - \left(\frac{x}{a}\right)^2\right) \; .$$

Possible choices for a function which is equal to zero at $y = -d, d$ are,

$$\cos\left(\frac{\pi}{2}\frac{y}{d}\right) \text{ or } \left(1 - \left(\frac{y}{d}\right)^2\right) \; .$$

Multiplying these choices together generates approximation functions which satisfy all the boundary conditions:

$$\begin{aligned} f(x, y) &= \cos\left(\frac{\pi}{2}\frac{x}{a}\right)\cos\left(\frac{\pi}{2}\frac{y}{d}\right) \; , \\ g(x, y) &= \left(1 - \left(\frac{x}{a}\right)^2\right)\left(1 - \left(\frac{y}{d}\right)^2\right) \; . \end{aligned}$$

Let us pick,

$$w(x, y) \approx C g(x, y) \; ,$$

as the approximate solution, where C is the undetermined coefficient. The total potential energy for this problem is,

$$\Pi_{\text{total}}(w) \Rightarrow \Pi_{\text{total}}(C) = \int_A \frac{1}{2} Gt\nabla w \cdot \nabla w \; dA - \int_A b_0 t w \; dA \; .$$

The gradient of our approximate form is

$$\nabla w = C \begin{bmatrix} \frac{2x}{a^2}\left(-1+\left(\frac{y}{d}\right)^2\right) \\ \frac{2y}{d^2}\left(-1+\left(\frac{x}{a}\right)^2\right) \end{bmatrix} ,$$

and

$$\nabla w \cdot \nabla w = C^2 \left\{ \frac{2x}{a^2}\left[-1+\left(\frac{y}{d}\right)^2\right]\right\}^2 + C^2 \left\{ \frac{2y}{d^2}\left[-1+\left(\frac{x}{a}\right)^2\right]\right\}^2 ,$$

which leads to the expression for Π_{total}:

$$\begin{aligned} \Pi_{\text{total}}(C) &= C^2 \int_A \frac{1}{2} Gt \left\{ \frac{2x}{a^2}\left[-1+\left(\frac{y}{d}\right)^2\right]\right\}^2 + \left\{ \frac{2y}{d^2}\left[-1+\left(\frac{x}{a}\right)^2\right]\right\}^2 dA \\ &\quad -C \int_A b_0 t \left(1-\frac{x}{a}^2\right)\left(1-\frac{y}{d}^2\right) dA \\ &= C^2 \mathbb{A} - C\mathbb{B} . \end{aligned}$$

Here we have denoted the definite integrals as $\mathbb{A}$ and $\mathbb{B}$ for compactness. The stationary condition on the potential energy yields:

$$\frac{\partial \Pi_{\text{total}}}{\partial C} = 2C\mathbb{A} - \mathbb{B} = 0,$$

from which C is determined as:

$$C = \frac{\mathbb{B}}{2\mathbb{A}} = \frac{5}{8}\frac{b_0}{G}\frac{a^2 d^2}{a^2+d^2} .$$

Thus the approximate solution is:

$$w(x,y) \approx \frac{5}{8}\frac{b_0}{G}\frac{a^2 d^2}{a^2+d^2}\left(1-\left(\frac{x}{a}\right)^2\right)\left(1-\left(\frac{y}{d}\right)^2\right) .$$

Remarks:

(1) As a simple check one can verify the dimensions of the solution. In this case

$$\frac{b_0}{G}\frac{a^2 d^2}{a^2+d^2} = \frac{[F/L^3]}{[F/L^2]}\frac{[L^2][L^2]}{[L^2]} = [L]$$

and the dimensions are seen to correctly be length.

(2) If we had used the cosine guess, $f(x,y)$, then the result would have been:

$$w(x,y) \approx \frac{64}{\pi^4}\frac{b_0}{G}\frac{a^2 d^2}{a^2+d^2}\cos\left(\frac{\pi}{2}\frac{x}{a}\right)\cos\left(\frac{\pi}{2}\frac{y}{d}\right) ,$$

which also has the correct units.

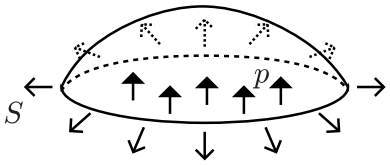


Fig. 7.3 Thin membrane problem under the action of a distributed transverse pressure $p(x, y)$ and a constant edge tension S.

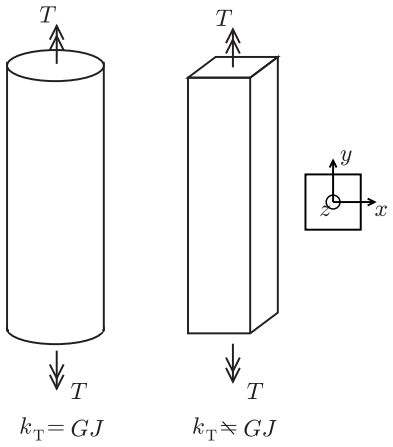


Fig. 7.4 Torsional stiffness problem for non-circular cross-sections.

7.1.2 Thin Membrane

The problem of solving for the deflection $w(x, y)$ of a thin membrane acted upon by a transverse pressure $p(x, y)$ and under constant tension S [F/L] has a total potential energy with a form identical to the anti-plane shear case. The setup of this problem is shown in Fig. 7.3. The total potential energy for such a system is

$$\Pi_{\text{total}}(w) \quad = \quad \int_A \frac{1}{2} S \nabla w \cdot \nabla w \; dA - \int_A p \; w \; dA \; ,$$

where $p(x, y)$ is a pressure applied in the direction orthogonal to the plane of the membrane. An identical procedure to the one described for anti-plane shear applies to thin membrane problems.

7.1.3 Non-circular solid cross section bars in torsion

In the torsion of bars with non-circular solid cross sections, one cannot use the assumptions made in the treatment of bars with circular cross-sections. In particular when the cross-section of the bar is non-circular, the cross-sections do not remain planar upon twisting; they warp out of plane. To solve such problems, one requires the equations of three-dimensional elasticity. The general theory associated with such problems is known as St.-Venant's theory of torsion. A central result of this theory is that the torsional stiffness of a non-circular torsion rod

$$k_T \quad := \quad \frac{T}{\phi'} \, ,$$

where T is the applied torque and ϕ' is the twist rate. The torque is given by a simple integral:

$$T \quad := \quad 2 \int_A \varphi \; dA \; .$$

The function φ in this expression is known as Prandtl's stress function. The governing potential for φ happens to be identical in form to the one for anti-plane shear; viz.,

$$\Pi_{\text{total}}(w) \quad = \quad \int_A \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \; dA - \int_A 2G\phi' \; \varphi \; dA \; ,$$

where G is the shear modulus. The boundary condition on φ is $\varphi = 0$ on the boundary of the domain A. Once φ is known, one can also obtain quantities other than T and k_T such as the torsional shear stresses σ_{xz} and σ_{yz} via

$$\begin{aligned} \sigma_{xz} \quad &= \quad \frac{\partial \varphi}{\partial y} , \\ \sigma_{yz} \quad &= \quad -\frac{\partial \varphi}{\partial x} \; . \end{aligned}$$

The procedure for solving such problems is identical to that described above.

7.1.4 Underlying equilibrium equations

Recall that in the application of the Principle of Stationary Potential Energy for one-dimensional problems, there is always an equilibrium equation whose solutions correspond to stationary points of the total potential energy. For example, with the tension-compression bar the governing differential equation for equilibrium is

$$(AEu')' + b = 0$$

and the corresponding potential energy is

$$\Pi_{\text{total}} \quad = \quad \int_0^L \frac{1}{2} AE(u')^2 \, dx - \int_0^L bu \, dx \; .$$

For the two dimensional problems that have been introduced, the underlying governing differential equation for equilibrium has not been mentioned up to now. For completeness we remark that these equilibrium equations all have the same form:

$$\blacksquare \nabla^2 w + \blacktriangle = 0 \; ,$$

where $\blacksquare$ and $\blacktriangle$ represent different constants depending on the mechanical problem at hand:

- Anti-plane shear: $\blacksquare = Gt$ and $\blacktriangle = bt \quad \Rightarrow \quad Gt\nabla^2 w + bt = 0.$
- Thin membrane: $\blacksquare = S$ and $\blacktriangle = p \quad \Rightarrow \quad S\nabla^2 w + p = 0.$
- Non-circular torsion: $\blacksquare = 1$ and $\blacktriangle = 2G\phi' \quad \Rightarrow \quad \nabla^2 w + 2G\phi' = 0.$

This form of equation is known as Poisson's equation and in certain cases can be solved directly through the method of separation of variables. However for general domains and in cases where the coefficients are not constants, approximate solutions are the norm as hand solutions are virtually impossible.

Example 7.2

Anti-Plane Shear

We illustrate here the proof that our multidimensional potential energy formulation is equivalent to the partial differential equations given above. We will do this for the anti-plane shear case. Consider a domain A with boundary ∂A, where w is stated to be zero on ∂A. In this case, $\mathcal{S} = \{w(x,y) \mid w = 0 \text{ for } (x,y) \in \partial A\}$ and $\mathcal{V} = \{\delta w(x,y) \mid \delta w = 0 \text{ for } (x,y) \in \partial A\}$. The minimization problem is

$$\min_{w \in \mathcal{S}} \Pi[w(x,y)] = \min_{w \in \mathcal{S}} \left[\int_A \frac{1}{2} Gt \nabla w \cdot \nabla w \; dxdy - \int_A btw \; dxdy \right] .$$

Note that if we know the true solution w then $\Pi[w] \leq \Pi[w+\alpha\delta w]$ for any $\alpha \in \mathbb{R}$. This implies that the function $h(\alpha) \equiv \Pi[w + \alpha\delta w]$ is minimized at $\alpha = 0$ for any $\delta w \in \mathcal{V}$ when w is the true solution. Thus we know that

$$\left.\frac{dh}{d\alpha}\right|_{\alpha=0} = 0\,.$$

Expanding this gives

$$\begin{aligned}
0 &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \left[\int_A \frac{1}{2} Gt\nabla(w+\alpha\delta w)\cdot\nabla(w+\alpha\delta w)\; dxdy \right. \\
&\qquad \left. - \int_A bt(w+\alpha\delta w)\; dxdy\right] \\
&= \int_A \frac{1}{2} Gt\left(\nabla w\cdot\nabla\delta w + \nabla\delta w\cdot\nabla w\right)\; dxdy - \int_A bt\delta w\; dxdy \\
&= \int_A Gt\nabla\delta w\cdot\nabla w\; dxdy - \int_A bt\delta w\; dxdy \\
&= \int_A Gt\left(\nabla\cdot(\delta w\nabla w) - \delta w\nabla^2 w\right)\; dxdy - \int_A bt\delta w\; dxdy \\
&= \int_{\partial A} Gt\underbrace{\delta w}_{0 \text{ on } \partial A}\nabla w\cdot\boldsymbol{n}\; ds - \int_A \delta w\left(Gt\nabla^2 w + bt\right)\; dxdy \\
&= -\int_A \delta w\left(Gt\nabla^2 w + bt\right)\; dxdy\,.
\end{aligned}$$

This last expression must hold true for all virtual displacements δw and thus one must necessarily have

$$Gt\nabla^2 w + bt = 0$$

at equilibrium. This proves that our potential energy formulation necessarily implies the partial differential equation form. One can also show the converse by simply reversing the steps shown. Thus the potential energy formulation is fully equivalent to the partial differential form.

7.1.5 Plate bending

As a last mechanical example of a system that can be described by a scalar-valued function of two variables one can consider the bending of a thin plate. In such problems, the transverse deflection of the plate is governed by a potential energy composed of the loads and the elastic strain energy of the plate. For isotropic linear elastic materials, the latter is given by:

$$\begin{aligned}
\Pi_{\text{elastic}} = \int_\Omega \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right. \\
\left. + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right] dA\,,
\end{aligned}$$

where $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of a plate with thickness h. If the loading is given by a pressure field, then the potential of the load is given by

$$\Pi_{\text{load}} = -\int_\Omega p(x,y)w(x,y)\,dA\,.$$

The total potential energy is more complex than our previous examples but its underlying features are the same. It is a scalar-valued functional[1] and its stationary points give rise to the equilibrium solutions. Given a domain Ω, loading $p(x,y)$, and boundary conditions on $w(x,y)$ one can proceed as above to find approximate solutions to the equilibrium equations.

[1] A functional is a function of a function.

Remarks:

(1) The underlying differential equation for w is the so-called bi-harmonic equation: $D\nabla^4 w = p$, where $\nabla^4(\cdot) = \nabla^2(\nabla^2(\cdot))$. It is the two-dimensional counter part to the Bernoulli-Euler beam equation and is sometimes known as the Kirchhoff-Love plate equation.

7.2 Piezometrically driven transport of ground water

In this section, we look briefly at a non-structural mechanics problem to illustrate the generality of the methods we have developed. In particular we briefly examine the equations governing the transport of ground water due to piezometric forces. To keep matters simple, we will restrict our attention to strictly two-dimensional domains. A basic problem in geotechnical engineering is that of ground water transport or seepage. Water, water content, and water flow play an important role in both natural and engineered soil systems. The basic variables that describe the relevant physics are the water content $c(x,y)$ (the ratio of local water mass to local soil mass) and the flux of water $\boldsymbol{q}(x,y)$ measured in dimensions of water volume per unit time per unit area. Flux is a vectorial quantity since flow has direction. For steady incompressible flow along a streamline in regions where there is no heat or work transfer to the fluid Bernoulli's equation tells us that:

$$\frac{v^2}{2g} + \frac{p}{\rho g} + z = h \text{ (a constant)}\,.$$

This is a statement of conservation of energy along a lossless streamline. The first term is known as the velocity head, the second the pressure head, and the third the elevation head. If there are frictional losses in the system then the right hand side can be augmented with a head loss term. In standard geotechnical settings it is safe to neglect the velocity head.

In seepage problems, the head is a function of position in the soil $h(x,y)$ and the flux of water is proportional to the gradient of the head

– water flows from high head to low head. The simplest model of this states that the relation is linear. This is known as Darcy's law:

$$\boldsymbol{q} = -k\nabla h\,,$$

where k is the permeability of the soil and it has dimensions of length per unit time. Typical values of k range from 1 cm/s for well drained conditions (say sands) to 10^{-4} cm/s for poor drainage (say very fine sands) to lower values for clays.

In order to determine the head, $h(x, y)$, we need a governing equation. This comes to us via conservation of mass. If we consider an arbitrary volume of soil, then any water that flows into the volume must also flow out under steady-state conditions. In equations, this says:

$$\int_A \boldsymbol{q}\cdot\boldsymbol{n}\,dA = 0\,,$$

where A is the surface of the volume and $\boldsymbol{n}$ is the outward normal to the volume. Using the divergence theorem, this tells us that for any volume:

$$\int_V \nabla\cdot\boldsymbol{q}\,dV = 0\,.$$

Since this must hold for any volume V, the integrand must be zero:

$$\nabla\cdot\boldsymbol{q} = 0\,.$$

This is a differential expression of conservation of mass. The same result can also be derived using a differential element construction. If we now combine this with Darcy's Law, then we obtain the governing equation for the head:

$$\nabla\cdot(k\nabla h) = 0\,,$$

which written out in two-dimensions (for constant k) says,

$$k\frac{\partial^2 h}{\partial x^2} + k\frac{\partial^2 h}{\partial y^2} = 0\,.$$

We recognize this to have the same functional form as we discussed for anti-plane shear, torsional warping, and membrane deflection. Thus by analogy, we can directly write a stationary (minimum) potential "energy" version of the seepage problem as

$$\min\Pi[h(x,y)] = \int_\Omega \frac{1}{2}k\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]dxdy\,.$$

To solve the minimization problem we can construct an expansion

$$h(x,y) = \sum_A h_A f_A(x,y)\,,$$

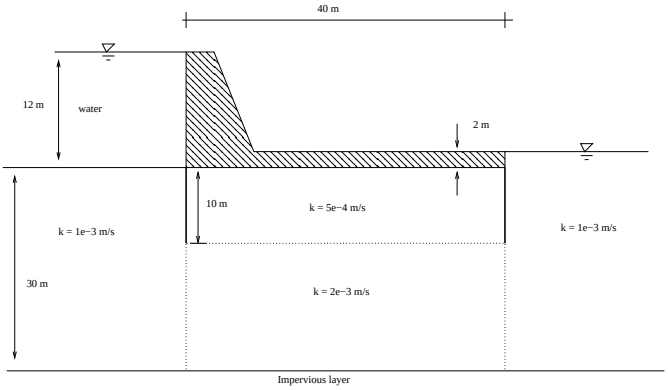


Fig. 7.5 Geometry of dam example.

where $f_A(x, y)$ are known functions as before and h_A are unknown parameters that need to be determined by minimizing the potential energy. The resulting matrix equations are simply:

$$\boldsymbol{K}\boldsymbol{h} = \boldsymbol{0}\,,$$

where $\boldsymbol{h}$ is the vector of head parameters and

$$K_{AB} = \int_\Omega k\nabla f_A \cdot \nabla f_B \, dxdy = \int_\Omega k\frac{\partial f_A}{\partial x}\frac{\partial f_B}{\partial x} + k\frac{\partial f_A}{\partial y}\frac{\partial f_B}{\partial y}\, dxdy\,.$$

By identifying boundary conditions and separating free from driven degrees of freedom, one can then solve these equations. For simple settings products of polynomials and trigonometric functions are suitable. For more general settings the use of local hat-like functions, i.e. those with compact support, are more feasible. From such a computation one obtains the head $h(x, y)$. This can then be used to find quantities like the pressure head ($h_{\text{pressure}} = h - h_{\text{elevation}}$) at various points and thus compute uplift forces; further, the magnitude of the gradient of the head can be computed $\|\nabla h\|$ to check for the relative magnitude of the forces on the soil particles.

Example 7.3

Seepage under a dam

As an application example, let us consider computing the seepage under a concrete dam with sheet piles. The geometry is shown in Fig. 7.5 with the water level of 12 m and varying soil permeabilities above an impervious layer. The boundary condition associated with the impervious layer is that there is no transport of water through it; i.e. $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on the surface $z = -32$ m. This boundary condition is also appropriate at the sheet piles and on the surfaces of the dam itself. Under the water at the soil interface, the total head is computed to be $h = 10$ m from Bernoulli's equation where the elevation head datum has been taken at the tailwater elevation (which makes the total head at the top surface of the water equal to 10 m. At the tailwater elevation the total head $h = 0$ m. To the right and the left we need to cut-off the domain to make it finite. For convenience we will assume that the flux is zero on these boundaries.

To compute the solution, we will break up the domain into triangular finite elements as shown in Fig. 7.6. This triangulation defines a set of natural hat-like functions which we can then use to minimize the system's potential energy. The results for the total head are shown in Fig. 7.7 and the streamlines due to ∇h with isocontours of head are shown in Fig. 7.8. The flux vectors are given in Fig. 7.9. All of these results are post-processed from the head field $h(x, y)$. A plot of the pressure head under the dam is shown in Fig. 7.10. The pressure head in this case is computed as $h_{\text{pressure}} = h - (-2)$. The integral of this quantity gives the uplift force on the dam. Figure 7.11 shows the magnitude of the piezometric gradient near the bottom of the sheet piles. The magnitude in this region is seen to higher than elsewhere in the domain and reflects the fact that the soil particles in this region are experiencing higher levels of shear due to water flow, than in other regions.

7.3 Virtual Work (Weak Form): Application to two dimensional problems

All of the problems treated in the previous section can also be addressed using virtual work (or weak equilibrium). In fact, if one uses the same approximation spaces for the primary unknown and the test functions, then one will recover the exact same discrete expressions. In cases, however, where there is no potential, then one needs to address the problem using virtual work/weak equilibrium.

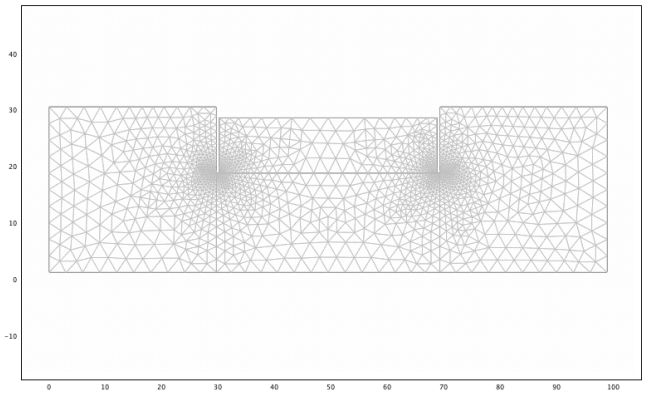


Fig. 7.6 Triangular mesh for computation.

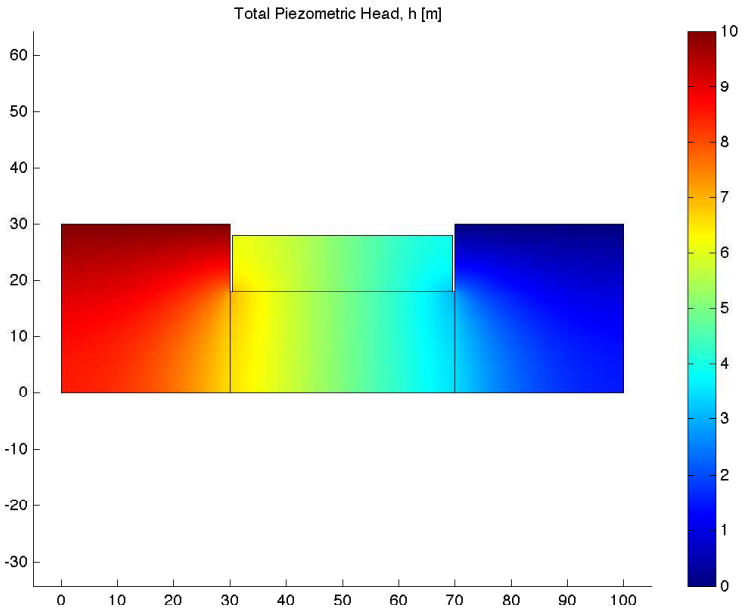


Fig. 7.7 Head distribution.

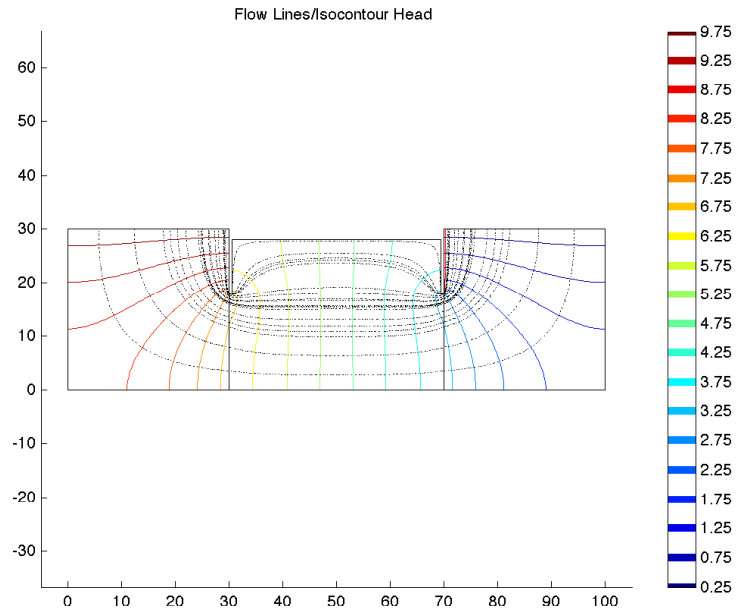


Fig. 7.8 Flow lines (dashed black lines) and isocontours of head (solid colored lines).

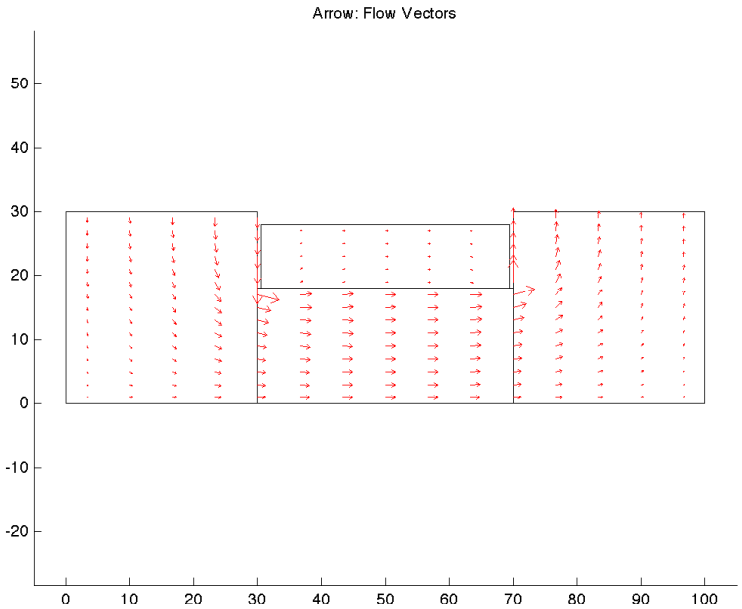


Fig. 7.9 Water flux.

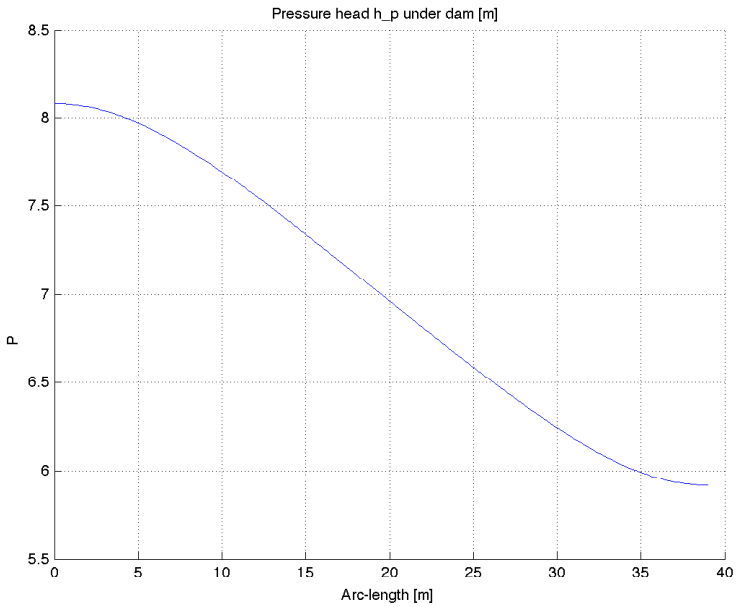


Fig. 7.10 Pressure head under dam.

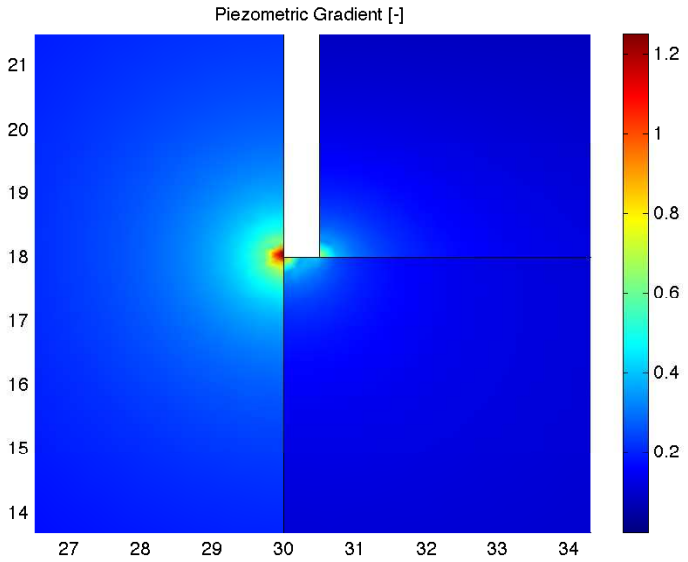


Fig. 7.11 Magnitude of piezometric gradient near the bottom of the sheet piles.

7.3.1 Advection diffusion for pollutant transport

As a concrete example, let us consider the problem of pollutant transport in a fluid (air for example). In such problems, we are interested in tracking the concentration of chemical species in space and time. The concentration, $c(\boldsymbol{x}, t)$, is measured as mass per unit volume, e.g. $\mathrm{mg/m^3}$. It is transported through the domain in two ways: (1) diffusion and (2) advection/convection. The two effects give rise to the total flux of the chemical species being tracked:

$$\boldsymbol{J} = \boldsymbol{J}_{\text{diffusion}} + \boldsymbol{J}_{\text{convection}} \,.$$

The dimensions of flux are mass per unit area per unit time. Random motion gives rise to diffusive fluxes and to first order they can be described by

$$\boldsymbol{J}_{\text{diffusion}} = -\boldsymbol{\varepsilon} \nabla c \,,$$

where $\boldsymbol{\varepsilon}$ is a matrix whose entries are known as the turbulent diffusivities; these are in a way material parameters of the whole system and can often depend upon position. The dimensions of a diffusivity is area per unit time. The advective flux is the material flux due to a moving fluid carrying along the chemical species. It is expressed as

$$\boldsymbol{J}_{\text{convection}} = c\boldsymbol{v} \,,$$

where $\boldsymbol{v}$ is the velocity of the fluid and can be a function of position (and time). The governing conservation law is the balance of mass. This is the same relation we considered in the seepage problem. In the steady state, this yields the conservation law:

$$-\nabla \cdot \boldsymbol{J} = 0 \,.$$

Combining the flux law with the mass balance equation gives the final (two-dimensional) advection-diffusion equation:

$$\boldsymbol{v} \cdot \nabla c = \frac{\partial}{\partial x}\left[\varepsilon_x \frac{\partial c}{\partial x}\right] + \frac{\partial}{\partial y}\left[\varepsilon_y \frac{\partial c}{\partial y}\right] \,,$$

where it has been assumed for simplicity that the off-diagonal terms in $\boldsymbol{\varepsilon}$ are zero and that $\boldsymbol{v}$ is a constant.

In order to focus on the essence of the problem, let us examine it in one-dimension where it reads:

$$v\frac{dc}{dx} = \frac{d}{dx}\left[\varepsilon \frac{dc}{dx}\right] \,.$$

Unlike the seepage problem, a potential formulation for this equation is not possible. In terms of the methodologies which we have developed to this point, our main option is to convert this equation into a virtual work or weak equilibrium/balance form. For concreteness let us suppose that we wish to solve this equation over the domain $[0, L]$ with a specified concentration at the left-end $x = 0$ of $c(0) = c_o$ and a specified diffusive

flux $J_{\text{diffusive}}(L) = J_L^D$ at the right-end $x = L$. The true solution lies in the solution space

$$\mathcal{S} = \{c(x) \mid c(0) = c_o\}\,.$$

To proceed let us define a test function space (space of virtual concentrations)

$$\mathcal{V} = \{\delta c(x) \mid \delta c(0) = 0\}\,.$$

Here we have followed the same pattern as before of requiring the test functions to be zero at points where we know the value of our primary unknown. To generate a weak form balance equation, we can multiply our governing balance equation by an arbitrary test function and then manipulate until we have it in an acceptable form. In this case, our steps produce the following.

$$\begin{aligned}
\delta c\, v\frac{dc}{dx} &= \delta c\frac{d}{dx}\left[\varepsilon\frac{dc}{dx}\right] \\
\int_0^L \delta c\, v\frac{dc}{dx} - \delta c\frac{d}{dx}\left[\varepsilon\frac{dc}{dx}\right] dx &= 0 \\
\int_0^L \delta c\, v\frac{dc}{dx} + \frac{d\delta c}{dx}\varepsilon\frac{dc}{dx}\, dx &= \left[\delta c\,\varepsilon\frac{dc}{dx}\right]_0^L \\
\int_0^L \delta c v\frac{dc}{dx} + \frac{d\delta c}{dx}\varepsilon\frac{dc}{dx}\, dx &= -\delta c(L) J_L^D\,. \qquad (7.1)
\end{aligned}$$

The basic (weak form) problem can then be stated as needing to find $c \in \mathcal{S}$ such that (7.1) holds for all $\delta c \in \mathcal{V}$.

As before to make this infinite dimensional search feasible, one can approximate the space of solutions and the space of test functions via finite dimensional subsets; i.e. we can create subsets $\hat{\mathcal{S}} \subset \mathcal{S}$ and $\hat{\mathcal{V}} \subset \mathcal{V}$ which are parameterized by a finite set of parameters:

$$\begin{aligned}
\hat{\mathcal{S}} &= \{c(x) \mid c(x) = \sum_i c_i f_i(x)\} \\
\hat{\mathcal{V}} &= \{\delta c(x) \mid \delta c(x) = \sum_j \delta c_j g_j(x)\}\,,
\end{aligned}$$

where the functions f_i and g_j are known and satisfy the necessary conditions for the approximation spaces to be subsets of the full spaces. If we choose $f_i(x) = g_i(x)$ then we obtain a Bubnov-Galerkin method. For the advection-diffusion equation this is known to not be optimal and in fact can lead to poor (unstable) results if too few terms are used in the expansions. A more common choice for this problem class is to pick different (though related) functions resulting in a Petrov-Galerkin method. A common choice would be to use linear hat functions for $f_i(x)$ and then to choose $g_i(x) = f_i + \text{sign}(v)df_i/dx$; this selection results in what is known as an up-winding method. It places weight in the weak form balance of mass to the *upwind* side of the flow. In this setting, the final discrete equations have the form $\sum_i K_{ji}c_i = F_j$, where

$$K_{ji} = \int_0^L g_j v\frac{df_i}{dx} + \frac{dg_j}{dx}\varepsilon\frac{df_i}{dx}\, dx$$

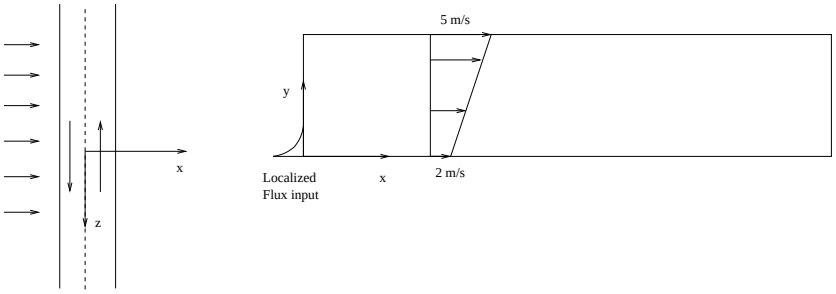


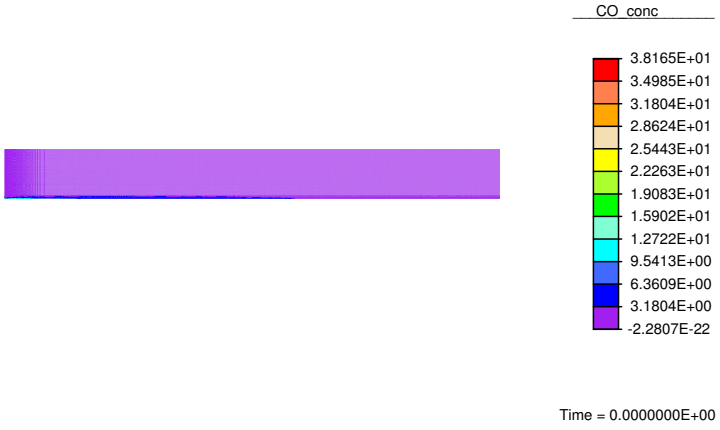
Fig. 7.12 Geometry and coordinate system for CO emission computation.

and F_j depends upon the particular boundary conditions assumed.

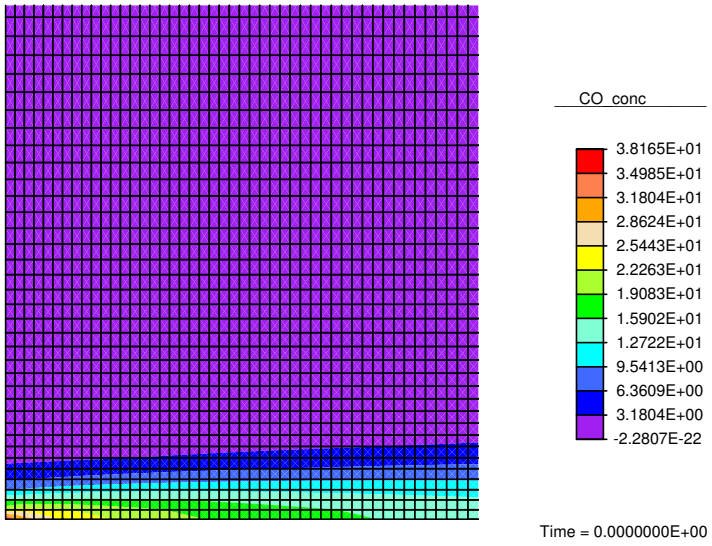
Example 7.4

Steady state CO concentration by a freeway As an example let us consider the situation described in Nazaroff and Alvarez-Cohen *Environmental Engineering Science* (2001, Ex. 7.D.2). In this problem we wish to determine the CO concentration distribution downwind from a freeway (in the steady state). On the freeway in question, 10,000 cars pass by in each direction per hour and are traveling at an average speed of 50 mi/hr. The vehicles are assumed to emit 20 gCO/mi. This emission diffuses away from the freeway as well as being convected away. For an analysis domain, we will look at a representative two dimensional vertical slice orthogonal to the freeway; see Fig. 7.12. Using the given data, one can show that the vehicles emit 70 mg/(m s) of CO into the domain. For the advection we will assume that the ground level wind speed is 2 m/s and that it increases linearly to 5 m/s at an elevation of 100 m (the top of our analysis domain). The turbulent diffusivity in the horizontal direction will be assumed as $\varepsilon_x = 0.1$ m^2/s and in the vertical direction we will assume that it varies parabolically as $\varepsilon_y = 0.1 + 0.1y(100 - y)/2500$ m^2/s. The dimensions of the analysis domain will be 1000 m by 100 m with zero flux boundary conditions top and bottom. At the left side, we will assume an inward boundary flux which is localized near ground level $\boldsymbol{J}(0, y) \cdot (-\boldsymbol{e}_x) = -70 \exp[-y/1.0]$. On the right side, we will assume a convective (only) boundary flux $\boldsymbol{J}_{\text{diffusive}}(1000, y) \cdot \boldsymbol{e}_x = 0$.

To create our approximation spaces we will use a two dimensional generalization of the hat functions that are built off quadrilaterals for the concentration field and their up-winding counterparts for the test functions. The quadrilaterals are made smaller near the emission source to increase the accuracy of the computation in that region (the region of high gradients). Figure 7.13(a) shows the concentration field for the overall domain using a 1000 by 100 grid (10,000 equations). A zoom view is shown in Fig. 7.13(b). Vertical and horizontal sections of the solution are shown in Fig. 7.14. The CO is seen to be localized near the ground level and then diffuses quite rapidly as one move away from the source. An examination of the flux vectors (not shown) of the CO close to the source reveals that the transport is mainly convective but the plume spreads vertically with diffusion. As noted the flux is composed of a convective component and a diffusive component. Looking at the separate components shows how the diffusive flux lifts the CO and mixes it in the vertical direction while the convective component moves it laterally without dispersion. Of importance for air quality standards is the ground level concentrations of CO and this is shown in Fig. 7.14(a) – teal-blue curve. The allowable 8hr exposure standard is 10 mg/m^3 which occurs $\sim$ 50 m from the freeway. At higher elevations the values drop off quickly; for example at 1.7 m elevation the concentration is always below 10 mg/m^3. If one recomputes the problem using emission values representative of 1960 values (roughly an order of magnitude higher than today's standards and four times higher than the number we have used in our analysis) then this distance increases to over 1000 m.

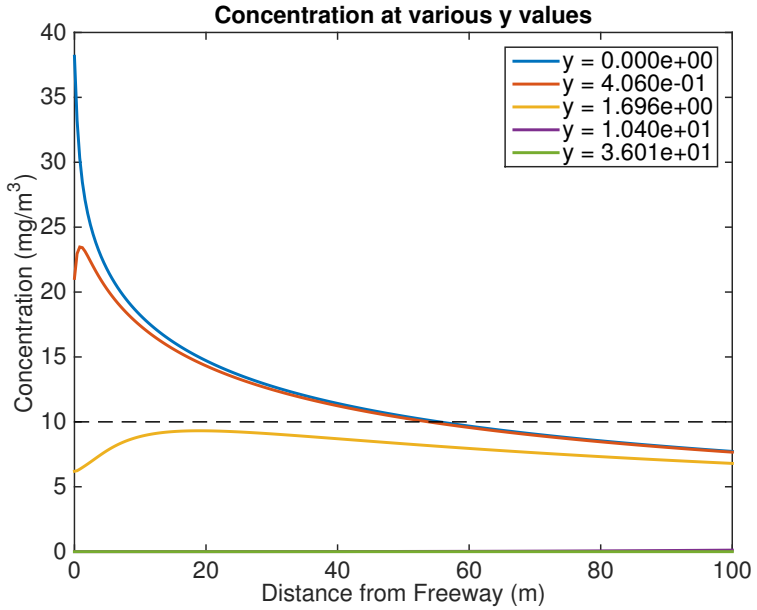


(a) Complete domain.

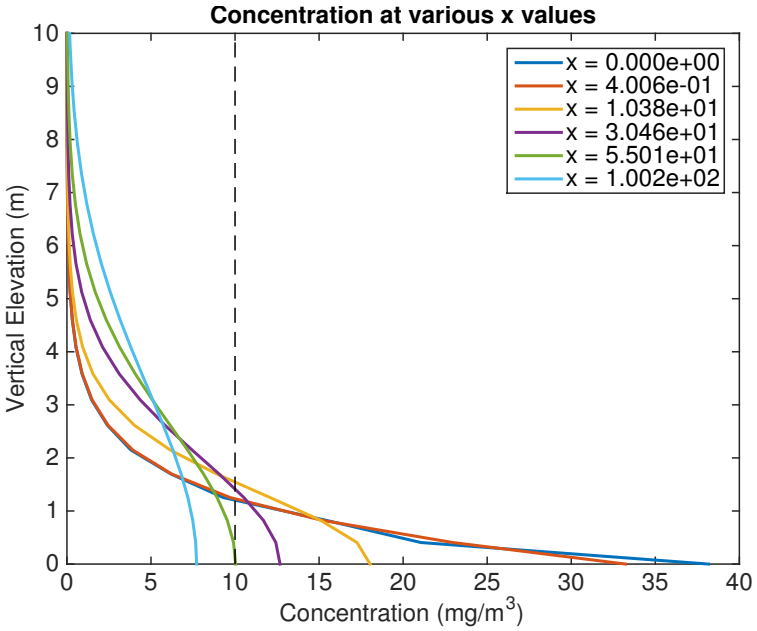


(b) Zoom view.

Fig. 7.13 Concentration field.



(a) Horizontal sections.



(b) Vertical sections.

Fig. 7.14 Concentration field sections.

Exercises

(7.1) Consider an elastic cylinder of outer radius R_o and inner radius R_i with length L. Assume that the outer radius is fixed (zero motion) and the inner radius is bonded to a rigid rod. A force F is applied to the end of the rod to move it an amount Δ. Using stationary potential energy find an expression for the deflection of the cylinder. Assume a one parameter solution (parameter C) of the form

$$w(r,\theta) = \Delta - \Delta\frac{r-R_i}{R_o-R_i} + C(r-R_i)(r-R_o) \in \mathcal{S}\,.$$

Approach this as an anti-plane shear problem. Also find the required force F. Note that $\tau_{rz} = G\gamma_{rz}$ and $\gamma_{rz} = \partial w/\partial r$ (for anti-plane shear).

(7.2) Consider a circular membrane of radius R which is restrained from deflection at its perimeter. The membrane is under a uniform tension of magnitude S and loaded with a uniform pressure p. Assume a deflection of the form:

$$w(r,\theta) = C\left[1-\left(\frac{r}{R}\right)^2\right]$$

and using stationary potential energy find an approximate expression for the center-point deflection. Note that in polar coordinates

$$\nabla w = \begin{pmatrix} \frac{\partial w}{\partial r} \\ \\ \frac{1}{r}\frac{\partial w}{\partial \theta} \end{pmatrix}$$

and that $\int_A(\cdot)dxdy \to \int_A(\cdot)rdrd\theta$.

(7.3) Consider a circular membrane of radius R which is restrained from deflection at its perimeter. The membrane is under a uniform tension of magnitude S and loaded with a point force P at its center. Assume a deflection of the form:

$$w(r,\theta) = C\left[1-\left(\frac{r}{R}\right)\right]$$

and using stationary potential energy find an approximate expression for the center-point deflection.

(7.4) Consider the membrane in Problem 7.2 but with a pressure loading of the form $p(r) = p_o\frac{r}{R}$. Find an expression for the displacement field using stationary potential energy using a guess of the form:

$$w(r,\theta) = \sum_{n=0}^{3} w_n\left(\frac{r}{R}\right)^n\,.$$

Since the guess does not automatically satisfy the kinematic boundary condition you can either add that constraint using a Lagrange multiplier or by simply eliminating one of the coefficients a priori and solving the minimization problem in terms of three unknown parameters instead of four. Note that the boundary conditions is $\sum_{n=0}^{3} w_n = 0$. Your result will actually be the exact solution.

(7.5) Consider a rectangular plate $\Omega = \{(x,y) \mid 0 \le x \le a \text{ and } 0 \le y \le b\}$. The plate has thickness h and is loaded at its center with a point force P (transverse to the plane of the plate). Assume that the plate is supported at its edges with simple supports (no displacement, free to rotate) and compute an approximate solution for the deflection of the plate of the form:

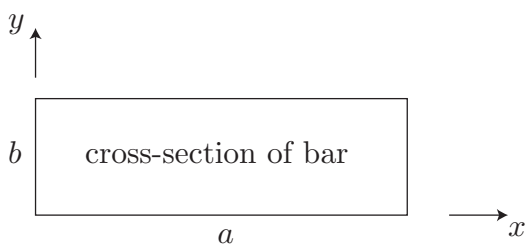
$$w(x,y) \approx C\sin(\pi x/a)\sin(\pi y/b)$$

using stationary potential energy.

(7.6) Consider a plate of dimension $2a\times 2b$ with clamped edges (zero motion and zero rotation) which is loaded with a uniform transverse pressure p_o. Assume an approximate solution in the form of a single period cosine function in x times a single period cosine function in y (coordinate origin at the center of the plate) and find an approximate expression for the center point displacement. [Hint: The exact solution $w(0,0) \approx 0.0759p_oa^4/Eh^3$ for $a/b = 1.5$ and $\nu = 0.3$. Your solution should be quite close to this.]

(7.7) Consider the potential energy for a stretched membrane of domain Ω with tension S and applied transverse pressure $p(x,y)$. Assume the membrane is fixed on its boundary $\partial\Omega$. Compute the variational derivative of the energy.

(7.8) Consider a rectangular bar with cross-sectional dimensions $a \times b$.



Determine an approximation for the torsional stiffness k_T by assuming an approximation of the form

$$\tilde{\mathcal{S}} = \{\varphi(x,y) \mid \varphi(x,y) = A\sin(\pi x/a)\sin(\pi y/b)\}$$

for the Prandtl stress function.

(7.9) Consider a square membrane with side length a which is clamped at its perimeter and subject to a uniform tension S and a transverse pressure $p(x,y)$.

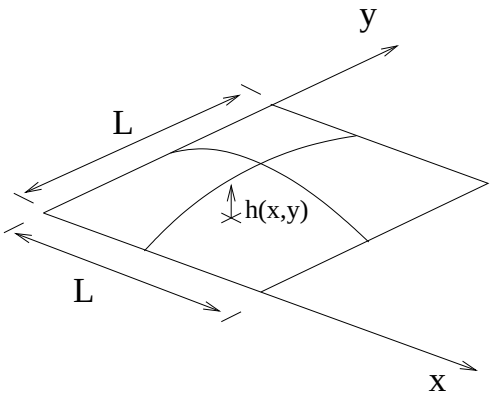
(a) Derive the weak form expression for this problem.

(b) Find a solution for the deflection of membrane by approximately solving the weak form. Use single parameter solution and test spaces.

(7.10) Bilayer lipid biomembranes are two dimensional plate-like structures, whose (elastic) bending energy is given by the so-called Helfrich energy functional:

$$\Pi_{\text{elastic}}[h(x,y)] = \int_A \frac{1}{2}B\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)^2 dxdy\,,$$

where A is the area of the membrane, B is the bending stiffness of the membrane (in units of energy), and $h(x,y)$ is the (transverse) deflection of the membrane as a function of x and y.

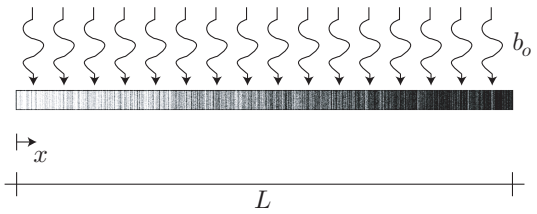


(a) Consider a square membrane of size $L \times L$ where $h = 0$ on the perimeter of the membrane and determine an approximate solution for the center deflection versus applied (uniform) normal pressure ($\Pi_{\text{load}} = -\int_A ph\ dxdy$). Assume $\hat{\mathcal{S}} = \{h(x,y) \mid h(x,y) = C\sin(\pi x/L)\sin(\pi y/L)\}$.

(b) If $L = 10$ μm, $B = 10^{-19}$ J, how much pressure is required to generate a 1 μm deflection?

(7.11) Consider Problem 6.13. What is the underlying governing differential equation for $w(x)$?

(7.12) A low-molecular weight substance γ is being injected into a one-dimensional reservoir at a constant rate b_o (g/m$^3 \cdot$ s). γ is being consumed by a chemical reaction at a spatially inhomogeneous rate $\alpha(x) = \alpha_1 + \alpha_2 x$ (1/s), where α_1, α_2 are given constants. It also diffuses laterally with a spatially inhomogeneous diffusivity $D(x) = D_1 + D_2 x$ (m^2/s), where D_1, D_2 are given constants.



If the concentration of γ is known at $x = 0$ and $x = L$, then the distribution of γ, $c(x) \in \mathcal{S} = \{c(x) \mid c(0) = c_o \text{ and } c(L) = c_L\}$ over the domain $[0, L]$, is the minimizer of the potential

$$\Pi[c(x)] = \int_0^L \frac{1}{2}D(x)(c')^2\ dx + \int_0^L \frac{1}{2}\alpha(x)c^2\ dx - \int_0^L b_o c\ dx$$

Determine the governing differential equation for $c(x)$.

(7.13) Consider the problem of Darcy flow (seepage) where the flux law has been modified to account for anisotropic permeability characteristics in the soil mass:

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = -\begin{bmatrix} k_x & k_{xy} \\ k_{xy} & k_y \end{bmatrix}\begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \end{pmatrix}.$$

Here k_x, k_{xy}, and k_y are given permeability constants. In this case the governing potential energy expression is of the form:

$$\Pi[h(x,y)] = \int_\Omega \frac{1}{2}\left[k_x\left(\frac{\partial h}{\partial x}\right)^2 + 2k_{xy}\left(\frac{\partial h}{\partial x}\right)\left(\frac{\partial h}{\partial y}\right) + k_y\left(\frac{\partial h}{\partial y}\right)^2\right] dxdy\,.$$

If one assumes an approximation of the form

$$h(x,y) = \sum_A h_A f_A(x,y)\,,$$

where $f_A(x,y)$ are known, then one can show that the linear equations governing the parameters h_A are given as $\sum_B K_{AB}h_B = 0$. Find an expression for K_{AB}.

Additional Reading

A

Statics:

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(3) Ottosen, N.S. and Petersson, H. (1992). *Introduction to the Finite Element Method*. Prentice-Hall, New York, NY.

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(5) Stewart, J. (2007). *Calculus: Early Transcendentals* (6th edn). Thompson, Brooks/Cole, Belmont, CA.

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(4) Slaughter, W. (2002). *The Linearized Theory of Elasticity*. Birkhäuser, Boston, MA.

(5) Ugural, A.C. and Fenster, S.K. (2011). *Advanced Mechanics of Materials and Applied Elasticity* (5th edn). Prentice-Hall, Upper Saddle River, NJ.

The finite element method:

(1) Hughes, T.J.R. (1987). *The Finite Element Method*. Prentice-Hall, Englewood Cliffs, NJ.

(2) Ottosen, N. and Petersson, H. (1992). *Introduction to the Finite Element Method*. Prentice Hall, Hertfordshire UK.

(3) Zienkiewicz, O.C., Taylor, R.L. and Zhu, J.Z. (2013). *The Finite Element Method: Its Basis & Fundamentals* (7th edn.) Butterworh-Heinemann, Oxford.

B

Dirac Delta Function

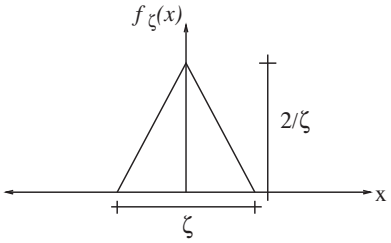

Fig. B.1 Distributed load representation of a localized force.

The Dirac Delta function from the study of ordinary differential equations is the proper mathematical representation of a point force (or other source of load). To see this, one must first observe that point forces are merely mathematical idealizations which we employ for convenience. In reality, it is impossible to apply a force at a point. Forces must be applied over finite areas. Figure B.1 shows one possible representation, $f_\zeta(x)$, of a distributed load that is localized in a region of width ζ near $x = 0$. Note that the total load represented by $f_\zeta(x)$ is given by

$$\text{Total force} = \int_{-\zeta/2}^{\zeta/2} f_\zeta(x)\, dx = \frac{1}{2}\zeta\frac{2}{\zeta} = 1, \tag{B.1}$$

independent of ζ. The idealization of a point force of magnitude 1 will then be given by $f_\zeta(x)$ in the limit as ζ goes to zero. We define this limit as $\delta(x)$; i.e.

$$\delta(x) = \lim_{\zeta\to 0} f_\zeta(x) \tag{B.2}$$

and call this function the *Dirac delta function.*

As defined, the Dirac delta function has the following indefinite integration property

$$\int \delta(x)\,\mathrm{d}x = H(x) + C, \tag{B.3}$$

where $H(x)$ is the Heaviside step function defined by:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0\,. \end{cases} \tag{B.4}$$

It is also useful to introduce the the Macaulay bracket notation, where angle brackets have the following special meaning:

$$\langle x\rangle = \begin{cases} 0 & x < 0 \\ x & x \geq 0. \end{cases} \tag{B.5}$$

With these definitions one can deduce the following useful integration rules:

$$\int H(x)\,\mathrm{d}x = \langle x\rangle + C \tag{B.6}$$

$$\int \langle x\rangle^n\,\mathrm{d}x = \frac{1}{n+1}\langle x\rangle^{n+1} + C. \tag{B.7}$$

Note that the definition we have introduced for the Dirac delta function also possesses the familiar property that for a continuous function

$g(x)$,

$$\int_{0^-}^{0^+} g(x)\delta(x)\,\mathrm{d}x = g(0). \tag{B.8}$$

Laboratory Exercises

C

Much of the utility of the topics presented in this book is related to their use in a computational setting. Throughout, the theory has been developed with an eye towards computer implementations. To gain a full appreciation and command of the concepts, computational exercises and, in particular, programing exercises are essential. In this appendix, one will find directed laboratory exercises covering the main topics presented in the preceding chapters. For most of the laboratory exercises, one starts with partially completed programs (written in MATLAB) and these are available for download from:

`http://www.ce.berkeley.edu/~sanjay/programs`

Each lab is designed to be comfortably completed within a single supervised three-hour computer laboratory session with the exception of the last laboratory assingment which is designed for two three-hours sessions.

C.1 Solution of traditional problems using BVP4c

The objective of this lab is to expose you to solving differential equations which govern mechanical phenomenon, not through hand calculations, but through numerical methods. As you will see in your studies, as the mechanical system becomes more and more complex, hand solutions will become more and more difficult to obtain; in certain cases analytical closed form solutions may not even be accessible. By using numerical methods, even when these closed form solutions are not available one will still be able to obtain good approximations for the exact solution. Good numerical methods let you control the magnitude of the permissible errors in your approximate solutions.

In this lab you will be using the software MATLAB and its built in functions to solve the mechanical problem associated with bars in tension and compression.

In this lab you will be asked to conduct the following things:

(1) Understand the procedure of using MATLAB to solve the tension-compression bar problem.

(2) Use it to solve a set of boundary value problems.

C.1.1 Tension-compression bar

Governing differential equation

The governing equations for a tension-compression bar are the following,

- Equilibrium

$$\frac{dR}{dx} + b(x) = 0 \tag{C.1}$$

  The distributed load $b(x)$ can vary along the length of the bar.

- Kinematics

$$\varepsilon = \frac{du}{dx} \tag{C.2}$$

- Constitutive relation

$$\sigma \quad = \quad E(x)\varepsilon \tag{C.3}$$

  The Young's modulus $E(x)$ can vary along the length of the bar.

- Resultant definition

$$R \quad = \quad A(x)\sigma \tag{C.4}$$

  Here we have assumed a simple system, where the stress σ is constant across any cross-section. The cross-sectional area $A(x)$ can vary along the length of the bar.

These four relations can be combined to obtain a single equation representing equilibrium in terms of the displacement,

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + b = 0 \ . \tag{C.5}$$

In order to utilize the solver in MATLAB, one must convert the governing equations into first-order form,

$$\boxed{\frac{d\boldsymbol{y}}{dx} = \boldsymbol{f}(\boldsymbol{y}, x),} \tag{C.6}$$

where $\boldsymbol{y}$ is a vector of unknown variables, and $\boldsymbol{f}$ is a vector of known functions depending on $\boldsymbol{y}$ and the position x. For the tension-compression bar, it is convenient to choose the variables u and R as the unknown variables (since the boundary conditions are typically given in terms of u and R). In order to expand from second order form to first order form we can re-introduce R into (C.5). This yields a system of two coupled

equations,

$$\begin{aligned} \frac{dR}{dx} + b &= 0, \\ R &= EA\frac{du}{dx}\,. \end{aligned}$$

The two equations can be rewritten as,

$$\frac{d}{dx}\begin{bmatrix} u \\ R \end{bmatrix} = \begin{bmatrix} \dfrac{R}{E(x)A(x)} \\ -b(x) \end{bmatrix}.$$

By defining,

$$\boxed{\begin{aligned} \boldsymbol{y} &:= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u \\ R \end{bmatrix}, \\ \boldsymbol{f}(\boldsymbol{y}, x) &:= \begin{bmatrix} f_1(\boldsymbol{y}, x) \\ f_2(\boldsymbol{y}, x) \end{bmatrix} = \begin{bmatrix} \dfrac{y_2}{E(x)A(x)} \\ -b(x) \end{bmatrix}, \end{aligned}} \tag{C.7}$$

one obtains the desired first-order form,

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{y_2}{E(x)A(x)} \\ -b(x) \end{bmatrix}.$$

Boundary condition

To solve the differential equation, one must apply boundary conditions. For the second-order differential equation (C.5), one requires 2 boundary conditions.

In order to apply boundary conditions in the solver in MATLAB, one must define a function which returns a residual measuring how much the boundary conditions are not satisfied; a residual of zero implies that the boundary conditions are satisfied exactly. The function has the form,

$$\boxed{\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \boldsymbol{0}}$$

where $\boldsymbol{g}$ is a vector depending on the value of $\boldsymbol{y}$ evaluated at the boundary points $x = a$ and $x = b$; Here we assume the problem is defined on the interval (a, b).

To clarify the form of the function, consider the boundary condition,

$$\begin{aligned} u(0) &= u_0, \\ R(L) &= R_L, \end{aligned}$$

where the displacement is known as u_0 at the end point $x = 0$, and the force is known as R_L at the end point $x = L$. The residual vector measuring the extent to which the boundary conditions are violated can be written as

$$\begin{bmatrix} u(0) - u_0 \\ \\ R(L) - R_L \end{bmatrix} .$$

Using the correspondence between u, R and $\boldsymbol{y}$ defined in (C.7), one defines $\boldsymbol{g}$ as,

$$\boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(L)) := \begin{bmatrix} g_1(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ \\ g_2(\boldsymbol{y}(0), \boldsymbol{y}(L)) \end{bmatrix} = \begin{bmatrix} y_1(0) - u_0 \\ \\ y_2(L) - R_L \end{bmatrix} . \quad \text{(C.8)}$$

Numerical method

To solve the differential equation in first-order form (C.6), we will use the built-in MATLAB function `bvp4c`. The following point is important to keep in mind when using numerical methods to solve differential equations.

A numerical method always gives approximate solutions! unless you get very lucky!

Only in special cases does one obtain an exact solution when using a numerical method. There are cases when it seems like one has the exact solution, but this is only because the solution has been obtained to a high-degree of accuracy. Thus when invoking a numerical method, one must specify the degree of accuracy one desires in the approximate solution. The accuracy of the approximate solution $\boldsymbol{y}_{\text{approx}}$ can be measured by the relative accuracy, defined as,

$$\frac{||\boldsymbol{y}_{\text{approx}} - \boldsymbol{y}_{\text{exact}}||}{||\boldsymbol{y}_{\text{exact}}||} ,$$

where $\boldsymbol{y}_{\text{exact}}$ is the exact solution.

In the case of using `bvp4c` to solve the differential equation (C.6), there are three parameters one can adjust to determine the attained accuracy in the approximate solution. Let us assume we would like to solve the differential equation on the interval (a, b). The three parameters are,

(1) x_i $(i = 1, \ldots, N)$: The points at which you want to satisfy the differential equation, where N is the total number of points and $x_1 = a$ and $x_N = b$.

(2) *RELTOL*: `bvp4c` will return an approximate solution $\boldsymbol{y}_{\text{approx}}$ which satisfies the relation,

$$\left\| \frac{d\boldsymbol{y}_{\text{approx}}(x_i)}{dx} - \boldsymbol{f}(\boldsymbol{y}_{\text{approx}}(x_i), x_i) \right\| \leq \left\| \boldsymbol{f}(\boldsymbol{y}_{\text{approx}}, x_i) \right\| RELTOL ,$$

for $i = 1, \ldots, N$. Thus this controls the relative error in the solution at the points x_i.

(3) *ABSTOL*: `bvp4c` will return an approximate solution $\boldsymbol{y}_{\text{approx}}$ which satisfies the relation,

$$\left\| \frac{d\boldsymbol{y}_{\text{approx}}(x_i)}{dx} - \boldsymbol{f}(\boldsymbol{y}_{\text{approx}}(x_i), x_i) \right\| \leq ABSTOL\,,$$

for $i = 1, \ldots, N$. Thus this controls the absolute error in the solution at the points x_i.

The number and location of the points x_i must be placed so that one will be able to sufficiently represent the behavior of the solution on (a, b). A smaller value of *ABSTOL* and *RELTOL* will lead to a more accurate solution but will in general require more time to obtain the solution. Typically, *ABSTOL* and *RELTOL* are chosen between the values of 1×10^{-1} and 1×10^{-16}. The setting of tolerances must always take into account the fact that computations on computers are always subject to finite precision limits due to the fact that numbers are stored in computers using a fixed number of bits.

C.1.2 MATLAB solution method with `bvp4c`

To solve boundary value problems in MATLAB, we will use the built-in ODE solver `bvp4c`. `bvp4c` take 4 arguments and returns a structure that contains the solution.

```
SOL = BVP4C(ODEFUN,BCFUN,SOLINIT,OPTIONS)
```

Type `help bvp4c` in MATLAB to see more information on the function. In short, the first argument is a pointer to a function which computes the right-hand side of the first order form of the ODE, the second argument is a pointer to a function which computes the boundary condition residual, the third argument is a structure with information to start the computation, and the fourth argument is a structure with solution options.

To utilize this solver in MATLAB, one must go through the following steps.

(1) Convert the differential equation into first-order form (C.6) and choose the unknown variables in the vector $\boldsymbol{y}$. Using this representation, construct the function `ODEFUN`.

(2) Construct a function which returns a the boundary condition residual as in (C.8). Using this representation, construct the function `BCFUN`.

(3) Use the function `BVPINIT` (built-in in MATLAB) to construct an initial solution structure `SOLINIT`.

(4) Use the function `BVPSET` to construct an options structure `OPTIONS`, which determines the degree of accuracy to which the solution is obtained.

(5) Pass the arguments `ODEFUN` and `BCFUN` as function handles and the structures `SOLINIT` and `OPTIONS` into the function `BVP4C`.

In the following function and structure explanations, *nvar* defines the length of the vector $\boldsymbol{y}$ defined in (C.6).

ODEFUN

```
[F] = ODEFUN(X,Y)
```

- This function serves the purpose of computing $\boldsymbol{f}(\boldsymbol{y}, x)$ mentioned in (C.6).
- *INPUT:*

  `X` :Scalar value defining the position x
  `Y` :Vector $(nvar \times 1)$ representing $\boldsymbol{y}$ evaluated at x.

- *OUTPUT:*

  `F` :Vector $(nvar \times 1)$ representing $\boldsymbol{f}$ evaluated at x using $\boldsymbol{y}$, i.e., $\boldsymbol{f}(\boldsymbol{y}(x), x)$.

- *EXAMPLE:* For the tension-compression bar in 1D with $E = 1$, $A = 1$ and $b(x) = \sin\left(\frac{\pi}{2}x\right)$, the function is defined as,

```
function [fxy] = bar1d_ode(x,y)

% -- Define material property and geometry
E = 1;
A = 1;
L = 1;

% -- Define distributed load
b = sin(x*pi/(2*L));

% -- Define function
fxy = [y(2)/(E*A);
                -b];
end
```

BCFUN

```
[RES] = BCFUN(YA,YB)
```

- To impose boundary conditions in MATLAB one must define a function $\boldsymbol{g}$ of the form,

$$\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)), \tag{C.9}$$

  where $\boldsymbol{y}(a)$ denotes the value of the function $\boldsymbol{y}$ at $x = a$, and $\boldsymbol{y}(b)$ denotes the value of the function $\boldsymbol{y}$ at $x = b$. This function returns a vector of residuals measuring the degree to which the boundary conditions are not satisfied; the function $\boldsymbol{g}$ takes the value $\boldsymbol{g} = \boldsymbol{0}$ if the value of $\boldsymbol{y}$ at $x = a$ and $x = b$ exactly satisfy the boundary conditions.

- *INPUT:*

  `YA` :Vector $(nvar \times 1)$ defining the value of $\boldsymbol{y}$ at position $x = a$
  `YB` :Vector $(nvar \times 1)$ defining the value of $\boldsymbol{y}$ at position $x = b$

- *OUTPUT:*

  `RES` :Vector $(nvar \times 1)$ representing $\boldsymbol{g}$.

- *EXAMPLE:* For the tension-compression bar in 1D with boundary conditions,

$$u(0) = 0,$$
$$R(L) = 1 \ .$$

  One defines the function $\boldsymbol{g}$ in the following way,

$$\boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(L)) := \begin{bmatrix} y_1(0) - u_0 \\ y_2(L) - R_L \end{bmatrix} \ .$$

```
function [res] = bar1d_bc(ya,yb)

% -- Boundary Conditions (BC)
%    u: displacement
%    f: force

ua = 0;
fb = 1;

res= [ya(1)-ua;
      yb(2)-fb];

end
```

BVPSET

```
OPTIONS = BVPSET('RelTol',RELTOL,'AbsTol',ABSTOL)
```

- To set the degree of desired accuracy in the approximate solution, one must construct an options structure `OPTIONS` to pass to `bvp4c`.
- *INPUT:*

  `RELTOL` : See Section C.1.1 for definition.
  `ABSTOL` : See Section C.1.1 for definition.

- *OUTPUT:*

  `OPTIONS` : A MATLAB structure.

BVPINIT

`SOLINIT = BVPINIT(X,YINIT)`

- One must construct a structure `SOLINIT`, defining initial parameters of the solution, to pass to `bvp4c`.
- *INPUT:*

  `X` : Vector defining the points at which one desires the solution.
  `YINIT` : Vector ($nvar \times 1$) defining a constant initial guess for each component of the solution.

- *OUTPUT:*

  `SOLINIT` : A MATLAB structure.

C.1.3 Lab Exercises

Function handles

If you are unfamiliar with function handles in MATLAB, try the exercise in Section C.1.4 regarding usage of function handles in MATLAB. Make sure you understand the importance of adding the `@` symbol.

Download files

(1) Download the file `bar1d.m`.

Tension-compression bar

(1) Execute the file,

```
>> bar1d
```

This should give you a plot showing the displacement $u(x)$ and the internal force $R(x)$. The solution is to a problem where

$$\begin{aligned} A &= 1\,, \\ E &= 1\,, \\ L &= 1\,, \\ b(x) &= \sin\left(\frac{\pi}{2L}x\right), \end{aligned}$$

subject to boundary conditions:

$$\begin{aligned} u(0) &= 0\,, \\ R(L) &= 0\,. \end{aligned}$$

The exact solution to this problem can be obtained by solving the differential equation by hand to yield:

$$\begin{aligned} u(x) &= \left(\frac{2}{\pi}\right)^2 \sin\left(\frac{\pi}{2L}x\right), \\ R(x) &= \left(\frac{2}{\pi}\right) \cos\left(\frac{\pi}{2L}x\right). \end{aligned}$$

Check the solution at a few points in the interval to determine the error in the numerical solution; think about how this relates to the error tolerances that you specified. Note that you can use the MATLAB function `deval` to evaluate the solution structure at a set of desired points.

(2) Change the file so that you obtain the solution for the problem,

$$\begin{aligned} A &= 1\,, \\ E &= 1\,, \\ L &= 1\,, \\ b(x) &= 0\,, \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0\,, \\ R(L) &= 1\,. \end{aligned}$$

What should the shape of the displacement field look like?

(3) Change the file so that you obtain the solution for the problem,

$$\begin{aligned} A &= 1\,, \\ E &= 1\,, \\ L &= 1\,, \\ b(x) &= \sin\left(\frac{2\pi}{L}x\right)\,, \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0\,, \\ u(L) &= 0\,. \end{aligned}$$

Does the solution make sense?

(4) Change the file so that you obtain the solution to the problem,

$$\begin{aligned} A &= 1\,, \\ E &= 1\,, \\ L &= 1\,, \\ b(x) &= \delta(x - L/2)\,, \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0\,, \\ u(L) &= 0\,. \end{aligned}$$

Does the solution make sense? Note, you will need to "define" the "meaning" of the delta function for MATLAB, since it is not a built-in function; see Appendix B or S. Govindjee *Engineering Mechanics of Deformable Solids* Example 2.8, page 28, for details on the delta function.

C.1.4 MATLAB tips

To use the MATLAB `bvp4c` solver, one must know how to use function handles. In basic programming, one learns how to pass "numbers" into a function. Analogous to this one can also pass a "function" into a function. This is done through the use of function handles or pointers.

The simple exercise below illustrates how they are used. We will combine the two functions:

- A function which takes a number as an argument and returns its 3rd power.
- A function which takes a function as an argument and displays the function value evaluated at 3.

Exercise

(1) Write a function called `cubic` which takes a number `x` as an argument and returns its 3rd power,

```
function y = cubic(x)
    y = x^3;
end
```

and save this as the file `cubic.m`.

(2) Write a function which takes a function `func` as an argument and prints the value of `func` evaluated at 3,

```
function eval_print(func)
    func(3)
end
```

and save this as the file `eval_print.m`.

(3) In the command window, execute the function `eval_print` with `cubic` as the input argument

```
>> eval_print(cubic)
```

You will see that this gives an error. The proper way of executing this is:

```
>> eval_print(@cubic)
```

The `@` tells MATLAB that the argument `cubic` is a function. Thus whenever MATLAB requires a function handle, one must be careful not to omit the `@`.

C.2 Using BVP4c for beam bending problems

The objective of this lab is to apply the skills learned in Lab 1 to the case of beam bending. You will first modify your program so that it is able to solve beam bending problems and plot appropriate results. Then you will apply the program to two simple problems for which hand solutions are well-known. The program from this lab is also useful for solve some of the exercises in Chapter 1.

C.2.1 Beam equations in first order form

Governing differential equations: The governing equations for a beam in bending are (see Fig. 1.8 for the sign convention):

- Equilibrium

$$\frac{dV}{dx} + q(x) = 0, \tag{C.10}$$

$$\frac{dM}{dx} + V = 0\,, \tag{C.11}$$

  where $q(x)$ is the distributed load.

- Kinematics

$$\theta = \frac{dv}{dx}, \tag{C.12}$$

$$\kappa = \frac{d\theta}{dx}\ . \tag{C.13}$$

- Effective constitutive relation

$$M \quad = \quad E(x)I(x)\kappa \tag{C.14}$$

  The Young's modulus $E(x)$ and area moment of inertia $I(x)$ can vary along the length of the beam.

These relations can be combined to obtain a single equation representing equilibrium in terms of the displacement,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2v}{dx^2}\right] = q\ . \tag{C.15}$$

This relation is effective for hand solutions as seen in Chapter 1. In order to utilize the solver in MATLAB, one needs to employ the governing equations in first-order form:

$$\boxed{\frac{d\boldsymbol{y}}{dx} = \boldsymbol{f}(\boldsymbol{y}, x),} \tag{C.16}$$

where $\boldsymbol{y}$ is the vector of unknown variables, and $\boldsymbol{f}$ is a vector of known functions depending on $\boldsymbol{y}$ and the position x.

For beams in bending, we choose the variables v, θ, M, and V as the unknown variables (since the boundary conditions are typically enforced on these quantities). Recall from the solution of systems of equations that one must have the same number of equations as variables. Thus we will need 4 differential equations from the governing equations above. These can be obtained by eliminating the variable κ from Eqns. (C.10-C.14). This yields the four equations:

$$\begin{aligned} \frac{dV}{dx} + q(x) &= 0, \\ \frac{dM}{dx} + V &= 0, \\ \theta &= \frac{dv}{dx}, \\ M &= E(x)I(x)\frac{d\theta}{dx} \ . \end{aligned}$$

These can be rewritten as:

$$\frac{d}{dx}\begin{bmatrix} v \\ \theta \\ M \\ V \end{bmatrix} = \begin{bmatrix} \theta \\ \dfrac{M}{E(x)I(x)} \\ -V \\ -q(x) \end{bmatrix} .$$

By defining

$$\begin{aligned} \boldsymbol{y} &:= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} v \\ \theta \\ M \\ V \end{bmatrix}, \\ \boldsymbol{f}(\boldsymbol{y}, x) &:= \begin{bmatrix} f_1(\boldsymbol{y}, x) \\ f_2(\boldsymbol{y}, x) \\ f_3(\boldsymbol{y}, x) \\ f_4(\boldsymbol{y}, x) \end{bmatrix} = \begin{bmatrix} y_2 \\ \dfrac{y_3}{E(x)I(x)} \\ -y_4 \\ -q(x) \end{bmatrix}, \end{aligned} \tag{C.17}$$

one obtains the desired first-order form,

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ \dfrac{y_3}{E(x)I(x)} \\ -y_4 \\ -q(x) \end{bmatrix} .$$

Boundary conditions: To solve the system of differential equations, one must apply boundary conditions. For the fourth-order differential equation (C.15), one requires 4 boundary conditions.

In order to apply boundary conditions in the solver in MATLAB, one needs to define a function which returns a residual measuring the the boundary condition violation; a residual of zero implies that the boundary conditions are satisfied exactly. The function has the form,

$$\boxed{\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \mathbf{0}}$$

where $\boldsymbol{g}$ is a vector-valued function depending on the value of $\boldsymbol{y}$ evaluated at the boundaries $x = a$ and $x = b$.

To clarify the form of the function, consider a cantilever beam with an end shear; assume the end at $x = 0$ is built-in and the end at $x = L$ has the applied end shear $\bar{V}_L$, then

$$\begin{aligned} v(0) &= 0, \\ \theta(0) &= 0, \\ M(L) &= 0, \\ V(L) &= \bar{V}_L \ . \end{aligned}$$

The vector $\boldsymbol{g}$ defining the boundary condition residual is then given as

$$\boldsymbol{g} = \begin{bmatrix} v(0) - 0 \\ \theta(0) - 0 \\ M(L) - 0 \\ V(L) - \bar{V}_L \end{bmatrix} .$$

Using the correspondence between v, θ, M, V and $\boldsymbol{y}$ defined in Eqn. (C.17), one can define $\boldsymbol{g}$ as:

$$\boxed{\boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(L)) := \begin{bmatrix} g_1(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_2(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_3(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_4(\boldsymbol{y}(0), \boldsymbol{y}(L)) \end{bmatrix} = \begin{bmatrix} y_1(0) - 0 \\ y_2(0) - 0 \\ y_3(L) - 0 \\ y_4(L) - \bar{V}_L \end{bmatrix} . \quad \text{(C.18)}}$$

C.2.2 First Steps

Starting from the file from Lab 1:

(1) Edit the function names in the file to make them refer to beams instead of bars. This is not necessary but is stylistically important – for example, bar1d → beam1d, etc.

(2) Edit the function beam1d so that the number of variables will be 4.

(3) Edit beam1d_ode to implement the beam bending equations.

(4) Edit beam1d_bc to implement beam bending boundary conditions.

(5) Edit beam1d_plot to create 2 additional sub-plots so that you have one plot for each of the four solution fields v, θ, M, V. Make sure that you have the correct plot labels.

Test your program: Use your program to solve the following problems.

(1)

$$\begin{aligned} E &= 200 \text{ kN/mm}^2, \\ I &= 10^4 \text{ mm}^4, \\ L &= 1000 \text{ mm}, \\ q(x) &= q_0 = -10 \text{ N/mm}, \end{aligned}$$

with boundary conditions

$$\begin{aligned} v(0) &= 0, \\ M(0) &= 0, \\ v(L) &= 0, \\ M(L) &= 0 \, . \end{aligned}$$

This is a beam with both ends pinned and a uniform distributed load of 10 N/mm. Is the displacement at the center of the beam what you expect from the classical solution; viz.

$$v(L/2) = \frac{5}{384} \frac{q_0 L^4}{EI} \; ?$$

Use `deval( )` to properly evaluate the solution. Do all four plots correspond to your intuition?

(2) Change the file so that you obtain the solution for the problem,

$$\begin{aligned} E &= 30 \times 10^6 \text{ psi}, \\ I &= 25 \text{ in}^4, \\ L &= 36 \text{ in}, \\ q(x) &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} v(0) &= 0, \\ \theta(0) &= 0, \\ M(L) &= 0, \\ V(L) &= -10 \text{ kip} . \end{aligned}$$

This is a cantilever beam with a load applied at the end. What should the shape of the displacement look like? Does the tip displacement match the expected result $PL^3/3EI$? Use `deval( )` to properly evaluate the solution. Do all four fields match your expectations for this problem?

C.3 Truss lab 1

The objective of this lab is for you to set up the necessary data structures for defining a truss and to use them to generate the compatibility matrix for a truss and the resulting stiffness matrix too. The results of this lab will also be used for the next lab where we will program the boundary conditions and loads into the file and solve the resultant equilibrium equations.

C.3.1 Truss equations synopsis

- Equilibrium

$$\boldsymbol{A}^T\boldsymbol{R} = \boldsymbol{F}\,,$$

- Compatibility/Strain-displacement relation

$$\boldsymbol{\varepsilon} = \left\lceil \frac{1}{L} \right\rfloor \boldsymbol{A}\boldsymbol{u}\,,$$

- Constitutive relation

$$\boldsymbol{\sigma} = \lceil E \rfloor \boldsymbol{\varepsilon}\,,$$

- Resultant definition

$$\boldsymbol{R} = \lceil A \rfloor \boldsymbol{\sigma}\,.$$

The combined equilibrium equation solely in terms of the nodal displacements is

$$\boxed{\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}\,,}$$

where $\boldsymbol{K} = \boldsymbol{A}^T \lceil AE/L \rfloor \boldsymbol{A}$. Recall further that the dimensions of $\boldsymbol{A}$ are $b \times 2n$, where b is the number of truss bars and n is the number of nodes in the truss. Each row of $\boldsymbol{A}$ has two non-zero (vectorial) entries. For a generic row r in $\boldsymbol{A}$, the non-zeros are located in columns associated with the two nodes associated with bar r. These entries are $\boldsymbol{e}^T_{n_1 n_2}$ and $\boldsymbol{e}^T_{n_2 n_1}$, where the first goes into the column(s) associated with node n_2 and the second into the column(s) associated with node n_1. The vectors themselves are the unit vectors connecting nodes n_1 and n_2.

C.3.2 Download

Download the two lab files. `plotmesh_truss.m` is a plotting routine which you will use in your program. There is no need to edit it.
`truss_student.m` is the file you will need to edit. In this file there are several locations identified with the text `COMPLETE AS APPROPRIATE`. At these points you will need to provide the requisite code.

C.3.3 Walk Through

Location of truss nodes

You will first need to define the locations of the nodes. This will be stored in a $2 \times n$ matrix `truss.node`. The first row of this matrix will contain the x-coordinates of the nodes and the second row the y-coordinates. Each column will thus represent the x, y coordinates of the node and there will be one column per node. Set up this matrix for the two bar truss shown in Fig. 2.14.

Definition of the bars

You will next need to define which nodes are connected to each other by bars. This information will be stored in a $2 \times b$ matrix `truss.conn`. Each column of `truss.conn` will correspond to a bar and the entry in the first row will be the node number for the first node for the bar and in the second row will be the node number for the second node for the bar. Set up this matrix for the truss in Fig. 2.14 where bar 1 is the diagonal bar and bar 2 is the vertical bar.

Checkpoint

At this point comment out the rest of the file below the call to `plotmesh_truss` and run it. You should get a plot of the truss. Go back and add a third bar which is horizontal and re-run to make sure things are working ok. Once satisfied, remove the third bar.

Initialize $\boldsymbol{A}$

Determine the size that $\boldsymbol{A}$ should be from the information in the structure `truss` and initialize.

Compute and insert entries into $\boldsymbol{A}$

We will now loop over the bars and construct the compatibility matrix using the information we have set up in `truss.node` and `truss.conn`. In the loop, r refers to the generic bar r.

End to end vector Using the information in `truss`, compute the end to end vector from the first node of bar r to the second node of bar r. Do not normalize yet. The result will go in `en1n2` as a 2×1 vector.

Compute bar length Compute the length of the bar and store in `L`.

Compute AE/L Set up an array of AE/L values for the bars. For simplicity, assume $AE = 10$ for every bar. Store in `AEoL`. There is line under the allocation line for $\boldsymbol{A}$ to preallocate `AEoL` that you should now properly complete. The preallocations are for efficiency.

Normalize Normalize the length of `en1n2` and store the result back into `en1n2`.

Compute the end to end vector in the other direction Compute the normalized end to end vector from the second node to the first node and store in `en2n1`.

Insert into $\boldsymbol{A}$ For the bar r set up row r in $\boldsymbol{A}$ by inserting the transpose of `en1n2` and `en2n1`. This is the tricky part. For example $\boldsymbol{e}_{n_1 n_2}^T$ is a 1×2 matrix. It will need to be inserted into the two columns associated with node n_2. As a concrete hint, suppose the first node of a bar 4 was node 1 and the second node was node 3. Then one needs to insert the first component of $\boldsymbol{e}_{31}^T$ into row 4 column 1 and the second component into row 4 column 2. Further one needs to insert the first component of $\boldsymbol{e}_{13}^T$ into row 4 column 5 and the second component into row 4 column 6.

Compute K

Compute $\boldsymbol{K}$. Hint: To turn the vector of AE/L values into a diagonal matrix use the command `diag`.

C.3.4 Verify

Two bar example

Verify that your code is producing the correct expressions for $\boldsymbol{A}$ and for $\boldsymbol{K}$ for the two bar truss from Fig. 2.14 with $AE = 10$. The answers are:

$$\boldsymbol{A} = \begin{bmatrix} -0.7071 & -0.7071 & 0 & 0 & 0.7071 & 0.7071 \\ 0 & 0 & 0 & -1.0000 & 0 & 1.0000 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 3.5355 & 3.5355 & 0 & 0 & -3.5355 & -3.5355 \\ 3.5355 & 3.5355 & 0 & 0 & -3.5355 & -3.5355 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.0000 & 0 & -10.0000 \\ -3.5355 & -3.5355 & 0 & 0 & 3.5355 & 3.5355 \\ -3.5355 & -3.5355 & 0 & -10.0000 & 3.5355 & 13.5355 \end{bmatrix}$$

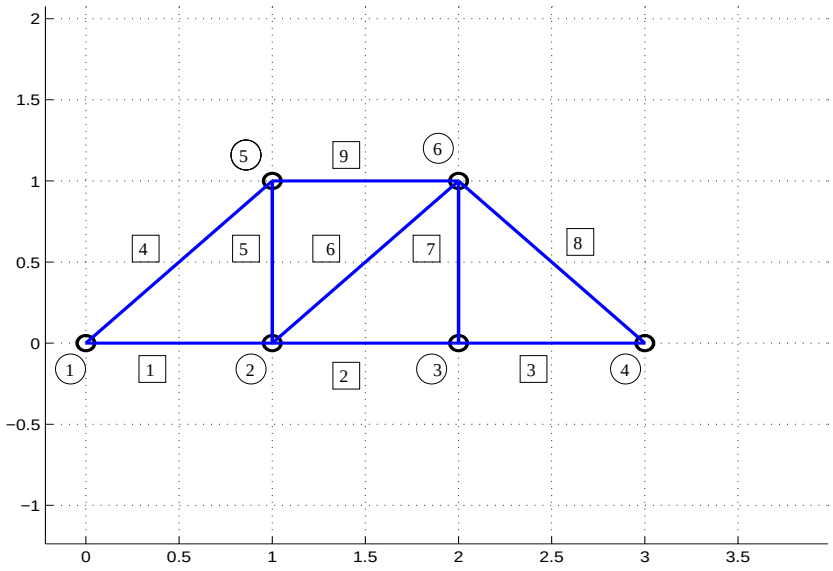

Fig. C.1 Truss program test geometry.

Test

Use your code to input the truss shown in Fig. C.1; assume $AE = 10$ and compute $\boldsymbol{A}$ and $\boldsymbol{K}$. Spot check various entries in the matricies to verify that they are correct.

C.4 Truss Lab 2

The objective of this lab is for you to complete the truss program which you started in the previous lab. In this lab you will add boundary condition data to your program and set up the active equations and solve them.

C.4.1 Boundary Conditions

The relevant technical details of the imposition of boundary conditions on a truss are provided in Sec. 2.1.3 on page 28. The primary data that needs to be specified for each node in the truss is which degrees of freedom (dofs) are specified, are part of $\boldsymbol{u}_d$, and which degrees of freedom are subject to applied forces, even if zero, and thus are part of $\boldsymbol{u}_f$. Additionally, one needs to specify the values of applied forces and imposed displacements. To achieve this you will create three additional entries in the `truss` structure from Lab 3.

(1) `truss.bc` is a $2 \times n$ matrix. Each column corresponds to one node. The first entry is for the x-dof and the second for the y-dof. If a dof has a prescribed displacement, then the matrix entry is set to 1 and if the dof has a prescribed force (even if zero) the matrix entry is set to 0. Because in typical applications, few dofs have prescribed displacements, the matrix is usually initialized to be all zeros and then 1s are inserted where needed.

(2) `truss.u` is a $2 \times n$ matrix. Each column corresponds to one node. The first entry is for the x-displacement and the second for the y-displacement. If a dof has a prescribed displacement, then the matrix entry is set to the value of the imposed displacement. Only the values associated with boundary code 1 in `truss.bc` are used; the rest are ignored.

(3) `truss.f` is a $2 \times n$ matrix. Each column corresponds to one node. The first entry is for the x-force and the second for the y-force. If a dof has a prescribed force, then the matrix entry is set to the value of the imposed force. Only the values associated with boundary code 0 in `truss.bc` are used; the rest are ignored.

C.4.2 Download

Download the lab files. `plotmesh_truss_L4.m` and `plot_defo_truss.m` are plotting routines which you will use in your program. There is no need to edit them. `truss_student_L4.m` is the file you will need to edit. In this file there are several locations identified with the text `COMPLETE AS APPROPRIATE`. At these points you will need to provide the requisite code.

C.4.3 Walk Through

Setting the boundary condition codes

You will first need to set the boundary condition codes for the nodes. These will be stored in a $2 \times n$ matrix `truss.bc`. The first row of this matrix will contain the x-direction boundary codes of the nodes and the second row the y-direction boundary codes. Each column will thus represent the x, y-direction boundary codes of each node and there will be one column per node. Set up this matrix for the truss in Fig. 2.14. Note that nodes 1 and 2 will have boundary codes $(1, 1)^T$ and node 3 will have boundary codes $(0, 0)^T$.

Set the imposed displacements

You will next need to define the driven displacement values at the node/dof combinations where you set a value of 1 in `truss.bc`. The given file already zeros out the `truss.u` array so there is nothing more to do for this test case. If non-zero values were specified as driven support motions, then one would have to set the corresponding numbers in this array.

Set the imposed forces

You will next need to define the imposed force values at the node/dof combinations where you set a value of 0 in `truss.bc`. The given file already zeros out the `truss.f` array so you will only need to set the forces on nodes with applied loads. For the test case there is a horizontal force at node 3. Thus one needs to set column 3 of `truss.f` to $(1, 0)^T$; i.e. a unit horizontal force.

Checkpoint

Now, comment out all the lines below this point, run your program and then call `plot_truss_L4(truss)`. This should produce a plot showing the appropriate supports and loads. Change node 1 to have a horizontal roller support, re-run, and re-plot. Make sure the plot is correct. If every thing is working, set the boundary codes back to being a pin support at nodes 1 and 2 and uncomment the lines which you commented out. Note that sometimes you will need to increase the size of the plot on your screen to distinguish between pins and rollers.

Stiffness matrix

The code to generate the stiffness matrix $\boldsymbol{K}$ is set up and should correspond to what you did in the prior lab. There is no need to edit this part of the code.

Extract the sub-matrices for the driven and free degrees of freedom

The degrees of freedom associated with the driven degrees of freedom are flagged by a 1 in `truss.bc`. To find the corresponding equation numbers one first needs to reshape the matrix into a $2n \times 1$ array. This is accomplished by using MATLAB's `reshape` command. The relevant equations numbers are then extracted using the `find` command. This sets up the needed index arrays `idd` and `idf`. This is already done for you in the code. The sub-matrix $\boldsymbol{K}_{ff}$, for example, is extracted using the statement `Kff = K(idf,idf)`. Complete the statements to extract $\boldsymbol{K}_{dd}, \boldsymbol{K}_{fd},$ and $\boldsymbol{K}_{df}$.

Extract the forces on the free degrees of freedom

To extract the relevant forces, we first reshape the forces into a $2n \times 1$ vector and then use the `idf` index array to find the relevant values and store them in `Ff`. This is done for you already.

Solve for the free displacements

We will now solve for $\boldsymbol{u}_f$. To do so we first reshape the displacement array into a $2n \times 1$ matrix. The solution $\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}[\boldsymbol{F}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d]$ is then stored in `u(idf)`. The reshape is done for you in the code already. Complete the line that computes `u(idf)`.

Compute the unknown reaction forces

Evaluate the unknown reactions forces and store them in `F(idd)`. The relevant relation is $\boldsymbol{F}_d = \boldsymbol{K}_{df}\boldsymbol{u}_f + \boldsymbol{K}_{dd}\boldsymbol{u}_d$.

Reshape and store result in `truss`

One now reshapes the $2n \times 1$ vectors `u` and `F` to store the solution into `truss.u` and `truss.f`, respectively. This is already done for you in the code.

Plotting

To plot the deformed truss, you can call the function `plot_defo_truss(truss,scale)`, where `scale` is a magnification factor for plotting the displacements (which are often too small to be seen without magnification/scaling).

Test 1

Run your program on the two bar truss in Fig. 2.14. A scale factor of 1 is plenty for this example since $AE = 10$ and the load is 1. Does the plot look correct? Print out `truss.u` to see the nodal displacements. Print out `truss.f`. Are the forces correct? Note you can easily check them by hand for this problem since the system is statically determinate.

Test 2

Use your code to input the truss shown in Fig. C.2. Assume the distances are given in milli-meters and that node 2 is pinned, node 5 has a vertical roller, and node 4 has a horizontal roller. Let each bar be 30 mm in diameter and be made of steel $E = 210$ kN/mm^2. At node 1 assume there is an imposed load of $-10.0\boldsymbol{e}_y$ kN. Find the displacement of node 1, the reaction force at node 2, and plot the deformed truss using an appropriate scale factor to make the motion of the truss clear.

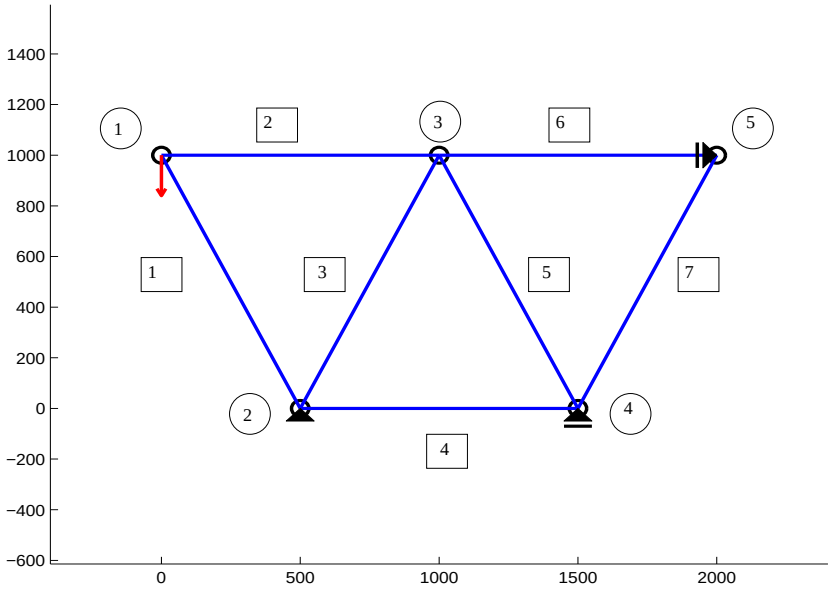

Fig. C.2 Test problem for truss program.

C.5 Truss Lab 3

In this lab you will use your completed assignment from the prior lab to solve two truss design questions.

C.5.1 Problem 1

Consider the truss shown in Fig. C.3. Assume that $L = 2$ ft, the bars are all solid round stock mild steel ($E = 30 \times 10^6$ psi), the upper bars have diameter 0.75 in, the lower bars have diameter 1.0 in and the diagonal and vertical bars have diameter 0.5 in. Assume that $F_1 = 2.0F_2 > 0$.

(1) Find the smallest value of F_2 at which a bar in the truss reaches the yield stress $\sigma_Y = 40$ ksi.
(2) Find the deflections at the locations of the applied loads when yield first occurs.
(3) Which bar yields first?

Answers should be reported correctly to 3 significant digits.

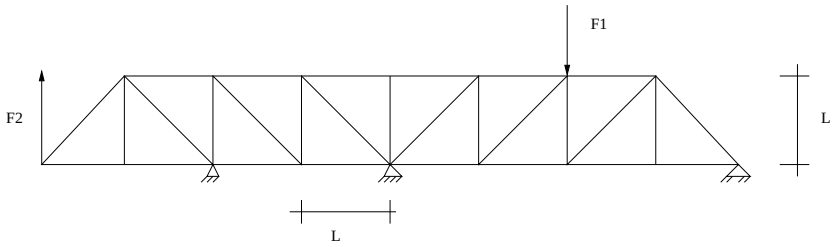

Fig. C.3 Multiply supported truss.

Hints:

(1) See Lab 3 for a synopsis of the truss equations.
(2) The problem is linear and thus you can exploit superposition, if you want.

C.5.2 Problem 2

Consider the truss shown in Fig. C.4, where all bars have modulus $E = 100$ kN/mm^2 and area $A = 100$ mm^2. Assume that $F = 500$ N. First, compute the state of the stress in the truss and observe that the forces in the upper and lower bars are increasing from the right to the left. Now try and redesign the truss so that the forces in the upper and lower bars are approximately uniform. Do this by moving only the free nodes while keeping the length of the structure at 999 mm. The supports should also remain fixed.

Hints:

(1) Just as in a beam you can think of the "internal moment" in this system as being linear.

(2) This "internal moment" is primarily carried by the forces in the upper and lower cords.

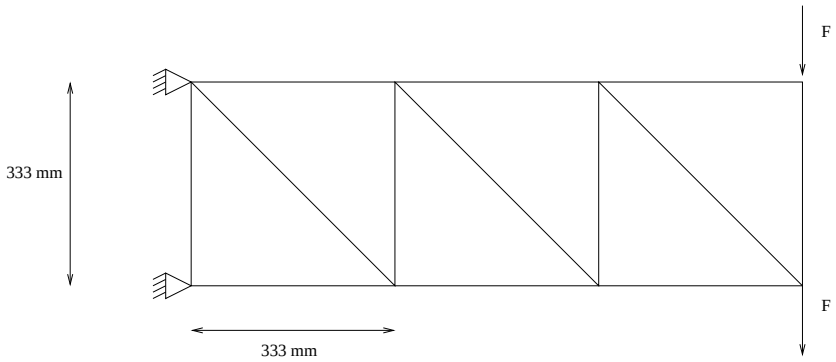

Fig. C.4 Cantilevered truss with end-load.

C.6 Potential energy: Visualization

The objective of this lab is to develop a understanding of the principle of stationary potential energy through the visualization of the potential energy in 1 and 2 degree of freedom systems.

C.6.1 Principle of stationary potential energy

Define the following quantites

Π_{total} : Total potential energy of the mechanical system,
Π_{elastic} : Elastic energy in the mechanical system,
Π_{load} : Energy due to the load

and define

$$\Pi_{\text{total}} := \Pi_{\text{elastic}} + \Pi_{\text{load}} .$$

Assume that all the energy quantites noted above depend on N variables or displacements, $u_1, \cdots, u_N$, i.e.,

$$\Pi_{\text{total}}(u_1, \cdots, u_N) .$$

By defining the vector $\boldsymbol{u}$,

$$\boldsymbol{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} ,$$

we can denote the dependence as

$$\Pi_{\text{total}}(\boldsymbol{u}) .$$

The principle of stationary potential energy states:

> A mechanical system is in an equilibrium state
> $\Leftrightarrow$
> Π_{total} is stationary.

More concretely this implies the following:

> A mechanical system is in an equilibrium state at $\widehat{\boldsymbol{u}}$
> $\Leftrightarrow$
> $\dfrac{\partial \Pi_{\text{total}}}{\partial u_i}(\widehat{\boldsymbol{u}}) = 0$ for $i = 1, \ldots, N$.

This principle allows us to look for the equilibrium states of a mechanical system by looking for the stationary points of its potential energy. This principle also tells us that if there are no stationary points, then there are no states of equilibrium.

C.6.2 Potential energy for linear mechanical systems

For linear mechanical systems under the action of dead-loads, Π_{total} can be written as

$$\Pi_{\text{total}} = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F}\,,$$

where $\boldsymbol{K}$ is an N-by-N matrix and $\boldsymbol{F}$ is a size N vector. We will assume here and throughout that $\boldsymbol{K}$ is a symmetric; i.e. $\boldsymbol{K} = \boldsymbol{K}^T$.

Example: Spring with an end-load

Consider a spring (spring constant k), fixed at one end and subject to a dead-load F at the other end. This is an example of the case of $N = 1$. Denote the displacement at the loaded end by u. The potential energy of the spring and the potential energy of the load are

$$\begin{aligned}\Pi_{\text{spring}} &= \frac{1}{2}uku = \frac{1}{2}ku^2\,,\\ \Pi_{\text{load}} &= -uF\end{aligned}$$

and thus

$$\Pi_{\text{total}} = \frac{1}{2}ku^2 - uF\,.$$

Example: Bar with two loads

Consider a mechanical system consisting of an elastic bar of length L fixed at $x = 0$ and subject to two loads, F_1 at the point $x = a$, and F_2 at the end $x = L$. Denote the displacement at $x = a$ as u_1 and the displacement at $x = L$ as u_2. Employing the knowledge that the displacement is linear between the loads, one obtains the following expression for the potential energies:

$$\begin{aligned}\Pi_{\text{bar}} &= \frac{1}{2}\frac{EA}{a}u_1^2 + \frac{1}{2}\frac{EA}{L-a}\left(u_2 - u_1\right)^2\,,\\ \Pi_{\text{load}} &= -F_1u_1 - F_2u_2\,.\end{aligned}$$

After some manipulation, the total potential energy can be expressed as,

$$\begin{aligned}\Pi_{\text{total}} &= \frac{1}{2}\begin{bmatrix}u_1 & u_2\end{bmatrix}\begin{bmatrix}\frac{EA}{a}+\frac{EA}{L-a} & -\frac{EA}{L-a}\\ -\frac{EA}{L-a} & \frac{EA}{L-a}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} - \begin{bmatrix}u_1 & u_2\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix}\\ &= \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F}\,.\end{aligned}$$

C.6.3 Solutions for linear mechanical systems

The behavior of the solutions for linear mechanical systems can be understood by looking at the properties of the matrix $\boldsymbol{K}$ (assumed symmetric). There are several important cases:

(1) $\boldsymbol{K}$ is (symmetric) positive definite (all eigenvalues are positive):
- There is only one stationary point which is also a minimum.
- The minimum corresponds to a stable equilibrium state.

(2) $\boldsymbol{K}$ has both positive and negative eigenvalues :
- There is only one stationary point which is a saddle point (not a minimum or maximum).
- The system is unstable at this point.

(3) $\boldsymbol{K}$ has a zero eigenvalue:
- There is at least one stationary point.
- The system can be at most neutrally stable at this(these) points.

C.6.4 Exercise

Download files

Download the files `plotenergy1v.m` and `plotenergy2v.m`.

Potential energy for a 1 degree of freedom case

The function `plotenergy1v.m` plots the total potential energy for a 1 degree of freedom linear mechanical system. The potential energy for this system is

$$\Pi_{\text{total}} = \frac{1}{2}Ku^2 - uF\,,$$

where K and F are scalars. This expression corresponds to the total potential energy for a spring (spring constant K) with end-load F.

You should be able to run the function with the following lines,

```
>> K = 1;
>> F = 1;
>> param.u_range = [-2,2];
>> param.e_range = [-2,2];
>> plotenergy1v(K,F,param);
```

to obtain the Figure C.5.

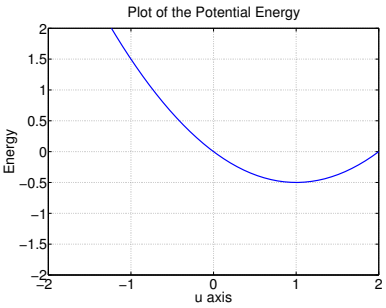

Fig. C.5 Sample potential energy for single degree of freedom system – $K = 1$ and $F = 1$.

(1) For each of the following cases
   (a) $K = 2, F = 1$
   (b) $K = -2, F = 1$
   (c) $K = 0, F = 1$

   plot the energy and identify key features on the graph; additionally, answer the following questions:

- Does the system have a solution (equilibrium point)?
- Does the potential energy have a stationary point?
- If the system has a solution calculate it by hand. What does this solution correspond to in the plot (locate it). Is the solution a maximum, minimum, or saddle state at this point?
- If the system has an equilibrium point, is this point stable or unstable?

Potential energy for a 2 degree of freedom system

The function `plotenergy2v.m` plots the total potential energy for a 2 degree of freedom linear mechanical system as well as its countour plot and gradient. The potential energy for this system is

$$\Pi_{\text{total}} = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F},$$

where $\boldsymbol{K}$ is a 2-by-2 matrix and $\boldsymbol{F}$ is a 2-by-1 vector.

(1) **Potential energy for a 2 spring system**

(a) Derive the expression for the total potential energy of the system constructed from 2 springs shown in Figure C.6. The stiffness of the springs are k_1 and k_2, the displacement and load at the two nodes are u_1, F_1, u_2, F_2, respectively. Write down the expression for $\boldsymbol{K}$ for this system.

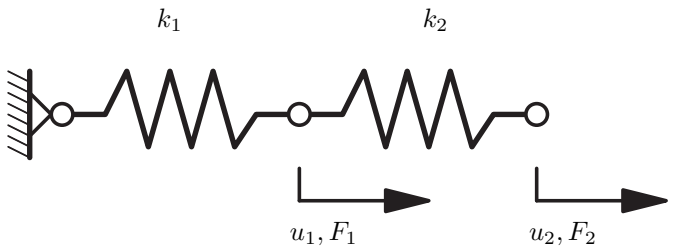

Fig. C.6 2 degree of freedom system

You should be able to run the plotting code with the following lines,

```
>> % K = DEFINE AN APPROPRIATE 2-BY-2 MATRIX;
>> % F = DEFINE AN APPROPRIATE LOAD VECTOR;
>> param.u1_range = [-2,2]; % Adjust as needed
>> param.u2_range = [-2,2]; % Adjust as needed
>> plotenergy2v(K,F,param);
```

(b) For each of the following cases,

(i) $k_1 = k_2 = 1, F_1 = 0, F_2 = 1$

(ii) $k_1 = -1, k_2 = 1, F_1 = 1, F_2 = 1$

(iii) $k_1 = k_2 = -1, F_1 = 1, F_2 = 1$
plot the energy using the function `plotenergy2v.m` and answer the questions:

- Does the system have a solution (equilibrium point)?
- Does the potential energy have a stationary point?
- If the system has a solution calculate it by hand. What does this solution correspond to in the plot. Is the solution a maximum, minimum, or a saddle state at this point?
- If the system has an equilibrium point, is this point stable or unstable? (HINT: Compute the eigenvalues of $\boldsymbol{K}$.)

(2) **Potential energy for a shallow truss structure** The potential energy for the shallow truss structure shown in Figure C.7 is given as,

$$\Pi_{\text{total}} = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F},$$

where

$$\boldsymbol{K} = \frac{2EA}{(1+a^2)^{3/2}}\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}.$$

For this problem assume $EA = 1$ and $F_1 = 0, F_2 = 1$. Comment on the behavior of this structure as a decreases from $a = 1$ to $a = 0$. Comment in terms of stability of the structure.

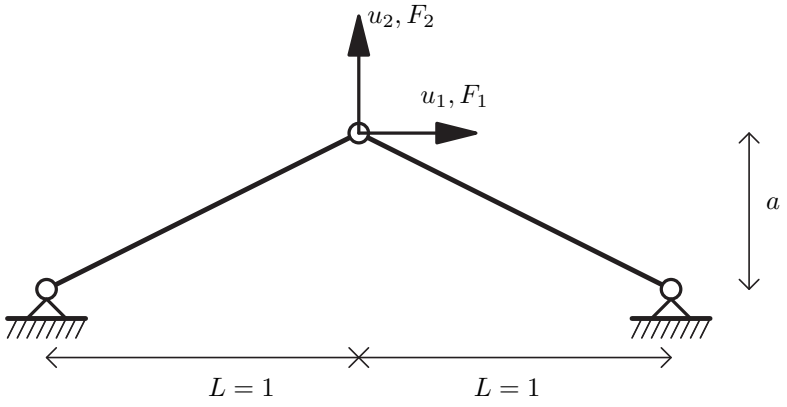

Fig. C.7 Shallow truss

C.6.5 Additional Exercise

Find the paper S. Govindjee, "Stability analysis of bay bridge saddle configuration," *ASCE Journal of Structural Engineering*, **136** 1613-1618

(2010).[1] This short paper describes an analysis of a structural repair system that was used to fix a broken eye-bar on the Oakland-San Francisco Bay Bridge in 2009.

(1) Read the entire paper and in your own words briefly summarize what is presented in the paper (a half-page paragraph should be sufficient).

(2) Verify that the results of the paper are correct by reproducing Figure 3 in the paper. [Note: the value of k is not needed. Simply plot Π/k. This is sufficient to understand the behavior of the system.]

[1] http://dx.doi.org/10.1061/(ASCE)ST.1943-541X.0000245

C.7 Ritz's method: Hat functions

The objective of this lab is to program a Ritz solution using a special set of Ritz functions known as hat functions. These function underly the most important numerical method for solving a very wide variety of problems in science and engineering – viz., the *finite element method*. Further, they posses good properties in terms of the degree to which they are linearly independent.

C.7.1 Model Problem

The model problem we will work with is an elastic tension-compression bar which has an imposed displacement at the left-end, an applied force at the right-end, and a constant distributed load. Thus the problem we wish to solve looks like:

$$AEu'' + b = 0\,, \tag{C.19}$$

where $u(0) = 0.03$, $AE\frac{du}{dx}(l) = F_{\text{app}} = -2000$, $b(x) = 4000$ and $AE = 30 \times 10^6$ – all in US customary units. The space of trial solutions for this problem is given by

$$\mathcal{S} = \{u(x) \mid u(0) = 0.03\}\,. \tag{C.20}$$

As we will be automating the solution for this problem we will work with a solution space that does not *a priori* enforce the kinematic boundary conditions. This implies that the potential energy expression we will work with will need to include the potential energy of the support reaction at $x = 0$. The needed expression for the total potential energy which describes this problem is

$$\Pi(u(x)) = \int_0^l \frac{1}{2}AE(u')^2\,dx - \int_0^l bu\,dx - F_{\text{app}}u(l) - R_0u(0)\,, \tag{C.21}$$

where $l = 2$. For later use we will consider breaking the length l into *nel* (equal) sized pieces (called elements); see Fig. C.8. In this way each integral can be written as a sum over the elements:

$$\int_0^l (\cdots)\,dx = \sum_{L=1}^{nel} \int_{x_L}^{x_{L+1}} (\cdots)\,dx\,. \tag{C.22}$$

C.7.2 Linear Hat Functions

As an approximation we will assume that

$$u(x) = \sum_{I=1}^{n} c_I g_I(x)\,, \tag{C.23}$$

where

$$g_I(x) = \begin{cases} \frac{x - x_{I-1}}{\Delta x} & x_{I-1} < x < x_I \\ \frac{x_{I+1} - x}{\Delta x} & x_I < x < x_{I+1} \\ 0 & \text{otherwise}\,. \end{cases} \tag{C.24}$$

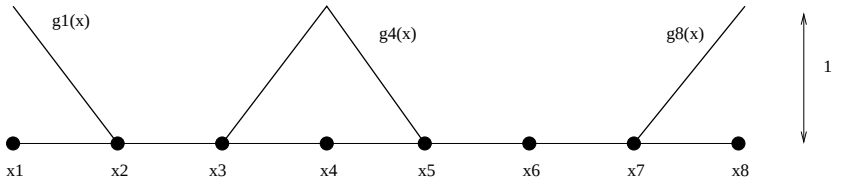

Fig. C.8 Example hat functions $g_1(x)$, $g_4(x)$, and $g_8(x)$.

The points $x_I = (I-1)\Delta x$ are called the nodes and $\Delta x = l/(n-1)$, where n is the number of functions in our Ritz expansion. Note that each function is equal to one at the node associated with its index and is zero at all the other nodes – $g_I(x_J) = \delta_{IJ}$. An important implication of this is that the parameters c_I represent the displacements at the nodes and, in particular, we already know $c_1 = 0.03$.

The intervals between the nodes, as already mentioned, are called the elements and there are $nel = n-1$ of them. Figure C.8 demonstrates what these functions look like, where we have used the example of $n = 8$ and thus $\Delta x = 2/7$ and $nel = 7$. Note that the functions on the ends are non-zero over just one element whereas those in the interior are non-zero over two elements.

C.7.3 Discrete Equations

Inserting the Ritz expansion into the functional and taking its derivative with respect to an arbitrary parameter yields the system of equations:

$$\sum_{J=1}^{n} K_{IJ} c_J = F_I \,, \tag{C.25}$$

where

$$K_{IJ} = \int_0^l g_I' A E g_J' \, dx \tag{C.26}$$

and

$$F_I = \int_0^l g_I b \, dx + F_{\text{app}} g_I(2) + R_0 g_I(0) \tag{C.27}$$

Because the linear hat functions have compact support, most of the terms in the stiffness matrix are zero. In fact $K_{IJ} = 0$ unless $|I-J| \leq 1$.

To efficiently compute the integrals and implement them in a clean fashion in code, the integrals are computed by computing the contributions from each element and then assembling them into the global stiffness K_{IJ}. Thus

$$K_{IJ} = \int_0^l g_I' A E g_J' \, dx = \sum_{L=1}^{nel} \int_{x_L}^{x_{L+1}} g_I' A E g_J' \, dx \,, \tag{C.28}$$

where nel is the total number of elements. The term inside the summation is known as the element stiffness and in our case it has only four

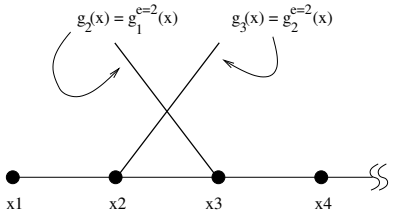

Fig. C.9 Local and global numbering scheme for hat functions over element $e = 2$.

non-zero entries. These occur for $I, J \in \{L, L+1\}$. Thus when computing the contribution from a given element, say element L, we usually form a two-by-two matrix with entries:

$$k_{ij}^e = \int_{x_L}^{x_{L+1}} g_i^{e'} AE g_j^{e'} \, dx \,, \tag{C.29}$$

where $i, j \in \{1, 2\}$ and $g_1^e = g_L$ and $g_2^e = g_{L+1}$. In this way k_{ij}^e, the so-called element stiffness matrix, is a two-by-two matrix that contributes to the global stiffness matrix. For example, consider the second element, $e = 2$, then $k_{11}^{e=2}$ would contribute to as $K_{22} = K_{22} + k_{11}^{e=2}$, $K_{23} = K_{23} + k_{12}^{e=2}$, etc. The advantage of this scheme is that the *element stiffness matrix* has a form that repeats for all elements and thus only needs to be programmed once; see Fig. C.9.

We treat the right hand side in a similar fashion; i.e. we form an element right-hand side and assemble it into the global right-hand side. So considering element L, we have

$$f_i^e = \int_{x_L}^{x_{L+1}} g_i^e b \, dx \, + F_{\text{app}} g_i^e(2) + R_0 g_i^e(0) \,. \tag{C.30}$$

Here as before $i \in \{1, 2\}$ and $g_1^e = g_L$ and $g_2^e = g_{L+1}$. This two-by-one vector is then assembled into the global right-hand side. For example, consider a case where one has 4 elements. Then for element three $F_3 = F_3 + f_1^{e=3}$ and $F_4 = F_4 + f_2^{e=3}$. Note that F_{app} will only contribute to the last element and R_0 will only contribute to the first element.

Exercise 1

Compute, by hand, an expression for the matrix elements k_{ij}^e for a generic element. Your result should be a two-by-two matrix and it will be symmetric.

Exercise 2

Compute an expression for f_i^e for a generic element. Your result should be a two-by-one vector. Remember that the applied end force will only affect the very last element and the support reaction will only affect the first element.

Exercise 3

Download the file `lab7_student.m` and program your expressions into it to solve the given problem. If you have done it correctly you will find that your approximate solution is *exact* at the nodes.

(1) Test it for $n = 2$ and $n = 3$ to verify this.

(2) How many elements do you need for the error to vanish in the "eyeball" norm?

Exercise 4

Make a log-log plot of the relative L^2 error versus n. How many terms are required to reduce the relative error to 10^{-6}? Hint: Compute the integrals by performing numerical quadrature over the elements and then add up the result.

C.8 Buckling: System of rigid-bars

The objective of this lab is to understand how the principle of stationary potential energy can be applied to understand the behavior of buckling phenomena. In this lab we will focus on the case of a discrete system consisting of rigid elements and springs.

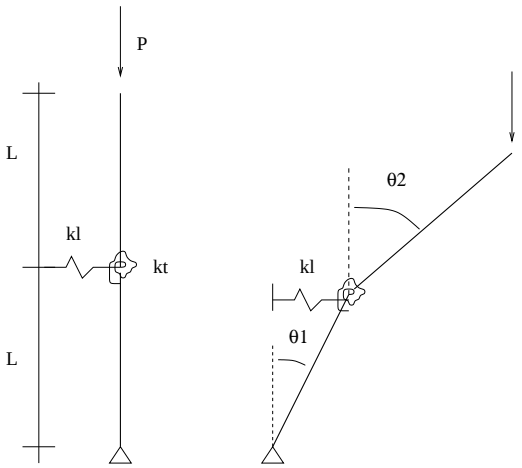

Fig. C.10 Two rigid-bar structure with two springs.

C.8.1 Geometry

The geometry we will work with has two degrees of freedom as shown in Fig. C.10. The two degrees of freedom that we will work with are the rotation angles of the bars from vertical. The bars are joined with a linear torsional spring with spring constant k_t which has dimensions of Force $\times$ Length. Further the middle joint is restrained by a linear translational spring with spring constant k_l which has dimensions of Force / Length.

C.8.2 Potential Energy

The potential energy for this system can be expressed as

$$\Pi(\theta_1,\theta_2) = \frac{1}{2}k_l L^2[(1-\cos(\theta_1))^2 + \sin^2(\theta_1)] + \frac{1}{2}k_t(\theta_2-\theta_1)^2 \\ -PL(2-\cos(\theta_1)-\cos(\theta_2))\,. \tag{C.31}$$

This expression is valid independent of the magnitude of the rotations.

Non-dimensionalization

In many problems, it is useful to non-dimensionalize expressions to remove redundant parameters from the problem and to improve performance in numerical calculations. The dimension of energy is Force $\times$

Length, thus a convenient parameter for non-dimensionalization is k_t. This yields

$$\Pi(\theta_1,\theta_2) = \frac{1}{2}e[(1-\cos(\theta_1))^2+\sin^2(\theta_1)] + \frac{1}{2}(\theta_2-\theta_1)^2 \\ -\lambda(2-\cos(\theta_1)-\cos(\theta_2))\,, \tag{C.32}$$

where the potential energy is now non-dimensionalized (divided by k_t), $e = k_l L^2/k_t$ is a non-dimensional ratio of spring stiffnesses, and $\lambda = PL/k_t$ is a non-dimensional load parameter.

Exercise 1

Approximate the energy to quadratic order in the rotations. Keeping up to quadratic order terms in the potential energy leads to linear order equilibrium equations. Hint:

$$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \tag{C.33}$$

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \cdots \tag{C.34}$$

Exercise 2

Compute the equilibrium equations associated with your approximate energy and arrange them in the matrix form $(\boldsymbol{K}-\lambda\boldsymbol{G})\boldsymbol{\theta} = \boldsymbol{0}$.

Exercise 3

Download the MATLAB files `animate8.m`, `plotpe.m`, `plotv.m`, and `potentiale.m`. These files include plotting routines as well as a file which computes the potential energy of the system for large deformations.

Compute (using Matlab) the buckling loads and modes using `[V,D]=eig(K,G)`. To help you visualize the results you should use the routine `plotv.m`. Typing `help plotv` explains how to use the routine. For example to plot the first buckling mode in red type:

```
>> plotv(V,D,1,L,'r')
```

where L is the length of the bars (choose 10 for example) and the vectors and eigenvalues are stored in `V` and `D`. Type `help eig` or `doc eig` for a description of the output of `eig`.

(1) Assuming $e = 0.1$, what is the critical non-dimensional buckling load? and corresponding bucking mode?

(2) Assuming $e = 5.0$, what is the critical non-dimensional buckling load? and corresponding bucking mode?

(3) Assuming $e = 100.0$, what is the critical non-dimensional buckling load? and corresponding bucking mode?

C.8.3 Energy Landscape

For every value of the load parameter, the potential energy possesses stationary points that represent the possible equilibrium states of the system. An equilibrium point can either be a minimum point (i.e. stable), a saddle point (i.e. unstable in one direction, stable in another), or a maximum point (i.e. unstable).

Exercise 4

Consider the case of $e = 10$. Make plots for the following three load cases using `plotpe.m`. Type `help plotpe` to see the arguments for this routine. For each case, identify all the equilibrium points on the graph and label them as stable or unstable.

(1) $\lambda = 0.80\lambda_{\text{cr}}$.
(2) $\lambda = 5.00\lambda_{\text{cr}}$.
(3) $\lambda = 30.0\lambda_{\text{cr}}$.

Note, `plotpe.m` plots the full non-linear energy from Eq. (C.32) and not the quadratic approximate energy.

C.8.4 Evolution of equilibrium

To be able to visualize the development of the system's equilibrium states one can animate the progression of the equilibrium states with increasing loads. To do so, use the routine `animate8.m`. The routine takes two arguments: (1) `e` the stiffness ratio and (2) `L` the bar lengths. Before being able to use the routine, you will need to edit the routine to add expressions for $\boldsymbol{K}$ and $\boldsymbol{G}$ where indicated.

To use type:

```
>> animate8(e,L)
```

where `e` and `L` are set to appropriate values.

The routine will plot the buckling modes was well as a surface plot of the energy and a contour plot of the energy. The contour plot will additionally have the directions of the linearized buckling modes superposed – red being the first mode and blue the second mode. The animation plots the state of energy of the system for ten load values below the critical load, then ten load values between the two buckling loads, and finally ten values above the second buckling load. Hitting any key on the keyboard will advance the animation – alternately one can edit `animate8` to have the program automatically sweep through the loading states by replacing `pause` with `drawnow` near the bottom of the routine.

Exercise 5

Animate the system for the case $e = 5$ and $L = 10$.

(1) By observing the plot, describe the expected evolution of the system as the load is increased from zero to the highest value.

(2) Are there ever any stable equilibrium states associated with the second buckling mode?

C.9 Buckling: Beams

The objective of this lab is to apply the principle of stationary potential energy for the solution of beam buckling problems.

Fig. C.11 Cantilever beam with compressive axial load.

C.9.1 Buckling load: A cantilever beam (clamped-free case)

Consider the cantilever beam showin in Fig. C.11, which is subjected to a compressive axial load. The potential energy for this system is

$$\Pi(v(x)) = \int_0^L \frac{1}{2} EI(v'')^2 \, dx - P \int_0^L \frac{1}{2}(v')^2 \, dx \,. \tag{C.35}$$

If we assume the deflection has the polynomial form

$$v(x) = \sum_{i=1}^{N} c_i \left(\frac{x}{L}\right)^{i+1} , \tag{C.36}$$

then the kinematic boundary conditions at $x = 0$ are automatically satisfied.

Exercise 1

If we insert this approximation into the expression for the potential energy, then we can write the governing equilibrium equations as $(\boldsymbol{K} - \lambda \boldsymbol{G})\boldsymbol{c} = \boldsymbol{0}$, where $\boldsymbol{K}$ comes from the bending energy term, $\boldsymbol{G}$ comes from the potential for the load, and $\boldsymbol{c}$ is the vector of c_is divided by the length. Here $\lambda = PL^2/EI$ is a non-dimensional load value. Show that

(1)
$$K_{ij} = \frac{(i+1)(j+1)ij}{i+j-1} \,, \tag{C.37}$$

(2)
$$G_{ij} = \frac{(i+1)(j+1)}{i+j+1} \,. \tag{C.38}$$

Exercise 2

Download the file `lab9_1_student.m`. This file partially implements a solution to this buckling problem. Where indicated complete the file as follows:

(1) Fill in the appropriate expressions for $\boldsymbol{K}$ and $\boldsymbol{G}$.
(2) Extract the critical non-dimensional buckling load λ_{cr} and store in `approx`.
(3) Extract the corresponding eigenvector and store in `c`.

Exercise 3

(1) What is the error with just one term in the approximation?

(2) What is the error with 3 terms in the approximation?

(3) Make a plot of error versus the number of parameters in the approximation for $n = 1, 2, \ldots, 10$. Use semilog axes (log error versus number of terms).

C.9.2 Beam with a transverse load and axial compression

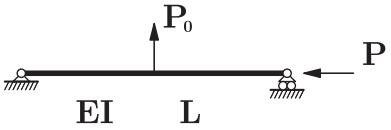

Fig. C.12 Simply supported beam with transverse load and axial compression.

Consider the beam shown in Fig. C.12. It is subjected to an axial compression P and a transverse load P_o. The potential energy for this system is given by

$$\Pi(v(x)) = \int_0^L \frac{1}{2} EI(v'')^2 \, dx - P \int_0^L \frac{1}{2}(v')^2 \, dx \; - P_o v(x_o) \,, \qquad \text{(C.39)}$$

where x_o is the location where the transverse load is applied. If we assume the deflection has the trigonometric form

$$v(x) = \sum_{i=1}^{N} c_i \sin(i\pi x/L) \,, \qquad \text{(C.40)}$$

then the resulting discrete equations have the form $(\boldsymbol{K} - P\boldsymbol{G})\boldsymbol{c} = \boldsymbol{F}$, where the stiffness $\boldsymbol{K}$ and geometric stiffness $\boldsymbol{G}$ matrices have a very simple form (they are diagonal).

Exercise 4

(1) Determine expressions for K_{ij} and G_{ij}.

(2) Determine an expression for F_i.

(3) Since your matrices are diagonal, one can find an expression for c_i by hand. Do so.

Exercise 5

Download the file `lab9_2_student.m`. This file partially computes a solution to this problem. Complete the file as follows:

(1) Where indicated implement your expression for c_i.

(2) Where indicated complete the expression that evaluates the deflected shape for plotting purposes.

Exercise 6

(1) Find and plot the deflection of the beam for a transverse load $P_o = 3$ and axial load $P = 0.2P_{\text{Euler}}$, where $x_o = 3L/5$.

(2) Plot the midspan deflection of the beam (with $P_o = 3$) versus $P \in [0,\, 0.95P_{\text{Euler}}]$. When you make your plot, use $v(L/2)$ for the abscissa and P the ordinate.

(3) Add to this plot curves for $P_o = 1$ and $P_o = 5$.

Provide a detailed description of what the plots tell you.

C.10 Bubnov-Galerkin method: Beams with Hermite polynomials

The objective of this lab is to program a virtual work based solution using a special set of Bubnov-Galerkin functions known as Hermite polynomials. These function underly the most important numerical method for solving beam problems – the finite element method. Unlike the polynomials we have been using, they produce matricies with reasonable properties as the number of approximation terms grows. To keep things simple will assume a linear elastic material but the ideas and methodology are quite general.

C.10.1 Model Problem

The model problem we will work with is a linear elastic beam which has an imposed displacement at the right-end, is built-in at the left-end and is subjected to a constant distributed load. Thus the problem we wish to solve looks like:

$$\frac{d^2M}{dx^2} = q \tag{C.41}$$

$$M = EI\kappa \tag{C.42}$$

$$\kappa = v'' \tag{C.43}$$

where $v(0) = 0$, $\theta(0) = 0$, $v(L) = 0.1$, $M(L) = EIv''(L) = 0$, $q(x) = q_o = -3000$, $EI = 120 \times 10^6$ with $L = 30$ – all in US customary units.

Exercise 1

Sketch the problem described above.

C.10.2 Function spaces

To deal with the boundary conditions we will employ the methodology we have used several times this semester of ignoring kinematic boundary conditions in the formal derivation of the matrix equations. We will then impose the boundary conditions at the end by separating the free degrees of freedom from the driven ones; see Lab 7. In this setting, this means

$$\mathcal{S} = \{v(x) \mid \text{no restrictions}\} \tag{C.44}$$

and

$$\mathcal{V} = \{\delta v(x) \mid \text{no restrictions}\} \tag{C.45}$$

C.10.3 Weak Form/Virtual Work Equation

For the given problem the virtual work equation reads:

$$\begin{aligned} \int_0^L \delta v''(x) EI v''(x)\, dx &= \int \delta v(x) q(x)\, dx \\ &- \delta v(0) V(0) - \delta v'(0) M(0) \\ &+ \delta v(L) V(L) + \delta v'(L) M(L) \end{aligned} \tag{C.46}$$

C.10.4 Hermite Hat Functions

In the spirit of the linear hat functions which we used before, we will assume an expansion for the displacement and the virtual displacement as:

$$v(x) = \sum_{J=1}^{n} v_J g_J(x) + \theta_J h_J(x)\,, \tag{C.47}$$

$$\delta v(x) = \sum_{I=1}^{n} \delta v_I g_I(x) + \delta\theta_I h_I(x)\,, \tag{C.48}$$

where as before I, J index a set of nodes that breaks up the domain into a set of $n-1$ elements. The parameters in this setting are v_J and θ_J and they also represent the beam displacement and rotation at the nodes. The functions have the following properties:

(1) $g_J(x)$ has unit value at the node it is associated with. It has zero slope there. At the neighboring nodes it has zero value as well as zero slope.

(2) $h_J(x)$ is zero at the node it is associated with but it has unit slope there. At the neighboring nodes it has zero value as well as zero slope.

The functions are depicted in Fig. C.13. In the graph there is a node at -1, 0, and $+1$. The upper graph shows $g_J(x)$ for the node at 0 and the lower graph $h_J(x)$ for the node at 0. Outside of the domain of the two elements attached to a node the functions are identically equal to zero.

The precise definitions in the general case are given by:

$$g_J(x) = \begin{cases} \frac{3\zeta^2}{\Delta x^2} - \frac{2\zeta^3}{\Delta x^3} & x_{J-1} < x < x_J \\ 1 - \frac{3\xi^2}{\Delta x^2} + \frac{2\xi^3}{\Delta x^3} & x_J < x < x_{J+1} \\ 0 & \text{otherwise}\,, \end{cases} \tag{C.49}$$

where $\xi = x - x_J$ and $\zeta = x - x_{J-1}$ and

$$h_J(x) = \begin{cases} -\frac{\zeta^2}{\Delta x} + \frac{\zeta^3}{\Delta x^2} & x_{J-1} < x < x_J \\ \xi - \frac{2\xi^2}{\Delta x} + \frac{\xi^3}{\Delta x^2} & x_J < x < x_{J+1} \\ 0 & \text{otherwise}\,. \end{cases} \tag{C.50}$$

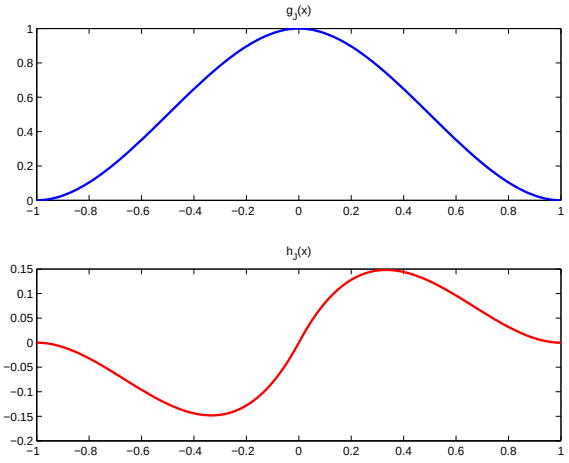

Fig. C.13 Hermite polynomials for a node located at 0 with neighboring nodes at -1 and $+1$.

Again, the points $x_J = (J-1)\Delta x$ are the nodes and $\Delta x = L/nel$, where $nel = n-1$ is the number of elements. Note, as before, the functions on the ends are non-zero over just one element whereas those in the interior are non-zero over two elements.

C.10.5 Discrete Equations

The discrete equations are arrived as by plugging our expansions into the virtual work equation, separating out the coefficients of the virtual motion, and noting that the remainder must be zero. The resulting relations are given by

$$\sum_{J=1}^{n} \boldsymbol{K}_{IJ}\boldsymbol{c}_J = \boldsymbol{F}_I\,, \tag{C.51}$$

where

$$\boldsymbol{K}_{IJ} = \int_0^L EI \begin{bmatrix} g_I'' g_J'' & g_I'' h_J'' \\ h_I'' g_J'' & h_I'' h_J'' \end{bmatrix} dx \tag{C.52}$$

and

$$\boldsymbol{F}_I = \int_0^L \begin{bmatrix} g_I q \\ h_I q \end{bmatrix} dx - \begin{bmatrix} g_I(0)V(0) \\ h_I'(0)M(0) \end{bmatrix} + \begin{bmatrix} g_I(L)V(L) \\ h_I'(L)M(L) \end{bmatrix} \tag{C.53}$$

and

$$\boldsymbol{c}_J = \begin{pmatrix} v_J \\ \theta_J \end{pmatrix}. \tag{C.54}$$

Note that for our particular problem $M(L)$ is known; it is zero. The other end reactions are unknowns and can be found from the 'right-hand side' entries associated with the (three) driven degrees of freedom.

Because the Hermite hat functions have compact support most of the terms in the stiffness matrix are zero. In fact $\boldsymbol{K}_{IJ} = \mathbf{0}$ unless $|I-J| \leq 1$. It should also be observed that we already know that $v_1 = 0.0$, $\theta_1 = 0.0$, and $v_n = 0.1$ by the boundary conditions.

To efficiently compute the integrals and implement them in a clean fashion in code, the integrals are computed by computing the contributions from each element and then assembling them into the global stiffness $\boldsymbol{K}_{IJ}$. Thus

$$\begin{aligned} \boldsymbol{K}_{IJ} &= \int_0^L EI \begin{bmatrix} g_I'' g_J'' & g_I'' h_J'' \\ h_I'' g_J'' & h_I'' h_J'' \end{bmatrix} dx \\ &= \sum_{A=1}^{nel} \int_{x_A}^{x_{A+1}} EI \begin{bmatrix} g_I'' g_J'' & g_I'' h_J'' \\ h_I'' g_J'' & h_I'' h_J'' \end{bmatrix} dx\,, \end{aligned} \tag{C.55}$$

where nel is the total number of elements. If we consider the contribution for a single element with, say nodes A and $A+1$, then the only contributions come from $I, J \in \{A, A+1\}$. This results in four (block) non-zero values for $\boldsymbol{K}_{IJ}$. Thus when computing the contribution from a given element, say element A, we usually form a two-by-two block matrix (four-by-four regular matrix) with block entries:

$$\boldsymbol{k}_{ij}^e = \int_{x_A}^{x_{A+1}} EI \begin{bmatrix} g_i^{e\,\prime\prime} g_j^{e\,\prime\prime} & g_i^{e\,\prime\prime} h_j^{e\,\prime\prime} \\ h_i^{e\,\prime\prime} g_j^{e\,\prime\prime} & h_i^{e\,\prime\prime} h_j^{e\,\prime\prime} \end{bmatrix} dx\,, \tag{C.56}$$

where $i, j \in \{1, 2\}$ and $g_1^e = g_A$ and $g_2^e = g_{A+1}$ (similarly for h_i^e). In this way $\boldsymbol{k}_{ij}^e$ is a two-by-two block matrix that contributes to the global stiffness matrix. For example, consider a case where one has 4 elements. Then for element two $\boldsymbol{k}_{11}^{e=2}$ would contribute to as $\boldsymbol{K}_{22} = \boldsymbol{K}_{22} + \boldsymbol{k}_{11}^{e=2}$, $\boldsymbol{K}_{23} = \boldsymbol{K}_{23} + \boldsymbol{k}_{12}^{e=2}$, etc.

We treat the right hand side in a similar fashion; i.e. we form an element right-hand side and assemble it into the global right-hand side. So considering element A, we have

$$\boldsymbol{f}_i^e = \int_{x_A}^{x_{A+1}} \begin{bmatrix} g_i^e q \\ h_i^e q \end{bmatrix} dx - \begin{bmatrix} g_i^e(0) V(0) \\ h_i^{e\prime}(0) M(0) \end{bmatrix} + \begin{bmatrix} g_i^e(L) V(L) \\ h_i^{e\prime}(L) M(L) \end{bmatrix}, \tag{C.57}$$

Here, as before, $i \in \{1, 2\}$ and $g_1^e = g_A$ and $g_2^e = g_{A+1}$ (simlarly for h_i^e). This two-by-one (block) vector is then assembled into the global right-hand side. For example, consider a case where one has 4 elements. Then for element three $\boldsymbol{F}_3 = \boldsymbol{F}_3 + \boldsymbol{f}_1^{e=3}$ and $\boldsymbol{F}_4 = \boldsymbol{F}_4 + \boldsymbol{f}_2^{e=3}$.

Exercise 2

The block matrix elements $\boldsymbol{k}_{ij}^e$ for a generic element can be computed by hand. Put together, the result is a two-by-two (block) matrix (or

four-by-four scalar matrix) and it will be symmetric. The final result is:

$$\boldsymbol{k}^e = \frac{EI}{\Delta x^3}\begin{bmatrix} 12 & 6\Delta x & -12 & 6\Delta x \\ & 4\Delta x^2 & -6\Delta x & 2\Delta x^2 \\ & & 12 & -6\Delta x \\ \text{sym.} & & & 4\Delta x^2 \end{bmatrix} . \tag{C.58}$$

Verify the first scalar entry of this matrix – i.e. verify that $k^e_{11} = 12EI/\Delta x^3$. For this exercise, it is helpful to note that over a single element there are only 4 non-zero functions. If the first node is at 0 and the second at a, then the two associated with the left most node are:

$$g_{\text{left}}(x) = 1 - \frac{3x^2}{a^2} + \frac{2x^3}{a^3} \tag{C.59}$$

$$h_{\text{left}}(x) = x - \frac{2x^2}{a} + \frac{x^3}{a^2} \tag{C.60}$$

and the two associated with the right most node are:

$$g_{\text{right}}(x) = \frac{3x^2}{a^2} - \frac{2x^3}{a^3} \tag{C.61}$$

$$h_{\text{right}}(x) = -\frac{x^2}{a} + \frac{x^3}{a^2} . \tag{C.62}$$

Exercise 3

The block vector entries $\boldsymbol{f}^e_i$ for a generic element can also be computed by hand. The result is a four-by-one scalar vector with the following entries.

$$\boldsymbol{f}^e = \frac{q_o \Delta x}{12}\begin{bmatrix} 6 \\ \Delta x \\ 6 \\ -\Delta x \end{bmatrix} . \tag{C.63}$$

At the end elements there are contributions from the boundary terms. For element 1, there is the additional contribution

$$\begin{bmatrix} -V(0) \\ -M(0) \\ 0 \\ 0 \end{bmatrix} . \tag{C.64}$$

For element *nel* (the last element), there is the additional contribution

$$\begin{bmatrix} 0 \\ 0 \\ V(L) \\ M(L) \end{bmatrix} . \tag{C.65}$$

Verify Eq. (C.63).

Exercise 4

Download the file `lab10_student.m` and complete the program. There is also a plotting program `evalbeam.m` to download; once downloaded, type `help evalbeam` to learn how to use.

Exercise 5

Simplify the loading by setting $q_o = 0$. The problem is then that of a cantilever beam with an end-shear which has the well-known solution. Use this special case to check that your program is correct. You should get an exact answer for any number of elements (even just one element). Look at all aspects of the solution. Forces, moments, displacements, and rotations to make sure that they are correct.

Exercise 6

Verify that your program converges by checking that the displacement solution converges as the number of nodes increases when you turn the distributed loading back on. Note that with these approximation functions one can increase the number of parameters without the difficulties that arise with our simple polynomials from the earlier labs.

Exercise 7

Where does the maximum bending moment occur for the problem? and what is its value?

Exercise 8

Add a mid-span pin support and solve the problem with your code. Where does the maximum bending moment occur now? and what is its value?

Exercise 9

Create a modified program that solves the buckling problem using these approximation functions and re-compute the answers to some of the prior lab and homework questions.

Sample Syllabus

D

At Berkeley we teach the entire contents of this text in a single semester. The pace is quite comfortable and allows plenty of time for discussion. Each week there are two one-hour lectures and a three-hour computer laboratory session. These are organized as follows:

(1) Week 1
   (a) Introduction and overview
   (b) Review: Tension-Compression bar as a second order differential equation
      - No Lab

(2) Week 2
   (a) Review: Torsion of a circular bar as a second order differential equation
   (b) Review: Beam bending as a fourth order differential equation
      - (Lab 1) BVP4c in Matlab applied to tension-compression bars

(3) Week 3
   (a) Equilibrium of Trusses: Introduction
   (b) Equilibrium matrix and compatibility matrix
      - (Lab 2) BVP4c in Matlab applied to beams

(4) Week 4
   (a) Truss Stiffness Matrix
   (b) Energy Definitions: Power, Work, Conservative Forces, Conservative Systems
      - (Lab 3) Construction of compatibility matrix for truss program

(5) Week 5
   (a) Energy conservation: trusses and expressions for torsion and bending.
   (b) Energy conservation: examples.
      - (Lab 4) Static condensation and solution for nodal displacements

(6) Week 6
   (a) Energy conservation example; Potential Energy

(b) Potential energy example, potential energy truss
- (Lab 5) Utilization of truss program from Lab 4 to analyze a truss.

(7) Week 7
(a) Potential energy truss, Approximate potential energy
(b) Midterm exam
- No Lab

(8) Week 8
(a) Approximate potential energy, the method of Ritz
(b) Approximate potential energy, the method of Ritz worked example with program
- (Lab 6) Visualization lab for potential energy in 1 and 2 degree of freedom systems.

(9) Week 9
(a) Revisit rigid buckling with potential energy
(b) Revisit beam buckling with potential energy
- (Lab 7) Potential energy solution with hat functions: mini-FEA code

(10) Week 10
(a) Buckling with supports and distributed loads
(b) Response of beams with axial compression; intro to virtual work (bars)
- (Lab 8) Buckling lab with rigid bars

(11) Week 11
(a) Virtual Work; Principle of V.W.; approximate V.W.
(b) Virtual Work expressions for torsion and bending
- (Lab 9) Buckling of beams using polynomial and trignometric expansions with potential energy approximations

(12) Week 12
(a) Introduction to multidimensional problems.
(b) Example two dimensional mechanical system
- (Lab 10) Virtual work solutions of beam problems using Hermite polynomials: mini-FEA code (part 1)

(13) Week 13
(a) Example application to seepage and Darcy's flow by potential energy methods.
(b) Example application to pollutant transport by advection-diffusion and weak equilibrium.
- (Lab 10 continued) Virtual work solutions of beam problems using Hermite polynomials: mini-FEA code (part 2)

(14) Week 14
   (a) Advection-diffusion example continued
   (b) Review, final exam discussion, evaluations
      - No labs

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