A Guide to Advanced Trigonometry

Before starting with Grade 12 Double and Compound Angle Identities, it is important to revise Grade 11 Trigonometry. Special attention should be given to using the general solution to solve trigonometric equations, as well as using trigonometric identities to simplify expressions. With the general solution it is important to know that in CAPS we no longer use the 'quadrant method', but only the rules for general solution stated below.

General Solution according to CAPS:

- If $\sin x = a,$ -1 ≤ a ≤ 1,

  Then $x = \sin^{-1} a + k360^\circ$ *or* $x = 180^\circ - \sin^{-1} a + k360^\circ \quad k \in Z$
- If $\cos x = a,$ -1 ≤ a ≤ 1,

  Then $x = \pm\cos^{-1} a + k360^\circ \quad k \in Z$
- If $\text{TAN}\, x = a, a \in \mathbb{R},$

  Then $x = \tan^{-1} a + k180^\circ \quad k \in Z$

Grade 11 Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = 1 - \cos^2\theta$
- $\cos^2\theta = 1 - \sin^2\theta$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Important to know and to remember

- If $\sin A = \sin B$

  Then $A = B + k360^\circ$ *or* $A = 180^\circ - B + k360^\circ \quad k \in Z$
- If $\cos A = \cos B$

  Then $A = B + k360^\circ$ *or* $A = -B + k360^\circ \quad k \in Z$
- If $\tan A = \tan B$

  Then $A = B + k360^\circ \quad k \in Z$
- If $\sin A = \cos B$

  Then rewrite as either $\sin A = \sin(90^\circ - B)$ *or*

  $\cos(90^\circ - A) = \cos B$

Once Grade 11 has been revised we can move on to Grade 12 Trigonometry. It is recommended that an identity or formula is taught one at a time and practised well. Start with the Compound Angle formulas, explain how $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ is proved and then use it to derive the other identities, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$. As it is stated in the CAPS document 'Accepting $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ derive the other compound angle identities.'

Now do examples using the Compound Angle formulas starting with basic examples and progressing to more difficult ones.

Then move on to Double Angle formulas. $\sin 2A = 2\sin A.\cos A$. Explain how to prove $\cos 2A = \cos^2 A - \sin^2 A$ and hence that the other formulas can be derived, $\cos 2A = 2\cos^2 A - 1$, $\cos 2A = 1 - 2\sin^2 A$. Again do adequate examples only using the Compound Angle Formula.

Complete your teaching of this section by doing exercises where both Compound and Double angle Identities are used in equations, to prove identities and to simplify expressions.

It is important to encourage pupils to work through past examination papers in preparation for their own examinations.

Video Summaries

Some videos have a 'PAUSE' moment, at which point the teacher or learner can choose to pause the video and try to answer the question posed or calculate the answer to the problem under discussion. Once the video starts again, the answer to the question or the right answer to the calculation is given.

Mindset suggests a number of ways to use the video lessons. These include:

- Watch or show a lesson as an introduction to a lesson
- Watch of show a lesson after a lesson, as a summary or as a way of adding in some interesting real-life applications or practical aspects
- Design a worksheet or set of questions about one video lesson. Then ask learners to watch a video related to the lesson and to complete the worksheet or questions, either in groups or individually
- Worksheets and questions based on video lessons can be used as short assessments or exercises
- Ask learners to watch a particular video lesson for homework (in the school library or on the website, depending on how the material is available) as preparation for the next days lesson; if desired, learners can be given specific questions to answer in preparation for the next day's lesson

1. **Revision of General Solution and Identities**

   This video revises the general solution of trigonometric equations and trigonometric identities.

2. **Identities and Equations**

   In this video, the Compound Angle Identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ is proved, and other identities derived from it. They are used in various examples.

3. **Using the Compound Angle Identities**

   Examples are done where only the Compound Angle Identities are used. These examples include proving identities and simplifying expression.

4. **Double Angle Identities**

   The double angle identities are introduced and proven.

5. **Using the Double Angle Identities**

   Examples are done where only the Double Angle Identities are used. These examples include proving identities and simplifying expression.

6. **Revising the Sine, Cosine and Area Rules**

   This video revises the sine, cosine and area rules. It then applies these rules to Grade 12 level problems.

7. **3D Trigonometric Problems**

   This video applies all of the skills learnt in Advanced Trigonometry to three dimensional problems.

Resource Material

Resource materials are a list of links available to teachers and learners to enhance their experience of the subject matter. They are not necessarily CAPS aligned and need to be used with discretion.

1. Revision of General Solution and Identities	http://mathsfirst.massey.ac.nz/Trig/TrigGenSol.htm	Notes and examples on using general solutions.
	http://mathsfirst.massey.ac.nz/Trig/TrigGenSol.htm	Notes and examples on using general solutions
	http://www.youtube.com/watch?v=EFktnRYXh78	A video on using general solution.
2. Identities and Equations	http://www.math.wfu.edu/Math105/Trigonometric%20Identities%20and%20Equations.pdf	Notes on using identities to solve trigonometric equations.
	http://www.youtube.com/watch?v=1xKo1Bqgv38	A video on using identities to solve trigonometric equations.
	http://www.purplemath.com/modules/proving.htm	Notes on and examples on using identities to solve trigonometric equations.
3. Using the Compound Angle Identities	http://www.mathsrevision.net/advanced-level-maths-revision/pure-maths/trigonometry/compound-angle-formulae	Examples and notes on using the compound angle formulas.
	http://www.education.gov.za/LinkClick.aspx?fileticket=ibmDNaU7kbA%3D&tabid=621&mid=1736	Examples and notes on using the compound angle formulas.
	http://library.leeds.ac.uk/tutorials/maths-solutions/video_clips/trg_geom/trigonometry/solving_trig_equations4_compoundangleformulae.html	A video on compound angle formulas.
4. Double Angle Identities	http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-doubleangle-2009-1.pdf	Notes and examples on using the double angle formulas.
	http://www.youtube.com/watch?v=mP6wujhQ8js	A video on the double angle formulas.
	http://www.brightstorm.com/math/precalculus/advanced-trigonometry/the-double-angle-formulas/	Notes and examples on using the double angle formulas.
	http://www.reddit.com/r/math/comments/1cilin/i_need_help_proving_identities_involving_double/	Notes and examples on using the double angle formulas.
5. Using the Double Angle Identities	http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-doubleangle-2009-1.pdf	Notes and examples on using the double angle formulas.
	http://www.youtube.com/watch?v=mP6wujhQ8js	A video on the double angle formulas.
	http://www.brightstorm.com/math/precalculus/advanced-trigonometry/the-double-angle-formulas/	Notes and examples on using the double angle formulas.
	http://www.reddit.com/r/math/comments/1cilin/i_need_help_proving_identities_involving_double/	

6. Revising the Sine, Cosine and Area Rules	http://www.mathstat.strath.ac.uk/basicmaths/332_sineandcosinerules.html	Shows step by step usage of Sine and Cosine Rules.
	http://www.bbc.co.uk/bitesize/standard/maths_ii/trigonometry/sin_cosine_area_triangle/revision/3/	An example of how to use the Area Rule.
	http://everythingmaths.co.za/grade-11/06-trigonometry/06-trigonometry-05.cnxmlplus	Summary of the three rules and questions that require the use of all three.
	http://www.banchoryacademy.co.uk/Int2%20Trigonometry%20worksheet.pdf	Questions involving all three rules.
7. 3D Trigonometric Problems	http://everythingmaths.co.za/grade-12/04-trigonometry/04-trigonometry-05.cnxmlplus	Everything Maths textbook chapter on applications of trigonometry. It includes worked examples of trigonometry in 3D.

Task

Question 1

Give the general solution for:

$\cos\theta = -0{,}766$

Question 2

Prove that:

$$\tan x = \frac{1-\cos 2x - \sin x}{\sin 2x - \cos x}$$

Question 3

Prove that:

$\sin(30^\circ + x) + \sin(30^\circ - x) = \cos x$

Question 4

Solve for x:

$\sin(3x + 50^\circ) + \cos(2x - 10^\circ) = 0$

And hence determine x if $x \in [-180^\circ;180^\circ]$

Question 5

Solve for A:

$4\cos^2 A + 2\sin A\cos A - 1 = 0$

Question 6

If $\theta \in [0^\circ;180^\circ]$, solve for θ, correct to one decimal place: $3\cos 2\theta = -2{,}34$

Question 7

Prove the identity:

$$\frac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta} = \frac{\cos\theta}{\sin\theta + 1}$$

Question 8

If $\tan 40^\circ = k$, express $\dfrac{2\sin 20^\circ.\cos 20^\circ}{2 - 4\cos^2 20^\circ}$ in terms of k.

Question 9

Find the general solution of θ, correct to one decimal place:

$\cos 2\theta + 2\sin 2\theta + 2 = 0$

Question 10

Simplify the following: $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$

Question 11

Prove the identity: $\frac{\cos 2x + \cos^2 x + 3\sin^2 x}{2 - 2\sin^2 x} = \frac{1}{\cos^2 x}$

Question 12

Answer this question without using a calculator. If $\sin 54^0 = p$, express each of the following in terms of p:

12.1 $\tan 54^0$

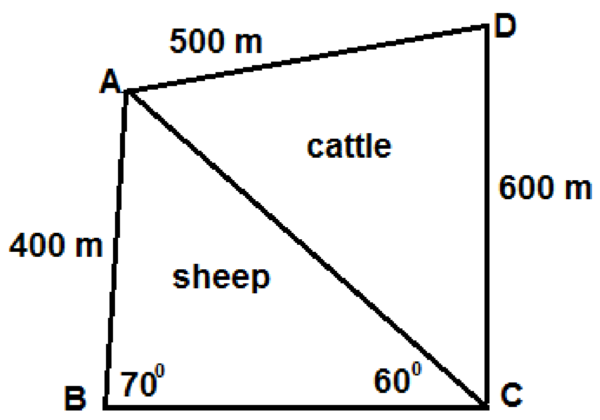
12.2 $\sin 306^0$

12.3 $\tan^2 144^0$

12.4 $\cos 108^0$

Question 13

Use the diagram to answer the questions.



Determine

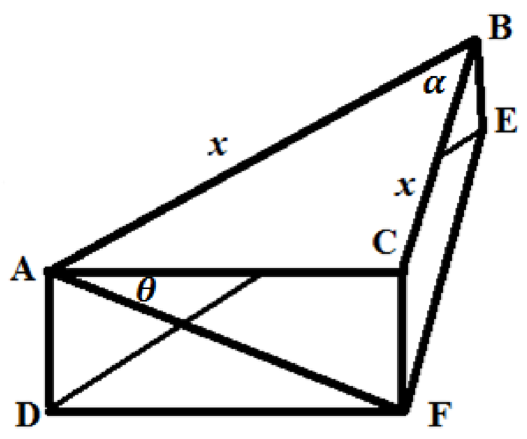
13.1. The length of the fence needed to stretch from point A to C.

13.2. The size of angle D

13.3. Hence, the area or the piece of land used for the cattle. (ΔADC)

Question 14

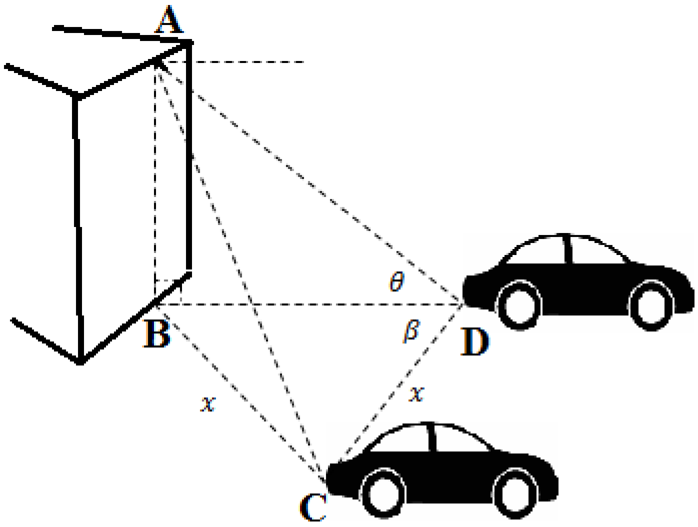
The upper surface of the prism is an isosceles triangle with $BA = BC = x$ and $A\hat{B}C = \alpha$. Furthermore, $C\hat{A}F = \theta$.



14.1. Write down an expression for AC in terms of α and x using the cosine rule.

14.2. Prove that the length of AF is given by $AF = \frac{x.\sqrt{2(1 - cos\alpha)}}{cos\theta}$

Question 15



A surveillance camera is placed at point A. It shows two cars parked outside the building. The angle of elevation of A from D is θ. Car C is equidistant from Car D and the building. Let x denote the distance DC and $C\hat{D}B = \beta$. Prove that AB $=2x.cos\beta.tan\theta$

Task Answers

Question 1

Give the general solution for:

$$\cos\theta = -0{,}766$$

$$\theta = \pm 139{,}99^\circ + k360^\circ \quad k \in Z$$

Question 2

Prove that:

$$\tan x = \frac{1-\cos 2x - \sin x}{\sin 2x - \cos x}$$

RHS

$$\frac{1-\cos 2x - \sin x}{\sin 2x - \cos x}$$

$$= \frac{1-\left(1-2\sin^2 x\right)-\sin x}{2\sin x\cos x - \cos x}$$

$$= \frac{2\sin^2 x - \sin x}{\cos x\left(2\sin x - 1\right)}$$

$$= \frac{\sin x(2\sin x - 1)}{\cos x\left(2\sin x - 1\right)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$\therefore LHS = RHS$$

Question 3

Prove that:

$$\sin(30^\circ + x) + \sin(30^\circ - x) = \cos x$$

LHS

$$\sin(30^\circ + x) + \sin(30^\circ - x)$$

$$= \sin 30^\circ.\cos x + \cos 30^\circ.\sin x + \sin 30^\circ.\cos x - \cos 30^\circ.\sin x$$

$$= 2\sin 30^\circ.\cos x$$

$$= 2(\frac{1}{2}).\cos x$$

$$= \cos x$$

$$\therefore LHS = RHS$$

Question 4

Solve for x:

$\sin(3x+50^\circ)+\cos(2x-10^\circ)=0$

And hence determine x if $x\in[-180^\circ;180^\circ]$

$\sin(3x+50^\circ)=-\cos(2x-10^\circ)$

$\sin(3x+50^\circ)=-\sin[90^\circ-(2x-10^\circ)]$

$\sin(3x+50^\circ)=-\sin(100^\circ-2x)$

$\sin(3x+50^\circ)=\sin(2x-100^\circ)$

$\therefore 3x+50^\circ=2x-100^\circ+k360^\circ$ **or** $3x+50^\circ=180^\circ-(2x-100^\circ)+k360^\circ$

$\therefore x=-150^\circ+k360^\circ$

$3x+50^\circ=180^\circ-2x+100^\circ+k360^\circ$

$5x=230^\circ+k360^\circ \quad k\in Z$

$5=46^\circ+k72^\circ \quad k\in Z$

$\therefore x=46^\circ;118^\circ;-26^\circ;-98^\circ;-170^\circ;-150^\circ$

Question 5

Solve for A:

$4\cos^2 A+2\sin A\cos A-1=0$

$4\cos^2 A+2\sin A\cos A-(\sin^2 A+\cos^2 A)=0$

$3\cos^2 A+2\sin A\cos A-\sin^2 A=0$ (Trinomial)

$3\cos^2 A+2\sin A\cos A-\sin^2 A=0$

$(3\cos A-\sin A)(\cos A+\sin A)=0$

$3\cos A=\sin A$ or $\cos A=-\sin A$

$3=\dfrac{\sin A}{\cos A}$ $\qquad \tan A=-1$

$\therefore A=71{,}57^\circ+k180^\circ$ or $\therefore A=-45^\circ+k180^\circ$

$\therefore A=180^\circ-45^\circ+k180^\circ$

$\therefore A=135^\circ+k180^\circ \quad k\in Z$

Question 6

If $\theta\in[0^\circ;180^\circ]$, solve for θ, correct to one decimal place: $3\cos 2\theta=-2{,}34$

$3\cos 2\theta=-2{,}34$

$\cos 2\theta=-0{,}78$

$2\theta=\pm\cos^{-1}(-0{,}78)+k360^\circ$

$2\theta=\pm 141{,}26^\circ+k360^\circ$

$\theta=\pm 70{,}6^\circ+k180^\circ \quad k\in Z$

Question 7

Prove the identity:

$$\frac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta} = \frac{\cos\theta}{\sin\theta + 1}$$

LHS

$$\frac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta - \cos\theta}{\sin\theta - (1 - 2\sin^2\theta)}$$

$$= \frac{\cos\theta(2\sin\theta - 1)}{2\sin^2\theta + \sin\theta - 1}$$

$$= \frac{\cos\theta(2\sin\theta - 1)}{(2\sin\theta - 1)(\sin\theta + 1)}$$

$$= \frac{\cos\theta}{\sin\theta + 1}$$

$$\therefore LHS = RHS$$

Question 8

If $\tan 40^\circ = k$, express $\frac{2\sin 20^\circ.\cos 20^\circ}{2 - 4\cos^2 20^\circ}$ in terms of k.

$$\frac{2\sin 20^\circ.\cos 20^\circ}{2 - 4\cos^2 20^\circ}$$

$$= \frac{\sin 2(20^\circ)}{2(1 - 2\cos^2 20^\circ)}$$

$$= \frac{\sin 40^\circ}{-2(2\cos^2 20^\circ - 1)}$$

$$= \frac{\sin 40^\circ}{-2(\cos 40^\circ)}$$

$$= \frac{\tan 40^\circ}{-2}$$

$$= \frac{k}{-2}$$

Question 9

Find the general solution of θ , correct to one decimal place:

$$\cos 2\theta + 2\sin 2\theta + 2 = 0$$
$$\cos^2\theta - \sin^2\theta + 4\sin\theta\cos\theta + 2(\sin^2\theta + \cos^2\theta) = 0$$
$$\cos^2\theta - \sin^2\theta + 4\sin\theta\cos\theta + 2\sin^2\theta + 2\cos^2\theta = 0$$
$$\sin^2\theta + 4\sin\theta\cos\theta + 3\cos^2\theta = 0$$
$$(\sin\theta + 3\cos\theta)(\sin\theta + \cos\theta) = 0$$

$\sin\theta = -3\cos\theta$ or $\sin\theta = -\cos\theta$

$\tan\theta = -3$ $\qquad$ $\tan\theta = -1$

$\theta = -71{,}6^\circ + k180^\circ$ or $\theta = -45^\circ + k180^\circ \quad k \in Z$

Question 10

Simplify the following: $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$

$$= \frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$$
$$= \frac{\sin(3x - x)}{\sin x \cos x}$$
$$= \frac{\sin 2x}{\sin x \cos x}$$
$$= \frac{2\sin x \cos x}{\sin x \cos x}$$
= 2

Question 11

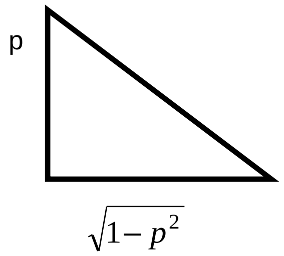
Prove the identity: $\dfrac{\cos 2x + \cos^2 x + 3\sin^2 x}{2 - 2\sin^2 x} = \dfrac{1}{\cos^2 x}$

LHS

$$= \frac{\cos^2 x - \sin^2 x + \cos^2 x + 3\sin^2 x}{2(1 - \sin^2 x)}$$
$$= \frac{2\cos^2 x + 2\sin^2 x}{2(1 - \sin^2 x)}$$
$$= \frac{2(\cos^2 x + \sin^2 x)}{2(1 - \sin^2 x)}$$
$$= \frac{2(1)}{2(\cos^2 x)}$$
$$= \frac{1}{\cos^2 x}$$
$$\therefore LHS = RHS$$

Question 12

12.1 (Pythagoras)



$\tan 54^\circ$

$= \dfrac{p}{\sqrt{1-p^2}}$

12.2 $\sin 306^0$

$= -\sin 54^0$

= - p

12.3 $\tan^2 144^0$

$= \dfrac{\sin^2 144^\circ}{\cos^2 144^\circ}$

$= \dfrac{\sin^2 (90^\circ + 54^\circ)}{\cos^2 (90^\circ + 54^\circ)}$

$= \dfrac{\sin^2 54^\circ}{\cos^2 54^\circ}$

$= \dfrac{1-p^2}{p^2}$

12.4 $\cos 108^0$

$= \cos 2(54^0)$

$= 1 - 2\sin^2 54^0$

$= 1 - 2p^2$

Question 13

$\dfrac{AC}{\sin B} = \dfrac{400}{\sin 60^\circ}$

$\dfrac{AC}{\sin 70^\circ} = \dfrac{400}{\sin 60^\circ}$

$AC = \dfrac{400}{\sin 60^\circ}.\sin 70^\circ$

$AC = 434{,}03m$

$d^2 = a^2 + c^2 - 2ab\cos D$

$(434{,}03)^2 = (600)^2 + (500)^2 - 2(600)(500)\cos D$

$188616{,}46 = 610000 - 600000\cos D$

$-421383.54 = -600000\cos D$

$0,7023059 = \cos D$

$D = 45,39^\circ$

Area of $\Delta ADC = \frac{1}{2}ac\sin D$

$\Delta ADC = \frac{1}{2}(600)(500)\sin 45,39^\circ$

$\Delta ADC = 106785,52m^2$

Question 14

$AC^2 = x^2 + x^2 - 2x.x\cos\alpha$

$AC^2 = 2x^2 - 2x^2\cos\alpha$

$AC = \sqrt{2x^2 - 2x^2\cos\alpha}$

$AC = x\sqrt{2(1-\cos\alpha)}$

$\frac{AF}{\sin 90^\circ} = \frac{AC}{\sin(90^\circ - \theta)}$

$AF = \frac{AC}{\sin(90^\circ - \theta)}$

$AF = \frac{x\sqrt{2(1-\cos\alpha)}}{\sin(90^\circ - \theta)}$

Question 15

$C\hat{B}D = B$

$\therefore B\hat{C}D = 180^\circ - 2B$

In ΔBCD

$\frac{BD}{\sin(180^\circ - 2B)} = \frac{x}{\sin B}$

$BD = \frac{x\sin 2B}{\sin B}$

$BD = \frac{x2\sin B\cos B}{\sin B}$

$BD = 2x\cos B$

In ΔABD

$\frac{AB}{BD} = \tan\theta$

AB = BD $\tan\theta$

= 2x cos B $\tan\theta$

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