INTERNATIONAL ECONOMIC REVIEW
Vol. 32, No. 2, May 1991

A GENERAL EQUILIBRIUM ANALYSIS OF OPTION AND STOCK MARKET INTERACTIONS*

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The traditional pricing methodology in finance values derivative securities as redundant assets that have no impact on equilibrium prices and allocations. This paper demonstrates that when the market is incomplete primary and derivative asset markets, generically, interact: the valuation of derivative and primary securities is a simultaneous pricing problem and primary security prices depend on the contractual characteristics of the derivative assets available. In a version of the Mossin mean-variance model we analyze an equilibrium in which a call option (derivative asset) is traded and the equilibrium stock price (primary asset) increases when the options market is opened.

1. INTRODUCTION

Well established valuation models in finance price derivative securities (securities whose payoffs depend on other traded assets) by arbitrage. In these complete market settings the payoff on a derivative security can be reproduced by some portfolio of traded assets. In the absence of arbitrage, its value must therefore be equal to the value of the replicating portfolio. In this approach the prices of primary securities are exogenously specified and are independent of the contractual features of the derivative securities. This paper considers a general equilibrium model of an incomplete financial market in which diverse investors trade a primary security (a stock) and a derivative security (a call option written on the stock). In this context we demonstrate that the option and the stock market, generically, interact. The value of the stock almost always depends on the contractual characteristic of the option contract (its exercise price). Conversely, the stock value cannot be taken as exogenously given when a newly introduced option is being valued. In a version of the classic mean-variance model of Mossin (1969) we demonstrate that the value of the underlying stock increases when an option contract is introduced in the market.[2]

* Manuscript received January 1989.

[1] We are indebted to Mark Machina and especially Heraklis Polemarchakis for their detailed comments on earlier drafts of the paper. The paper has also benefitted from the suggestions of Larry Benveniste, Philip Dybvig, Philippe Jorion, Richard Kihlstrom, Karl Shell, Marti Subrahmanyam and Suresh Sundaresan. We are also grateful to two anonymous referees for their comments. Earlier versions of the paper were presented at the French Finance meetings (1987), the European Finance meetings (1987) and the Econometric Society Winter meetings (1987). The work of the first author was supported in part by the Faculty Research supplement of the Graduate School of Business, Columbia University.

[2] The ability to complete the market by issuing sufficiently many options has been pointed out in many studies (Ross 1976, Hakansson 1978b, Breeden and Litzenberger 1978, John 1984 and Green and Jarrow 1987, among others). Yet little is known about the pricing consequences of incomplete financial markets.

The canonic model for pricing derivative securities rests on the foundations laid by Black and Scholes (1973) and further developed by Merton (1973). Typically these models take as primitives the stochastic processes followed by the prices of the primary securities and cast the analysis within a complete markets setting. In the Black-Scholes option pricing model, for instance, the stock price follows a Geometric Brownian Motion process and the rate of return on the instantaneous bond is a constant. Since there is a linear relationship between the changes in the price of the stock (the underlying primary security) and the sources of uncertainty (the Brownian Motions), the arrival of the relevant information can be duplicated by a strategy involving trading in the primary security. The bond, furthermore, provides a riskless vehicle for transfers of capital over time.

In the context of the Black and Scholes model, the market completeness assumption requires a particular resolution of the uncertainty that may be an inaccurate representation of observed price time series. Lumpiness in the releases of information by managers, for instance, induces discontinuous components in prices. More generally, even within the context of continuous processes for primary securities, the coefficients of the model may be generated by information sources of dimensionality greater than the dimensionality of the space spanned by marketed primary securities. In that context a duplicating portfolio cannot be constructed. Derivative securities, and in particular options, increase the span of the payoff space and will be traded in equilibrium provided there is sufficient diversity among investors.

In this paper we consider an economy with an incomplete market in which a stock, a call option written on the stock and a bond are available. For a generic set of endowments the value of the stock depends on the exercise price of the option. The intuition for the result is straightforward: the relative prices of the assets depend on the equilibrium allocation of the commodity, which in turn depends, if markets are incomplete, on the linear subspace spanned by the payoffs of the assets. If there is enough diversity among agents to support trade in the option, the option contract affects the subspace spanned and therefore the equilibrium price of the stock.[3] Our analysis identifies ranges for the option exercise price over which option innovations will leave the span unaltered and the value of the stock unchanged.

Broadly interpreted, our analysis demonstrates that the dependence between the valuation of primary and derivative assets is a robust property of economies with incomplete markets. When markets are incomplete financial innovation causes, for

For instance Hakansson (1979a) states: "So we find ourselves in the awkward position of being able to derive unambiguous values only for redundant assets and unable to value options which have social value."

[3] Examples can be constructed in which the equilibrium allocation of the commodity is generically (in endowments) affected by changes in the exercise price of the option contract, yet the value of the stock is immune to those changes; for instance, when all agents have von Neumann-Morgenstern preferences with linear date zero utility and quadratic date one utility. In this example options will be held in equilibrium to hedge the random date one endowment of the commodity, but the price of the stock is independent from the option exercise price since this economy aggregates. This example is a pathologic case since minimal diversity among agents (for instance, the presence of an agent with power utility) will restore an interaction.

a generic set of endowments, a reallocation of consumption across investors: only particular endowments' configurations (for instance if endowments are Pareto optimal) may lead to the absence of trades once the new contract is available. If there is sufficient agent diversity the prices of all assets will then change and reflect the contractual characteristics of the new derivative asset created.

In a second stage we specialize the economy to a version of the Mossin (1969) setting to derive sharper results on the consequence of an option innovation. In this classic setting with quadratic von Neumann-Morgenstern preferences over date one consumption (there is no date zero consumption) we suppose that there are two classes of investors who disagree on the downside potential of the stock, i.e, they differ in beliefs by a mean-preserving spread on the lower tail of the stock's payoff distribution. Under these conditions the introduction of an option increases the equilibrium price of the stock and, consequently, decreases the volatility of the stock rate of return.[4]

At first blush, when an option is introduced, one might expect investors to reduce their demand for the stock and instead purchase some of the new option. Were this to be the case, the price of the stock would fall. The flaw in this reasoning is that the option is complementary to the stock at the aggregate level and not a substitute for it. In our model investors with a high risk assessment have a relative preference for a portfolio that pays off for large values of the stock payoff since they place a higher likelihood on extreme realizations of the stock payoff. To achieve their desired payoff pattern they sell the stock and buy the option. That is, they view the option as a substitute for the stock. The low risk assessment investors, on the other hand, view the stock as a complement for the option in the sense that they buy more of the stock and sell the option. The second class of investors has a stronger reaction to the change in the market structure as a result of their lower risk assessment. This causes the aggregate demand for the stock to increase: the option complements the stock at the aggregate level. It follows that the stock is more valuable in the presence of an option; its price increases. The volatility of the stock rate of return decreases so that the introduction of the option contract stabilizes the stock market.

In Section 2 of the paper we describe the structure of the economy and define the competitive equilibrium. Section 3 provides a generic analysis of the interactions between the option and the stock market. In particular, we identify precise conditions under which the interaction cannot be ignored in pricing problems. Section 4 specializes the analysis to an economy with quadratic von Neumann-Morgenstern preferences and limited diversity of beliefs in which the effects of an option innovation can be analyzed. Conclusions and extensions are discussed in the last section.

[4] In an economy without random date one endowments and composed of agents with diverse quadratic utility functions but homogeneous beliefs two-funds separation holds, inside assets (zero supply) are not traded and primary and derivative asset markets do not interact. More generally, the markets fail to interact in economies where two-funds separation holds, for instance for families of preferences in the HARA class (Rubinstein 1974). As noted by Dybvig and Ingersoll (1983) the introduction of options results in a failure of mean-variance pricing when investors' preferences are sufficiently diverse.

2. THE ECONOMIC MODEL

We consider a single good, pure exchange economy with one period (two dates, zero and one). The uncertainty is described by a finite space of states of nature Ω with generic element $\omega = 1, \dots, \Omega$. The uncertainty resolves at date one; date-events are denoted by 0 and $(1, \omega)$, $\omega = 1, \dots, \Omega$.

A single, perishable good is available at each date-event. A commodity bundle is,

$$c = (c(0), c(1)) = (c(0), \dots, c(1, \omega), \dots),$$

a vector in $\mathcal{R}^{\Omega+1}$. The commodity is taken as the numeraire; its price is set equal to one at all dates-events.

The financial market is composed of three real assets: a primary security (the stock), a call option written on the stock and a riskless bond. At date zero markets open and trades take place. At date one the uncertainty resolves and securities pay off.

The stock is a claim against the output of an exogenous productive technology. It is in positive supply and has a payoff contingent on the state of nature. Let $S(\ )$: $\Omega \to \mathcal{R}_+$ denote its payoff; $S = (S(1), \dots, S(\omega), \dots)$ is a vector in $\mathcal{R}^{\Omega}_{++}$, the positive orthant of $\mathcal{R}^{\Omega}$.

The option is in zero supply (inside asset) and has a payoff dependent on the payoff of the stock, $g(\omega) \equiv (S(\omega) - X)^+ \colon \Omega \to \mathcal{R}_+$ where $(S(\omega) - X)^+ \equiv \max\{S(\omega) - X, 0\}$ and $X \geq 0$ represents the exercise price or strike price of the option; $g = (g(1), \dots, g(\omega), \dots)$ is in $\mathcal{R}^{\Omega}_{++}$.

The riskless bond is also in zero supply (inside asset) with payoff equal to R.

Since these securities are real assets the payoffs $S(\omega)$ and $g(\omega)$ are homogeneous of degree one in the prices of the commodity and the bond pays off R units of the commodity in each state. The prices for the stock, the option and the bond are respectively $p = (p_s, p_o, p_b)$. Aggregate supplies of assets are $(\bar{x}_s, 0, 0)$.

The following assumptions are made.

ASSUMPTION 2.1. $\Omega > 3$.

ASSUMPTION 2.2. $S(\Omega) > S(\Omega - 1) > X > S(2) > S(1)$.

Since three assets are available the first assumption guarantees that the market is incomplete. Assumption 2.2 guarantees that the equity and the bond are not perfect substitutes. It also restricts the range of possible exercise prices for the call option: it implies that the option has a positive payoff in at least two states (Ω and $\Omega - 1$) and at most in $\Omega - 2$ states. Our analysis will demonstrate that the stock and the option markets will cease to interact when the call option's exercise price fails to satisfy Assumption 2.2 (see Remark 3.2).

The $\Omega \times 3$-dimensional matrix of asset payoffs is then,

$$R(X) = [S, (S - X)^+, R],$$

a function of the call exercise price. Its column span determines the set of attainable allocations. To prevent the reallocations of revenue from varying discontinuously (with X) we restrict the domain of call exercise prices:

$$\mathscr{X} = \{X\colon S(\omega) \neq X, \ \omega = 1, \dots, \Omega\}.$$

Given Assumption 2.2, this domain is nonempty and open.

A portfolio is $x = (x_s, x_o, x_b)$ where x_s, x_o and x_b represent respectively the shares holdings of the stock, option and bond. Asset prices do not allow for arbitrage if and only if $R(X)x > 0 \Rightarrow p'x > 0$.[5] To eliminate the possibility of arbitrage, we restrict the domain of asset prices (Geanakoplos and Polemarchakis 1986):

$$\mathscr{P}(X) = \{p\colon \text{for some } \pi = (\pi(1), \dots, \pi(\omega), \dots) \in \mathscr{R}^{\Omega}, \ p = \pi' R(X)\}.$$

This domain is a nonempty open set.

Agents in this economy consume and choose a portfolio at date zero and consume at date one. Agent h, $h = 1, \dots, H$ is characterized by his utility function u^h over consumption bundles, his endowment of the consumption good, a nonnegative commodity bundle $\bar{c}^h$ and his endowment of shares, a portfolio $\bar{x}^h = (\bar{x}_s^h, 0, 0)$. Endowments of the stock are nonnegative. There are no endowments of the option and the bond. An admissible portfolio demand involves nonnegative holdings of the stock, but possibly short positions in the option and the bond. An admissible consumption demand is a nonnegative commodity bundle.

ASSUMPTION 2.3. $H > 3$.

ASSUMPTION 2.4. *For $h = 1, \dots, H$, the utility function u^h is continuous, monotonically increasing and strictly quasi-concave. On the interior of its domain of definition it is twice continuously differentiable; the gradient Du^h is positive ($Du^h \gg 0$) and the matrix of second partials D^2u^h is negative definite on the orthogonal complement ($[Du^h]^{\perp}$) of Du^h. For a sequence c_n, $n = 1, \dots$, in the interior of the consumption set with limit c on the boundary,*[6]

$$\lim_{n\to\infty} c_n = c \Rightarrow \lim_{n\to\infty} c_n' Du^h(c_n)/\|Du^h(c_n)\| = 0.$$

ASSUMPTION 2.5. *For $h = 1, \dots, H$, $\bar{c}^h + \bar{x}^h S \gg 0$.*

Assumption 2.3 ensures that there is enough diversity in the economy for the option to be held. Assumption 2.4 is standard. The negative definiteness of the matrix of second partials as well as the boundary behavior of preferences guarantee the differentiability of the excess demand function over an appropriately restricted domain of economies, exercise prices for the option and asset prices. Assumption

[5] The symbol $'$ denotes the transpose of a vector or a matrix.

[6] For a matrix $A \in \mathscr{R}^{\Omega+1}$ the notation $\|A\|$ denotes the usual matrix-norm (Mas-Colell 1985, p. 15).

2.5 guarantees that the no trade allocation is an interior point of the consumption set.

In our analysis the payoff vector S, the asset structure and the agents' endowments of shares are held fixed. An economy is thus an array of initial endowments of consumption bundles and preferences, $e = (\dots, (u^h, \bar{c}^h), \dots)$. The set of economies is an open set denoted by $\mathscr{E}$.[7] We say that a property holds generically if it holds for an open and dense set of economies (i.e., a set of full Lebesgue measure).

Given prices p the budget set of investor h is defined by $\mathscr{B}^h(\bar{c}^h, X, p) = \{(c^h, x^h) \in \mathfrak{R}_+^{\Omega+1} \times \mathfrak{R}_+ \times \mathfrak{R}^2 : c^h(0) + x_s^h p_s + x_o^h p_o + x_b^h p_b \le \bar{x}_s^h p_s + \bar{c}^h(0), c^h(1, \omega) \le x_s^h S(\omega) + x_o^h g(\omega) + x_b^h R + \bar{c}^h(1, \omega)\}$.

Agents choose consumption and assets so as to maximize their utility subject to the budget constraint,

$$\operatorname*{Max}_{c,x} u^h(c) \quad \text{s.t.} \quad (c, x) \in \mathscr{B}^h(\bar{c}^h, X, p), \qquad h = 1, \dots, H.$$

To ensure regularity of the demand behavior of agents we restrict the domain of economies, call exercise prices and asset prices,

$$\mathscr{D} = \{(e, X, p) : e \in \mathscr{E}, \qquad X \in \mathscr{X}, \qquad p \in \mathscr{P}(X)\}.$$

This domain is a nonempty open set. By Assumption 2.5 there exists $(c, x) \in \mathscr{B}^h(\bar{c}^h, X, p)$ such that $c^h \gg 0$, $h = 1, \dots, H$.

On the set $\mathscr{D}$ there exists a unique solution to the individual optimization problem,

$$(c^h, x^h)(e, X, p), \qquad h = 1, \dots, H.$$

The individual demand function for date zero consumption and assets is continuously differentiable, satisfies Walras law and the boundary condition,

$$\lim_{n \to \infty} p_n = p \quad \text{and} \quad p \in \partial\mathscr{P} \Rightarrow \lim_{n \to \infty} \|(c^h(1), x^h)(e, X, p_n)\| = \infty.$$

For $(e, X) \in \mathscr{E} x \mathscr{X}$, a competitive equilibrium is a price vector p and an allocation of the commodity and the securities $\{(c^h, x^h), h = 1, \dots, H\}$ such that,

(i) markets clear: $\Sigma_h x_s^h = \bar{x}_s$, $\Sigma_h x_o^h = 0$, $\Sigma_h x_b^h = 0$,

(ii) individuals behave rationally: (c^h, x^h) is maximal in $\mathscr{B}^h(\bar{c}^h, X, p)$ for $h = 1, \dots, H$.

The competitive allocation corresponding to the equilibrium price $p(e, X)$ is written $(c^h(0; e, X, p(e, X), c^h(1; e, X, p(e, X))), h = 1, \dots, H$.

Given our assumptions a competitive equilibrium exists (Geanakoplos and Polemarchakis 1986).

[7] We consider an open set, $\mathscr{C} \in \mathfrak{R}_{++}^{(\Omega+1)H}$, of endowments such that $\mathscr{C}$ is bounded and bounded away from zero. Similarly, we consider an open set, $\mathscr{W} \in \mathfrak{R}^H$, of preferences constructed as follows. For each u^h satisfying Assumption 2.4 add a small multiple $a^h v$ of any smooth function v to construct a utility $w^h = u^h + a^h v$ satisfying the same assumption. The space of utility functions $w^h(a^h)$, $h = 1, \dots, H$, is then a finite dimensional manifold $\mathscr{W} \in \mathfrak{R}^H$. The set of economies is $\mathscr{E} = \mathscr{W} \times \mathscr{C}$.

3. OPTION AND STOCK MARKET INTERACTIONS

The option and the stock market interact when the valuation of the stock depends on the contractual characteristic of the call option: its exercise price. To demonstrate the interaction between the two markets we need to show that for $X^1 \neq X^2$, distinct exercise prices, the corresponding equilibrium prices of the stock, $p_s^1(e, X^1) \in \mathcal{P}(X^1)$ and $p_s^2(e, X^2) \in \mathcal{P}(X^2)$, are distinct.

PROPOSITION 3.1. *Suppose that Assumptions* 2.1 *through* 2.5 *hold. Then, generically, the stock and option markets interact.*

PROOF OF PROPOSITION 3.1. We prove three auxilliary Lemmas. In the first we establish that equilibrium asset prices can be written as functions of the corresponding equilibrium allocation. This expression is reminiscent of "martingale" representation formulae (Harrisson and Kreps 1979). Then, to demonstrate the interaction between the markets we need only show that equilibrium allocations generically change when the option exercise price changes. This is accomplished in the next two Lemmas. First we show that different option exercise prices induce different asset spans under Assumptions 2.1 and 2.2. Second we demonstrate that different asset spans generate different equilibrium allocations for a dense set of economies.

We introduce the notation $D_1 u^h(c^h) = (\dots , D_1 u^h(c^h)(\omega), \dots)$ to represent the gradient of the utility function with respect to date one consumption, $c^h(1)$; a vector of dimension Ω. Similarly $D_0 u^h(c^h)$ is the derivative with respect to date zero consumption, $c^h(0)$.

LEMMA 3.1. *Let* $\{c^h(e, X, p(e, X)), h = 1, \dots , H\}$ *represent the equilibrium allocation. Equilibrium prices* $p = (p_s, p_o, p_b)$ *can be represented as,*

$$p_s = \sum_\omega \left[\sum_h D_1 u^h(c^h(e, X, p(e, X)))(\omega) \Big/ \sum_h D_0 u^h(c^h) \right] S(\omega) \tag{3.1}$$

$$p_o = \sum_\omega \left[\sum_h D_1 u^h(c^h(e, X, p(e, X)))(\omega) \Big/ \sum_h D_0 u^h(c^h) \right] g(\omega) \tag{3.2}$$

$$p_b = \sum_\omega \left[\sum_h D_1 u^h(c^h(e, X, p(e, X)))(\omega) \Big/ \sum_h D_0 u^h(c^h) \right] R. \tag{3.3}$$

In vector notation, $p = \Sigma_\omega \, [\Sigma_h \, D_1 u^h(c^h(e, X, p(e, X)))/\Sigma_h D_0 u^h(c^h)] R(X)$.

PROOF OF LEMMA 3.1. The necessary conditions for the agents' optima are,

$$-D_0 u^h(c^h)p + D_1 u^h(c^h)R(X) = 0, \; h = 1, \dots , H.$$

Evaluating these equations at the equilibrium allocations, summing over agents and solving for prices leads to the representation formulae in the Lemma. □

REMARK 3.1. The equilibrium price of the stock depends on the equilibrium allocation, preferences and the stock's payoff. To demonstrate the presence of a robust interaction with the option market we need to show that, generically, the equilibrium allocation is not invariant to changes in the option exercise price.

LEMMA 3.2. *Suppose that Assumptions* 2.1 *and* 2.2 *hold. Let $\langle R(X)\rangle$ denote the column span of the matrix $R(X)$. Then, for $X^1 \neq X^2 \in \mathcal{X}$, $\langle R(X^1)\rangle \neq \langle R(X^2)\rangle$.*

PROOF OF LEMMA 3.2. By Assumption 2.2 the option is not spanned by the stock and the bond. It follows that the spans $\langle R(X^1)\rangle$ and $\langle R(X^2)\rangle$ are identical if and only if the vectors $(S - X^1)^+$ and $(S - X^2)^+$ are colinear. This holds if and only if $S(\Omega) \geq X^1, X^2 \geq S(\Omega - 1)$ or $S(2) \geq X^1, X^2 \geq S(1)$. Assumption 2.2 rules out these configurations so that $\langle R(X^1)\rangle = \langle R(X^2)\rangle$ if and only if $X^1 = X^2$. □

REMARK 3.2. The restriction imposed by Assumption 2.2 now becomes transparent: it rules out situations where changes in the call exercise prices trivially have no effect on the asset span. As an illustration, suppose that $S = (1, 2, 3, 4, 5)$ and $R = 1$. Consider the exercise prices $X^1 = 4$ and $X^2 = 4.5$ with associated option payoffs $(S - X^1)^+ = (0, 0, 0, 0, 1)$ and $(S - X^2)^+ = (0, 0, 0, 0, .5)$. We trivially have $(S - X^2)^+ = .5\,(S - X^1)^+$, i.e. it is possible to replicate either of the two options by holding the appropriate quantity of the other option.

LEMMA 3.3. *Suppose that Assumptions* 2.1 *through* 2.5 *hold. Consider $X^1 \neq X^2 \in \mathcal{X}$, distinct call option exercise prices, along with their associated date one equilibrium consumption allocations $c^h(1;\, e,\, X^1,\, p(e,\, X^1))$ and $c^h(1;\, e,\, X^2,\, p(e, X^2))$, $h = 1, \dots, H$. Then,*

$$\{c^h(1;\, e, X^1, p(e, X^1)),\ h = 1, \dots, H\} \neq \{c^h(1;\, e, X^2, p(e, X^2)),\ h = 1, \dots, H\},$$

i.e., the corresponding competitive allocations are distinct.

PROOF OF LEMMA 3.3. Consider the equilibrium allocations $\{x^{h,1}(e,\, X^1,\, p(e, X^1)),\ h= 1, \dots, H\}$ and $\{x^{h,2}(e, X^2 p(e, X^2)),\ h = 1, \dots, H\}$ associated with the equilibrium prices $p(e,\, X^1) \in \mathcal{P}(X^1)$ and $p(e,\, X^2) \in \mathcal{P}(X^2)$. From Lemma 3.2, $\langle R(X^1)\rangle \neq \langle R(X^2)\rangle$ when $X^1 \neq X^2$. If the equilibrium allocations are such that dim $[\,..\,, x^{h,1}, ..\,] = 3$ or dim $[\,..\,, x^{h,2}, ..\,] = 3$ (i.e., the option is held by at least one agent in one of the two allocations) it must be the case that $R(X^1)x^{h,1} \neq R(X^2)x^{h,2}$ for some h, $h = 1, \dots, H$. It follows that the consumption allocations must be distinct since,

$$c^h(1;\, e, X^1, p(e, X^1)) = R(X^1)x^h(e, X^1, p(e, X^1)) + \bar{c}^h(1),$$

and

$$c^h(1;\, e, X^2, p(e, X^2)) = R(X^2)x^h(e, X^2, p(e, X^2)) + \bar{c}^h(1).$$

Hence to establish the result if suffices to show that the option is held: dim $[\,..\,, x^{h,1}, ..\,] = 3$ or dim $[\,..\,, x^{h,2}, ..\,] = 3$. Equivalently, it suffices that one of the equilibrium allocations span a 3-dimensional space: there exists three agents, say

$h = 1, 2, 3$ such that $\det [x^1(e, X, p(e, X)), x^2(e, X, p(e, X)), x^3(e, X, p(e, X))] \neq 0$.

Let $z(e, X, p)$ denote the aggregate excess demand for assets and define the determinant $\delta(e, X, p) \equiv \det [x^1(e, X, p), x^2(e, X, p), x^3(e, X, p)]$. We show that there exists an open and (Lebesgue-)dense set of economies, $\mathscr{E}^* \in \mathscr{E}$, such that for $e \in \mathscr{E}^*$, $X \in \mathscr{X}$, $p \in \mathscr{P}(X)$,

$$z(e, X, p) = 0 \Rightarrow \delta \neq 0.$$

Consider the excess demand function $z_{(e,X)}: \mathscr{D} \to \mathscr{R}^3$ and the augmented function $(z, \delta)_{(e,X)}: \mathscr{D} \to \mathscr{R}^4$, both parametrized by endowments, preferences and the call exercise price, (e, X). By a standard argument (Geanakoplos and Polemarchakis 1986, pp. 82–84) both functions are transverse to the origin:[8] $z_{\mathrm{T}} 0$ and $(z, \delta)_{\mathrm{T}} 0$. By the Transversal Density Theorem (Mas-Colell 1985, p. 45), there exists a set of economies and exercise prices of full Lebesgue measure, $(\mathscr{E} x \mathscr{X})^* \subset (\mathscr{E} x \mathscr{X})$, such that $(z, \delta)_{(e,X)\mathrm{T}} 0$ for $(e, X) \in (\mathscr{E} x \mathscr{X})^*$. The boundary behavior of the individual demand functions for first period consumption and assets implies that the set $(\mathscr{E} x \mathscr{X})^*$ is open. Since $\dim (\mathscr{P}(X)) = 3$, it then follows that $(z, \delta)_{(e,X)\mathrm{T}} 0$ if and only if $(z, \delta)^{-1}_{(e,X)}(0) = \emptyset$. Hence, on $(\mathscr{E} x \mathscr{X})^*$, $z_{(e,X)} = 0 \Rightarrow \delta_{(e,X)} \neq 0$. The projection $\mathscr{E}^* \equiv \operatorname{proj} (\mathscr{E} x \mathscr{X})^*$ if an open set of full Lebesgue measure. Thus, for $e \in \mathscr{E}^*$, the exercise price of the option affects the equilibrium allocations. This completes the proof of Lemma 3.3. □

REMARK 3.3. That the option exercise price generically affects the equilibrium allocations under our assumptions can be demonstrated when the preferences are held fixed, i.e., when economies are parametrized by endowments alone. However, it is possible to construct robust economies (to perturbations in endowments) where allocations are affected yet the value of the stock does not depend on the call exercise price. For instance when preferences are von Neumann-Morgenstern linear-quadratic: heterogeneity in endowments leads to holdings of options, yet since marginal utility is linear the economy aggregates and the value of the stock is invariant to the strike price (see equation (3.1)). Perturbations in the preferences of the agents populating the economy straightforwardly reintroduce the interaction between the markets.

The proof of Proposition 3.1 now follows by combining Lemmas 3.1 and 3.3. □

4. A MEAN-VARIANCE ECONOMY

We now specialize the economy to a version of the familiar Mossin (1969) setting. We first present the economy and describe its equilibrium (subsection 4.1). Then,

[8] See Mas-Colell (1985, pp. 42–45) or Geanakoplos and Polemarchakis (1986, pp. 80–82). Consider a smooth map f: $M \to N$ where M and N are smooth m- and n-dimensional manifolds belonging to a finite dimensional Euclidean space and let $0 \in N$. Then the map f is transverse to 0 ($f_{\mathrm{T}} 0$) if for all $x \in M$ with $f(x) = 0$, $Df(x)$ has full rank n. When $f_{\mathrm{T}} 0$ and $m \geq n$ then $f^{-1}(0)$ has dimension $m - n$. In particular when $m = n$, $f^{-1}(0)$ is a set of discrete points (dimension 0). When $f_{\mathrm{T}} 0$ and $m < n$ then $f^{-1}(0) = \emptyset$, the empty set.

we analyze the effects on the price of the stock of changes in the exercise price of the option or in the diversity among investors (subsection 4.2). Finally, we study the effects of an option introduction (subsection 4.3).

4.1. *A Mean-Variance Economy.* We assume that the economy is populated by two classes of investors, with high ($h = 1$) or low risk assessment ($h = 2$), who disagree only about the downside potential of the stock. Both classes have identical quadratic utility function of date one consumption,[9]

$$U^h(c^h(\omega)) = c^h(\omega) - k(c^h(\omega))^2, \qquad h = 1, 2, \tag{4.1}$$

where k denotes the common preference parameter and $c^h(\omega)$ is date one consumption (there is no consumption at date zero). Endowments of shares of the stock are identical, $\bar{x}_s^h = \bar{x}_s/2$, $h = 1, 2$. There are no endowments of the commodity.

Disagreement about the downside potential of the stock is of the following form. Investors' probability assessments, P^h, differ only over the lower tail of the stock's payoff in such a way that the moments,[10]

$$E^hS \equiv \int_\Omega S(\omega)P^h(d\omega)$$

$$E^h(S - X)^+ \equiv \int_\Omega (S(\omega) - X)^+ P^h(d\omega)$$

$$E^h[(S - X)^+]^2 \equiv \int_\Omega [(S(\omega) - X)^+]^2 P^h(d\omega)$$

$$E^h[S(S - X)^+] \equiv \int_\Omega [S(\omega)(S(\omega) - X)^+]P^h(d\omega),$$

$h = 1, 2$, are common to all individuals. Hence, the *only* heterogeneity allowed is with regard to the second moment of the stock payoff $E^hS^2 \equiv \int_\Omega S(\omega)^2 P^h(d\omega)$, $h = 1, 2$; investors of type 2 perceive less risk in the stock payoff, i.e., $E^1S^2 > E^2S^2$. One can think of the individuals in this economy as having beliefs that differ by a Rothschild-Stiglitz (1970) mean preserving spread on the downside potential of the firm, the set $\mathcal{S}^X = \{S: S \le X\}$.[11] When expectations are common to both classes of

[9] While we recognize the limitations of the these preferences (e.g., Arrow 1971, Chapter 3), they nevertheless possess the very significant advantage of facilitating fully computable solutions and the derivation of interesting comparative static results. Most, if not all, of the other preference forms from the HARA family (see, for instance, Rubinstein 1974) fail to yield closed form expressions for both the stock and option demands and analytically tractable expressions for equilibrium prices.

[10] The analysis in this section holds when the set Ω is a compact set (continuum of states).

[11] To illustrate this structure of beliefs, consider the case where $S(\omega)$ takes the values (1, 1.5, 2.5, 3, 5) at date 1 and agents have respective beliefs $P^1 = (0.25, 0, 0, 0.25, 0.50)$ and $P^2 = (0, 0.25, 0.25, 0, 0.50)$. Then, if the option on this stock has an exercise price of 4, it is easily verified that the two agents will agree

investors we ignore the superscript h; for instance we write ES. The variances of the stock payoff are $\sigma_h^2 \equiv E^h S^2 - (ES)^2$, $h = 1, 2$. The variance of the option payoff and the covariance between the stock and the option payoffs are common to both classes of investors and are respectively written as $\sigma_o^2 \equiv E[(S - X)^+]^2 - [E(S - X)^+]^2$ and $\sigma_{so} = ES(S - X)^+ - ESE(S - X)^+$.[12]

The nature of the heterogeneity we allow in this example is quite limited. It is important to stress that while greater heterogeneity in beliefs (for instance about the upside potential of the stock) and/or preferences will produce even greater interactions between the stock and option markets than that obtained in the subsequent sections, it unfortunately will also preclude the derivation of clear cut comparative static results.

The economy under consideration is described by the set of parameters $e = \{k, \bar{x}_s, ES, \sigma_1^2, \sigma_2^2, \sigma_o^2$ and $\sigma_{so}\}$. Since there is no date zero consumption we are free to normalize the bond price, $p_b = 1$. Since the utility function is quadratic we need to restrict the set of parameters to ensure the existence of a well behaved equilibrium. The set of admissible economies is $\mathscr{E} = \{e: p \geq 0, x_s^h \geq 0, c^h(e, X, p(e, X)) \geq 0$ (a.s.) and $DU^h(c^h(e, X, p(e, X))) > 0$ (a.s.), $h = 1, 2\}$. For the economy under consideration equilibrium prices can now be written as follows.

PROPOSITION 4.1. *Consider an admissible economy $e \in \mathscr{E}$ characterized by two classes of investors with identical quadratic utility von Neumann-Morgenstern preferences and identical endowments but with beliefs that differ by a mean preserving spread on the compact interval $\mathscr{S}^X$. Then, the equilibrium is,*[13]

$$p_s = R^{-1}[ES - \lambda(\delta\sigma_1^2 + (1 - \delta)\sigma_2^2)] \tag{4.2}$$

$$p_o = R^{-1}[E(S - X)^+ - \lambda\sigma_{so}] \tag{4.3}$$

$$x_s^h = (\alpha/\Delta^h)[\sigma_o^2(ES - p_s R) - \lambda\sigma_{so}^2], \qquad h = 1, 2 \tag{4.4}$$

$$x_o^h = (\alpha/\Delta^h)\sigma_{so}[p_s R - (ES - \lambda\sigma_h^2)], \qquad h = 1, 2 \tag{4.5}$$

where

on $E(S)$, $E(S - X)^+$, $E[(S - X)^+]^2$ and $E[S(S - X)^+]$ but will differ in their view of the stock's second moment, $E^1(S)^2 \neq E^2(S)$.

[12] This model is a special case of the economy analyzed in the previous sections. Indeed, the difference in beliefs can be reinterpreted as a heterogeneous state dependence in the preferences over date one consumption. More generally, economies with von Neumann-Morgenstern preferences and heterogeneous beliefs are special cases of the model with general preference structures. It follows that standard existence theorems apply (Debreu 1959, or Geanakoplos and Polemarchakis 1986).

[13] Consider the same economy where a set of options has been introduced so as to complete the market (full contingent claims economy). In this economy the demands for risky assets can be explicitly computed as $x^h = (M^h)^{-1}\mu^h$, where M^h is the matrix of second moments (of profit positions) and $\mu^h \equiv (E[S - p_s R], E^h[(S - X_1)^+ - p_{o1}R], \dots, E[(S - X_{(\Omega-2)})^+ - p_{o(\Omega-2)}R])$ is the vector of expected profits. For the diversity of beliefs assumed, M^h and μ^h depend on h, although some of the components are common to all agents. In this complete markets economy there is general interaction between the markets since some of the option exercise prices fall below the area of disagreement among agents. It follows, in particular, that the stock price does not admit a simple representation as in equation (4.2). Also, the value of the stock relative to its value (4.2) in the incomplete market economy is unclear.

(4.6) $$\lambda \equiv \bar{x}_s/[k^{-1} - \bar{x}_s ES],$$

(4.7) $$\delta \equiv \Delta^2/(\Delta^1 + \Delta^2),$$

(4.8) $$\Delta^h = E[(S-X)^+ - p_o R]^2 E^h[S - p_s R]^2 - (E[(S-X)^+ - p_o R][S - p_s R])^2, \qquad h = 1, 2$$

(4.9) $$\alpha \equiv (2k)^{-1} - (\bar{x}_s/2)p_s R.$$

In this economy the option price (4.3) is identical to the option price formula in a similar economy with homogeneous beliefs (the standard CAPM model). This is a consequence of the limited form of heterogeneity that we have introduced in which investors agree on all of the moments of the option's payoff. In contrast, the stock price formula (4.2) reflects the diversity in the risk assessments of investors, i.e., their disagreement about the downside potential of the stock. The appropriate measure of aggregate risk becomes a weighted average of the diverse variances where the weights sum to one. From equation (4.4) we have $x_s^1 \Delta^1 = x_s^2 \Delta^2$ so that the weight $\delta \equiv \Delta^2/(\Delta^1 + \Delta^2)$ can also be written as the ratio of the equilibrium allocation of the stock of high risk assessment agents to the total endowment in the economy, $\delta = (x_s^1/\bar{x}_s)$. The stock price depends on the option exercise price X since the weight δ depends on X. In the absence of diversity ($\sigma^2 \equiv \sigma_h^2$, $h = 1, 2$) the stock price becomes the classic CAPM formula, $p_s = R^{-1}[ES - \lambda\sigma^2]$, and is clearly independent of the option's strike price.

The terms Δ^1 and Δ^2 defined in equation (4.8) represent the determinants of the matrices of second moments associated with each class of investors. By the concavity of the utility function these determinants are strictly positive which implies that the weights δ and $1 - \delta$ belong to the open interval (0, 1). It also follows that the equilibrium allocations x^1 and x^2 satisfy the nonnegativity requirement, $x_s^h \geq 0$ (the no-short sales constraint).

The option demand function (4.5) shows that the extent to which the option is traded is related to the deviation of the equilibrium stock price p_s from the classic mean-variance price $R^{-1}[ES - \lambda\sigma_1^2]$ that would prevail if all investors had beliefs σ_1^2.

As we show next, when $\sigma_1^2 > \sigma_2^2$, investors with the greater risk assessment ($h = 1$) will move out of the more risky investment on a payoff basis (the stock) into the less risky investment (the option). In equilibrium they hold a long position in the option and are net sellers of the stock (i.e., $x_s^1 - \bar{x}_s^1 = x_s^1 - \frac{1}{2}\bar{x}_s < 0$).

COROLLARY 4.1. *Under the conditions of Proposition* 4.1, $\sigma_1^2 > \sigma_2^2 \Rightarrow$
(i) $x_o^1 > 0$ *and* $x_o^2 < 0$.
(ii) $x_s^1 - \frac{1}{2}\bar{x}_s < 0$ *and* $x_s^2 - \frac{1}{2}\bar{x}_s > 0$.

When the option cannot be spanned trade follows if there is sufficient investor diversity. Trading in the option market in turn alters the demands for the stock and produces a dependency of the stock price on the option exercise price. This dependency is analyzed further in the next subsection.

4.2. *Stock Value, Span Changes and Investor Diversity.* In this section we examine (i) the effect on the value of the stock of a change in the span (a change in the call exercise price), and (ii) the effect on equilibrium prices and quantities of increased diversity in investors' beliefs about the downside potential of the stock. Throughout the remainder of the paper we assume that the aggregate demand for the stock is decreasing in the stock price p_s in a neighborhood of equilibrium. Specifically let e denote the set of parameters describing the economy. Recall that $\mathscr{E}$ denotes the set of "admissible" economies. Let $p_s(e, X)$ denote the equilibrium stock price level. We define the set $\mathscr{E}^*$ as,

DEFINITION 4.1. $\mathscr{E}^* = \{e\colon e \in \mathscr{E}, (\partial x(p)/\partial p]|_{p_s(e,\ X)} < 0\}$.

Thus $\mathscr{E}^*$ is the set of "admissible" economies (indexed by their parameters) that produce an aggregate demand for the stock that is a decreasing function of the stock price in a neighborhood of equilibrium.[14]

First we examine the impact of the call exercise price X on the value of the stock. Clearly, by modifying the level of the exercise price one changes both the risk of the option relative to the stock (the covariance effect) and the intrinsic risk of the option (the variance effect). The modification in these risk properties of the option, in general, induces changes in the demands for the stock and hence its equilibrium price. For economies in $\mathscr{E}^*$ the effect of the option exercise price on the stock price is unambiguous as is stated in the next Corollary.

COROLLARY 4.2. *Let $e \in \mathscr{E}^*$. Then an increase (decrease) in the option exercise price results in a decrease (increase) in the stock price.*

The intuition behind this result and the ones that follow can be easily understood via a graphical device that has become standard in the field of the option pricing. Here, we adapt this graphical analysis to account for equilibrium considerations. Each of the figures below graphs the portfolio payoffs on the vertical axis against the payoff on the stock on the horizontal axis. Figure 1 represents the portfolio payoffs in the absence of trading in the stock and the bond market. Since investors are identically endowed with half of the aggregate supply of shares of the stock their portfolio payoffs at the end of the period are identical and equal to half the end of period total market value of the stock.

When only the stock and the bond are available the payoffs that can be constructed have a linear structure (straight lines in the payoff space). Figure 2 describes the geometry of attainable portfolio payoffs when an option is traded. With an option portfolio payoffs that have a triangular shape with an angle at the exercise price of the option can be constructed. The orientation of the triangle

[14] Consider an economy $e \in \mathscr{E}$ and let ρ_1 denote the correlation coefficient between the stock payoff and the option payoff for investor 1. Sufficient conditions for e to belong to $\mathscr{E}^*$ are (i) $\rho_1^2 \geq 1/3$, or (ii) $\rho_1^2 < 1/3$ and $k^{-1} > \bar{x}_s ES + \bar{x}_s \sigma_1 (2 - 3\rho_1^2)^{1/2}$, or (iii) $\max_\omega S(\omega) - ES \geq \sigma_1 2^{1/2}$. It is straightforward to show that many standard distributions (e.g. Uniform, Normal, etc.) satisfy the standard deviation bound in conditions (iii). Even if the standard deviation bound is violated the sets of conditions (i) or (ii) still guarantee the result. If all of the three sets of conditions fail it may still be the case that the result holds since we have only identified sufficient conditions and not necessary conditions.

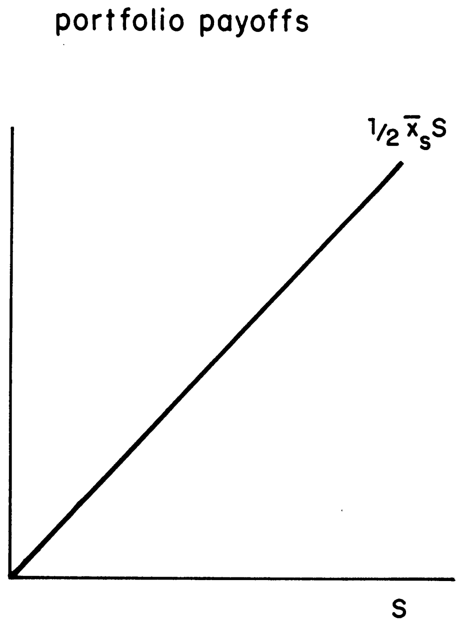

FIGURE 1

PORTFOLIO PAYOFFS IN THE ABSENCE OF TRADE (ENDOWMENTS).

(angle up or down) depends on the sign of the positions in the stock and option market. For instance, a long position in the option contract combined with a short position in the stock creates a downward orientation. The slopes of the sides of the triangle depend on the number of stocks and option contracts held. By selling shares of the stock and using the proceeds to buy sufficiently many option contracts an investor reduces the angle of the triangle making it more acute. This reallocation increases the portfolio payoffs for extreme payoffs of the stock.

Figure 3 represents the *equilibrium* portfolio payoffs in a two-investor economy with the stock, the bond and the option. Since in equilibrium the option and the bond are in zero net supply, the portfolio payoffs of the two classes of investors in Figure 3 are symmetric with respect to their endowment payoff. As demonstrated in Corollary 4.1 investors with a high risk assessment liquidate part of their stock position and purchase the option whereas the second class of investors performs the symmetric reallocation. This equilibrium outcome is intuitive in view of the geometry of portfolio payoffs in the presence of an option. Investors perceiving a high stock volatility place a higher likelihood on extreme payoffs of the stock and have a preference, relative to the other investors, for portfolios with a higher payoff in these states of nature. By purchasing the option and selling part of their endowed shares of the stock they create an equilibrium portfolio that achieves this preferred payoff pattern.

Figure 4 below provides the intuition for the result of Corollary 4.2. It graphs the changes in the portfolio demands of the two classes of investors as a result of the increase in the option's exercise price (the dashed lines refer to the initial allocation

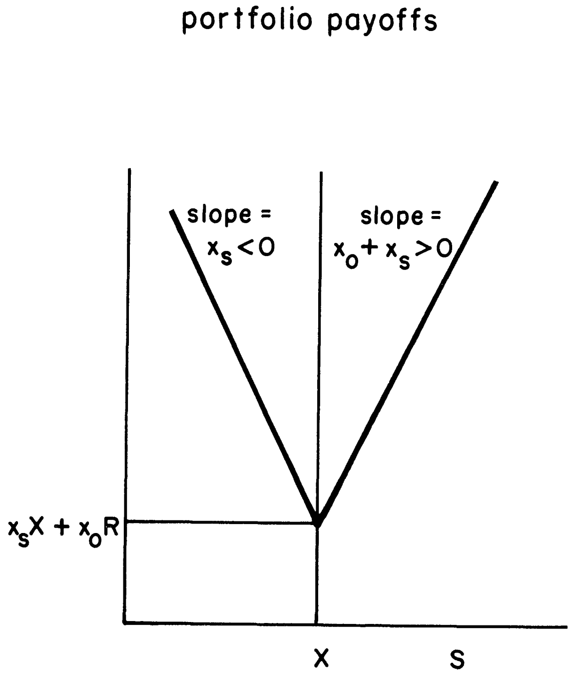

FIGURE 2

GEOMETRY OF PORTFOLIO PAYOFFS WITH THE STOCK, THE BOND AND OPTION.[15]

from Figure 3; the solid lines represent the demands for portfolios after the option's exercise price is increased to X'). Contrary to Figure 3 it focuses on the demand functions (before price adjustments) as opposed to final equilibrium allocations (which include price adjustments). This enables us to understand the price pressures that take place. An increase in the exercise price of the option, ceteris paribus, causes the angles of the portfolios constructed to move to the right. As a result the option becomes less useful in creating portfolios that enable investors to exploit their differences in risk assessments (recall that investors differ by a mean preserving spread on the lower tail of the stock payoff and therefore have a preference for payoff patterns with an angle as close as possible to the area of disagreement). Investors with a high risk assessment attempt to maintain a lower payoff on their portfolio in the area of disagreement. Since the exercise price of the option increases, they demand the stock and supply the option to achieve this goal. Investor with the low risk assessment seek to maintain a higher payoff in the area of disagreement and consequently supply the stock and demand the option. The aggregate demand for the stock decreases since the low risk investors are more sensitive to changes in the economic environment. The second class of investors is relatively more hurt by the parallel shift to the right in the portfolio payoffs (preserving the angle) that takes place when the exercise price is increased to X'.

[15] The portfolio payoff is $x_s S + x_o (S - X)^+ + x_b R$. For $S \leq X$ the portfolio payoff becomes $x_s S + x_b R$ since the option is out of the money.

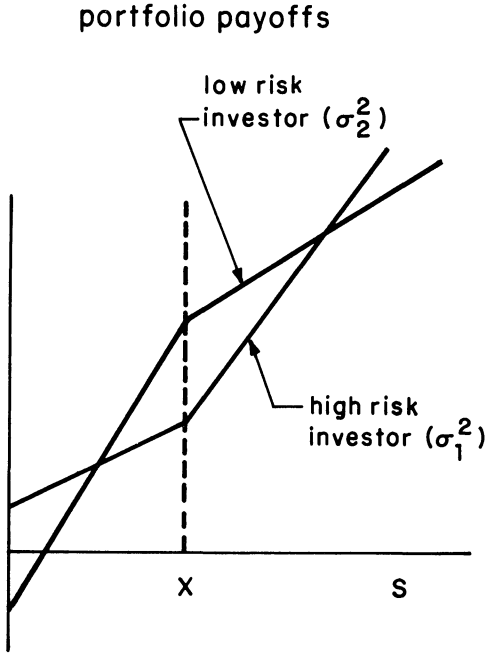

FIGURE 3

EQUILIBRIUM PORTFOLIO PAYOFFS.

This follows since their beliefs are concentrated on a smaller set of outcomes of the stock. The aggregate demand for the stock being downward sloping, the result follows.

Next, we examine the effect on equilibrium quantities of increased diversity in investors' beliefs. Given our definitions the following results hold.

COROLLARY 4.3. *Let $e \in \mathscr{E}^*$ and assume an increase in the dispersion of beliefs holding the (arithmetic) average fixed (i.e., $d\sigma_1^2 = -d\sigma_2^2$). Then the aggregate demand for the stock increases while the aggregate demand for the option falls. The price of the option is unchanged whereas the stock price increases.*

The intuition for this result is straightforward. Since the divergence in risk assessments increases, investors wish to enhance their preference for a particular payoff pattern. Investors of type I (II) demand (supply) more options and supply (demand) the stock. Since the absolute size of the change of beliefs is the same for both types, the initially low risk investors are more strongly affected. This being the case, aggregate demand increases resulting in an increase in the equilibrium stock price. Figure 5 details the changes in the demands following the increased divergence in the risk assessments.

The absence of an effect on the price of an option stands in contrast to the classic result that a change in the variance of the stock return changes the value of the option (Black and Scholes 1973 and Merton 1973). This lack of response is a direct

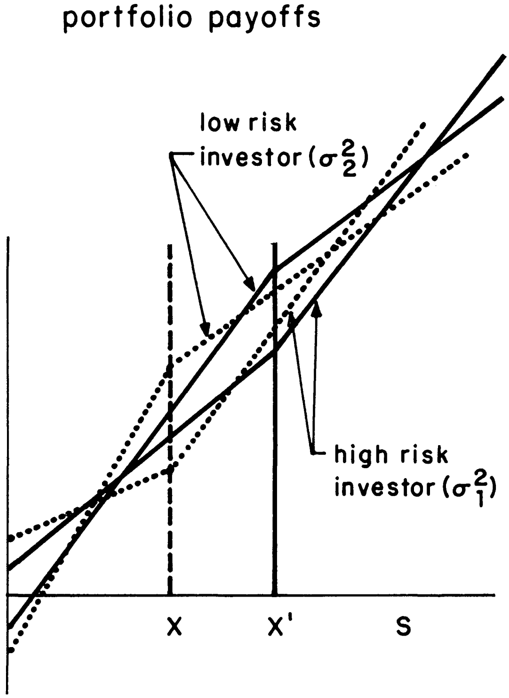

FIGURE 4

DEMAND FOR PORTFOLIO PAYOFFS AFTER EXERCISE PRICE CHANGE.

consequence of the limited form of disagreement under consideration, restricted to the downside potential of the stock. Since in our setting the expected payoff on the option as well as the covariance between the stock and the option are not affected the option price remains immune to the increased diversity.

In a similar vein, it is straightforward to analyze the effects of changes in the risk preference parameter k or the endowment level $\bar{x}_s$.

4.3. *Financial Innovation Via Option Contracts.* The first comparative static result in Corollary 4.2 straightforwardly enables us to assess the effect on the stock price of the introduction of an option contract. Indeed, we know from that corollary that the stock price decreases as the option exercise price increases. The limiting stock value attained (as the exercise price converges to $\max_\omega S(\omega)$) is the equilibrium stock price in the economy without the option contract. Hence, starting from this equilibrium position and introducing an option with an exercise price that lies above the area of disagreement between the two investors will increase the equilibrium stock price.[16]

[16] The increase in the stock price following the introduction of an option is not driven by the fact that quadratic preferences exhibit increasing absolute risk aversion, nor by the fact that there is only one stock in the market. Numerical examples with power utilities (constant relative risk aversion) and multiple assets can be constructed where the property holds as well. For instance, consider the following

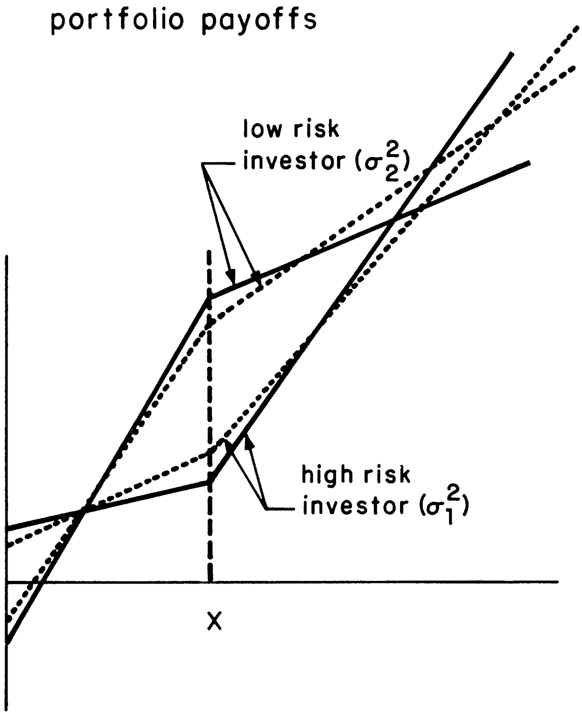

FIGURE 5

DEMAND FOR PORTFOLIO PAYOFFS AFTER INCREASED DIVERGENCE IN RISK ASSESSMENT (SOLID LINES).

COROLLARY 4.4. *Let $e \in \mathscr{E}^*$. Then introducing an option contract in the economy with a stock market and a bond market increases the equilibrium value of the stock and decreases the volatility of the stock rate of return.*

The introduction of the option in the financial market enables investors to construct the complex payoff patterns graphed in Figure 2 (when the stock and the bond are available only linear payoffs can be constructed). Under the assumptions of this section, investors with a high risk assessment have a preference (relative to the other investors) for portfolios that pay off for extreme realizations of the stock. This preferred pattern is achieved by demanding the option and offering the stock (see Figures 6 and 7 below). Investors of type II offer the option and demand the

economy: two investors have constant relative risk aversion of 1 and 3; two stocks are available with payoffs drawn from a joint normal distribution with means (1.1, 1.1), standard deviations (0.4, 0.4) and correlation of 0.8, truncated at plus or minus two standard deviations; the bond payoff and price are set equal to one; total supply of the stocks is (1, 1); endowments of the stock are symmetric (0.5, 0.5). Equilibrium prices of the two stocks are (0.9201, 0.9201). Now introducing an option on the first stock with exercise price $X = 1.1$ increases both stock prices to (0.9242, 0.9237). Additional increases are recorded when a second option on the second stock is introduced. Also, similar comparative static results hold when the parameters of the economy are varied.

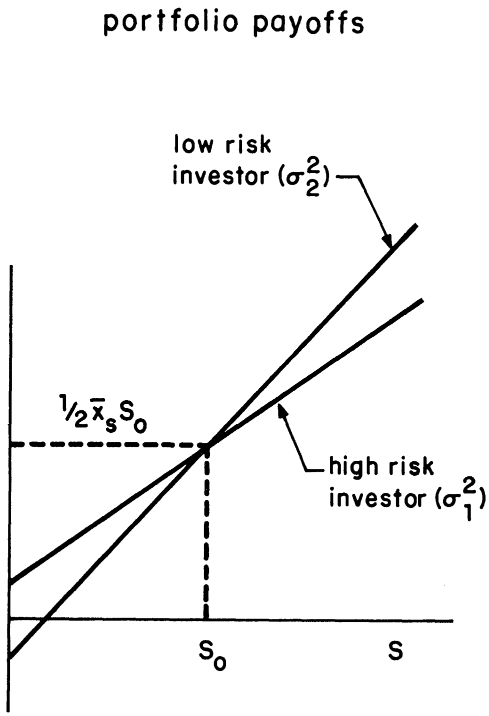

FIGURE 6

PORTFOLIO PAYOFFS WITHOUT THE OPTION.

stock. Since their reaction is stronger, however, the aggregate demand for the stock increases. This causes the stock price to increase.

When the price of the stock increases, the volatility of the stock rate of return perceived by each investor decreases. It follows that the introduction of the option market in this economic context stabilizes the stock market.[17] This is an important feature of the model in view of recent regulatory interest in the operation of derivative securities markets.

5. CONCLUDING REMARKS

In this paper we have demonstrated that, in incomplete markets, the valuation of derivative securities, generically, cannot be treated independently from the valuation of primary securities. Furthermore, in a version of the Mossin mean-variance economy where investors have diverse beliefs, we have shown that the value of the underlying stock increases when an option is introduced.

Three aspects of our analysis deserve further consideration. First, the presence of a robust interaction between the option and the stock market raises the question

[17] Since the payoffs on the stock are exogenous the only reduction in volatility that can be discussed in our model is in rates of return.

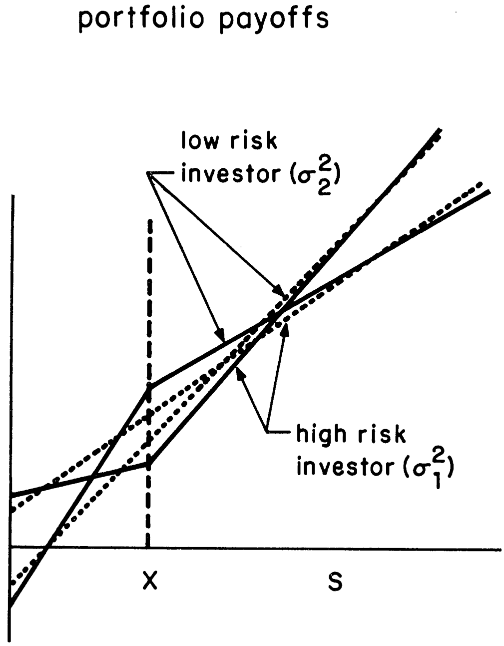

FIGURE 7

PORTFOLIO PAYOFFS WITH THE OPTION.

of the accuracy of arbitrage based option valuation formulae, such as the Black and Scholes model, as an approximation of the equilibrium value of an option. Furthermore, the increase in the type and number of contracts traded suggests that financial markets become more complete. Does it follow that the accuracy of arbitrage based models increases? Or is it the case that the market completion mechanism introduces discontinuities so that valuation errors increase when additional securities are introduced?

Our analysis also suggests that the increase in the stock price experienced when an option is created extends to some economies with diversity in preferences. Since this result has been empirically documented (Detemple and Jorion 1990) a characterization of the set of economies which possess the property will provide information on the mix of investors operating in option markets.

Lastly, our paper in accordance with the recent literature on incomplete markets takes the incompleteness of the market as exogenously given. An important generalization would formulate a process for asset creation and analyze the interactions between primary and derivative assets within an economy with endogenous market structure.

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APPENDIX

PROOF OF PROPOSITION 4.1. To simplify notation define the profits on a stock and option position as, $\gamma_s = S - p_s R$ and $\gamma_o = (S - X)^+ - p_o R$. The first order conditions are,

$$\alpha E\gamma_s = x_s^h E^h \gamma_s^2 + x_o^h E\gamma_s \gamma_o, \tag{A.1}$$

$$\alpha E\gamma_o = x_s^h E\gamma_s \gamma_o + x_o^h E\gamma_o^2, \tag{A.2}$$

where $\alpha \equiv (2k)^{-1} - (\bar{x}_s/2)p_s R$. Solving for x_s^h and x_o^h yields the demand functions,

$$x_s^h = (\alpha/\Delta^h)[E\gamma_o^2 E\gamma_s - E\gamma_s \gamma_o E\gamma_o] \tag{A.3}$$

$$x_o^h = (\alpha/\Delta^h)[-E\gamma_s \gamma_o E\gamma_s + E^h \gamma_s^2 E\gamma_o]. \tag{A.4}$$

To derive the option pricing formula (4.3), sum equation (A.2) over h and use $\Sigma\, x_s^h = \bar{x}_s$ and $\Sigma\, x_o^h = 0$, yielding, $2\alpha E\gamma_o = \bar{x}_s E\gamma_s \gamma_o$. Substituting the definition of α and rearranging leads to, $k^{-1}E\gamma_o = \bar{x}_s E\gamma_o S$. Finally using the definition $ES(S - X)^+ = ES\, E(S - X)^+ + \sigma_{so}$ produces the option price formula $p_o = R^{-1}\, [E(S - X)^+ - \lambda\sigma_{so}]$ displayed in equation (4.3), where $\lambda \equiv \bar{x}_s/[k^{-1} - \bar{x}_s ES]$.

Note also that $E\gamma_o = \lambda\sigma_{so}$ and consequently $E\gamma_s\gamma_o = \sigma_{so} + E\gamma_o E\gamma_s = \sigma_{so}[1 + \lambda E\gamma_s]$ and $E\gamma_o^2 = \sigma_o^2 + (E\gamma_o)^2 = \sigma_o^2 + \lambda^2\sigma_{so}^2$. Substituting these results in the demand for the stock and the option results in the demand functions (4.4) and (4.5).

Summing equations (A.1) leads to, $2\alpha E\gamma_s = \Sigma_h x^h E^h \gamma_s^2 = \Sigma_h x^h \sigma_h^2 + \bar{x}_s(E\gamma_s)^2$. Substituting for α and rearranging leads to the stock price (4.2). □

PROOF OF COROLLARY 4.1. (i) Substituting the stock price formula in the demand for the option leads to, $x_o^1 = (\alpha/\Delta^2)\ \delta\ \sigma_{so}\ \lambda\ [\sigma_1^2 - \sigma_2^2]$, and $x_o^2 = -x_o^1$. By the Cauchy-Schwarz inequality $\Delta^1 > 0$ and $\Delta^2 > 0$. In addition "admissibility" requires the market price of risk λ to be positive. It follows that $k^{-1} > \bar{x}_s ES > \bar{x}_s p_s R$, where the last inequality holds since $\delta \in (0, 1)$. Thus, $\alpha > 0$ and the result follows. (ii) Note that $x_s^1 = x_s^2(\Delta^2/\Delta^1)$ where $0 < \Delta^2 < \Delta^1$. Since $x_s^1 + x_s^2 = \bar{x}_s$ we get $0 < x_s^1 < (\bar{x}_s/2) < x_s^2 < \bar{x}_s$.

PROOF OF COROLLARY 4.2. We first demonstrate the following auxiliary lemma. Define the elasticity coefficient of a function $f(X)$ as $\xi_x(f) \equiv (\partial f/\partial X)/f$.

LEMMA A.1.

$$\operatorname{sgn}\,[\partial(x_s^1(p_s) + x_s^2(p_s))]/\partial X = -\operatorname{sgn}\,[\tfrac{1}{2}\,\xi_x(\sigma_o^2) - \xi_x(\sigma_{so})]$$

$$\operatorname{sgn}\,[\partial x_s^h(p_s)/\partial X] = \operatorname{sgn}\,[\tfrac{1}{2}\,\xi_x(\sigma_o^2) - \xi_x(\sigma_{so})](-1)^{h+1}, \qquad h = 1, 2$$

$$\operatorname{sgn}\,[\partial p_s/\partial X] = -\operatorname{sgn}\,[\tfrac{1}{2}\,\xi_x(\sigma_o^2) - \xi_x(\sigma_{so})].$$

PROOF OF LEMMA A.1. We first show the effect on the demand functions. Using equation (A.3) we have,

$$\begin{aligned}\partial x_s^1(p_s)/\partial X &= -(x_s^1/\Delta^1)(\partial\Delta^1/\partial X) + (\alpha/\Delta^1)[(\partial\sigma_o^2/\partial X)E\gamma_s - 2\lambda\sigma_{so}(\partial\sigma_{so}/\partial X)]\\ &= [\alpha/(\Delta^1)^2]\{-[\sigma_o^2 E\gamma_s - \lambda\sigma_{so}^2][[(\partial\sigma_o^2/\partial X) + 2\lambda^2\sigma_{so}(\partial\sigma_{so}/\partial X)]E^1\gamma_s^2\\ &\quad - 2\sigma_{so}(\partial\sigma_{so}/\partial X)[1 + \lambda E\gamma_s]^2] + [(\sigma_o^2 + \lambda^2\sigma_{so}^2)E^1\gamma_s^2\\ &\quad - \sigma_{so}^2[1 + \lambda E\gamma_s]^2][(\partial\sigma_o^2/\partial X)E\gamma_s - 2\lambda\sigma_{so}(\partial\sigma_{so}/\partial X)]\}.\end{aligned}$$

Straightforward, but lengthy rearrangements and simplifications now lead to,

$$\partial x_s^1(p_s)/\partial X = [\alpha/(\Delta^1)^2][\lambda\sigma_1^2 - E\gamma_s][-2\sigma_o^2(\partial\sigma_{so}/\partial X) + \sigma_{so}(\partial\sigma_o^2/\partial X)][1 + \lambda E\gamma_s]\sigma_{so}.$$

Since $p_s R = ES - \lambda(\delta\sigma_1^2 + (1-\delta)\sigma_2^2)$ where $\delta \in (0, 1/2)$, the result follows. Similarly it can be shown that $\partial x_s^2(p_s)/\partial X$ has the sign of $[\lambda\sigma_2^2 - ES + p_s R][-2\sigma_o^2(\partial\sigma_{so}/\partial X) + \sigma_{so}(\partial\sigma_o^2/\partial X)]$, where the first bracket is negative. The result for aggregate demand follows since the magnitude of $\partial x_s^2(p_s)/\partial X$ exceeds the magnitude of $\partial x_s^1(p_s)/\partial X$. To prove this statement observe that,

$$\partial x_s^h(p_s)/\partial X = (x_o^h/\Delta^h)[1 + \lambda E\gamma_s][-2\sigma_o^2(\partial\sigma_{so}/\partial X) + \sigma_{so}(\partial\sigma_o^2/\partial X)].$$

Since $x_o^1 = -x_o^2$, we have,

$$\begin{aligned}\partial[x_s^1(p_s) + x_s^2(p_s)]/\partial X = x_o^1[(1/\Delta^1) - (1/\Delta^2)][1 + \lambda E\gamma_s]&\\ \times[-2\sigma_o^2(\partial\sigma_{so}/\partial X) + \sigma_{so}(\partial\sigma_o^2/\partial X)].&\end{aligned}$$

The result follows from the fact that $x_o^1 > 0$ and $\Delta^2 < \Delta^1$.

To show the effect on the stock price use the implicit function theorem applied to, $\psi(p_s, X) \equiv \bar{x}_s - [x_s^1(p_s) + x_s^2(p_s)] = 0$. Clearly, $\partial p_s R/\partial X = -[\partial\psi/\partial X]/[\partial\psi/\partial p_s R]$, where $\partial\psi/\partial X = -\partial[x_s^1(p_s) + x_s^2(p_s)]/\partial X$ and $\partial\psi/\partial p_s R = -\partial[x_s^1(p_s) + x_s^2(p_s)]/\partial p_s R$. By our assumption on aggregate demand, $\partial\psi/\partial p_s R$ is positive. It follows that sgn $[\partial p_s R/\partial X]$ = sgn $[\partial[x_s^1(p_s) + x_s^2(p_s)]/\partial X]$. □

To complete the proof of Corollary 4.2 in the text note that,

$$-[\partial\sigma_o^2/\partial X]/\sigma_o^2 + 2[\partial\sigma_{so}/\partial X]/\sigma_{so} = [\partial\rho_h^2/\partial X][\sigma_h^2\sigma_o^2/\sigma_{so}^2],$$

where ρ_h denotes the correlation coefficient between the stock and the option payoff. Indeed,

$$\begin{aligned}\partial\rho_h^2/\partial X &= \partial[\sigma_{so}^2/\sigma_h^2\sigma_o^2]/\partial X\\ &= [\sigma_h^2\sigma_o^4]^{-1}[2\sigma_o^2(\partial\sigma_{so}/\partial X)\sigma_{so} - (\partial\sigma_o^2/\partial X)\sigma_{so}^2]\\ &= \sigma_{so}^2[\sigma_h^2\sigma_o^2]^{-1}[2(\partial\sigma_{so}/\partial X)/\sigma_{so} - (\partial\sigma_o^2/\partial X)/\sigma_o^2].\end{aligned}$$

It follows that $\partial p_s R/\partial X$ has the sign of $\partial\rho_h^2/\partial X$ which is also the sign of $\partial\rho_h/\partial X$ (since $\rho_h \geq 0$). It is easy to verify that this sign is negative. □

PROOF OF COROLLARY 4.3. The following partial derivatives are obtained,

$$[\partial x_s^1(p_s, p_o)/\partial\sigma_1^2]\, d\sigma_1^2 + [\partial x_s^2(p_s, p_o)/\partial\sigma_2^2]\, d\sigma_2^2$$
$$= -[x_s^1/\Delta^1][\partial\Delta^1/\partial\sigma_1^2]\, d\sigma_1^2 - [x_s^2/\Delta^2][\partial\Delta^2/\partial\sigma_2^2]\, d\sigma_2^2$$
$$= \{-[x_s^1/\Delta^1][\partial\Delta^1/\partial\sigma_1^2] + [x_s^2/\Delta^2][\partial\Delta^2/\partial\sigma_2^2]\}\, d\sigma_1^2,$$

where the second equality follows from the restriction $d\sigma_1^2 = -d\sigma_2^2$. Then using the result $x_s^2 = x_s^1(\Delta^1/\Delta^2)$ and the fact that $\partial\Delta^1/\partial\sigma_1^2 = \partial\Delta^2/\partial\sigma_2^2 = E\gamma_o^2$ obtained from equation (4.8) in the text, we get, $[E\gamma_o^2/(\Delta^1\Delta^2)^2]\, x_s^1\, \Delta^1[(\Delta^1)^2 - (\Delta^2)^2]\, d\sigma_1^2$, which is positive (when $d\sigma_1^2 > 0$) since $x_s^1 > 0$ and $\Delta^1 > \Delta^2$ ($\sigma_1^2 > \sigma_2^2$). Now from the aggregate demand for the option,

$$[\partial x_o^1(p_s, p_o)/\partial\sigma_1^2]\, d\sigma_1^2 + [\partial x_o^2(p_s, p_o)/\partial\sigma_2^2]\, d\sigma_2^2$$
$$= -(x_o^1/\Delta^1)(\partial\Delta^1/\partial\sigma_1^2)\, d\sigma_1^2 - (x_o^2/\Delta^2)(\partial\Delta^2/\partial\sigma_2^2)\, d\sigma_2^2$$
$$+ (\alpha/\Delta^1)E\gamma_o\, d\sigma_1^2 + (\alpha/\Delta^2)E\gamma_0\, d\sigma_2^2$$
$$= -x_o^1[(\Delta^1)^{-1} - (\Delta^2)^{-1}]E\gamma_o^2\, d\sigma_1^2 + \alpha[(\Delta^1)^{-1} - (\Delta^2)^{-1}]E\gamma_0\, d\sigma_1^2.$$

Substituting equation (A.4) for the option's demand x_o^1 we get,

$$= (\alpha/\Delta^1)[(\Delta^1)^{-1} - (\Delta^2)^{-1}]\{[E\gamma_s\gamma_o E\gamma_s - E\gamma_o E^1\gamma_s^2]E\gamma_o^2 + [E\gamma_o^2 E^1\gamma_s^2$$
$$- (E\gamma_s\gamma_o)^2]E\gamma_o\}\, d\sigma_1^2$$
$$= (\alpha/\Delta^1)[(\Delta^2 - \Delta^1)/(\Delta^1\Delta^2)]E\gamma_s\gamma_o\{E\gamma_s E\gamma_o^2 - E\gamma_s\gamma_o E\gamma_o\}\, d\sigma_1^2$$
$$= E\gamma_s\gamma_o x_s^1[(\Delta^2 - \Delta^1)/(\Delta^1\Delta^2)]\, d\sigma_1^2.$$

The option price (equation (4.3)) clearly does not depend on the diversity of the risk assessments. To find the effect on the stock price apply the implicit function theorem to the equilibrium condition, $\psi(p_s, X, \sigma_1^2, \sigma_2^2) \equiv x_s - [x_s^1(p_s) + x_s^2(p_s)] = 0$. Since $d\sigma_1^2 = -d\sigma_2^2$ we have,

$$[\partial p_s R/\partial\sigma_1^2]\, d\sigma_1^2 + [\partial p_s/\partial\sigma_2^2]\, d\sigma_2^2 = -\{[\partial\psi/\partial\sigma_1^2]\, d\sigma_1^2 + [\partial\psi/\partial\sigma_2^2]\, d\sigma_2^2\}/\{\partial\psi/\partial p_s R\}$$

which is positive given the results above and our assumption on aggregate demand.

Proof of Corollary 4.4. We have to show that the limit of the stock price as the exercise price of the option converges to $\max_\omega S(\omega)$ is the equilibrium price of the stock in the economy without the option market. But,

$$\Delta_1/\Delta_2 = \{E^1\gamma_s^2 - (E\gamma_s\gamma_o)^2/E\gamma_o^2\}/\{E^2\gamma_s^2 - (E\gamma_s\gamma_o)^2/E\gamma_o^2\}.$$

Using the option price formula (4.3) we can write $(E\gamma_s\gamma_o)^2/E\gamma_o^2 = [\lambda^2 + \sigma_o^2/\sigma_{so}^2]^{-1}[1 + \lambda E\gamma_s]^2$, where σ_o^2/σ_{so}^2 can be further reduced to $1/(\sigma_h^2\rho_h^2)$. When X converges to $\max_\omega S(\omega)$, ρ_h converges to zero and the ratio Δ^1/Δ^2 converges to $E^1\gamma_s^2/E^2\gamma_s^2$, the ratio in the economy without the option. Since the stock price is a weighted average of the investors' risk assessments where the weights depend only on the ratio Δ^1/Δ^2 the result follows. □

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