

A History of Trigonometry Education in the United States:
1776-1900

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ABSTRACT

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This dissertation traces the history of the teaching of elementary trigonometry in United States colleges and universities from 1776 to 1900. This study analyzes textbooks from the eighteenth and nineteenth centuries, reviews in contemporary periodicals, course catalogs, and secondary sources. Elementary trigonometry was a topic of study in colleges throughout this time period, but the way in which trigonometry was taught and defined changed drastically, as did the scope and focus of the subject.

Because of advances in analytic trigonometry by Leonhard Euler and others in the seventeenth and eighteenth centuries, the trigonometric functions came to be defined as ratios, rather than as line segments. This change came to elementary trigonometry textbooks beginning in antebellum America and the ratios came to define trigonometric functions in elementary trigonometry textbooks by the end of the nineteenth century.

During this time period, elementary trigonometry textbooks grew to have a much more comprehensive treatment of the subject and considered trigonometric functions in many different ways. In the late eighteenth century, trigonometry was taught as a topic in a larger mathematics course and was used only to solve triangles for applications in surveying and navigation. Textbooks contained few pedagogical tools and only the most basic of trigonometric formulas. By the end of the nineteenth century, trigonometry was

taught as its own course that covered the topic extensively with many applications to real life. Textbooks were full of pedagogical tools.

The path that the teaching of trigonometry took through the late eighteenth and nineteenth centuries did not always move in a linear fashion. Sometimes trigonometry education stayed the same for a long time and then was suddenly changed, but other times changes happened more gradually. There were many international influences, and there were many influential Americans and influential American institutions that changed the course of trigonometry instruction in this country. This dissertation follows the path of those changes from 1776 to 1900. After 1900, trigonometry instruction became a topic of secondary education rather than higher education.

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To Jacob, Michael, and John

CHAPTER I:

Introduction

Need for the Study

The history of mathematics education is in the focus of research today, both internationally and within the United States. The most notable work on the history of mathematics education in the United States is a two-volume *History of School Mathematics*, edited by George M. A. Stanic and Jeremy Kilpatrick, published by the National Council of Teachers of Mathematics (2003). The history of mathematics education was the topic of a study group at the International Congress on Mathematics Education in Copenhagen in 2004, and based on the experience of this study group the *International Journal for the History of Mathematics Education* was created. In recent years, important contributions to the history of mathematics education in the United States have been made by many researchers. These contributors include Amy Ackerberg-Hastings (2000) Leo Corry (2007), Eileen Frances Donoghue (2001, 2008), Gloriana Gonzalez and Patricio G. Herbst (2006), Dustin Jones (2008), Jeremy Kilpatrick (1994, 2003), Sharon L. Senk and Denisse R. Thompson (2003), George M. A. Stanic (2003), etc.

Understanding this history has implications for how mathematics is taught today (Cajori, 1890; Coxford and Jones, 1970; Butts, 1971). Furthermore, “we, as mathematics

educators, need a record of our history not simply to serve as a partial guide to present actions but to better understand who we are” (Stanic and Kilpatrick, 2003, p. 13).

Currently, there is concern about the state of trigonometry instruction. Current studies on trigonometry instruction include Fi, 2003; McMullin, 2003; Weber, 2005; Thompson, 2007; Bressoud, 2010, etc. A better understanding of the history of trigonometry education will aid researchers of mathematics education and mathematics educators (Coxford and Jones, 1971; Stanic and Kilpatrick, 2003), and it will “serve as a partial guide to present actions” (Stanic and Kilpatrick, 2003, p. 13).

Knowledgeable commentators note the need for a study on the history trigonometry instruction during the eighteenth and nineteenth centuries (Allen, 1977; Moritz, 1908; Bressoud, 2010), when trigonometry was taught mainly in colleges and universities (Cajori, 1890). Bressoud (2010) says, “I do not know why the definition of the trigonometric functions changed in the 19th century from the circle definition to that of the ratios.”

Although the history of trigonometry has received a significant amount of attention in literature (Van Brummelen, 2009; Maor, 1998; Stedall, 2008; Heath, 1981; McBrewster, Miller, and Vandome, 2009), the history of the *teaching* of trigonometry needs more study.

The history of higher education in the United States has received a great deal of study (Thwing, 1906; Thelin, 2004; Whitehead, 1973; Hofstadter, 1955; Brubacher, 1958; Cohen, 2010; Lucas, 1994; Rothblatt, 1993; Rudolf, 1991; Veysey, 1965; Butts, 1939), but these studies do not consider extensively the place of trigonometry education within the larger landscape of the history of higher education. Even though related

histories—the history of trigonometry and the history of higher education—have received a large amount of consideration in the literature, in the intersection of these histories lies a history that needs to be written.

The only book-length history of trigonometry education is a history of trigonometry in the United States and Canada from 1890-1970 (Allen, 1977), but it addresses mainly the twentieth century, and it only addresses trigonometry education in secondary schools. More study is needed to concentrate on the eighteenth and nineteenth centuries and on trigonometry education within institutions of higher education (Allen, 1977).

Purpose of the Study

The purpose of this study is to follow the history of trigonometry education in colleges and universities from 1776 to 1900. To achieve its purpose the study addresses the following research questions:

1. How did trigonometry textbooks change from 1776-1900:
 - a) in content? What topics were covered during this time period, and how do the topics change over time?
 - b) in approach? Namely, in what order are the topics presented, and with what emphasis on each topic?
 - c) in pedagogy? Particularly, what types of questions and problems are posed to students within the textbooks, are answers and/or solutions given, and how many? What other pedagogical techniques are used?

2. How did the contributions of Euler and others in the field of trigonometry influence the teaching of trigonometry in colleges and universities?
3. What were the major social and political factors affecting higher education during this time period, and how did these factors affect trigonometry education?
4. What individuals played major roles in trigonometry education during this time, and what influence did they have on the teaching of trigonometry?

Procedures of the Study

There are three major components to this study:

1. Analysis of textbooks;
2. Analysis of other primary sources such as course catalogs, final examinations manuscripts, and letters; and
3. Analysis of journal articles and other academic writings.

Analysis of Textbooks

The analysis of textbooks is the largest portion of the study. The textbooks were gathered from Louis Karpinski's *Bibliography of Mathematical Works Printed in America through 1850* (1940), *A History of Textbooks at the United States Military Academy at West Point* (Arney, 2001), and *The Teaching and History of Mathematics in the United States* (Cajori, 1890).

Textbooks are analyzed by coding each of the textbooks regarding the following items:

1. Method for defining trigonometric functions,

2. Topics addressed, including a complete listing and analysis of the theorems that are presented,
3. Order of topics, including which theorems are proved as the results of others,
4. Types and numbers of questions asked and whether solutions and/or answers are provided, and
5. Other pedagogical tools included in the textbook.

The coding and analysis of textbooks requires careful tabulation and comparison, but ultimately this is a qualitative comparison of the textbooks, and this analysis answers all parts of the first research question.

Analysis of Primary Sources Other than Textbooks

The analysis of other primary sources such as course catalogs, final examinations, letters, and manuscripts is the second component of this dissertation. Archives of Harvard University, William and Mary University, Yale University, Princeton University, the University of Pennsylvania, Columbia University, and the USMA at West Point, either physically or online, are used for understanding the mathematics curricula in a more robust way than that which is provided by textbooks alone. Also, collections of David Eugene Smith, an important mathematician and mathematics educator from this time, are used to gather information about the important players in the developments of trigonometry education.

Analysis of Journal Articles and Other Academic Writings

The analysis of journal articles and other academic writings is the third component of this dissertation. Although no extensive works treat the early history of trigonometry education, some scholarly research in the history of undergraduate mathematics in the United States is relevant to this dissertation. Also, academic writings relating to the history of higher education in the United States provide important information about the social and political forces present in colleges and universities during this time. There is also an analysis of mathematical writings that contributed to the development of trigonometry during this time period to see how such developments influenced the teaching of elementary trigonometry in colleges and universities. Finally, the teaching of trigonometry was debated in periodicals, and these are discussed and their arguments analyzed.

The results of this study are given chronologically, concluding with summary of findings. The summary of findings specifically answers the research questions. Throughout the historical account, the research questions guide the discussion.

CHAPTER II:

Literature Review

This study focuses on trigonometry education in the United States of America, beginning with the year 1776, the year of the Declaration of Independence, and concluding with the beginning of the twentieth century. For the time period from 1776 to 1900, trigonometry was a topic of study in higher education. While extant studies focus on the history of higher education and the history of mathematics education, none have as their major concern the history of trigonometry education. Allen's 1977 study, *The Teaching of Trigonometry in the United States and Canada: A Consideration of Elementary Course Content and Approach and of Factors Influencing Change, 1890-1970*, considers trigonometry education at a later time, when trigonometry was a topic of secondary education. A ten-year overlap exists between Allen's study and this study, but this study focuses on trigonometry in higher education, while Allen focuses on trigonometry education in secondary schools. From 1890 to 1900, trigonometry was often taught in colleges and universities but was also often taught in secondary schools. For this reason, both Allen's study and this study have good reason to include this decade, and the analysis in this study of 1890 to 1900 is entirely different from Allen's analysis.

Although a history of trigonometry education during this time is needed, related fields speak to the history of trigonometry education and give needed background

information for understanding the larger social and political forces that affected trigonometry education during the eighteenth and nineteenth centuries. This literature review focuses on three relevant areas—the history of mathematics education, the history of trigonometry, and the history of higher education—each of which contribute significant background and context to this study.

The section on the history of mathematics education considers two types of studies—those that are relevant to this study because of their content and those that are relevant to this study because of their methodology. First, there exist several references on the history of mathematics education that make direct contributions to this study because although they do not concern themselves with trigonometry education entirely, trigonometry education makes up a part of the history of mathematics education. From these references on the history of mathematics education, this study gains relevant information. Also included here is a discussion of Allen's 1977 study on the history of trigonometry education from 1890 to 1970.

Second, there is an examination of studies on the history of mathematics education that do not directly give information to this study but that inform the methodology of this study. Although these studies do not concern trigonometry education, they guide the historical methods that this study uses because histories of mathematics education must have nuanced methods that are somewhat different from other types of history. These studies serve as examples of historical methods that are used in this study. Furthermore, some of these studies discuss their methodologies, and these discussions provide a framework for understanding the methodologies appropriate for studying the history of mathematics education.

Following the section on literature about the history of mathematics education, there is a discussion of the literature about the history of trigonometry. The section on the history of trigonometry has three parts. The first part is a survey of the history of trigonometry from its beginnings in ancient Greece through the time this study considers. Because the development of trigonometry affects the teaching of trigonometry, it is essentially important to understand the history of trigonometry. Furthermore, until relatively recently, teachers of trigonometry were the trigonometers of their generation, and their students were the trigonometers of the next generation (Van Brummelen, 2009). It is a modern innovation, not an ancient practice, that teachers and mathematicians are distinct from each other, so separating history of mathematics education from history of mathematics is impossible for earlier times. In this way, studying the history of mathematics *is* studying the history of mathematics education.

Second, there is a review of David M. Bressoud's 2010 article entitled "Historical Reflections on Teaching Trigonometry." In this article, Bressoud uses the history of trigonometry to shed light on how trigonometry ought to be taught. Finally, there is a review of references on the history of trigonometry, the same references that were used as the sources for the survey of the history of trigonometry.

Finally, there appears a discussion of sources on the history of higher education. The section on the history of higher education provides a landscape in which this study situates the history of trigonometry education. Because trigonometry education during this time took place in colleges and universities, it is essential to understand the factors that affected higher education during this time. Many strong influences affected higher education from 1776 to 1900. First, political influences brought textbooks from Britain,

then France, and then Americans began writing their own textbooks, with French, British, German, and other European influences. Also, higher education has its own political forces, and understanding these sheds light on how certain colleges and universities affected others and what other factors affect colleges and universities. Understanding books on the history of higher education gives this study important background information. It also raises questions about the influences from higher education on trigonometry education that this study seeks to answer.

History of Mathematics Education

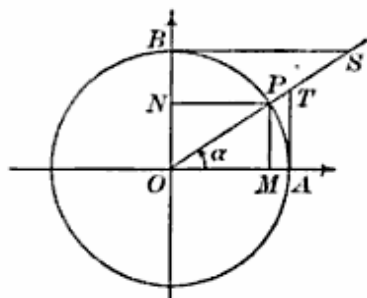
This section begins with a discussion of the histories of mathematics education that pertain directly to this study. Following that appears a discussion of the only other extant history of trigonometry education, Harold Allen's 1977 *The Teaching of Trigonometry in the United States and Canada: A Consideration of Elementary Course Content and Approach and of Factors Influencing Change, 1890-1970*, which is important as a history and also a discussion of Allen's methodology is relevant to this study. Finally, a discussion ensues of other histories of mathematics education whose methodologies serve as models for this study.

References that Directly Inform this Study

First, several histories of mathematics education pertain directly to this study. Florian Cajori's *The Teaching and History of Mathematics in the United States* (1890) is the most comprehensive, but also David Eugene Smith's "Early American Mathematics" (1933), Keith Hoskin's "Textbooks and the Mathematization of American Reality: the

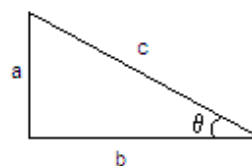
Role of Charles Davies and the US Military Academy at West Point” (1994), and Amy Ackerberg-Hasting’s *Mathematics is a gentleman’s art: Analysis and synthesis in American college geometry teaching, 1790-1840* (2000) directly address the topic of this study. Although other histories of mathematics education briefly mention issues related to this study, the aforementioned works are the most thorough when it comes to discussing trigonometry education from 1776-1900 or related topics, and the extant information is well covered by these texts. Understanding what these texts say about trigonometry education during this time period gives a clear picture of what is already known.

First, an explanation of the terminology necessary to understand the history of trigonometry education appears. The first way that trigonometry was taught and understood is known as the “line system.” The line system defines the trigonometric functions as line segments on a circle (see Figure 2.1). Later, trigonometry is taught and understood using the “ratio system,” where the trigonometric functions for angles between 0 and 90 degrees are defined as ratios of sides of a triangle (see Figure 2.2). These systems are explained in greater detail in the section on the history of trigonometry.



$PM = \text{sine}$ $NP = \text{cosine}$
 $TA = \text{tangent}$ $RS = \text{cotangent}$
 $OT = \text{secant}$ $OS = \text{cosecant}$

Figure 2.1¹



$\sin \theta = a/c$ $\cos \theta = b/c$
 $\sec \theta = c/b$ $\csc \theta = c/a$
 $\tan \theta = a/b$ $\cot \theta = b/a$

Figure 2.2

The most comprehensive resource on the history of mathematics education at the university level that addresses the time period of this study is Florian Cajori's *The Teaching and History of Mathematics in the United States* (1890). Cajori provides lots of important information to this study. Among the most important is detailed information about which textbooks were used during different time periods and the influences that affected these choices. That information is given in detail in the following chapters of this study. Aside from the information about textbooks, Cajori gives a great deal of insight

¹ This diagram is taken from Playfair, 1837, p. 223. By Playfair's definition as well as his contemporary authors, the trigonometric functions were not thought of as functions of angles, but rather, the line segments themselves were thought to be named sine, tangent, secant, and so on. This diagram will be used throughout the dissertation to demonstrate the line system.

into the history of mathematics education in American colleges and universities from their inception until his present day, the late nineteenth century.

Cajori shows that by the revolutionary period and even earlier, trigonometry was a well-established part of the higher education curriculum. By 1800, plane trigonometry for sophomores was a typical part of required courses at most colleges. the earliest influences on textbooks were English. For a short time in 1801 and following, textbooks by American authors were written and used almost exclusively in New England colleges, but soon the French influence took over mathematics textbooks. For a few years in some colleges and universities, French textbooks themselves were used without translation, but quickly translations of French textbooks appeared and were used for the greatest portion of the nineteenth century. However, although the influence of the French caused translations of French authors to replace translations of Euclid's geometry (to which plane trigonometry was added), Cajori says, "It has been said of American writers that, while they have given up Euclid, they have modified Legendre's Geometry so as to make it resemble Euclid as much as possible" (p. 156). By this account, it is not entirely clear how much influence French authors truly had on the teaching of trigonometry, since American authors were creating, in some cases, very liberal translations of these texts. This study explores how influential French mathematics was in great depth in the following chapters.

In addition to discussing outside influences on American higher mathematics education, Cajori also discusses influences from within the United States, especially the influence of the US Military Academy at West Point and West Point's prolific author of mathematics textbooks, Charles Davies. West Point became a center for science and

mathematics that produced many of the nation's mathematics professors, which caused both Davies' and West Point's influence to grow considerably, especially between 1830 and 1860.

Cajori's history also discusses the way in which the ratio system for defining trigonometry overtook the line system. Although the ratio system was known, understood, and used by mathematicians since the latter part of the eighteenth century, American textbooks were slow to adopt the new system. Benjamin Peirce was one of the first to write an American textbook that included the ratio system. Peirce, professor at Harvard, wrote in 1835 an *Elementary Treatise on Plane Trigonometry* within a series of textbooks. Rev Thomas Hill (1881) speaks of his books as follows, "They were so full of novelties that they never became widely popular, except, perhaps, the Trigonometry; but they had a permanent influence upon mathematical teaching in this country; most of their novelties have now become commonplace in all textbooks."² This demonstrates first that Peirce's textbooks contained new mathematical ideas, including the ratio system for defining trigonometry. However, it also shows that these new ideas took several decades to become prevalent and encountered much criticism initially, although eventually the newer ideas were accepted.

In reference to Peirce's trigonometry text mentioned above, Cajori writes:

"The ratio system in trigonometry was used before this by Hassler in his masterly, but ill-appreciated, work on Analytic Trigonometry, and also by Charles Bonnycastle in his Inductive Geometry. But this system met no favor among teachers" (p. 135). Cajori here suggests that the reason for rejecting the ratio system initially was a pedagogical one

² Memorial Collection, by Moses King, 1881, quoted by Cajori, 1890.

because teachers did not favor the new system. This study further explores teachers' objections to the ratio system and debates between teachers over the different systems in later chapters. Cajori goes on to say, "The most popular works on trigonometry, such as the works of Davies and Loomis, as also those of Smyth, Hackley, Robinson, Brooks, and Olney, adhered to the old and obsolete 'line system,' and it was not till within comparatively recent years that the 'ratio system' came to be generally adopted. The old 'line system' was brought to America from England, but the English discarded it earlier than we did. In 1849 De Morgan wrote that the old method of defining trigonometric terms was universal in England until *very lately*" (p. 135).

Cajori again shows the reluctance of Americans to move from the ratio system to the line system, demonstrating that even the English, from whom the Americans inherited the line system, gave it up well before the Americans did, Cajori then goes on to say, "The final victory of the system in this country is due chiefly to the efforts of Peirce, Chauvenet, and their followers. It is significant that Loomis, in a late edition of his trigonometry, has been driven by the demands of the times to abandon the old system" (p. 135). Although Cajori stresses that Americans were very slow to give up the line system in favor of the ratio system, by the time he is writing in 1890, the ratio system had been well-established for several years.

Still, some authors were slower to give up the old line system than others. Cajori writes that some translators modified the French textbooks a great deal to make them more like the textbooks they had previously used. He identifies Loomis as the author who most modified Legendre to be like Euclid. Cajori says, "His trigonometry has been wedded to the old 'line system,' and it is only within the last two or three years that a

divorce has been secured” (p. 156). Cajori also says that Loomis’ texts were extremely popular and widely used, even though he was the slowest to give up the old line system.

At the beginning of the nineteenth century, Cajori’s history suggests, many new textbooks were written and there seemed to be concern with the best methods. However, a stagnation of progress existed in the mainstream of textbooks so that by 1890, Cajori believes that textbooks have finally moved away from antiquated methods of teaching. He says, “The mathematical teaching of the last ten years indicates a ‘rupture’ with antiquated traditional methods and an ‘alignment with the march of modern thought.’ As yet the alignment is by no means rectified. Indeed it has but barely begun. The ‘rupture’ is evident from the publication . . . of such trigonometries as Oliver, Wait, and Jones” (p. 156). This study explores this “antiquated tradition” and the subsequent break from it in detail in the coming chapters.

David Eugene Smith’s “Early American Mathematics” in 1933 addresses the history of mathematics education in colleges and universities. He writes the following:

“Before 1800 the colleges were generally content to use the textbooks of Great Britain and France; after that date America began to produce her own and, what was far more important, to produce men possessed of some native ingenuity. England, through the American Revolution, lost some of her control over the activities of the colleges which she had helped to establish. French mathematics began to exert a greater influence than before. Translations were made of textbooks by Bourdon, Legendre, Lacroix, and Biot, these replacing the English works on the same subjects.” (p. 227)

Therefore the translation of French textbooks and their replacement of English textbooks is well documented and often observed, and Smith’s observations agree with Cajori’s, although Smith’s view is of a strong French influence, while Cajori doubts the true influence of French textbooks, even though the translations of French textbooks were widespread.

Keith Hoskin's "Textbooks and the Mathematisation of American Reality: the Role of Charles Davies and the US Military Academy at West Point" (1994) discusses Charles Davies' important influence over mathematics textbooks in the United States during the nineteenth century. Davies' own translations of French textbooks were widely popular until the end of the nineteenth century. He discusses not only the popularity and widespread use of Davies' textbooks, but also in an attempt to explain why this happened, he discusses the influence of West Point and its graduates throughout American universities.

Hoskin discusses the pedagogical methods at West Point during Davies' time. He explains that West Point had strict requirements that its teachers follow the textbook extremely closely. Speaking about the new Superintendent of West Point when Davies first began as an assistant teacher of mathematics, Sylvanus Thayer, he says:

"Thayer set up from the outset a new academic hierarchy, run by an Academic Board made up of the Professors with himself as President, which was solely responsible for the choice of class-books, syllabus and improvements in those areas. He set up a prescribed "Manner of Giving Instruction," teaching to streamed sections, following the letter of the textbook section assigned, requiring every student to recite from the assigned section each day, giving a numerical mark to each, as it was formally stated in the Regulations a few years later, for each class the teacher "shall keep daily notes of its progress and relative merit, and at the end of each week shall report thereon to the Superintendent, according to form B."³

Thayer brought this format back from Europe, particularly from France's *École Polytechnique*, from which he also brought back a professor, Claude Crozet, as well as many French textbooks. This rigid format of instruction shows that textbooks were essentially important at West Point, and studying textbooks is an effective strategy to

³ There is no page number given here because in the current online access to the journal, the text is given as a continuous document rather than in its original numbered format.

study pedagogy during this time, because teachers had to follow the textbook extremely closely, and the textbook determined what the teachers taught.

Furthermore, Hoskin says that because West Point was “the leading scientific and mathematical college of the ante-bellum era, it produced a not insignificant number of the next generation of college mathematics professors, who therefore carried the West Point curriculum and pedagogy with them into a wider collegiate world.” Additionally, because West Point was the leading mathematical college in the United States during this time, other colleges and universities looked to West Point for an example of effective mathematics curriculum and pedagogy.

Hoskin goes on to argue that Charles Davies’ series of mathematics textbooks became extremely popular, both because they were written in a style that was conducive to teaching in the manner previously described and also because West Point was so influential during this time. Because Davies’ textbooks were written by West Point’s then professor of mathematics and used at West Point, other colleges and universities adopted them readily.

Stanley Guralnick wrote *Science and the Antebellum American College* (1975), a volume of the *Memoirs of the American Philosophical Society*, which contains a chapter on mathematics. He begins by describing the state of mathematics instruction in the eighteenth century. Before 1788, although mathematics instruction was well established in American colleges, professors had manuscript notebooks that students copied to have their own notebooks. In 1788, Nicholas Pike produced the first American mathematics textbook, and his textbook was approved by many university professors and presidents. After this text, many American textbooks were written following English models. Then,

in 1814, Jeremiah Day of Yale began a new wave of textbooks, these known for their clear explanations and logical progression. Guralnick says, “Day’s algebra was immediately accepted by all the colleges. In some, this and his subsequent volumes on other mathematical subjects were used for two decades or even longer” (p. 50). He then goes on to argue that Day began an American mathematical revolution toward justification and systematization that continued until 1836, when French mathematics took over completely in American colleges.

Guralnick then describes the influx of French mathematics. He says that although French mathematics was common in England as early as 1813, America was about ten years behind. John Farrar of Harvard was the first professor to translate and adapt French texts, but soon many American mathematics professors were doing so, and “the largest share of this market was finally, in 1834, captured for a generation by Charles Davies” (p. 51). The fifteenth edition of Davies’ *Elements of Geometry and Trigonometry* was printed in 1860, and this text was used until the end of the nineteenth century.

Guralnick goes on to discuss the change that took place among college students during this time. College students began to see the mathematics requirements as too strenuous and often asked for exemptions from the requirements, and they questioned the usefulness of learning mathematics for real life. Guralnick remarks, however, that in the span of one generation, the college mathematics curriculum went from what is now considered an eighth-grade level to what is now a college-sophomore level. Because of this radical shift, some discontent is not unreasonable. He concludes by saying that “to teachers, at least, it was clear that the applicability of the subject lay not in navigation, commerce, nor even mere reasoning, but in science”(59). This study delves deeper into

many of the themes and ideas Guralnick explores, including the influences on textbooks; influential professors, authors, and colleges; and the views of teachers and students regarding instruction.

Helena M. Pycior's chapter in *The History of Modern Mathematics* (1989) entitled "British Synthetic Vs. French Analytic Styles of Algebra in the Early American Republic" analyzes the algebra textbooks of Jeremiah Day of Yale, John Farrar of Harvard, Charles Davies of West Point, and Benjamin Peirce of Harvard to examine the factors that influence these texts. Although Pycior's study concerns algebra textbooks, it is relevant to this study because the same authors she analyzes also wrote popular trigonometry textbooks. She questions Cajori's model that says first influences were British, then French. Pycior concludes that Day's text has a British style, Farrar's has a French style, Davies mixes the two, and Peirce does not take from British or French styles but rather his text is written independently from these influences.

In her interpretations, Pycior focuses on the two methods of mathematical proof and structure that were often debated during this time, *analysis* and *synthesis*. Analysis refers to a style in which mathematics follows the intuitive path of discovery by using constructive proofs, while synthesis is a demonstrative style, where proofs are given in a concise way that often does not show how the author arrived at the proof. Synthesis allows *reductio ad absurdum*, while analysis does not. Analysis is the distinguishing feature of French style and influence while synthesis comes from the British. This study extends Pycior's study by examining whether her observations about the influences on mathematics textbooks hold true with respect to trigonometry, in addition to algebra.

Another work within mathematics education that has direct implications to this study is Amy Ackerberg-Hastings' 2000 dissertation entitled *Mathematics is a Gentleman's Art: Analysis and synthesis in American college geometry teaching, 1790-1840*. Ackerberg-Hastings says that overall, mathematics in the early United States of America has largely been ignored by historians of mathematics, who focus on the postbellum era, even though there were important developments in mathematics and influential mathematicians during this time. In her text, Ackerberg-Hastings extends Pycior's reinterpretation of the influences on textbooks with a focus on geometry textbooks. She goes on to focus on three of the most important mathematics professors and textbook authors of the antebellum era: Jeremiah Day of Yale, John Farrar of Harvard, and Charles Davies of West Point. Ackerberg-Hastings includes a biography of all three and considers each of their geometry textbooks, especially the roles of analysis and synthesis in their textbooks.

Although this study is not concerned with the teaching of geometry but rather with the teaching of trigonometry, Day, Farrar, and Davies also authored popular trigonometry textbooks. This study gains important insights from Ackerberg-Hastings' biographies. She outlines the textbooks that each of the three chose for his college and gives insights as to how, when, and why each began composing his own textbooks. She also gives information as to the number of printings and the extent of use of several of the textbooks. This study gives this detailed information as it becomes relevant in further chapters.

Allen's Teaching of Trigonometry

Next ensues a discussion of Harold Allen's 1977 *The Teaching of Trigonometry in the United States and Canada: A Consideration of Elementary Course Content and Approach and of Factors Influencing Change, 1890-1970*. Allen's work does not relate directly to this study because it considers a different time period and it considers secondary school rather than higher education. However, as the only other book-length history of trigonometry education in the United States, its content is important to consider. Also important is a consideration of the methodology of Allen's study, which gives insight for the methodology of this study. Allen's study concerns itself with the period of time in which trigonometry was taught in secondary schools, and no other historical study exists of trigonometry education during this time period for the United States or Canada.

Allen's study focuses primarily on textbooks, and Allen analyzes textbooks from this time in a quantitative manner. First Allen reviews the overall development of secondary education and develops subintervals of time in which to consider this history. For each subinterval, he examines influences on secondary education programs, especially mathematics offerings, and considers secondary mathematics programs and their trigonometric content. He identifies topics included in elementary trigonometry and develops a list of items that indicate content and approach (160 items total). He examines the inclusion and sequencing of those items in trigonometry textbooks from the United States and Canada (145 textbooks total). Allen also investigates the role of trigonometric topics in curricula in both the United States and Canada.

From his research, Allen concludes several historical trends. First, at the beginning of the time period that Allen considers, the late nineteenth and early twentieth centuries, trigonometry was primarily an elective course in high schools, but by the 1920s it was frequently paired with second-course algebra and was taken by more students. Right-angled trigonometry came to be part of the junior high mathematics program. Also, there was a shift from a numerical and geometrical emphasis to a functional and analytical emphasis. Finally, early textbooks contained no or few exercises or problems, while later textbooks incorporate these extensively.

Allen's study examines textbooks and does an exhaustive quantitative study of them. The conclusions he draws are interesting and helpful. Although Allen's study achieves its goal using quantitative methods, it may have been possible to achieve the same goal more efficiently by using a qualitative rather than a quantitative approach, and furthermore it may have been possible to attain additional results using a qualitative methodology. Ultimately, a quantitative approach may not be ideal for a historical study, especially because it cannot address some of the most interesting questions of history. In a quantitative study, it is difficult to determine who influenced the changes that took place, how these changes occurred, and why these changes were instituted. This study does not use a quantitative methodology, but instead it looks to other histories of mathematics education to inform its methodology.

Mathematics Education References for Methodology

The following texts within mathematics education are now considered, not for their direct implications to this study, but rather for their methodology in considering the

history of mathematics education. The first of such texts is Eileen F. Donoghue's chapter in *A History of School Mathematics* (2003) entitled "Algebra and Geometry Textbooks in Twentieth-Century America." The second, from the same *History* (2003), is "Pedagogy in Text: An Analysis of Mathematics Texts from the Nineteenth Century" by Karen D. Michalowicz and Arthur C. Howard. The final study to consider is Michael George's Teachers College-Columbia University dissertation *The History of Liberal Arts Mathematics* (2007). This study considers the establishment and development of the liberal arts mathematics course, a general term for mathematics courses offered in colleges and universities that are offered for students who are not majoring in mathematics or in a field that is closely related to mathematics.

In "Algebra and Geometry Textbooks in Twentieth Century America" (2003), Donoghue analyzes introductory level algebra and geometry textbooks in order to "trace the flow of stability and change" (p. 329) in these textbooks over the course of the twentieth century. She divided the century into twenty-year periods, and within each twenty-year period, she selected at least ten textbooks that were widely used and are representative of the time period to examine for her research. For each textbook she analyzed "the author's views or intentions," "the choice, sequencing, and presentation of topics," as well as "any special features." Although she analyzed at least ten textbooks for each period, Donoghue only discusses a few textbooks for each twenty-year period.

Donoghue first discusses several textbooks written by George "Bull" Wentworth, his son George Wentworth, and David Eugene Smith. Their textbooks were influential and widely used during the first two decades of the twentieth century. These textbooks offered standard approaches to algebra and geometry. In contrast, Donoghue then

discusses Meyer's *First-Year Mathematics for Secondary School*, which was innovative especially in its pedagogy but also in its content.

In the second two decades of the twentieth century, Donoghue discusses three texts that were all innovative compared to those from the first two decades. First, she discusses Hart's *Progressive High School Algebra* (1935) and Wells and Hart's *Progressive Plane Geometry* (1935), which contained a spiraling curriculum. She also discusses Sewnson's *Integrated Mathematics Series*, which was an integrated algebra and geometry curriculum. This textbook series introduced tracking to reach students of different levels. Donoghue gives the topics and the sequencing of this textbook series in detail because of its uniqueness among textbooks.

In the time from 1940 to 1960, Donoghue first discusses Welchons and Krickenberger's *Algebra, Book One* (1949), which was a traditional textbook that reflected little change from Bull Wentworth's textbooks. She also discusses Birkhoff and Beatley's *Basic Geometry* (1941), which incorporated line and angle measurement with deductive proof in order to simplify geometrical concepts. Schorling, Clark, and Smith's *Modern School Geometry* (1948) is the final high school text that Donoghue discusses for this time period. A traditional geometry textbook, *Modern School Geometry* distinguished itself by being less formal than some of its predecessors, like Wentworth's. Although Northrop's *Fundamental Mathematics* (1945) was intended to be a first year college-level textbook, Donoghue examines it because many students at this time were going to college at age sixteen after only two years of high school. This textbook had strong conceptual emphasis, and it included sections on logic, algebra, geometry, and coordinate geometry.

In her section on the 1960's and 1970's, Donoghue goes into great depth into the influence of School Mathematics Study Group (SMSG). "Distinguished mathematicians and outstanding high school teachers" (p. 365) comprised this group, which wrote a series of textbooks. The SMSG textbooks included an *Algebra* and a *Geometry*, and these textbooks had an informal, conversational tone—they were meant for students to read. The SMSG textbooks influenced other works from that time, most notably Dolciani's *Modern Algebra* and Dolciani's *Modern Geometry*, which were commercial textbooks (unlike the SMSG texts) but had much of the same structure and tone as the SMSG *Algebra* and *Geometry*.

Somewhat later, in the early 1970s, the Secondary School Mathematics Curriculum Improvement Study (SSMCIS) produced a different series of textbooks for grades 7 through 12 that restructured traditional mathematics instruction and attempted to raise the level of mathematics being taught in the United States.

In the decades from 1981 to 2000, Donoghue discusses the influence of the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989). She particularly emphasizes the role of the NCTM standards in the University of Chicago School Mathematics Project (UCSMP) and by the Consortium for Mathematics and Its Applications (COMAP). Each of these created a series of textbooks that were substantially different from their predecessors and integrated technology and real-world applications into classroom instruction. The COMAP textbooks differed from the UCSMP textbooks because they were based entirely on applications of mathematics in the context of students' future jobs and did not follow a traditional set of topics. UCSMP's textbooks, although they included many applications, had a more traditional

sequence of mathematics topics. Donoghue summarizes that in general, over the course of the century, algebra and geometry textbooks moved toward each other, and each started to incorporate ideas from the other field.

Donoghue's work on algebra and geometry textbooks in the twentieth century is an important model for this study because Donoghue uses both the textbooks and information about important organizations and individuals to trace the development of the mathematics curriculum during this time. When Donoghue analyzes textbooks, she discusses both their content and their pedagogy. She also considers the order of the topics and the ways that they build off each other. She also discusses the tone of the textbooks and how students were expected to use them. This study analyzes textbooks in the same ways that Donoghue does. The following chapter gives further detail on the methodology of this study.

“Pedagogy in Text: An Analysis of Mathematics Texts from the Nineteenth Century” by Karen D. Michalowicz and Arthur C. Howard (2003) is another example of methodology for this study to model. The authors consider elementary and secondary school mathematics texts and discuss the content and pedagogy in the texts. Although mathematics textbooks were published in the eighteenth century, they were printed for teacher use, rather than student use. In the nineteenth century, student texts were created, even for elementary grades, and these texts included pedagogical advice because most teachers at that time were not trained in teaching.

The authors outline three pedagogical methods used in nineteenth-century textbooks—“the *rule* method, the *inductive* method, and the *analytic* method” (p. 80)

where the latter two may be combined in a single textbook. The authors proceed to discuss several textbooks that embody each of these methods.

First, for the rule method, the authors describe in detail one of the most popular of such textbooks, and then they list or briefly describe many other such texts, including one Canadian and one Mexican text. The rule method is where the textbook provides rules for students to memorize along with many practice exercises so that students become proficient with implementing the rules.

Second, the authors discuss the inductive method, which “involved posing carefully graded questions that were intended to lead the pupil to a concept without the necessity of stating it” (p. 86). The authors mention many inductive textbooks and give detailed explanations of the pedagogy involved in this method, citing the prefaces of the textbooks. These textbooks contained extensive prefaces explaining the inductive method, making it clear that the method was controversial and not always implemented as the authors intended. Still, inductive method textbooks gained popularity through the nineteenth century, and many such textbooks were published in the United States and Canada.

The authors discuss the analytic method, which “presented an ‘operation’ with an accompanying analysis—that is, a detailed explanation of a particular way to think through the solution of the problem” (p. 90). The authors give as examples several textbooks that use the analytic method, some of which also included the inductive method.

Finally, the authors discuss the content of textbooks during the nineteenth century. In the early nineteenth century, arithmetic textbooks often included many

problems from commerce that were very practical for students' lives. British currency was still largely in use during this time, and when dealing with British currency, fractions and proportions are required because at that time each monetary denomination was a somewhat arbitrary fraction of the next. For example, one pound equals twenty schillings, and one schilling equals twelve pence. Textbooks, therefore, focused heavily on fractional and proportional thinking. There was little or no coverage of geometric topics at this time. In the mid-nineteenth century, commerce was still covered extensively in textbooks, but as commerce increasingly moved from local to long-distance, the problems of commerce changed, involving longitude to determine time zones, loss of cargo in a ship, investments by speculators, and so on. In the latter part of the nineteenth century, Federal money became available and with that development, decimals appeared in textbooks. The exchange of currency also became a popular topic, as did interest and banking and business transactions. In addition, geometry became a more common topic at this time.

The authors conclude by saying that teachers and textbook authors from the nineteenth century favored textbooks that emphasized conceptual understanding (those containing the analytical and inductive methods) and real-world applications. It is especially remarkable that the real-world applications changed as the world changed.

Michalowicz and Howard's study analyzes textbooks in a somewhat different manner from Donoghue's, with less depth and less attention to the texts, but more attention paid to understanding the overarching themes of all the textbooks during this time period. This study aims to do both types of textbook study—an in-depth consideration of many important textbooks as well as an analysis the general trends of

textbooks to gain an understanding of the overall pedagogy and content and how these changed over time.

In *The History of Liberal Arts Mathematics* (2007), Michael George says that prior to the twentieth century there was a prescribed mathematics sequence for all undergraduates. In the early twentieth century, more and more students were enrolling and a course developed as an alternative to the traditional mathematics course for students who were not studying mathematics or science. Beginning in the 1930s, textbooks were developed to give students an overview of mathematics. George's dissertation studies the history of this liberal arts mathematics course from its inception through the present day.

George looks at three areas of information: textbooks, journal articles and committee reports, and book-length academic works, including dissertations. He selected textbooks from the *American Mathematical Monthly's* list of published books if they intended to be used to provide college students with an overview of mathematics. George uses textbooks to trace the content and development of the courses. He uses journal articles as well as prefaces of textbooks to trace the authors' ideas about the purpose of the course. Book-length academic works and journal articles were used to map out the development of undergraduate curriculum. George tells the history chronologically. Finally, a quantitative study established categories for the topics in the textbooks and assigned a percentage of the text to each category for each textbook.

At the turn of the twentieth century, high school population was exploding, causing it to be impossible to teach such a large number of students the level of mathematics that was required for college entrance. At the same time, the increase in

students in secondary and post-secondary education and the Social Efficiency movement caused a turn toward “practical” education, for the development of skills rather than intellect. Concern for “general” students needing “practical” education gave rise to the liberal arts mathematics course. This course was intended for students to learn computation, financial planning, general mathematical vocabulary, and mathematical needs of civic education. Also, discontent emerged concerning the prior view that traditional mathematics gave rise to mental discipline. The new viewpoint was that mathematics disgusted students because they were learning boring information that was useless for their lives.

The first textbooks were a survey course in mathematics—less content, less depth, and more appealing to students. These courses were slow to be adopted by many colleges and universities, and those that resembled traditional mathematics more were adopted more readily. When the “new math” era began, liberal arts mathematics textbooks included set theory, logic, and axiomatic method. As “new math” died out, the liberal arts mathematics course evolved to a course that surveyed mathematics throughout history, the arts, and human culture, called “modern mathematics.” In the sixties the types of non-traditional mathematics courses exploded—history of mathematics, discrete mathematics, aesthetic mathematics, and so on. At the end of the twentieth century there was a shift toward quantitative reasoning and literacy.

The quantitative study showed that algebra-based mathematics declined from 45% to 10%, “modern mathematics” increased in the sixties and then declined in the seventies, and applied mathematics gradually increased in significance.

George concludes that from the outset of the liberal arts mathematics course until now, there is a shift from logic and rigorous mathematics toward appreciation for mathematics and a shift from mathematics in civilization to quantitative reasoning. Mathematics instructors need to understand the dualism between pure and applied mathematics and need to know what they are intending to teach, rather than teaching a “grab bag” of mathematical topics.

In this history, George studies textbooks as well as social and political factors affecting liberal arts mathematics courses, and he uses a variety of sources to gather this information. George’s study of textbooks does not go into much depth. This study uses the same types of sources as George’s study, however this study considers textbooks in much greater depth than George’s study does, and this study does not do a quantitative study of textbooks. It made sense for George to do a quantitative study of textbooks because liberal arts mathematics courses varied in the topics they studied. However, since trigonometry is the constant topic of this study, a quantitative study is not appropriate.

Overview of the History of Trigonometry

This section contains a short history of trigonometry from its beginnings to the timeframe relevant to this study. Although primitive forms of trigonometry were in existence previously, modern trigonometry originated with Hipparchus of Nicaea, also known as Hipparchus of Rhodes, a Greek astronomer, ca. 190-120 B.C. In order to make his calculations, Hipparchus computed a trigonometric table using the principle Pythagorean identity, a half angle formula, and the sine of sums and differences formula (Maor, 1998). Although Hipparchus’ writings have not been preserved, we know about

Hipparchus from later writers who reference his works. Hipparchus constructed a table of chords on a circle depending on the central angle of the arc bounding the chord. Half of this chord later became the sine function (Van Brummelen, 2009).

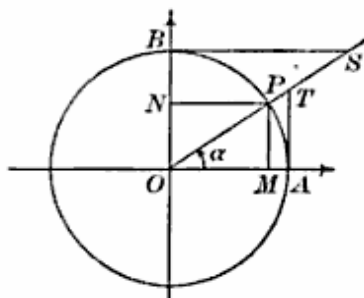
The next major trigonometric work belongs to Ptolemy of Alexandria (ca. 85-165 A.D), who wrote the *Almagest*. In it, he gives a table of chords that is accurate enough for most modern uses and shows how to use the table to solve any planar triangle (Maor, 1998). He discovered chord addition and subtraction formulas. Ptolemy also studied gnomon shadow lengths. A gnomon was simply a stick stuck vertically in the ground, and its shadow was measured depending on the angle of the sun. The study of the gnomon later became the tangent function (Van Brummelen, 2009).

After Ptolemy, Greek trigonometry had little development. It is not clear to what extent Indian trigonometry came from Greece through trade routes and to what extent it was Indian in origin. The earliest extant Indian works (approximately 500 CE) contain verses to help memorize formulas for calculations but very little in the way of reasoning behind the trigonometric laws. Even in the oldest Indian texts, tables of chords were not given but rather of half-chords, now known as the sine function. (Van Brummelen, 2009). Sine is the oldest of the modern trigonometric functions. The first time it was named, although not the first time it was used, was in Hindu work written in Arabic (ca. 510), which uses *jya-ardha* meaning chord-half, in time shortened to *jya* or *jiva*. When this was translated in to Latin, *jiva* was thought to be *jaib* (because Arabic uses mostly consonants and vowels are interpreted by context) and was translated *sinus*. From this we get the first trigonometric function, *sinus* or *sine*. Tangent and cotangent appear next in an Arabic

work in the ninth century. The six trigonometric functions used today appear for the first time together with their modern names in the sixteenth century (Maor, 1998).

When comparing Indian methods with Ptolemy's method for calculating trigonometric functions, Ptolemy's were superior, giving less error and giving values for every degree, whereas Indian methods gave values only for every third degree. Indian texts contained sum and difference formulas for sine as well as formulas for the sine of multiples of angles and halves and other divisions of angles. Indian texts also contained the law of sines (Van Brummelen, 2009). Finally, Madhava, a medieval Indian trigonometer, wrote texts that contained the Taylor series for sine and cosine, which allowed him to calculate sine accurately, and he had a series for π that would have allowed him to calculate it to arbitrary accuracy (Van Brummelen, 2009).

Soon after the founding of Islam in the late seventh century, translations of Indian trigonometry texts as well as of Ptolemy's *Almagest* appeared in the Muslim world. Muslim trigonometers chose to use the Indian sine rather than the Greek chord for the ease of calculations. They used versed sine, and implicitly used tangents and cotangents to calculate shadows of gnomons. Muslim trigonometers improved on trigonometric tables by calculating $\sin(1^\circ)$ more accurately than Ptolemy, but like Ptolemy they did so by determining upper and lower bounds. They discussed in the context of gnomons functions equivalent to the modern tangent, cotangent, secant, and cosecant, but at first these were relegated to gnomons and not used in trigonometry proper. Abu'lWafa's *Almagest* is the first text where all six of the modern trigonometric functions were brought together and defined in one diagram (see Figure 2.3) (Van Brummelen, 2009).



$$\begin{array}{ll}
 PM = \text{sine} & NP = \text{cosine} \\
 TA = \text{tangent} & RS = \text{cotangent} \\
 OT = \text{secant} & OS = \text{cosecant}
 \end{array}$$

Figure 2.3

By using all of the modern trigonometric functions, Abu'lWafa was able to make trigonometric calculations much easier, but only some of his colleagues accepted this change, while others continued relegating all but sine and cosine to the gnomon. Over time, however, the six trigonometric functions remained together. Abu'lWafa also proved what are today known as the Pythagorean identities ($\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, and $1 + \cot^2 x = \csc^2 x$) (Van Brummelen, 2009).

In the medieval West, there are clearly strong influences from both Greek trigonometry and Muslim trigonometry. Again, trigonometers in the West spent a great deal of time creating more accurate and more detailed sine tables. Western trigonometers, although they translated both Islamic and Greek texts, again did not use all the trigonometric functions together, but left tangent, cotangent, secant, and cosecant in the world of the gnomon (Van Brummelen, 2009).

Finally, at the University of Vienna, Regiomontanus (also known by his given name, Johann Muller), influenced by two teachers and astronomers, John of Gumbden and George Peurbach, wrote the first work of trigonometry that removed trigonometry from astronomy and made it into its own field. In his 1464 work, *De triangulis omniouis* (“On triangles of every kind“), he shows to solve all possible cases of triangles. In *De triangulis omniouis* Regiomontanus brought together all that was known about trigonometry at that time, and in doing so created a “rebirth of trigonometry in Europe” (Zeller, 1941). Contemporaries of Regiomontanus—Johann Werner, Peter Apian, and Nicolaus Copernicus—got their trigonometries from Regiomontanus (Zeller, 1941). His first trigonometry did not contain tangent, and therefore was not as advanced as some Arabic authors of the same time period, but later in his career, Regiomontanus constructed a table of sines that included a table that showed he knew tangents as well (Zeller, 1941). Throughout this time, there was still disagreement as to the names of the trigonometric functions and whether tangent, cotangent, secant, and cosecant were proper trigonometric functions (Van Brummelen, 2009).

After Regiomontanus, the next trigonometer who published significant advances in trigonometry was George Joachim Rheticus, who developed more accurate tables of sines as well as tables of tangents, secants, and all their complements in the mid-sixteenth century. Rheticus also did away with trigonometric functions that depended on the arc of a circle and constructed a right triangle where the trigonometric functions depended on the angles of the right triangle (Zeller, 1941).

Franciscus Vieta (also known as François Viète) followed Rheticus in medieval trigonometry. His treatise on trigonometry, published in 1579, pioneered the use of

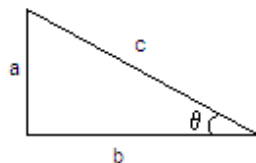
algebraic methods to advance trigonometry (Zeller, 1941). Van Brummelen (2009) says that Vieta, by creating symbolic algebra and applying this system to trigonometry, founded modern analytic trigonometry.

The next trigonometrist of importance was Thomas Finck who, in a 1583 work, was the first to introduce the terms “tangent” and “secant” to describe these functions, and he considered the trigonometric functions to be lines on a circle (as shown in Figure 2.1). Zeller’s history ends with Bartholomaeus Pitiscus, who at the turn of the sixteenth century wrote “the most outstanding treatise of trigonometry developed before the introduction of logarithms” and whom Zeller praises for his extraordinary “clarity of ideas and simplicity of form” (p. 112).

During this time, the end of the sixteenth century, trigonometry became analytic. Two mathematical developments during this time allowed for trigonometry to become analytic—symbolic algebra and analytic trigonometry. There is debate among historians as to who should receive the credit for moving from geometrical methods to algebraic methods in trigonometry. Maor (1998) says that although there were several key players in this process—Vieta, Rene Descartes, and Pierre de Fermat in the late sixteenth century and seventeenth century, and Roger Cotes, Abraham DeMoivre in the eighteenth century—it was Leonhard Euler’s *Introductio in Analysis Infnitorum* that made the shift complete. Out of necessity, when trigonometry became analytic and involved complex numbers, the trigonometric functions were thought of completely apart from their line representations and the circles on which they originated. Mathematically, then, the transformation of trigonometry was completed with the publication of Euler’s *Introductio* in 1748 (Maor, 1998).

Smith agrees with Maor that throughout the seventeenth century, there was a trend toward using algebraic rather than geometrical methods in trigonometry. He credits John Wallis, Isaac Newton, Thomas Fantel de Lagny, Jakob Bernoulli, Jakob Kresa, and Freidrich Christian Mayer with important developments toward this end.

Smith goes on to write that Freiderich Willhelm Opper (c. 1746) using algebra proved all the theorems of plane and spherical trigonometry from a few simple geometric theorems. Nevertheless, Euler overshadowed his success. In 1748, Smith writes, Euler makes his main contributions to trigonometry in his *Introductio in Analysis Infinitorum*. “It is here that trigonometry comes into its own as a distinct branch of mathematics. Here is created and perfected the formal language of the science.” Finally, Smith says that Simon Klugel first defined the trigonometric functions as ratios (see Figure 2.4).



$$\sin \theta = a/c$$

$$\cos \theta = b/c$$

$$\sec \theta = c/b$$

$$\csc \theta = c/a$$

$$\tan \theta = a/b$$

$$\cot \theta = b/a$$

Figure 2.4

Euler and others had thought of the trigonometric functions as ratios, but they did not define them as such.

Van Brummelen (2009) says that when Euler discovered the connections between trigonometry and differential equations as well as the connections between sine/cosine and exponential functions, he caused trigonometry to be “drawn into the library of functions” (p. 284).

Ultimately, no matter who gets the credit for it, the movement of trigonometry from the older line definitions to the modern ratio definitions has transformed the subject entirely. Trigonometry was formerly used only for astronomy and surveying. The advent of the ratio definitions, however, turned trigonometry into an analytic subject. In this way, trigonometry’s development separated its end immeasurably from its beginning.

“Historical Reflections on the Teaching of Trigonometry”

Before moving to a review of the references on the history of trigonometry, there is a discussion of an article is relevant to this study. This article, David M. Bressoud’s 2010 article entitled “Historical Reflections on Teaching Trigonometry,” is best discussed after discussing the history of trigonometry. It was published in the *Mathematics Teacher* in a shorter form than it was originally written. This review considers the original version of the article (Personal communication, September 22, 2010).

Bressoud explains a dichotomy between two types of trigonometry that are taught today—triangle trigonometry and circle trigonometry. Triangle trigonometry is where “angles are commonly measured in degrees and the trigonometric functions are defined as ratios of the sides of a right-angled triangle” (p. 1). Circle trigonometry is where “angles are commonly measured in radians and the trigonometric functions are expressed in terms of the coordinates of a point on the unit circle centered at the origin” (p. 1). In other words, triangle trigonometry corresponds to the “ratio system” and circle trigonometry is similar to the “line system.”

Bressoud suggests that problems exist with the current practice of teaching triangle trigonometry first, followed by circle trigonometry because this practice leads to student misconceptions. Bressoud comments that triangle trigonometry is taught first because it is thought to be simpler, even though history suggests just the opposite. He says the following about the development and teaching of triangle trigonometry, “Trigonometry arose in the study of the heavens among the classical Greeks, and this was

always circle trigonometry. It took over a thousand years before the first intimations of triangle trigonometry appeared, and it was not until the 16th century that became generally used as a tool for surveying. The switch in instructional emphasis from circle trigonometry to triangle trigonometry did not occur until the mid- to late-19th century” (p. 1). Bressoud decides to operate under Henry Poincare’s notion that the historical development of a science is a guide for its teaching. He goes on to discuss the origins and development of trigonometry “in order to reflect on how it should be taught.”

Bressoud begins with the original problem of trigonometry: “Given an arc of a circle, find the length of the chord that connects the endpoints of that arc.” The first evidence of trigonometry consists of tables of arc lengths and their corresponding chord lengths. Bressoud highlights the importance of the chord, which varies with the arc it subtends, to the modern day notion of function. The chord was one of the first places where there exist two different lengths that vary with respect to one another and take on all real values within a given range.

He argues that students have a difficult time conceiving of sine (which is half of that chord) as a periodically varying function when students are first taught to think of sine as “opposite over hypotenuse.” Bressoud shows some of the earliest developments of trigonometry, which were largely dependent on Euclidean geometry. He contends that teaching these theorems as they were discovered would be an excellent way to connect trigonometry to Euclidean geometry. Teaching this way would make it easier for students to remember the trigonometric formulas because they would understand where they came from, rather than simply memorizing them.

Bressoud discusses the appearance of the other trigonometric functions and the first evidence of the connection of trigonometry to the right triangle. He traces the root of the ratio definitions of the trigonometric functions to Johann Muller's (also known as Regiomontanus) question: "Given an acute angle and one side of a right-angle triangle, to find the length of one of the other sides." The easiest way to solve Muller's problem is to think of the trigonometric functions as ratios, which was probably the origin of using the trigonometric ratios as the *definitions* of the trigonometric functions. Bressoud suggests that it would not be much harder to solve this problem using similar triangles, and that would allow the preservation of circle trigonometry.

In addition to the problem of the ratio definitions, Bressoud argues that the radian measure of angles is a source of problems for students. Initially radians were used only to measure the distance of an arc, an idea which makes perfect sense geometrically. However, when angles began to be measured by radians, Bressoud argues, they ceased to be comprehensible. It is especially difficult to make sense of the radian measure of an angle when the angle in question is no longer on a circle (since trigonometric functions have been removed from the circle) and is now the acute angle on a triangle.

Bressoud concludes by listing some of the advantages of teaching circle trigonometry rather than triangle trigonometry. He lists many, including the following: it helps students understand the names of the trigonometric functions, in particular the arcsine, which is the arc corresponding to the sine, and so on; it connects trigonometry to Euclidean geometry; it helps students understand the trigonometric functions as a relationship between two continuously varying quantities; it simplifies the idea of the radian measure; and it prepares students better for calculus.

Commenting on the change to using the ratio definitions to introduce trigonometry, Bressoud says, “I do not know why the definition of the trigonometric functions changed in the 19th century from the circle definition to that of the ratios, but a reasonable guess is that it came about because practical applications were foremost in school mathematics, and students of trigonometry were far more likely to use it as surveyors, solving Muller-type problems, than as learners of calculus” (p. 18). Overall, Bressoud makes the case that triangle trigonometry is a convenient shortcut in teaching and learning trigonometry, but like most shortcuts, causes more problems down the road than it solved in the first place. This study seeks to give a more complete answer to why the change was made to teach triangle trigonometry rather than circle trigonometry.

Resources on History of Trigonometry

This section considers resources on the history of trigonometry that discuss the development of trigonometry from its invention to the timeframe relevant to this study. Extant resources on the history of trigonometry are few. This review discusses *The Mathematics of the Heavens and the Earth* by Glen Van Brummelen (2009), *Trigonometric Delights* by Eli Maor (1998), and *The Development of Trigonometry from Regiomontanus to Pitiscus* by Mary Claudia Zeller (1941). Van Brummelen covers the history of trigonometry from its beginnings to 1550 while Zeller’s history covers the developments in trigonometry in medieval Europe. Maor discusses interesting selections from the history of trigonometry, without writing a chronological history of trigonometry. All these pieces of the history of trigonometry do not comprise the entire history of trigonometry. To supplement these, in the David Eugene Smith Professional Collection at

the Rare Books and Manuscripts Library at Columbia University, there exists a collection of notes Smith made regarding the history of trigonometry. In these notes, Smith considers more modern developments in trigonometry. In order to get the most complete picture of the histories of trigonometry possible, a discussion of all these texts ensues, omitting when any text discusses developments in spherical trigonometry because spherical trigonometry is not relevant to this study.

The history of trigonometry is also discussed in sections of several larger histories of mathematics, but these provide neither new information nor a new perspective from those in the aforementioned texts on the history of trigonometry. For that reason, this review does not discuss histories of trigonometry within larger histories of mathematics.

In *The Mathematics of the Heavens and the Earth* (2009), Glen Van Brummelen begins by defining trigonometry. The modern forms of trigonometry such as the trigonometric functions and even the word trigonometry come from medieval times or later, but the roots of these ideas come from far earlier. Van Brummelen says that trigonometry is “the *systematic* ability to convert back and forth between measures of angles and of lengths” (p. 9). He begins by describing the work of Hipparchus, Ptolemy, and Indian trigonometry. He details the different authors’ methods for calculating sine. Next, Van Brummelen discusses trigonometric developments in the Muslim world, and then he moves to the West until 1550.

Van Brummelen focuses on trigonometric developments in different areas of the world at different but sometimes overlapping time periods. Because of the lack of clear communication between different areas of the world, developments in trigonometry happened in non-linear fashion. Van Brummelen also highlights the strong connection

between astronomy and trigonometry during trigonometry's early development, which caused trigonometry to develop not independently for its own sake, but only as a tool for another purpose. Since Van Brummelen's history ends in 1550, it focuses on the early development of trigonometry, just as it was coming to be its own discipline.

In his 1998 *Trigonometric Delights*, Eli Maor explores interesting topics in the history of trigonometry. The text is not a thorough history, but rather a collection of historical snippets. Maor begins with ancient Egypt, where what is now the cotangent was used to measure a pyramid. He then describes the history of angle measure, with the degree measure originating with the Babylonians and the radian measure coming along in the nineteenth century in order to rid formulas of the factor $\pi/180$.

Maor goes on to discuss the relationship between line segments and angles, or in other words, elementary plane trigonometry. Maor says "elementary plane trigonometry—roughly speaking, the trigonometry known by the sixteenth century—concerns itself with the quantitative relations between angles and line segments, particularly in a triangle." Maor describes the development of the six trigonometric functions and then how trigonometry became analytic beginning at the end of the sixteenth century.

Mary Claudia Zeller's 1944 Ph.D. dissertation *The Development of Trigonometry from Regiomontanus to Pitiscus* is a history of trigonometric developments in medieval Europe. Zeller studies the works of trigonometry written during this period in great detail, and shows the development of trigonometry in the West. She does not discuss Eastern developments in trigonometry, but gives background to her history that shows the developments by Greeks, Muslims, and Hindus prior to the Renaissance.

Zeller's history of the medieval European developments of trigonometry gives important information for the time leading up to the focus of this study, and her history shows that the definitions, names, and forms of the trigonometric functions underwent considerable variations during the middle ages.

Generally, information on the history of trigonometry since the middle ages is lacking (Van Brummelen, 2009), but David Eugene Smith wrote a number of notes on the history of trigonometry, where he gives significant insights into the development of trigonometry.⁴ These notes provide information on important contributors to the field of trigonometry in modern times.

History of Higher Education

This section addresses the history of higher education. Because trigonometry education from 1776 to 1900 took place almost exclusively within institutions of higher education, it is necessary to understand what historians have written about higher education. These resources provide the necessary background information as this study considers one aspect of higher education. Furthermore, histories of higher education are relevant to the history of trigonometry education because the social and political forces that affected higher education as a whole certainly affected trigonometry education as well.

This section discusses four approaches to studying the history of higher education. Although there are many resources on the history of higher education (Thwing, 1906; Hofstadter, 1955; Burbacher and Rudy, 1958; Whitehead, 1963; Veysey, 1965;

⁴ David Eugene Smith. Notes on the history of trigonometry. David Eugene Smith Professional Collection, Box 92, Rare books and manuscripts, Columbia University. This collection appears to be notes toward an article or the chapter of a book that Smith never wrote.

Westmeyer, 1985; Rudolph, 1991; Lucas, 1994; Thelin, 2004; Cohen and Kisker, 2010), they are too numerous to discuss here, and many of them are quite similar with respect to the years prior to 1900, so little new information would be gained by discussing them all. In order to get a representative portrait of the history of higher education, this review discusses four of the histories.

First, a discussion ensues of Thwing's 1906 *A History of Higher Education*.

Thwing's history is foundational, and the other works all reference this text. It was the first history of higher education, and it remained the only major history for half a decade. Furthermore, it studies precisely the time period that is of interest to this study, as well as an earlier time period. It concludes at the same time this study concludes and therefore is not consumed with times irrelevant to this study.

Second, this review discusses Hofstadter's 1955 *American Freedom in the Age of the College*, which examines many of the political structures in place in American colleges until 1860 and how they affected academic freedom and academic progress in American colleges. Hofstadter's text is important to discuss because it gives a unique look at some of the social and political factors that affected American colleges during the time of this study. Hofstadter's history is not a complete history in the same way that Thwing's is, but it focuses on one area—academic freedom—and explores it in great depth.

Third, this review discusses Thelin's *A History of American Higher Education* (2004). Thelin's history is important to discuss because it reexamines previous histories like Thwing's, which have a top-down approach and mostly consider presidents and professors, and discusses aspects of the history and sources that had not previously been

considered. Since his history is unlike previous histories, it provides a new and different perspective.

Finally, there is a discussion of Cohen and Kisker's *The Shaping of American Higher Education: Emergence and Growth of the Contemporary System* (2010). Like Thelin, Cohen and Kisker's history is not as much an original work of history as it is a synthesis of other works. Cohen and Kisker's work is useful to examine because rather than summarizing historical developments in higher education through time, they instead divide the history of higher education into eight topics and discuss the developments in different historical eras with respect to each of the topics. They also provide an overall synthesis of the historical developments within each topic. The division of the history of higher education into topics makes Cohen and Kisker's history unique and important, because they are able to address each topic in more detail and with more coherence than a history that simply considers events chronologically. Their discussion of the developments within each topic and through the eras makes Cohen and Kisker's text an important companion to texts like Thwing's and Hofstadter's.

By discussing these four texts, this literature review has a complete picture of the extant histories of higher education, because these texts all offer different perspectives. These four texts are representative of the histories of higher education that currently exist.

Thwing's 1906 *A History of Higher Education in America* is a broad overview of college and university education in America from the founding of Harvard College in 1636 to the turn of the twentieth century. Thwing gives a very detailed look at the beginnings of higher education in the colonial era, where he follows the founding and the development of the first three colleges—Harvard, William and Mary, and Yale—very

closely. Here he shows the deeply English roots of the first American colleges. He traces the founding of the next three colleges—Princeton, Pennsylvania, and Columbia—through their more diversely European founders and influences. He shows that these three colleges had a close community with each other because of their proximity both in time and space.

During the revolutionary and post-revolutionary period, those educated in the American institutions of higher education were declaring the United States of America independent and leading the fight for its independence. He argues that the France gave a good deal of money and resources to the United States in the years after the Declaration of Independence, 1778 and after, and the influence this bought France was largely educational. French language began to be taught in colleges, and French textbooks for other subjects were widely used. Thwing specifically discusses the French influences on mathematics textbooks because during this time many French textbooks were either used or translated for institutions of American higher education. About replacing English textbooks with French textbooks, he says:

“The improvements in textbooks and in methods of instruction have in this period been largely due to French influence. The place which English mathematics had held from the foundation of the first college down to the beginning of the last century was complete. Upon general grounds the superiority of French to English mathematics came to be recognized near the beginning of the nineteenth century. English authors gave way to French in many of the best colleges. The translation of Laplace by Bowditch, begun in 1829, quickened the study of French mathematics in America. . . . In 1820 Farrar published translation of LaCroie’s Trigonometry; in 1832 Benjamin Peirce became Professor of Mathematics and Natural Philosophy in Harvard College. Professor Peirce has been called by Sir William Thomson the founder of higher mathematics in America. He was both an algebraist and an astronomer.” (p. 303)

French influence was also causing American colleges to have a more “centralized method of organization and instruction.” Clearly, there was a strong French influence on mathematics curriculum and instruction during the revolutionary and post-revolutionary periods. This influence has been observed widely in histories of higher education (Burbacher and Rudy, 1958; Lucas, 1994; Rudolph, 1991; Vesey, 1965).

Thwing goes on in his process of tracing influences on American higher education as he examines the German influence, which followed the French influence. Many American students studied abroad in Germany from 1815-1885, and brought back with them a great deal of German influence, which was waning by the turn of the twentieth century, when Thwing was writing his history. Most of the German influence on higher education took place in the final three-quarters of the nineteenth century. Thwing’s discussion both of French and German influences is essential to understanding the development of mathematics education in higher education.

Richard Hofstadter’s *American Freedom in the Age of the College* (1955), gives an account of academic freedom in American colleges, ending his history in 1860. Although Hofstadter focuses on American higher education, he begins his history with the medieval university and describes how it came to be independent, which set the stage for later colleges and universities to be politically independent.

American colleges were founded with some of the same elements of the English and European colleges and universities—elements that did not promote as much academic freedom. However, Hofstadter’s history goes on to show the unique structure and organization that American colleges developed by the mid-eighteenth century, distinct from English or other European models. At this time, American higher education

consisted of colleges rather than universities, they were dispersed in location, rather than clustered at one academic center like Oxford and Cambridge, and they had a “system of lay government” (p. 114).

In many ways, the eighteenth century American college had progressed significantly from its colonial roots. Hofstadter summarizes, “On the whole the colonial elite need not have been ashamed of its educational achievement, for the colonial colleges, with all their weaknesses, made remarkable gains during the eighteenth century, not only in the direction of higher standards but of greater liberality. The sponsorship of an enlightened aristocracy has often been identified with such gains in American higher education” (p. 151).

In the nineteenth century, however, he argues that there was an extreme slowing of the academic progress, even to the point of backsliding in some areas. He argues that this regression happened because “the sponsors of collegiate education, instead of developing further the altogether adequate number of institutions that existed in 1800, chose to establish new institutions far beyond the number demanded by the geography of the country” (p. 209). This study examines whether this more general trend of slowing academic progress during the nineteenth century affected trigonometry and the teaching of trigonometry.

In his 2004 *A History of American Higher Education*, John Thelin tells a comprehensive history of higher education from colonial times to the end of the twentieth century. Although by this point there have been many histories of American higher education, he argues that there is need for another history because there have been several recent histories that provide some insight into higher education. However, these are more

narrowly focused, and these histories have not been synthesized into one comprehensive history. Thelin's focus is not his own historical scholarship, but rather he synthesizes the works of other historians pertaining to the history of higher education. Much of this history calls into question the traditionally-held notions of the history of higher education, in part because this history includes not only the "great deeds of great men" but also the "informal yet powerful memories of the students" (p. xx).

Thelin's history begins with colonial colleges, but considers their history in a manner completely dissimilar to that of Hofstadter or Thwing. He examines the ways in which the Universities with colonial colleges later portrayed their heritage, and then shows these portrayals largely to be false. For example, he notes that many of Harvard and Yale's portrayals of their colonial roots hearken to Oxford and Cambridge. He argues that in the most important ways, Harvard and Yale were completely different from Oxford and Cambridge, making their self portraits faulty.

In the post-revolutionary period, Thelin shows how the political climate in the new United States allowed and encouraged the explosion of colleges and universities. Thwing and Hofstadter also note this point, but Thelin goes further to show that the granting of charters for colleges was not necessarily given only to colleges who were worthy of granting academic degrees. He argues that even though it has been traditionally believed there was a very high failure rate for these new colleges, new research suggests that they were actually very resourceful and suggests a much higher survival rate.

In the time from 1860-1890, Thelin argues that although the traditional belief is that war stops progress (and this is true for most colleges and universities in the South), in the North the Civil War and reconstruction allowed colleges and universities to push

through progress in areas where they were previously unsuccessful. This was true for the Morrill Land Grant Act of 1862 as well as for the education of women and the diversification of programs offered at institutions of higher education.

From 1890 to 1910, Thelin argues that although this was a great boom of higher education both in finances and in popularity, it was also a time in which higher education as a whole was trying to decide what distinguished a “great American university,” and institutions of higher education were vying for this prestige. At the same time, Americans increasingly looked to higher education to advance their and their family’s prestige.

Overall, Thelin’s history of higher education is not as much a complete history as it is a retelling of portions of history that are popularly misunderstood. As such, it is a valuable contribution to the history of higher education, especially to clarify misconceptions that are often proliferated. However, it should not be seen as a complete history itself, especially for the earlier years it discusses. Thelin does not address mathematics education in any significant detail, but mentions it often as an important part of the curriculum.

Arthur Cohen and Carrie B. Kisker’s *The Shaping of American Higher Education: Emergence and Growth of the Contemporary System* (2010) is a synthesis of trends in American higher education from colonial times through the modern day. The authors organize the discussion by dividing the history into six eras and then discussing the major trends by grouping them into topics. It discusses each of the topics in chapters that focus on the six eras. It focuses much more on a synthesis of the historical trends than a careful documentation of the history. Within each era, the discussion includes eight topics—

societal context, institutions, students, faculty, curriculum and instruction, governance and administration, finance, and research and outcomes.

Throughout the eras, Cohen and Kisker show the overarching trends that have developed within each topic. In the societal context, they show how the nation and economy have grown, increasing American society's need for and expectations of higher education. Institutions have changed from fairly homogenous to extremely diverse. They have also changed from colleges with a singular purpose to a wide variety of institutions with each institution fulfilling many purposes and the entire system of higher education fulfilling a vast number of purposes. In terms of the students attending institutions of higher education, the trend has constantly been toward allowing access for a greater number and a greater diversity of students. For the faculty, however, the trend has in some ways been opposite that of students. Whereas the first faculty of colleges had varying backgrounds and qualifications, they have shifted toward a much greater degree of professionalism.

As the students and faculty have changed, Cohen and Kisker argue, so has the curriculum and instruction. For a significant span in the beginning of higher education in America, curriculum and instruction were uniform for all students and were focused on studying the classics with a strong influence from the church. Over time, higher education broke away from this type of curriculum, and the curricula at institutions of higher education became varied for different students at different institutions. Also, the focus became increasingly on preparation for the future vocation of the students. In terms of governance and administration, although churches founded the institutions of higher education, over time they became increasingly secular. In terms of finance, higher

education started out being entirely privately funded and over time there was more and more public funding. One main trend in the outcome of higher education has been present even from colonial times is a trend toward “individual mobility” (p. 4). Research in higher education has tended to focus increasingly on “societal and economic development” (p. 4).

Overall Cohen and Kisker take the history of higher education and synthesize the trends through time in different eras. They show the ways in which the issues that higher education faces today are very different from the issues it has faced in the past, as current trends are often different from the trends that higher education has followed in previous eras. In terms of mathematics education, the trend that Cohen and Kisker highlight is the continued emphasis placed on mathematics as one of the pillars of higher education from the colonial era to the current era. The mathematics curriculum was virtually universal for all undergraduates until the era of WWII.

Summary and Synthesis

Although little research exists on the history of trigonometry education itself, there are three main areas that provide background information for this study—the history of mathematics education, the history of trigonometry, and the history of higher education.

Within the history of mathematics education, this study examines other histories of mathematics education that are relevant to the study. Additionally, it examines a few studies that are not directly relevant, but that provide examples of methodology that are useful for this study. The following chapter discusses the methodology for this study.

The section on the history of trigonometry is split as well. There is a summary of the history of trigonometry, a review of an article that uses the history of trigonometry to suggest how it should be taught, and there is a review of references on the history of trigonometry.

Finally, there is a discussion of several histories of higher education, each of which represents a different approach to studying the history of higher education. There are so many histories of higher education that it would be unreasonable to review them all, so instead four are chosen. Together they are representative of the larger group of histories of higher education.

The review of this literature leaves three areas open for this study to explore: the movement from the “line system” to the “ratio system” for teaching trigonometry, the social and political factors that determined trigonometry textbook adoption in colleges and universities, and the degree to which the French and other Europeans influenced the teaching of trigonometry in the United States of America.

From the “Line System” to the “Ratio System”

The history of trigonometry reveals that as trigonometry advanced in to an analytic field, beginning in the late sixteenth century and concluding in the mid-eighteenth century, mathematicians stopped defining trigonometry geometrically and started defining it algebraically. In the mid- to late-eighteenth century, mathematicians abandoned the “line system” in favor of the “ratio system” for defining trigonometry (Maor, 1998; Zeller, 1941; Smith). However, college professors—the teachers of trigonometry—did not make this change until in some cases over a century later, and no

obvious explanation for exists for this delay. A delay of even a generation would be understandable and even expected, but a delay of over one hundred years is difficult to fathom.

Additionally, it is of interest to this study that although English mathematics was abandoned at least partly because of the superiority of French mathematics, still the English adopted the “ratio system” several decades before the Americans did (Cajori, 1890). This study seeks to understand the hesitation Americans had in adopting the “ratio system” more thoroughly. This study explores in great depth the development of this change when it eventually happened and explores when, where, and why it took place. This study also investigates Bressoud’s claim that the ratio system was instituted in the nineteenth century because it was the most convenient method for solving the most common problems of the day.

The Process of Textbook Adoption in Colleges and Universities

One possible explanation for professors’ reluctance to teach using the “ratio system” rather than the “line system” is a lack of available textbooks using this method. However, this is cannot be the entire explanation. There were textbooks in existence that used the “ratio system,” but they were not popular and were not widely adopted (Cajori, 1890). What kept them from being popular? How were textbooks chosen, and what factors contributed to certain textbooks’ enduring popularity? Ackerberg-Hastings (2000) explores these questions to some degree with respect to geometry textbooks. This study explores these questions in depth concerning trigonometry textbooks in the following chapters.

The French and European Influences

There are differing views on the influences in trigonometry teaching during the time from 1776-1900. Particularly, many historians point to the French influence on textbooks in the late eighteenth and early nineteenth centuries (Ackerberg-Hastings, 2000; Pycior, 1993; Thwing, 1906). However, Cajori (1890) calls into question whether this influence made much of a difference because translators made changes to the French texts, making them more like the English texts they had formerly used. This study examines these texts in detail and analyzes how substantially and in what ways the French influenced the teaching of trigonometry in the United States.

Furthermore, the French influence during the revolutionary and post-revolutionary periods in higher education is well-documented in histories of higher education, as is the German influence that followed the French (Thwing, 1906; Cohen and Kisker, 2010; Brubacher and Rudy, 1958; Lucas, 1994). However, within histories of mathematics education, the French influence is often discussed, but there is little discussion of any German influence (Cajori, 1890; Smith, 1933; Hoskin, 1994). This study seeks to address and resolve this disconnect.

Overall, the review of relevant literature gives important background information and answers some questions, but most of all gives rise to many areas where there is a need to do further research. This study explores and develops these areas in greater depth.

CHAPTER III:

Methodology

The methodology of this study consists of three main parts—methods for historical research, methods for researching the history of mathematics education, and, in particular, methods of textbook analysis.

Methods for Historical Research

Foucault (1969) and Howell and Prevenier (2001) describe the role of the modern historian. Foucault says that the role of the modern historian is to take documents and by describing and comparing them turn them into monuments. Howell and Prevenier say that although primary sources of history can never be fully reliable, the historian's job is to make them into a meaningful story that helps to explain the past and the connection of the current to the past.

Although no historical study can be completely without bias and no historical study can represent the entirety of the past accurately, this study strives to be impartial by considering a variety of types of sources as well as a variety of sources within those types.

The largest group of sources is textbooks. The textbooks were chosen using Louis Karpinski's *Bibliography of Mathematical Works Printed in America through 1850*

(1940), *A History of Textbooks at the United States Military Academy at West Point* (Arney, 2001), and Cajori's *The Teaching and History of Mathematics in the United States*. Karpinski's *Bibliography* lists books printed as well as the number of printings. Knowing the number of printings gives insight as to how influential and widespread different textbooks were. This study gives preference to those works that had the greatest number of printings.

The textbooks used at the USMA at West Point are given special consideration. Arney's *History of Textbooks* shows not only which books were printed often, but it also gives the dates that the textbooks were used. This allows makes it possible to see which textbooks were given up in favor of others, and which textbooks were enduring parts of the curriculum. As was discussed in the previous chapter, West Point's mathematics education was very influential because West Point produced many of the mathematics professors that ended up teaching at colleges and universities across the country (Cajori, 1890; Hoskin, 1994; Ackerberg-Hastings, 2000). Because of West Point's great influence, this study considers all of the textbooks that were used there for at least three years.

Finally, Cajori's *Teaching and History* gives information about which textbooks were used at certain colleges and also tells about some of the most influential textbooks and their authors. Cajori mentions textbooks only because of their importance, so if Cajori tells about a trigonometry textbook, then this study analyzes it.

Another type of source is the archived collections of university documents as well as collections of writings of important professors of mathematics. The archives of Columbia College (now Columbia University) were consulted and course catalogs from

1886 to 1900 were examined. For further information on Columbia College, the author consulted *From King's College to Columbia, 1746-1800*, by David C. Humphrey (1976).

Online archives of Harvard University, Yale University, and Princeton University were examined for sources such as university catalogs, reading lists, and final examinations from trigonometry courses.

The David Eugene Smith Professional Collections, residing at the Columbia University Rare Books and Manuscripts Library was consulted. Box 92, *Notes on the history of trigonometry*, was examined thoroughly for relevant materials.

Journal articles and periodicals are the final type of source this study considers. Just as current journals contain cogent discussions of mathematics education issues, so do journals from earlier time periods. Such articles may give voice to some of the less well-known teachers, who would otherwise be silent in a study that focuses on the most important mathematics professors and textbook authors.

Any history involves many voices, and in order to understand the history most completely, the historian must listen to them all (Howell and Prevenier, 2001). This study considers information from these three types of sources so that many different voices of different types of people are heard.

Methods for History of Mathematics Education

From the founding of the United States of America, trigonometry was an integral component of higher education (Cajori, 1890). By the turn of the twentieth century, trigonometry had begun to be included in secondary schooling for many students (Allen, 1977). During the time in which trigonometry education was focused in colleges and

universities, its teaching underwent many important changes (Cajori, 1890). This study traces the history of the teaching of elementary trigonometry from 1776 to 1900.

Perhaps most significantly, it was during this time that the trigonometric functions went from being defined as line segments on a circle to being defined as functions of angles (Cajori, 1890). In the late eighteenth century and beyond, the definitions of the trigonometric functions were given by line segments on a circle, the “line system,” which originated from the ancient Greeks’ and Arab’s conceptions of trigonometry. By the turn of the twentieth century, trigonometric functions were defined as ratios with real number arguments, “the ratio system,” and the line segments formerly thought of as definitions were relegated to line representations as a visual aid (Wentworth, 1897) or, in some cases, were absent altogether (Anderegg and Roe, 1896).

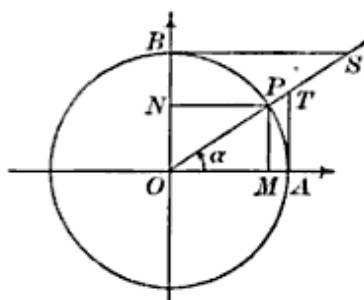
Today, these line representations have all but disappeared from the teaching of trigonometry, where the line representations of sine and cosine are the scarce remnants of what were not long ago the foundational definitions of trigonometry. In fact, most teachers of trigonometry today are not even familiar with the line representations of secant, tangent, cosecant, and cotangent (Carter, Zimmerman, and Jain, 2009).

Because this study considers the teaching of mathematics, it uses methodology that is particular to the history of mathematics education. The most common method that is used to study pedagogical differences in the history of mathematics education is to differentiate between two or three different pedagogical methods, describe them, list the characteristics of each method, and then categorize the primary sources (usually textbooks) by which method they use. Pycior (1993) and Ackerberg-Hastings (2000) do this with *analysis* and *synthesis* in algebra and geometry textbooks. Michalowicz and

Howard (2003) categorize elementary mathematics textbooks according to those that follow “the *rule* method, the *inductive* method, and the *analytic* method” (p. 80).

Likewise, this study differentiates between the “line system” and the “ratio system” for defining and teaching trigonometry.

The line system and the ratio system were described in detail in the literature review. For the purposes of this discussion of methodology, the systems’ characteristics can be outlined as follows. The line system defines the trigonometric functions as line segments on a circle that change as the arc they subtend changes (as seen in Figure 3.1), and trigonometric identities and theorems follow from those definitions on the circle. For example, the identities that are now known as the Pythagorean identities can all be found by observing right triangles in the circle. To illustrate, in Figure 3.1, substitute the trigonometric functions into the Pythagorean theorem and let $OA = 1$. ΔOMP gives $\sin^2 \alpha + \cos^2 \alpha = 1$, ΔOAT gives $\tan^2 \alpha + 1 = \sec^2 \alpha$, and ΔORS gives $1 + \cot^2 \alpha = \csc^2 \alpha$.



$$PM = \text{sine}$$

$$NP = \text{cosine}$$

$$TA = \text{tangent}$$

$$RS = \text{cotangent}$$

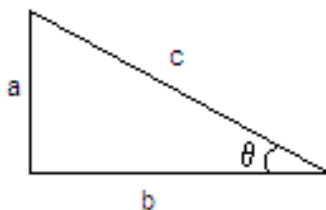
$$OT = \text{secant}$$

$$OS = \text{cosecant}$$

Figure 3.1

Other trigonometric theorems and identities are proved using these geometric definitions. In the case of the Pythagorean identities, the proof is easier in the line system, but this is certainly not true for all theorems and identities.

The ratio system defines the trigonometric functions as ratios of sides of a right triangle (see Figure 3.2). Further theorems and identities are proved algebraically. What are now known as the Pythagorean identities can again be used as an example. In the ratio system, letting $c = 1$ and substituting the trigonometric functions into the Pythagorean theorem for the defining triangle (see Figure 3.2) gives $\sin^2 \theta + \cos^2 \theta = 1$. Divide each term of this first Pythagorean identity by $\cos^2 \theta$ to get $\tan^2 \theta + 1 = \sec^2 \theta$, and divide each term by $\sin^2 \theta$ gives $1 + \cot^2 \theta = \csc^2 \theta$.



$$\sin \theta = a/c \quad \cos \theta = b/c$$

$$\sec \theta = c/b \quad \csc \theta = c/a$$

$$\tan \theta = a/b \quad \cot \theta = b/a$$

Figure 3.2

This study's textbooks analyses seek to classify textbooks as to whether they use the line system, the ratio system, or a combination of both. This study characterizes textbooks not only by how they define the trigonometric functions, but also by how they prove common trigonometric theorems. As the following chapters demonstrate, some textbooks follow the line system strictly, some use ideas from both systems, and other textbooks follow the ratio system strictly. While the historical trend is that as time progressed textbooks stopped using the line system and started using the ratio system, the change happened gradually.

Methods for Textbook Analysis

Although it cannot be certain that an accurate study of textbooks is also an accurate study of teaching, Gray (1948) and Hoskin (1994) suggest that during the nineteenth century, trigonometric textbooks were followed so carefully that examining textbooks not only reveals the content of the course but also reveals the method of its teaching. The USMA at West Point even went so far as to have rules laid out for the teachers dictating that they had to follow the textbook extremely closely (Hoskin, 1994). As was mentioned in the first section of this chapter, West Point was extremely influential in mathematics education during this time, so if West Point instructors followed the text closely, then this was very likely happening at other colleges and universities as well. For the aforementioned reasons, textbook analysis can be beneficial.

How does this study engage in textbook analysis?

This study looks to an important historian of mathematics education who analyzes textbooks, Eileen F. Donoghue. In an introduction to one of her studies, Donoghue (2003) writes, “This chapter focuses on the textbooks themselves and what they may reveal about the mathematics that was taught and learned. The discussion of the authors’ views or intentions, of the choice, sequencing, and presentation of topics, and of any special features provides some sense of the textbook as an instructional tool. In a broad way, the approach is somewhat akin to archaeology, in which the textbooks are viewed as remnant artifacts of classroom practice” (p. 329-330). Not only does this study take up Donoghue’s approach on the usefulness of studying textbooks, but it also uses her ideas about how to analyze textbooks. This study uses coding to analyze the textbooks in each of the following areas:

1. Method for defining trigonometric functions,
2. Topics addressed, including a complete listing and analysis of the theorems that are presented,
3. Order of topics, including which theorems are proved as the results of others,
4. Types and numbers of questions asked and whether solutions and/or answers are provided, and
5. Other pedagogical tools included in the textbook.

This study also considers the introductions to the textbook where available and where applicable to gain insight about the authors’ intentions. By analyzing textbooks in these ways, this study follows Donoghue’s methods for meaningful textbook analysis.

Summary

This chapter details the methodology this study uses. Because this study is one of the history of mathematics education, three types of methods are employed. First, methods for studying history are used to make sure there is as little bias as possible and that this history represents all relevant viewpoints. Second, a frequently-used method in the history of mathematics education are employed—this study categorizes the pedagogy of trigonometry education into two systems, and these two systems guide the analysis in this study. Finally, because studying textbooks is an important piece of this study, there is a discussion of methods of analyzing textbooks. This study analyzes textbooks based on the content, the presentation, the order of topics, questions and problems given for the students, and special pedagogical features. These three methodologies guide the research and analysis in this study.

CHAPTER IV:

The Early Establishment of Trigonometry Education

This chapter focuses on trigonometry education from 1776 to 1820. The textbooks included in this chapter were printed either in the eighteenth century or the first two decades of the twentieth century and were in use during these times as well. In general, textbooks are included in a time span based on their publication dates, but some of the oldest textbooks that are included were printed before 1776. They are considered because they were still in use in 1776 and beyond.

During this time span, colleges were using textbooks from England, Scotland, and the United States of America. Although England and Scotland are both part of Great Britain, the textbooks as well as the teaching of mathematics were substantially different from each other (Ackerberg-Hastings, 2000). This chapter first compares textbooks from each country among themselves, and then there is a comparison of the countries' textbooks one to another. It is helpful to group the countries' textbooks because each country's textbooks are similar. Comparing the three countries' textbooks is also helpful because, in general, English textbooks were used earliest, followed by Scottish and then American textbooks. Because of this chronology, making comparisons between the countries also shows how trigonometry education was changing over time.

A Note on Terminology

The terminology used during the eighteenth and nineteenth centuries in trigonometry textbooks was very different from that which is used today. So that it can be most comprehensible today, this study will use modern terminology. For formulas given in the list of terminology, refer to figure 4.1.

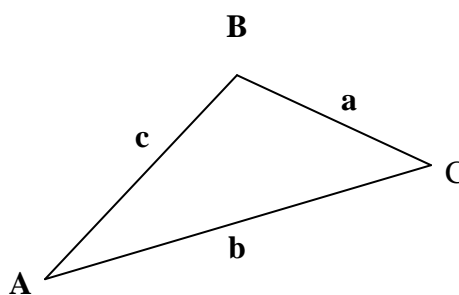


Figure 4.1

The following terminology will be used:

1. The *Pythagorean identities* refer to $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, and

$$1 + \cot^2 x = \csc^2 x;$$

2. The *principal Pythagorean identity* is $\sin^2 x + \cos^2 x = 1$;

3. The *law of sines* says that in triangle ABC (Figure 4.1) the sine of an angle of a triangle is to the side opposite the angle as the sine of another angle of that triangle is to the side opposite that angle, $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.

4. The *law of cosines* says that in triangle ABC (Figure 4.1), $c^2 = a^2 + b^2 - 2ab \cos C$.

5. There is a law that states “The Sum of the Legs [adjacent to] any Angle of a Plane Triangle is to their Difference, as the Tangent of half the Sum of the Angles opposite to those legs is to the Tangent of half their Difference” (Ward, 1747, p. 478). This law is obsolete in trigonometry today, but for the purposes of this study it will be called the *law of tangents*. The law of tangents can be expressed symbolically as follows (see Figure 4.1):

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\tan\left(\frac{1}{2}(A-B)\right)}$$

6. Another law shows the proportionality of the greatest side of a triangle to the sum of the other two sides as the difference of those two sides is to the difference of the two parts of the longest side created by constructing a perpendicular to the longest side. For the purposes of this study, this law is called the *law of proportionality*.

The law of proportionality can be expressed symbolically as follows:

$$\frac{c}{a+b} = \frac{a-b}{d-e} . \text{ See figure 4.2.}$$

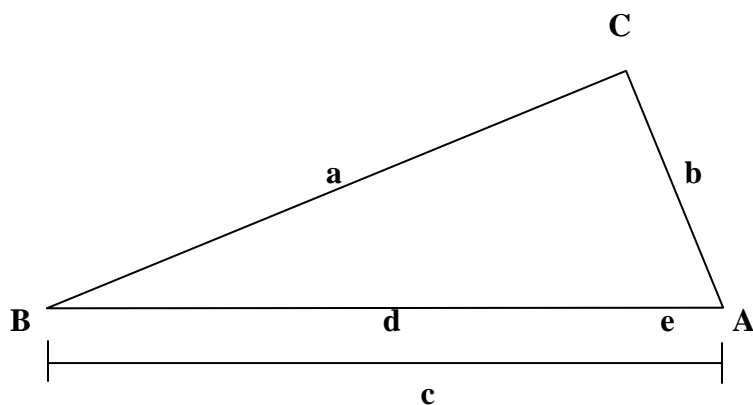


Figure 4.2

7. The *inverse relationships of the trigonometric functions* are $\sin x = \frac{1}{\csc x}$,
 $\sec x = \frac{1}{\cos x}$, and $\tan x = \frac{1}{\cot x}$.

On the Construction of Trigonometric Tables

During this time, trigonometers and students of trigonometry used trigonometric tables to find the values of particular trigonometric functions. Because all calculations relied on them, the construction of trigonometric tables was important. Most textbooks devoted a lengthy section to explaining how the author calculated the trigonometric tables.

Through the late eighteenth century and most of the nineteenth century, the calculation of trigonometric tables was basically the same in all textbooks. For the purpose of efficiency, the method is explained here, and unless otherwise noted, each author's method for calculating the trigonometric tables is as follows.

First, the sine of 30° is used as a starting point, since its value can be determined exactly. The sine of half that angle is determined using the half angle formula, and half that angle, and so on. That continues until sine of $1'$ is determined to ten decimal places of accuracy. This extreme accuracy is possible because as the angle becomes very small, the sine of the angle becomes very nearly the angle itself. The difference between the sines of angles very near to $1'$ are so small that the sine of $1'$ can be determined with a great deal of accuracy.

After determining sine of $1'$, angle addition formulas and the symmetry of the circle are used to construct the remainder of the trigonometric table, so that the trigonometric functions of every degree and minute are listed in the trigonometric table with great accuracy.

Beginning in the late nineteenth century, trigonometric tables were constructed with as much accuracy as desired using the analytic power series for sine and cosine, but the trigonometric tables that were constructed in the way that is explained above are accurate enough even for modern uses of trigonometry (Van Brummelen, 2009).

Most textbooks referred to these trigonometric tables as “natural” trigonometric tables, and most textbooks also included logarithmic trigonometric tables. The logarithmic tables gave not the values of the trigonometric functions themselves, but instead the values of the logarithms of the trigonometric functions. Using logarithmic tables allowed those making computations to add and subtract, rather than multiply and divide. That helped computations become easier, especially because trigonometric computations almost always included decimals to the ten-millionths place and beyond.

Eighteenth and Early Nineteenth Century Teaching of Trigonometry

In the eighteenth century, the main influences on trigonometry education in America were British. Initially, there was a strong English influence. At the same time and somewhat later, there was a Scottish influence. In the early nineteenth century, American authors began writing trigonometry textbooks, although their works were clearly under heavy English and Scottish influence. Still, there can be seen significant

differences between the English, Scottish, and American textbooks that American colleges were using during the eighteenth and early nineteenth centuries.

During this time period, it is worth noting that much of the focus of trigonometry texts is on computation of the trigonometric functions themselves and on ways of making other trigonometric computations easier.

Logarithmic computation became popular during this time as well, because it was impractical constantly to have to multiply and divide decimals that go to the ten-millionths place and beyond. By using logarithms, it is possible to use addition and subtraction rather than multiplication and division, which makes the computations involved in solving trigonometric problems much more reasonable. Because of these difficulties, time was spent within trigonometry texts on the use of logarithms to compute trigonometric formulas, and this causes the focus of trigonometry textbooks to be very different to those of more recent times, when these calculations became trivial because of the widespread use of calculators.

English Textbooks

The first textbooks containing trigonometry known to be used in American colleges were English. The earliest of the English textbooks was John Ward's *Mathematics*, which was printed in London in several different editions. It was in use as a textbook at Harvard College from 1726-1738 (Cajori, 1890). Yale College used Ward's text for as many as fifty years during the eighteenth century, not replacing it until 1801 (Ackerberg-Hastings, 2000). Dartmouth College also used it as a textbook, and the University of Pennsylvania used this text as a reference (Cajori, 1890). This text is not by

any means exclusively a trigonometry textbook, but it does contain elements of elementary trigonometry. In Ward's 1724 edition, he defines sine and tangent within the section titled "Elements of Geometry" as follows: "...finding the *Quantity of Angles*, which is done by the Help of *Right Lines*, called *Sines* and *Tangents*, the Length whereof are *Calculated* to every *Degree* and *Minutes* of a *Quadrant* by much Labour" (p. 356). Then he shows how to find the sine or cosine of an angle.

After this, when showing how to find the tangent, he gives the diagram shown in Figure 4.3 within the text (p. 359):

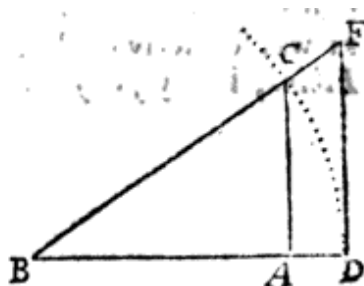


Figure 4.3

and says, "For supposing $BC=BD$ Radius, AC the *Sine* of the Arch CD . Then BA is the *Co-sine*, and FD is the *Tangent* of the same Arch. But $BA:CA::BD:FD$, &c." He says that tangent can be found if sine and cosine are both known, and that if sine is known that cosine can be found and vice versa since $\sin^2 \theta + \cos^2 \theta = 1$. He says, shortly after this, "Perhaps it may here be *expected*, that I should have shew'd and *Demonstrated* (or at least have inserted) the *Proportions* from whence the foregoing *Equations* for making *Sines* were *produc'd*; but I have *Omitted* that, as also their *Use* in *computing* the *Sides*

and *Angles of plain*⁵ *Triangles by the Pen only, (viz. without the Help of Tables)* for the *Subject* of another *Discourse* hereafter, if *Health* and *Time* permit.” After this short section on trigonometry, Ward uses sine and tangent occasionally in other parts of the text, such as the part on conic sections.

Finally, in the chapter entitled, “The Arithmetik of Infinites apply’d to Superfices and Solids,” he briefly defines versed sine, and then includes a table of versed sines and uses it in many of the following problems. He also uses versed sines in the following and final chapter of the text, entitled, “Of Practical Gauging.”

Ward’s pedagogical model for the text is as follows: first he gives a theorem, and then he gives a demonstration of that theorem, which is an example of a problem that he solves completely. In Ward’s text, there are no problems or exercises for students to solve. All diagrams are given within the text, but diagrams are rare in this text.⁶

In a 1747 edition of the same text, Ward includes “A Supplement not in any Former Editions of this Book. Containing the History of Logarithms, with several easy methods of constructing the tables of the logarithms and sines, &c. Also the Demonstration of the Axioms and the Doctrine of plain Trigonometry. Extracted from the Philosophical Transactions and Works of Dr. Keil, Ronayne, Ward, &c.” In this supplement, Ward gives a more thorough treatment to trigonometry than that which is given in the original text. He defines sine and cosine according to the line definitions, and then shows how to find sine and cosine using the trigonometric tables. He explains how

⁵ Before standardized spelling was developed for the English language, “plain” is often used to mean what today we spell “plane.”

⁶ Information about how and where the diagrams are located within the text will be grouped with pedagogical information because it is pedagogically relevant, even though the printing technology, rather than the author’s pedagogical idea, determines whether diagrams are given within the text or at the end.

the trigonometric tables are calculated, and defines tangent proportionally with sine and cosine.

After that, Ward has a section entitled, “Plane Trigonometry Definitions.” Here he defines the measurement of angles as well as all of the trigonometric functions in this order: sine, tangent, secant, versed sine, and finally their co functions. After is the following, in this order:

- an explanation of the solution of right triangles in seven cases by letting the radius of the circle be equal to a helpful part of the right triangle, depending on the given information
 - given the angles and one leg, find the other leg
 - given the angles and one leg, find the hypotenuse
 - given the legs, find the angles
 - given the legs, find the hypotenuse
 - given one leg and the hypotenuse, find the angles
 - given one leg and the hypotenuse, find the other leg
 - given the hypotenuse and the angles, find a leg⁷
- a discussion of the solution of oblique triangles, including
- the law of sines
- the law of tangents
- the law of proportionality

⁷ Most textbooks that come later contain only four cases for the solution of right triangles, where the cases are separated by the given information. Ward also separates the cases by the information sought. In later textbooks, the general format is as follows: given the legs, find the hypotenuse and the angles. The task becomes finding everything that is unknown, rather than simply one piece of the missing information.

- the solution of oblique triangles in six cases using the law of sines, law of tangents, and the law of proportionality
 - given two sides and an angle not included between them, find the other non-included angle
 - given two sides and an angle not included between them, find the third side
 - given two angles and a non-included side, find the other non-included side
 - given two sides and the included angle, find the other angles
 - given two sides and the included angle, find the other side
 - given all the sides, find all the angles⁸

Pedagogically, this supplement of Ward's text follows somewhat of a different model than his original text. In the supplement, there are no example problems worked out for the student to see. The supplement mainly consists of definitions, propositions, theorems, and corollaries.

The fact that Ward's 1747 *Mathematics* includes a supplement devoted to trigonometry that was not included in his 1724 edition shows that trigonometry was being given an increasingly important role in mathematics curriculum during the eighteenth century.

The next known textbook containing trigonometry to be used in the United States was John Keil's⁹ *The Elements of Plain and Spherical Trigonometry*, which was printed

⁸ Other textbooks also give the cases for solving oblique triangles differently. In Ward's textbook, if one case is used to find a piece of information about a triangle, a later case can then be used to find the rest of the information that is needed. In that way, the cases build upon each other.

in Dublin in 1726. Although the text was printed in Dublin, Keil was professor of astronomy at Oxford, so the greatest influence on the text is considered to be English. This text was used at the college of Pennsylvania (today the University of Pennsylvania) in 1758 and beyond (Cajori, 1890).

In Keil's text, he first defines trigonometric functions geometrically. Then he gives, in this order:

- Pythagorean theorem
- principal Pythagorean identity
- double angle and half angle formulas
- sum and difference formulas for trigonometric functions.
- compares sines to tangents
- sine of one minute
- infinite series of Newton for sine and cosine
- other Pythagorean identities, which he justifies using a geometric argument (using the line definitions of the trigonometric functions)
- $\tan(x) = \sin(x) / \cos(x)$
- law of sines
- law of tangents
- law of proportionality

Following these definitions, theorems, laws and identities is an explanation of how to solve a right triangle, given various information. The strategy he employs is to let the

⁹ There is a discrepancy as to the spelling of Keil's name. The text prints his name as Keil, but elsewhere he is known as Keill.

radius equal one of the given sides, and then use where the trigonometric functions naturally fall with respect to the radius to solve for what is missing. He also gives an explanation of how to solve oblique triangles by using the law of sines, law of tangents, and the law of proportionality.

Keil's pedagogical model for the text is as follows: he gives definitions then propositions, often in a problem-solution model, as well as theorems and corollaries. There are no examples of problems worked out for students to see, and neither are there exercises or problems for the student. All diagrams are given at the end of the text.

The last English textbook to consider is Charles Hutton's *A Course of Mathematics* (in two volumes), first published in London in 1798 and later published in the United States in 1812. Charles Hutton was first a professor at the Royal Military Academy in England and later a professor at the United States Military Academy (USMA) at West Point. The USMA at West Point used Hutton's *Mathematics* (in two volumes) from 1802-1823 (Shell-Gellasch, 2001). Hutton's text was first printed in many editions in London before it was printed in America. Hutton was also professor of mathematics at West Point. In the first volume of his text, he includes a chapter on plane trigonometry. In the second volume of his text, he includes a chapter on plane trigonometry considered analytically.

First, in the first volume's introductory chapter on plane trigonometry, Hutton defines the trigonometric functions geometrically. He first defines sine, tangent, secant, and versed sine, each as a function of an arc, and then he defines the co functions as functions of their complements. He includes both natural and logarithmic tables for sine, tangent, and secant. He also discusses the general tendencies of these functions as their

arcs change. He shows how to find natural sine and cosine using power series. Then using this information he shows how to compute tangents and secants by similar triangles given in the definitions of the trigonometric functions. From this discussion, he gets the Pythagorean identities as well as the inverse relationships of the trigonometric functions. He also discusses assuming that the radius is one and then gives the identities that follow from that assumption.

Hutton then divides the solution of triangles into three cases. He gives the law of sines followed by five exercises for students. The first exercise is solved first by geometric construction and then arithmetically, and at this time he also discusses the advantages and disadvantages of finding trigonometric functions logarithmically. Following this, he gives four more exercises, two where the answer is given without a solution, and two where neither answer nor solution is given. Afterward, he gives the law of tangents and the law of cosines. For each, he solves one exercise fully and then provides several more exercises, some with answers but no solutions given.

After these three laws, Hutton gives the solution for all right triangles, using proportions of sides and trigonometric functions. As a corollary, he reduces these proportions to simpler equations by assuming that the radius is equal to one. As another corollary, he gives proportions that are similar to the “ratio system,” although they are not the trigonometric ratios in their finished form. He proves that when given a right triangle, unknown parts can be found using proportions: “As radius is to either leg of the triangle; so tangent of its adjacent angle, to its opposite leg; and so secant of the same angle, to the hypotenuse” (p. 391). These proportions are not the trigonometric ratios in their finished form, but they contain the same basic ideas.

Following this, he gives two exercises, one with the solution given both geometrically and arithmetically, and one with answer only given. Next, Hutton includes a section on the most useful trigonometric formulas, containing the sum and difference formulas for sine and cosine as well as half and double angle formulas. Finally, he includes a section called, “The Application of Plane Trigonometry to the Determination of Heights and Distances,” which contains problems concerning angles of depression and elevation.

Plane Trigonometry is the last topic in Volume One of Hutton’s Course of Mathematics, and the first chapter in Volume Two is a treatment of “Plane Trigonometry considered analytically.” He introduces this by saying:

“There are two methods which are adopted by mathematicians in investigating the theory of Trigonometry: the one *Geometrical*, and the other *Algebraical*. In the former, the various relations of the sines, cosines, tangents, &c. . . . are deduced immediately from the figures to which the several enquiries are referred; each individual case requiring its own particular method, and resting on evidence particular to itself. In the latter, the nature and properties of the linear-angular quantities (sines, tangents, &c.) being first defined, some general relation of these quantities, or of them in connection with a triangle, is expressed by one or more algebraically equations; and then every other theorem or precept. . . . is developed by the simple reduction and transformation of the primitive equation.” (p. 1)

Hutton goes on to explain the advantages and disadvantages of each method. The advantage of the geometric method, or the “line system,” he says, is that it keeps the “objects of inquiry” (p. 2) in front of the student at all times, preventing errors and making the steps one must take clear and straightforward. The geometric method is quick to achieve the initial formulas of trigonometry, but it is difficult with this method to produce all the formulas that are desired. In the algebraic method, or the “ratio system,” on the other hand, the most elementary formulas are difficult to develop, but once these are established, it is easy to come by multitudes of other useful formulas. However,

Hutton says, since there is little connection in the algebraic method to the principles on which trigonometry is based, it “requires frequent checks to prevent any deviation from the truth” (p. 2).

Using the geometric method to prove the theorems of trigonometry forces students to use the definitions of the trigonometric functions constantly. Using the definitions constantly causes it to be difficult to make mistakes but also causes it to be difficult to prove some of the theorems of trigonometry. Hutton emphasizes that the use of the algebraic method is better suited for more advanced students of trigonometry because by using algebra, students move away from the basic definitions of the trigonometric functions, so students must check back with the functions often to make sure they have not made a mistake.

Hutton defines the trigonometric functions geometrically, with the same definitions as are given in volume one. He then gives the “ratio system” and then by algebra he derives many of the same formulas as he already proved in volume one as well as others. This study will not consider his chapter on analytic trigonometry in detail because the focus is limited to elementary trigonometry. However, it is interesting and informative to hear what Hutton says about geometric versus algebraic methods of trigonometry and to bear in mind that even in his chapter on analytic trigonometry he begins with the geometric definitions of the trigonometric functions.

When considering pedagogy, this is the earliest example of a text that includes exercises for the student to complete individually. In all cases, Hutton’s exercises require students to replicate a problem he has solved in the text using a different set of given values. Hutton gives students three levels of assistance with exercises; in the first level,

students are provided with the full solution, in the second level they are given the answer but no solution, and in the third level students are given neither the solution nor the answer. As one of the first to include exercises for students, Hutton shows that he intended students to use and learn from the textbook. He saw the textbook as a pedagogical tool, and expected students to use it by completing exercises that were given in the textbook.

Additionally, Hutton completes many of his examples in more than one way when possible. This shows a true attentiveness to student understanding, and a desire to promote not only computational ability but also a grasp of the connections between the concepts and the procedures. Finally, Hutton writes rules that students need to memorize in a poetic style, offset from the text, seemingly to allow easier memorization. The law of sines is shown in figure 4.4 as an example:

Given three such parts, that an angle and its opposite side shall be two of them ; to find the rest.

In any plane triangle, the sides are proportional to the sines of their opposite angles.* That is,

As one side	:
Is to another side	::
So is sin. angle opp. the former	:
To sin. angle opp. the latter.	

Figure 4.4

In Hutton's text, diagrams are given frequently within the text.

Scottish Textbooks

Along with the English textbooks, American colleges also used Scottish textbooks. The first of the Scottish textbooks, Simson's *Euclid*, was in widespread use by American colleges in the late eighteenth and early nineteenth centuries (Ackerberg-Hastings, 2000). From its founding in 1795 until 1822, the University of North Carolina used Robert Simson's *Elements of Euclid*, mainly a geometry text to which is attached a treatise on plane and spherical trigonometry (Cajori, 1890). Additionally, Jeremiah Day purchased Simson's *Euclid* for Yale until 1804 (Ackerberg-Hastings, 2000).

Robert Simson was a professor of mathematics at the University of Glasgow from 1711 to 1761, during which time he translated Euclid's *Elements of Geometry* from the Greek text. In later editions, a treatise on plane and spherical trigonometry is added to the *Elements* (Trail, 1812).

In his treatise, Simson first defines the following:

- the measurement of angles by degrees, minutes, and seconds,
- sine, versed sine, tangent, and secant geometrically
- the complement of an angle and the co functions

After the definitions, he shows the following properties, laws, and theorems:

- the reciprocal relationship between pairs of trigonometric functions (he says, "the radius is a mean proportional between the co-sine and secant of any angle ABC")
- the relationships of the sides of a right triangle to the trigonometric functions of the acute angles of that right triangle (these are similar to the trigonometric ratios, but are not fully developed as such)

- the law of sines
- the law of tangents
- the law of cosines
- the law of proportionality
- the solution of right triangles in five cases
 - given the two legs, find the angles opposite them
 - given one leg and the hypotenuse, find the non-right angles
 - given one leg and one angle, find the other leg*
 - given one leg and one angle, find the hypotenuse*¹⁰
 - given the hypotenuse and one angle, find the side opposite the angle
- the solution of oblique triangles in four cases
 - given the angles and one side, find the other two sides
 - given two sides and an angle opposite one of them, find the other angles
 - given two sides and the included angle, find the other angles
 - given the three sides, find the angles

Pedagogically, Simson first defines trigonometry, proves many theorems that will be needed, and then he shows how to use the theorems to find the solutions of triangles. He works out no examples, and he gives no exercises for students to practice. He uses diagrams frequently, but all diagrams are given at the end of the text.

John Playfair's *Elements of Euclid* was another extremely influential Scottish textbook. Playfair, like Simson, was Scottish. He was the University of Edinburgh chair of mathematics from 1785 to 1805, and while he held that professorship, he published his

¹⁰ In most texts, the cases with the * are combined into a single case as follows: given one leg and one angle, find the other leg and the hypotenuse.

Elements, the first edition in 1795. In many cases Playfair's *Euclid* replaced Simson's *Euclid* because it was seen to be a clearer text that was better organized (Ackerberg-Hastings, 2000). After purchasing Simson's *Euclid* for Yale until 1804, Jeremiah Day then purchased Playfair's *Euclid* until at least 1818 (Ackerberg-Hastings, 2000).

Additionally, Dartmouth College replaced Ward's *Mathematics* with Playfair's *Euclid*, using it at least until 1834 (Cajori, 1890). Finally, Playfair's *Euclid* is the first text known to be in use at the College of New Jersey (later Princeton University) and was used there at least until 1850 (Cajori, 1890). Like Simson's *Euclid*, Playfair's text is primarily a geometry text, but it has a treatise on plane and spherical trigonometry attached. Playfair says in the preface to his 1813 text that he admires Simson's text greatly, especially Simson's translation of Euclid, but that Playfair attempts to improve on this by writing his text to be most useful to students of mathematics (Playfair, 1860).

Playfair begins his treatise by defining and showing the following:

- the arc as the measure of an angle
- trigonometric functions defined geometrically in the following order: sine, versed sine, tangent (noting that the tangent of half a right angle is equal to the radius), and secant
- the sine, tangent, secant of an arc are the same as those of its supplement
- the complement and the co functions
- the product of tangent and cotangent is the square of the radius, as is the product of cosine and secant (proved using parallel lines)

After these definitions and properties, Playfair gives the following series of propositions:

- the trigonometric ratios (by examining the relationships within a right triangle)

- the law of sines
- the law of tangents
- the law of proportionality
- the law of cosines

Afterward, Playfair has a section on the rules of trigonometric calculation, where he proposes to solve all possible plane triangles. He begins with right triangles, and shows how to solve them in four cases. Then he shows how to solve oblique triangles, also in four cases. Finally, Playfair discusses the construction of trigonometric tables, including how to find the trigonometric functions of an arc of one minute. He then gives the sum and difference formulas, which allow the tables to be constructed from known values of trigonometric functions.

Pedagogically, Playfair first gives definitions, then a series of propositions, and finally he proposes general problems and shows how to solve them. Playfair does not solve any examples, nor does he give any exercises for students to solve. In the 1860 reprint of the 1813 edition, there are numerous diagrams given within the text, but it was probably not the case in all of the editions of Playfair's *Euclid* that the diagrams were given within the text.

American Textbooks

The next known textbooks that American colleges used were written by American authors and printed in the United States of America. The first of these came from the beginning of the nineteenth century. In 1801 Harvard used Samuel Webber's *Mathematics* (Cajori, 1890). In 1801, Yale also began using Webber's text, finally

replacing Ward's *Mathematics*. Yale used Webber's text until 1815 (Ackerberg-Hastings, 2000). Webber was the Hollis Professor of Mathematics at Harvard and later the president of Harvard, and the textbook, rather than being his own writing, is a compilation of "the best authors." He credits Hutton and Bonnycastle for the majority of the work, including the sections involving trigonometry (Webber, 1808). John Bonnycastle was an English professor of Mathematics at the Royal Military Academy in Woolwich, and Charles Hutton is the author of the previous text.

Webber first shows how to construct sine, tangent, and secant in a chapter on geometry. At this time, he defines these as well as versed sine in the footnotes on the constructions. Within a few pages, he concludes his chapter on geometry, and begins a chapter on plane trigonometry. Here he defines the following:

- the measurement of angles by degrees
- the complement of an angle
- the supplement of an angle
- sine, versed sine, tangent, and secant geometrically
- cosine, cotangent, and cosecant (defined as the sine, tangent, and secant of the complement)

Webber then explains both natural and logarithmic trigonometric tables and briefly tells how they are used.

Webber then moves on to explain how to solve triangles. He goes through the same trigonometric laws as the other authors, as follows:

- law of sines
- law of tangents

- law of proportionality

Within each law, he gives three methods for solving triangles; arithmetic computation is one option among construction and instrumental measurement of the triangle. He explains how to use each of the three methods for each trigonometric law. For each law, after the explanations of the three methods, Webber gives an example of a triangle with an angle and two sides given, which he solves using all three methods. Following his example he gives several exercises for practice, each with the answer given but no explanation of the solution. After that, he demonstrates how to solve a right triangle given the angles and a leg in the same manner, but with no additional exercises.

Pedagogically, Webber's text is similar to that of Hutton. Webber also gives exercises for students to complete, although he gives answers to all the exercises whereas Hutton gives answers only to some. Webber, like Hutton, shows more than one solution for the same example when it is possible to solve the example in more than one way. Webber also offsets rules that students must memorize in a verse, so that students can memorize them more easily. Finally, in Webber's text, diagrams are given within the text and are frequent throughout the text.

The next important text to examine is Jeremiah Day's *Mathematics*. Yale College used Day's series on mathematics at least from 1815 to 1848, which includes *A Treatise of Plane Trigonometry* (Cajori, 1890), which was first printed in 1815 and had approximately nine editions printed in total (Ackerberg-Hastings, 2000). Day began as a tutor of mathematics at Yale in 1798, becoming professor of mathematics and natural sciences in 1803 and president of Yale in 1817. Day's series on mathematics was extremely influential in American colleges and it shaped mathematics education in the

nineteenth century. Furthermore, he tried to write his textbooks in such a way that students could understand and use the texts (Ackerberg-Hastings, 2000).

Day began his treatise by saying that the object of trigonometry is to measure the lengths of the sides and the angles of triangles. He first defines and demonstrates the following:

- the measures of angles
- the complement of an angle and other basic terms
- the trigonometric functions with geometric definitions
- the trigonometric co-functions
- the inverse relationships of the trigonometric functions
- the Pythagorean identities by geometry
- the trigonometric functions of special angles that are equal to the radius (for example, cosine of 0° and sine of 90°)
- the use of both natural tables and logarithmic tables

After the basic definitions and identities, Day discusses the solutions of right triangles in detail. Within the text, there are explanations and then examples to follow with solutions completely worked out. After the discussion, there are six exercises for practice, with no solutions or answers given. Following that, Day explains how to use the Pythagorean theorem to solve a right triangle if two sides are given. Three exercises accompany this explanation with answers but no solutions given. After that, Day shows how to use logarithms to find difference of two perfect squares with one exercise (answer given). Afterward, he begins his discussion of how to solve oblique triangles with triangles that can be solved by each of the following laws:

- law of sines
- law of tangents
- law of proportionality

He explains how to solve oblique triangles in each of these cases. After each explanation, Day gives two or three exercises, one with the answer but no solution given, and the remaining example(s) with neither answer nor solution. After the section is concluded, he gives four exercises for practice, with neither solutions nor answers given. For these four exercises, the student not only has to work the problem alone but also has to decide which law(s) to use to solve the triangles.

Day then discusses geometrical construction of triangles using the plane scale. He has a description of and demonstrates the use of Gunter's scale, which was a special ruler used in navigation to help calculate trigonometric values, logarithms, and so on. The next section is the first principles of trigonometric analysis (in this section Day credits Euler's Analysis of the Infinite, Hutton's Mathematics, Lacroix's Differential Calculus, Mansfield's essays, Legendre's, Lacroix's, Playfair's Cagnoli's, and Woodhouse's Trigonometry). In this section, he discusses the following:

- the signs of the trigonometric functions in different quadrants
- the extreme values of the trigonometric functions
- sum and difference formulas
- half-angle formulas
- double and multiple angle formula
- Heron's formula
- the development of the trigonometric tables

After that, he gives methods for calculating astronomical triangles whose side lengths are given by their logarithms.

Pedagogically, Day's format is similar to Webber. After giving examples with their solutions, he gives exercises for practice, some with answers only and some without answers. Day also offsets passages in verse that students need to memorize. Unlike Webber, Day does not give more than one solution to his problems. In Day's text all diagrams given at end of text.

Trends in Textbooks—English, Scottish, and American

The earliest authors here examined are Ward and Keil, both of whom were English mathematicians. Their texts were written 1724 and 1726, respectively, and they follow similar formats, especially pedagogically. In particular, the 1747 supplement to Ward's text that contains a more thorough treatment of trigonometry is very similar pedagogically to Keil's text. This is not surprising, however, because Ward credits Keil in his supplement, whose ideas he borrowed for this additional text. Both Keil in his text and Ward in his supplement give definitions and propositions often in a problem-solution format. An example of the problem-solution format for the propositions is as follows from Keil's text:

“Proposition III:

Problem.

The sine DE of any Arc DB being given, to find DM or BM the Sine of half the Arc.

DE being given, CE (by the last *Prop.*) will be given, and accordingly EB which is the Difference between the Cosine and Radius. Therefore DE, EB, being given in the Right-angled Triangle DBE, there will be given DB, whose half DM is the Sine of the Arc $DL = \frac{1}{2}$ the Arc BD.” (p. 7)

Although a figure is not included in the text, Figure 4.5 accompanies Proposition III.

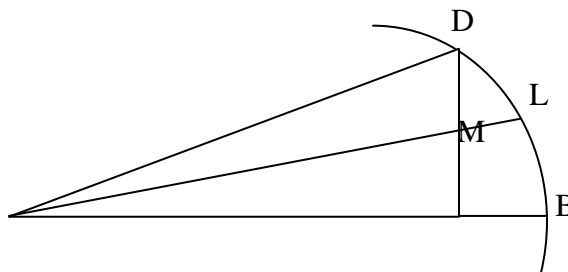


Figure 4.5

Thus he presents a problem that one may want to solve trigonometrically, and then solves it generally. However, neither Ward in his supplement nor Keil in his text give examples of specific problems with their solutions. Their entire trigonometry texts are given in general, with no actual triangles solved or examples using numbers given. They show how to solve triangle ABC where certain information is given, but they do not say, for example, let side $AB = 3.5$, and so on.

In contrast to his supplement, Ward does give specific problems and their solutions in his original text. The following is an example of how to find the tangent of an arc when the sine of that arc is given:

“Let the Sine of $19^{\circ}13'$ (before found) be given, *viz.* $0,3291415=S$. To find T the *Tangent* of the same *Arch*.

First $0,3291415 \times ,3291415 = 0,108334127 = SS$.¹¹

¹¹SS simply means S^2 or Side². At that time, it was not possible to use a superscript, so powers were notated by repeating the base the appropriate number of times.

Again $1 - 0,108334127 = 0,891665873 = 1 - SS$.

Then $0,891665873 \cdot 0,108334127 = 0,1214963252$

And $\sqrt{0,1214963253} = 0,3485632 = T$, the *Tangent* of $19^\circ 13'$. As was requir'd." (p. 360)

Although in this way Ward's original text is pedagogically somewhat more developed, this text is extremely limited in its treatment of trigonometry. The section on trigonometry in Ward's original text almost entirely focuses on calculating the trigonometric functions.

In content, Keil and Ward (here consider the 1747 supplement, since Ward's original text is so thin in its treatment of trigonometry) both have as their end the solution of all triangles, right and oblique, if adequate information is given, and they both take similar approaches to the solution of triangles. Also, both base the solution of oblique triangles on the law of sines, the law of tangents, and the law of proportionality. Keil's text gives more depth than that of Ward; Keil includes the infinite series of Newton for sine and cosine, the double and half angle formulas, the sum and difference formulas, and the non-principal Pythagorean identities, all of which Ward does not. Other than these topics that Keil includes and Ward does not, the content of these texts is very similar.

The content of Hutton's textbook does not differ much from its predecessors, but it differs greatly in pedagogy. The one way in which it does differ from Ward's and Keil's textbook is that it includes both natural and logarithmic tables, as well as an explanation of how to use both.

Hutton's text is structurally different from its predecessors because the end of the discussion is not the solution of triangles. The end of Hutton's text, by contrast, is the trigonometric ratios. This is interesting because the trigonometric ratios grow to play an extremely important part in trigonometry and here they are seen as the ultimate theorem

of a trigonometric text. It is especially significant because the solution of triangles, which many of the authors of these texts see as the entire reason for taking up the study of trigonometry, moved to be central in the text, and the trigonometric ratios have taken its place as the end of the study.

The structure of Hutton's text is also different from the previous texts because rather than giving all the theorems and trigonometric laws together and then explaining how to solve all triangles using all of the laws, they are given within the discussion of solving triangles. For example, Hutton first proves the law of sines and then there is an example of a triangle where two sides and a non-included angle are given. Hutton then demonstrates how to use the law of sines to solve this triangle. He goes on to prove the law of tangents and then gives an example of a triangle that can be solved by the law of tangents, and so on. This structural difference, although it is not known why Hutton chose to structure his text in this way, has pedagogical implications. Structuring the text in this way allows students to apply and practice the law they have learned before they learn the next law. In this way, the structure of Hutton's text is a pedagogical change.

The greatest difference between Hutton's text and its predecessors are the direct pedagogical changes. Hutton makes four important advancements in pedagogy in his text. First, he includes example problems and works them out for students to follow. Second, he works out each of these example problems in three ways, to show the variety of methods for solving the problems and to demonstrate that all methods yield the same result. Third, he gives students exercises with answers only so that they can practice and check their answers to make sure they have completed the exercises properly. Hutton's text contains in total seven exercises with answers but no solutions given, and four

exercises with neither answer nor solution. Finally, Hutton offsets rules that students need to memorize in verse to make them easier to memorize.

After examining these three English authors, this study next examines two Scottish authors—Robert Simson and John Playfair. Each wrote a textbook containing the *Elements of Euclid*, to which are attached treatises on plane and spherical trigonometry.

Pedagogically, Simson and Playfair are very similar to each other and similar to Keil and Ward. They give definitions, theorems, propositions, and then they solve problems in general cases. They do not work out examples for students, nor do they give students exercises with which to practice.

In terms of content, Simson and Playfair are also very similar to each other. They both have as their end the solution of all triangles, and they both give the solutions of triangles in similar ways. To solve right triangles, Simson uses the relationships of the sides of a right triangle to the acute angles of that same triangle, which are very similar to but not as well-developed as the trigonometric ratios. Playfair first develops the trigonometric ratios fully and then uses them to solve right triangles. For solving oblique triangles, both Simson and Playfair use the law of sines, the law of tangents, the law of cosines, and the law of proportionality. The only other difference in content is that Playfair includes a section on the development of trigonometric tables, including the sum and difference formulas, where Simson does not.

Webber and Day were the first American authors of trigonometry textbooks that were used in United States colleges. The use of American textbooks increased greatly during the beginning of the nineteenth century, in part because at this time shipping from

Europe was difficult because of the Napoleonic Wars and later because of the War of 1812. This caused American colleges to become more self-reliant when it came to producing textbooks (Ackerberg-Hastings, 2000).

Webber's text is, in many ways, very similar to the English authors. This is not surprising, however, because Webber says his text is a compilation of "the best authors," among whom are Ward, Keil, and Hutton. Hutton's text covers more content than that of Webber. Hutton includes a discussion of how each of the trigonometric functions change as the arc changes. He also derives the Pythagorean identities as well as the inverse relationships of the trigonometric functions and includes the law of cosines where Webber does not. Finally, Hutton deduces the trigonometric ratios after he discusses the solutions of triangles, and he includes a section on problems having to do with angles of elevation and depression.

In structure, Webber's text is similar to Hutton's, especially because for the solution of triangles. They both give one law, then an example of a triangle that must be solved using that law, and then moves onto the next law. Unlike Hutton's text, the solution of triangles is the end of the discussion of trigonometry. In this way, Webber is more similar to Ward and Keil than to Hutton.

Pedagogically, Webber's text is again similar to Hutton's. Webber also gives specific examples fully solved as a demonstration for students and then gives exercises with answers only given for students to practice. His text contains a total of ten exercises. Like Hutton, Webber also solves the same example using more than one method. Finally, Webber also offsets passages for memorization in verse. Although they are very similar pedagogically, Hutton at times shows a greater pedagogical development than Webber in

the types of assistance given in exercises because Hutton has three levels of assistance on exercises—solution, answer only, and neither answer nor solution—while Webber has only the first two levels.

Day's text includes significantly more content than Webber's. He includes the inverse relationships of the trigonometric functions as well as the Pythagorean identities immediately after the definitions of the trigonometric functions. He also discusses trigonometric functions of special angles, the use of the plane scale and Gunter's scale, the first principles of trigonometric analysis, and Heron's formula. He explains how to solve astronomical triangles whose side lengths are given in logarithms. Finally, he does not include the trigonometric ratios.

Structurally, Day's text is also somewhat different from Webber's. Day says at the outset of his text that the aim of trigonometry is to solve triangles, and the solution of triangles is basically at the end of his text, although he has attached after that a discussion of the main ideas of trigonometric analysis.

Pedagogically, Day's text is more involved for the student than Webber's. In addition to having exercises immediately following examples of how to solve different types of triangles, Day also gives a set of problems at the end of each section that have to do with everything in the section. In total, Day gives twelve exercises with answers provided and ten exercises without answers, meaning that he gives about twice as many exercises as Webber. Along with Webber, Day sets off in verse passages that students must memorize. Unlike Webber, however, Day does not give more than one solution even if it would be possible to solve an example in more than one way, although he does solve problems with both trigonometric functions and with logarithms of trigonometric

functions, to show both of these methods of computation. Webber, in contrast, show the solutions by arithmetic computation as well as geometric construction. Although they do have some differences, the American writers make significant pedagogical developments, and they are all consciously moving toward writing textbooks that are more pedagogically sophisticated.

At the same time, it must be noted that pedagogical advancements happened chronologically. The English texts were the oldest. In fact, at the time that Ward and Keil's texts were printed and used, most often students did not have their own copy of the textbook. The teacher only had a copy of the textbook, and students wrote the textbook in their notebooks. This happened because of the difficulty of printing books at that time (Cajori, 1890). However, as printing presses became more sophisticated, it became possible for students each to have a copy of the textbook. It makes sense that at that time, textbook authors began to include exercises and problems for students to solve.

Over time, textbook authors were thinking more and more about the students that were reading and using the textbooks. At the beginning of the American republic, textbooks were references only, but as time went on, textbooks contained more and more pedagogy. They were written to be used not merely for reference but also as a tool to help students learn trigonometry.

At the end of the eighteenth century, the strongest influence in American textbooks was British. The earliest textbooks came from England, and slightly later textbooks came from Scotland. Although they are both a part of Great Britain, the schools of mathematical thought differed somewhat between England and Scotland (Ackerberg-Hastings, 2000). A couple of American authors wrote textbooks, but British and

American authors were very closely related at this time. For example, one of the British authors was Charles Hutton, but when he published his textbook, he was a professor at USMA at West Point. He was born in England and before he was a professor at West Point, he was a professor at the Royal Military Academy in England. Even though he wrote his *Course of Mathematics* while in the United States, he was still English.

British and British-influenced trigonometry textbooks were heavily geometry based. The line system was used to define the trigonometric functions and to prove the theorems and formulas. British textbooks often did not prove all the theorems and formulas they put forth, however. In general, British textbooks placed more emphasis on the procedures of solving triangles than on justifying every formula.

After students had their own textbooks, British textbooks included exercises for students to practice, and they set off in italics the formulas students needed to memorize. British and British-influenced authors had a strong sense of pedagogy.

Summary

As time went on, textbooks used in American colleges moved from English to Scottish to American. Although these textbooks have many similarities, there is considerable advancement in the textbooks as well. First, there is a shift to include more and more topics within trigonometry. Over time, the content of trigonometry textbooks included a wider variety of applications, and also in one case the end of trigonometry stopped simply being the solution of triangles. While the solution of triangles was still a goal of trigonometry and was included in textbooks, the ratio system for trigonometric functions became the end of trigonometry instead. This shows that trigonometry was

moving away from simply being only geometric and computational and toward an algebraic system.

Pedagogically, the English and Scottish textbooks were very similar, but there were significant advancements from the English and Scottish textbooks to the American textbooks. Moving from the British to the American textbooks, the major advancements in pedagogy are as follows:

- there are examples solved in the textbook
- the same example is often solved in more than one way
- there are exercises for students (sometimes with answers given)
- laws that students have to memorize are given in verse and set off from the text

These advancements mark a great difference in the attention the authors pay to students' learning, especially the learning that happens when a student is studying the textbook.

The advancements in book-printing technology encouraged these pedagogical advances and made them possible. Each of the textbook authors was himself a professor of mathematics, so he certainly used pedagogical tools when teaching students. However, at a time that roughly coincides with advances in printing presses that allowed books to be printed more easily and more cheaply, these professors decided to include in their textbooks the same pedagogical tools they used when teaching students.

However it was not chronology alone that dictated pedagogical advancements within textbooks. As the next chapter makes clear, French authors in the mid-nineteenth century did not have these pedagogical tools in their textbooks even though they were writing much later when the printing technology was widely available. These developments not only happened because of chronology and technology, but also they

were British and American in origin. British and American textbook authors were more concerned with pedagogy in textbooks than European textbook authors of the same time. In the next chapter, this study examines the changes in both pedagogy and content that took place during the middle of the nineteenth century.

CHAPTER V:
Antebellum Trigonometry Education

This chapter focuses on trigonometry education in the antebellum period, from 1820 to 1860. In antebellum America, colleges and universities most often used trigonometry textbooks written either by French or American authors, and many of those written by Americans were translations and adaptations of French textbooks. During the American Revolutionary War and following it for a considerable amount of time, France gave the United States of America a great deal of financial assistance, and this financial assistance bought France influence in the American educational system (Thwing, 1906).

American colleges and universities began using French textbooks, particularly mathematics textbooks, instead of those written by British or American authors (Cajori, 1890). Initially, colleges and universities used French textbooks themselves, which was feasible because students were also learning the French language. Soon, however, American mathematics professors began translating French textbooks, and colleges began using these translations instead.

Some colleges and universities were still using British textbooks, but since these were discussed in the preceding chapter, this chapter contains no further discussion of these textbooks. There were also a couple of American textbooks written during this

time, without any clear outside influences. A discussion of these follows the discussion of French textbooks.

Finally, there are two book reviews of one of the American textbooks, and these reviews reveal a debate among professors of trigonometry as to the best way to define and teach trigonometry. One reviewer of a textbook agrees with the author of the textbook that the line system should be abandoned, touting the merits of the ratio system. Another reviewer and professor of trigonometry disagrees, saying that understanding the line system is necessary for understanding the rest of trigonometry. This debate will be discussed in great detail at the end of this chapter.

A Note on Terminology

The terminology used during the nineteenth century in trigonometry textbooks was very different from that which is used today. So that it can be most comprehensible today, this study will use modern terminology. In addition to the terminology defined in Chapter 4, the following terminology will be used:

1. *Reference angles* are angles between 0 degrees and 90 degrees that are used to find the trigonometric functions of larger angles, found using the symmetry of the trigonometric functions on the circle.
2. The *trigonometric ratios* are as follows: In a right triangle ABC with sides a, b, and c, where side a is opposite angle A and so on where angle C is the right angle and side c is the hypotenuse, $\sin A = a/c$, $\cos A = b/c$, and $\tan A = a/b$.
3. Expressions that represent the products of the trigonometric functions include the following:

- $\sin(a + b) + \sin(a - b) = 2 \sin a \cos b$
- $\sin(a + b) - \sin(a - b) = 2 \cos a \sin b$
- $\cos(a + b) + \cos(a - b) = 2 \cos a \cos b$
- $\cos(a + b) - \cos(a - b) = 2 \sin a \sin b$

Early to Mid-Nineteenth Century Teaching of Trigonometry

The Antebellum period was a time of transition for the teaching of trigonometry in the United States. Prior to this, the main influences on the teaching of trigonometry were British, and there were a few American authors who wrote textbooks as well, although arguably their main influence was British. Over time, many innovations occurred in terms of the pedagogy of textbooks. Trigonometry moved from a part in larger mathematics textbooks to its own textbook, and more trigonometry content was added to textbooks. Although trigonometry grew in importance and pedagogy in textbooks advanced, the style of teaching trigonometry did not change much during this time.

Moving into the Antebellum teaching of trigonometry, many things changed. The primary influence during this time was French, and French mathematics differed drastically from British (Cajori, 1890; Pycior, 1989). Algebra took on an increasingly large role in trigonometry, and the order of topics presented in trigonometry textbooks changed drastically, giving preference to the algebraic side of trigonometry rather than the geometric side. Although there were many changes in the approach to teaching trigonometry, the antebellum period saw little in terms of pedagogical development.

French Textbooks

First this study analyzes one of the French textbooks that colleges and universities used without translation. Using French textbooks directly was fairly rare, and after American authors wrote translations and adaptations of French textbooks, those were used instead.

The USMA at West Point used Sylvestre Francois Lacroix's *Traite Elementaire de trigonometrie rectiligne et physique* from 1825 to 1832. During this time, the French text was used without translation (Shell-Gellasch, 2001), which was possible because all students at this time were required to learn the French language as well (Cajori, 1890).

In his text, Lacroix first defines the trigonometric functions using the line system, in the following order: sine, cosine, versed sine, tangent, secant, cotangent, and cosecant. Following the line definitions, he defines co-functions as the function of the complement. Lacroix goes on to derive the following identities and formulas:

- the inverse relationships of the trigonometric functions
- principle Pythagorean identity
- sum and difference formulas, with proof
- double and multiple angle formulas
- half angle formulas
- sine and cosine of 90 degrees
- the length of an arc is always longer than its sine and shorter than its tangent

Following these initial formulas, Lacroix suggests letting the radius equal one, and shows how the identities and formulas get simpler from there. Moving forward in the

text, however, he does not assume the radius is one, even though he appreciates the way it simplifies many of the formulas.

Lacroix goes on to find specific values of trigonometric functions. He calculates the sines of very small arcs, for use in computations of larger sines, and he finds exact values for sine and cosine of 30 and 60 degrees, depending on the radius. Here, Lacroix has a discussion of when the sine and cosine of larger angles are equal to those of smaller angles, so as to simplify computations. This is similar to the modern notion of reference angles. He also discusses the behaviors of the trigonometric functions as the angle moves around the circle. Often when trying to make the calculations convenient, Lacroix assumes that the circumference of the circle is π , making the diameter one. Finally, Lacroix uses the half-angle formula to find the exact value for sine of 45 degrees. The sine of 45 degrees is easily determined in other ways, so Lacroix likely demonstrates the half-angle formula on this angle to show that the formula works properly.

After discussing specific values of trigonometric functions, Lacroix shifts his focus to the solution of triangles. Here he derives the trigonometric ratios, and from the trigonometric ratios, he proves the following laws:

- law of sines
- law of cosines
- alternate law of cosines that is useful with logarithms

Lacroix uses these when he gives several examples of the solution of triangles, both right and oblique.

In terms of pedagogy, Lacroix's diagrams are given at the end of the text, and there are no exercises or problems for students to do on their own, although there are examples solved in the text.

Translations and Adaptations of French Textbooks

Although colleges in the United States used French textbooks without translation for a short time, they moved quickly into using English translations of French textbooks. Though they were called translations, many of these textbooks were adapted considerably, rather than simply translated.

The first of such textbooks is David Brewster's translation of Adrien Marie Legendre's *Trigonometry*. The USMA at West Point used this textbook from 1832 to 1839. Because the translations done during this time were liberal translations and adaptations (Cajori, 1890), it is reasonable to consider Brewster to be the author, even though the text is based on Legendre's French text. Also, Cajori (1890) and others refer to Brewster as the author, making it clear that this is the common practice.

Brewster first gives the line definitions of the trigonometric functions. He defines sine, tangent, secant, and versed sine, and then he defines the co-functions as functions of the complement.

After the definitions, Brewster discusses special angles and the general tendencies of the trigonometric functions, the signs of trigonometric functions in each of the four quadrants of the circle, and the signs of the functions of negative angles ($\sin(-x) = -\sin(x)$, for example). Brewster shows that the sine of any angle can be reduced to the sine of an angle of 90 degrees or fewer, using the symmetry of the circle. Furthermore,

the cosine of an angle can be reduced to the sine of its complement, and other trigonometric functions can be reduced to sines as well by formulas that he discusses later in the textbook. Afterward, he proves the following theorems:

- The sine of an arc is half the chord which subtends a double arc, and he uses this theorem to find that the sine of 30 degree is equal to half the radius.
- The principle Pythagorean identity, and he uses it to find the cosine of 30 degrees.
- The inverse relationships of the trigonometric functions. Brewster uses these formulas to deduce the signs of tangent, cotangent, and cosecant in the four quadrants.
- The remaining Pythagorean identities, proved algebraically from the principle Pythagorean identity.
- The sum and difference formulas for sine and cosine.
- Double and half angle formulas for sine and cosine.
- Formulas for reducing trigonometric functions into a single one.
- Law of tangents.
- Sum, difference, double, and triple angle formulas for tangent, with proof.
- Setting the radius equal to one, Brewster calls the resulting trigonometric functions the *natural* sines, cosines, etc.
- If a natural table were known, then it would be easy to find trig functions with other radii, since they will be proportional.

Brewster then explains that if the trigonometric lines themselves were used, then it would be necessary to do a lot of multiplying and dividing in calculations, so it is easier to use logarithms of sines, cosines, etc. in tables where the radius is 10^{10} and therefore its logarithm is 10. He explains the method used to calculate a table of natural sines, and he

explains a table of logarithms. He gives an explanation in detail on how to use a table of logarithmic sines to find any trig function of a “given arc or angle.”

The section on finding values of trigonometric functions is followed by a section on solving rectilinear triangles. In this section, Brewster begins by giving the following theorems, each with an accompanying proof:

- Ratio definitions of sine and cosine.
- Ratio definition of tangent.
- Law of sines.
- Law of cosines.
- Law of tangents.

Using these five basic theorems, Brewster explains how to solve different types of triangles:

- Solution of rectilinear triangles by means of logarithms. He discusses how to solve missing parts of triangles using a table of logarithms.
- Solution of right angled triangles, containing an explanation accompanied by two examples solved and one example with the answer only.
- Solution of rectilinear triangles in general, in four cases:

Case 1. Given a side and two angles of a triangle, to find the remaining parts.

Gives an explanation followed by two examples, one is solved and one has only the answer.

Case 2. Given two sides of a triangle, and an angle opposite one of them, to find the third side and the two remaining angles. Gives two examples, one solved, one gives only the answer.

Case 3. Given two sides of a triangle, with their included a two angle, to find the third side and two remaining angles. Gives an explanation, then two examples, one solved, one gives only the answer.

Case 4. Given the three sides of a triangle, to find the angles. Gives an explanation, then two examples, one solved, one gives only the answer.

Brewster follows this section on the solution of triangles with a section on applications. He gives eight exercises, first three with solutions, next five with answers only. These exercises include problems with immeasurable distances (such as the distance across a lake) and problems with finding the angle of elevation or the angle of depression. The applied exercises are the last section of Brewster's *Legendre*.

In terms of pedagogy, Brewster's textbook is similar to its American predecessors in terms of the other exercises; some exercises are solved completely, while others are given with answer only. These exercises are simple practice problems where students must reproduce a procedure or use a formula as has been demonstrated previously in the textbook. Diagrams are given frequently and within the text.

However, Brewster's *Legendre* makes one large leap forward with its addition of real-life applications using trigonometry. Previous to Brewster, the only textbook author to include applications was Hutton, and his applications were limited to angle of depression and angle of elevation problems only. The addition of applications in Brewster's textbook is significant. Not only do the applications show how trigonometry can be used in real life, but they also cause students to use theorems from all throughout the textbook, helping them to make connections within the textbook.

One example of an angle of depression problem that Brewster includes is as follows: “Wanting to know the distance between two inaccessible objects which lie in a direct line from the bottom of a tower of 120 feet in height, the angles of depression are measured and found to be, of the nearest, 57° ; of the most remote $25^\circ 30'$: required the distance between them” (p. 251). Brewster also includes other types of applications, for example the following problem: “From a station P there can be seen three objects, A, B, and C, whose distances from each other are known, viz. $AB=800$, $AC=600$, and $BC=400$ yards. There are also measured the horizontal angles, $APC=33^\circ 45'$, $BPC=22^\circ 30'$. It is required, from these data, to determine the three distances, PA, PC, and PB” (p. 251).

The inclusion of applications shows that pedagogy was becoming a greater concern for textbook authors. The next textbook discussed is also a translation of Legendre’s *Trigonometry*, but there are significant differences between the two versions.

Charles Davies translated and adapted a version of Legendre’s *Trigonometry* that was used at USMA at West Point from 1839-1881 (Shell-Gellasch, 2001), and at colleges and universities across the country during that time (Cajori, 1890). For example, Georgetown College used this textbook from 1860-1873, Columbia College used it through most of the nineteenth century, until 1889. In 1839, Dartmouth started using Legendre’s *Trigonometry* instead of Playfair’s *Euclid* and continued to do so through most of the nineteenth century. “The influence of the Military Academy at West Point was now beginning to be felt at Dartmouth” (Cajori, 1890, p. 166). Fifteen editions of Davies’ *Legendre* were printed. The last edition was printed in 1860, and colleges continued to use the textbook until the end of the nineteenth century (Guralnick, 1975). This study considered the 1858 edition, but changes between the editions are minimal.

Davies was involved in the translating and adapting of Brewster's *Legendre* and is named as a coauthor of that text. In some ways Davies' *Legendre* could be considered another version of Brewster's *Legendre*, but there are significant differences between the two textbooks. The differences will become clear through the analysis and will be discussed in the summary.

Trigonometric functions in Davies' *Legendre* are given by their line definitions. Unlike his predecessors, Davies sets the radius equal to one immediately, instead of doing so after showing many of the formulas. After giving the definitions, Davies does the following:

- Derives the principle Pythagorean identity
- Proves the inverse relationships of the trigonometric functions
- Observes the Pythagorean identity involving secant and tangent from a right triangle within the diagram he used to define the trigonometric functions
- Proves the Pythagorean identity involving cosecant and cotangent using algebra

Algebraic signs of circular functions. First Davies discusses how the signs of sine and cosine change when the angle in question is in the different quadrants. Using the formulas that relate the other trigonometric functions to sine and cosine, Davies finds the signs of the other trigonometric functions. Davies goes on to discuss how the magnitudes of sine, cosine, and the other trigonometric functions change as the angle moves around the circle. He also includes the function values of all the trigonometric functions when the angle is a multiple of 90 degrees.

Reference Angles. Following his more general discussion of the signs and magnitudes of the trigonometric functions, Davies uses formulas to reduce all the trigonometric

functions of large angles to angles in the first quadrant, so that it is only necessary to know the sine and cosine of angles between and including 0 and 90 degrees.

At this point, Davies gives several theorems:

- The sine of an arc is equal to one half the chord of twice the arc. Davies uses this to find sine of 30 degrees; sine, cosine, and tangent of 45 degrees; and cosine of 60 degrees.
- Trigonometric ratios for sine, cosine, tangent, and cotangent.

Note: When Davies' gives the trigonometric ratios as a theorem, he says the following: "The ratios of the two preceding articles are important, as all the other relations are deduced at once from them. In fact, these ratios are given, by many writers, as the definitions of the circular functions to which they are respectively equal." Although Davies does not himself use the ratio definitions, he recognizes that many of his contemporary authors do so.

- Sum and difference formulas, double angle formulas, and half angle formulas.
- The sine of the difference of two arcs is to the sines, as the sum of the sines to the sine of the sums.

Introduction of the Radius into Trigonometric Formulas. Although Davies previously set the radius equal to one for all the trigonometric formulas, here he goes back and gives all the trigonometric ratios with the radius included in the formulas, in case it is needed to calculate a trigonometric function where the radius is not equal to one. At this time, setting the radius equal to one was certainly not universal.

Davies then explains how both the natural and logarithmic trigonometric tables are created, shows how to use them, and gives several exercises for students to practice

doing so. He discusses the relations between the sides and functions of the angles of oblique angled plane triangles as follows:

- Law of sines
- Law of tangents
- Law of cosines
- “The cosine of half of either angle is equal to the square root of half the sum of the three sides, into half the sum minus the side opposite the angle, divided by the rectangle of the two adjacent sides.”
- “The sine of half of either angle is equal to the square root of half the sum of the three sides minus one of the adjacent sides, into half the sum minus the other adjacent side, divided by the rectangle of the two adjacent sides.”

Solution of right-angled triangles. Using four cases, Davies explains how to solve right triangles.

Case 1. Given the hypotenuse and one acute angle. Explains, gives one exercise solved, and one exercise with answer only.

Case 2. Given a side adjacent to the right angle and either acute angle. Explains, gives one exercise solved, and one exercise with answer only.

Case 3. Given the two sides about the right angle. Explains and gives two exercises with answers only.

Case 4. The hypotenuse and one other side being given. Explains but gives no exercises.

Solution of Oblique-Angled Triangles. Using four cases, Davies explains how to solve non-right triangles.

Case 1. Given one side and either two angles.

Case 2. Given two sides and an angle opposite one of them.

Case 3. Given two sides and their included angle.

Case 4. Given the three sides.

For each case, Davies explains how to solve a triangle with that type of information, and he gives one exercise solved and one exercise with answer only.

After explaining how to solve both right and oblique triangles, Davies' textbook contains a section that shows how to apply solving triangles to heights and distances. This section includes what today are called angle of elevation problems and immeasurable distance problems. He gives explanations first, and follows them with six exercises for students. The exercises give the answers only.

Pedagogically, Davies' textbook is very similar to Brewster's. There are exercises within each type of problem, some that show the solution, and some with answers only given. Also, there is a section on applications at the end of the textbook that applies the types of problems the textbook addresses to real-life situations. There is no need to give examples of Davies' application problems because they are quite similar to Brewster's. In Davies' textbook, all diagrams occur within the text.

Liberal Translations and Adaptations of French Textbooks

Although Brewster's, Davies' and many other authors' textbooks were called translations of French textbooks, often it was not the case that the author merely translated a French author's textbook. Most often, and certainly in the cases of Brewster and Davies, French textbooks were used as a model, but the American author would add

examples and exercises for students to complete, re-structure passages, add topics, and occasionally delete topics they found unnecessary.

Legendre says about a translation into English of one of his texts, July 3, 1832: “Your work is not merely a translation of with a commentary; I regard it as a new edition, augmented and improved, and such a one as might have come from the hands of the author himself, if he had consulted his true interest, that is, if he had been solicitously studious of being clear” (Cajori, 1890, p. 105).

Elias Loomis, connected with Yale but not professor there (Cajori, 1890), wrote a series of mathematics texts, including *Plane and Spherical Trigonometry* in 1848. These texts were not always the most mathematically accurate, but they were understandable to beginners. Cajori says about Loomis’ texts “It has been said of American writers that, while they have given up Euclid, they have modified Legendre’s *Geometry* so as to make it resemble Euclid as much as possible. This applies to Loomis with greater force, perhaps, than to any other author. His trigonometry has been wedded to the old ‘line system,’ and it is only within the last two or three years that a divorce has been secured” (Cajori, 1890, p. 156).

Although French mathematics was extremely influential during this time period, textbook authors who translated French textbooks did not merely translate the words. Instead, they used the French textbooks as a basis to write their own textbooks, each adding to the French textbook their own style and pedagogy. In many cases, their style and pedagogy resembled the British because it was the strongest influence until the French. Still, there was certainly a strong French influence during this time.

American Textbooks

There were three major textbooks written by American authors during the Antebellum period. The first was written by Ferdinand Rudolf Hassler. In 1826, he wrote *Elements of Analytic Trigonometry: Plane and Spherical*. This textbook was published by the author. Although it is called *Elements of Analytic Trigonometry* and this study only considers works of elementary trigonometry, this study will consider Hassler's text as an elementary trigonometry textbook because that is how it was meant to be used. Hassler intended the textbook to be used for beginners, not as an advanced study of analytic trigonometry.

Hassler's *Elements of Analytic Trigonometry* was the first textbook in the United States that used the ratio system to define trigonometric functions. Cajori writes the following about Professor Hassler:

“Hassler's teaching power must have been hampered somewhat by his limited acquaintance with the English language. While at West Point he began writing his ‘Elements of Analytic Trigonometry,’ published by him in 1826. It was written in French and translated for publication by Professor Renwick. . . .

Hassler was, no doubt, the first one to teach *analytic* trigonometry in this country—the first one to discard the old ‘line system.’” (1890, p. 84).

Although Hassler's textbook is considered as an American textbook because it was written and published in the United States, Hassler was Swiss. Because he had little proficiency with the English language (Cajori, 1890), Hassler originally wrote the textbook in French and had it translated before publication. In some ways it makes more sense to compare Hassler's text to French textbooks because the Western European

countries' textbooks had more in common than Hassler's textbook did with American textbooks.

Before considering the content of Hassler's textbook, it is important to look to his introduction to see what he says about his own textbook. Hassler goes to great lengths in the introduction to make sure that line segments on the circle are not taken as their definitions, especially emphasizing that the line representations are not the trigonometric functions themselves. He says:

“The names that are given to the several ratios, that exist among the sides of a right angled triangle, taken by pairs, are purely conventional, although the terms have in part been deduced from geometric considerations, having reference to the circle. But it is of the greatest importance carefully to avoid confounding the lines, that correspond to these ratios, or trigonometrical functions, when represented in a circle, with these ratios themselves.” (p. 6)

By saying this, he trying to divorce the trigonometric functions from the line segments that previously defined them. Hassler takes great care to emphasize that it is ratios that define the trigonometric functions, even though there are lines that correspond to the functions, they should not be confused with the functions themselves.

Hassler also comments on the reason for teaching trigonometry in this way. He says that “it was the desire of introducing into the course of mathematics at the United States' military academy at West-point [*sic*], the most useful mode of instruction in this branch, that led me to the preparation of this work” (p. 6). Hassler goes on to explain exactly why he believes this to be the “most useful mode of instruction,” and it is mainly a desire to present the whole of trigonometry in the easiest possible way. He says that

“this mode of proceeding appears to lead to the desired aim with the least labour of intellect, and thus is the most easy way to the final end; which is, to present to the reader a full system of this branch of mathematics in such a way as to furnish every necessary element for the solutions of trigonometry, both plane and spherical.” (p. 7)

After the introduction is completed, Hassler included comments within the text to reinforce the ideas he presented in the introduction. When defining the trigonometric functions as ratios, he says “these several functions are known by names, whose origin and signification are of no importance; but it is the more important, that we fully and precisely understand their value and mutual relations” (p. 14). Here again, Hassler makes it clear that he has no use for the line system. He embraces the ratio system wholeheartedly.

He says it is “the combination of these ratios give the whole of that multitude of trigonometric functions, that enable us to solve every question in trigonometry, and which are perpetually applied in analysis” (p. 14). Hassler argues that there is no need for the line system because all of trigonometry can be solved and every question answered using the ratio system alone, and it can be done more easily using the ratio system. For these reasons, he rejects the line system and uses the ratio system completely throughout his textbook.

The first part of Hassler’s *Elements of Analytic Trigonometry* is entitled “Analysis of trigonometric functions.” The definitions of the trigonometric functions are given completely in ratios. Hassler defines sine, cosine, tangent, secant, cosecant, and cotangent. He mentions versed sine and co-versed sine only to say that they are useless and will not be defined here. Hassler then shows the following:

- $\tan x = \sin x / \cos x$, etc. proved using the ratio definitions
- the multiplicative identities
- the Pythagorean identities, derived using the Pythagorean theorem
- various algebraic manipulations of the Pythagorean identities
- extreme values of the trigonometric functions
- the solution to right angled plane triangles

At this point, Hassler introduces the unit circle and uses it to determine the signs of the trigonometric functions in the four quadrants, and then he proves the following:

- sum and difference formulas
- half-angle formulas
- general formulas for multiples of angles
- formulas for compound angles where one angle is determinate
- formulas that will be helpful for calculus

After these formulas, Hassler provides some explanation as to the construction of trigonometric tables.

The next part of Hassler's *Elements of Analytic Trigonometry* is plane trigonometry. In this part, Hassler demonstrates how to solve all cases of oblique plane triangles using the formulas derived in Part 1 and how to calculate areas of plane triangles given different types of information.

In terms of pedagogy, Hassler does not give any examples, exercises, or problems. His explanations are purely theoretical, and no numerical values are given in the textbook to illustrate how to solve a particular type of problem. Diagrams are given frequently within the text.

The second American textbook from the Antebellum period was written by Benjamin Peirce, who was a professor of mathematics at Harvard University. Peirce's *Plane and Spherical Trigonometry* was first published in 1838, and a new edition was published in 1852. It was one in a series of mathematics textbooks that Peirce wrote.

Plane and Spherical Trigonometry was used at Harvard beginning when it was published in 1838. "There was a lot of discussion and debate about whether Peirce's textbooks should be used because they were said to be too difficult for beginners to understand. This matter was studied by a committee in 1848. However, Peirce's textbooks continued to be used for quite a while" (Cajori, 1890, p. 141). Harvard used Peirce's *Trigonometry* through 1870.¹² Following that, elementary plane trigonometry became a requirement for admission.¹³

The following discussion is of an edition of Peirce's *Plane and Spherical Trigonometry* that was published in 1852, although the changes between the first edition and this edition were minimal.

Peirce begins with the goal of trigonometry—that is, the solution of triangles. He defines sine, tangent, and secant followed by the co-functions. Peirce uses the "ratio system" to define the trigonometric functions. Using algebra, Peirce derives the following:

- inverse relationships of the trigonometric functions
- $\tan x = \sin x / \cos x$, etc. proved using the ratio definitions
- Pythagorean identities, proven using algebra

¹² Harvard University Archives, Online Collection. The Harvard University Catalog, 1852-1854 through 1866-1870.

¹³ Harvard University Archives, Online Collection. The Harvard University Catalog, 1872-1873

There are then three exercises, one with the solution and two with answers only.

At this time, Peirce proves as a theorem the line representations of sine, cosine, tangent, and secant, showing their locations on a unit circle. About this he says:

“The preceding theorems have been adopted by most writers upon trigonometry, as the definitions of sine, cosine, tangent, and secant, except that the radius of the circle has not been limited to unity. By not limiting the radius to unity, the sines &c. have not been fixed values, but have varied with the length of the radius; whereas their values, in the system here adopted, are the fixed ratios of their values as ordinarily given to the radius of the circle in which they are measured. Thus, if R is the radius, we have $\sin, \cos, \&c$ in the common system = $R \cdot \sin, \cos, \&c$. in this system.”

Peirce does not discuss his reasons for defining the trigonometric function using the ratio system, although he does acknowledge that he is unique at this time for doing so. He also does not explain why he limits the radius to unity. His explanations seem to be given only so that it is clear to the reader that there is no discrepancy between his trigonometric functions and those of other authors.

At this time, Peirce finds the sine of very small angles, followed by three exercises, one with the solution and two with answers only. Peirce discusses solving right triangles given different information. For this section, he includes three exercises with solutions and five more with answers only. Peirce then gives the following general formulas:

- addition and subtraction formulas
- double angle formulas
- trig functions of special angles (180, 270, 360, 30, and 45 degrees)
- the trig functions of the supplements, $90 + \text{an angle}$
- the trigonometric functions of the negative angles

- theorems concerning how each trigonometric function changes as the angle increases or decreases.

Next, there is a chapter on solving oblique triangles. The format of this chapter is that Peirce proves a theorem that allows the solution of a triangle with certain information given. Next, he shows how to solve a triangle where that theorem is applicable. Finally, he gives exercises between two and six exercises, one with the solution shown, and the rest with answers only. He follows this format for the following theorems and types of triangles:

- law of sines
- solve a triangle when two angles and one side are given
- solving a triangle with two sides and one angle given, in cases
- law of tangents
- solve a triangle when two sides and the included angle are given
- law of cosines
- solve a triangle when three sides are given

Because Peirce has then achieved his goal of solving all plane triangles, this concludes his discussion of Plane Trigonometry.

Pedagogically, Peirce has more tools for students than Hassler. He gives many exercises, some with the solution shown and some with answers only. However, he only includes simple practice problems for students. Unlike Brewster and Legendre, Peirce does not include applications. In the same textbook on *Plane and Spherical Trigonometry*, Peirce does include a treatise on Surveying and Navigation that does apply many of the theorems of plane trigonometry. However, these applications are within the

realm of a treatise on surveying and navigation and do not come within the treatise on plane trigonometry. For the sake of consistency, they will not be considered here.

Diagrams in Peirce's textbook are given frequently within the text.

Elias Loomis wrote the third important antebellum American trigonometry textbook in 1848, *Elements of Plane and Spherical Trigonometry*. 60,000 copies of it were printed before Loomis came out with a significantly revised edition (Loomis, 1886). The revised edition will be discussed in the following chapter of this dissertation. The text discussed here is the twenty-fifth edition, printed in 1873, but the changes before the 1886 edition were minimal (Loomis, 1886).

Loomis begins by defining trigonometry as “the science which teaches how to determine the several parts of a triangle from having certain parts given” (p. 20). Loomis defines the following:

- Degree measure of angles
- Complement of an angle
- Supplement of an angle
- Sine
- Versed sine
- Tangent
- Secant
- Cosine
- Cotangent
- Cosecant

The definitions of the trigonometric functions are all given as lines segments on a circle. Following the definitions of the trigonometric functions, Loomis proves the inverse relationships of the trigonometric functions using similar triangles on the circle. He shows a trigonometric table and explains how to use it along with a logarithmic trigonometric table and how to use that. He gives several examples and shows how to solve them.

After the discussion of trigonometric tables, Loomis has a section on solving right triangles. He discusses letting a convenient leg of the triangle equal the radius of the circle, which allows the unknown side to be equal to one of the known trigonometric functions. He explains how to solve right triangles in four cases, depending on what information is given. For each case, he gives one example with the solution and another with answer only. After the four cases are all explained, Loomis gives six examples for practice, where the student has to determine which of the four cases applies to the exercise. At the end of the section, there is an explanation of how to find the third side of a right triangle, using manipulations of the Pythagorean theorem.

After explaining how to solve right-angled triangles, Loomis has a section on solving oblique-angled triangles. He begins this section by giving the following theorems, each with proof:

- Law of sines
- Law of tangents
- Law of proportionality

In a similar format to the previous section, Loomis explains how to solve oblique-angled triangles in three cases. Each case has one example solved, one with answer only, and at the end of the section, there are six exercises for practice, with no answers.

After explanations of how to solve triangles, Loomis has a section on tools used for drawing trigonometric constructions. Using a few simple tools, students then complete exercises in trigonometric construction. This is similar to the modern practice of geometric constructions, except that there are a few extra tools.

Loomis goes on to discuss how the trigonometric functions change as the associated arc changes, including discussing the signs of the trigonometric functions in the different quadrants and the functions of negative angles.

The discussion of how the trigonometric functions change is followed by a discussion of the following trigonometric formulas, each with proof:

- Sine and cosine of sums and differences
- Sine and cosine of double arc
- Sine and cosine of half arc
- Expressions for the products of sines and cosines
- How to construct a trigonometric table beginning with how to find $\sin(1')$,

Pedagogically, Loomis' textbook has all the tools that were common at this time. Diagrams are given frequently within the text. Theorems and important definitions are italicized to highlight their importance. Loomis gives many exercises, some solved in detail as an example, some with answers only given, and some at the end of a section with no answers given. These ending exercises are particularly challenging because

students have to decide which laws or methods to use in order to solve them, since they are not paired directly with the topic to which they relate.

Loomis does not give applications like Brewster and Legendre, but he does give sections of exercises at the end of a chapter. For example, after showing how to solve oblique triangles by cases, with exercises following each case, Loomis has another section of exercises where students are not told which case to use. This section of exercises demonstrates a particular attention to students' learning because it requires students to consider how best to solve the problem, rather than simply follow a pattern that has been demonstrated by the textbook author.

In terms of content, Loomis' textbook is less like his contemporaries and more like his predecessors. Not only does he define the trigonometric functions as lines on the circle, but he also does not give the trigonometric ratios at all, and he does not let the radius equal one to simplify calculations.

New Pedagogy Takes Shape

In the previous chapter, a clear progression of pedagogical techniques emerged that authors used to help students to get the most out of their textbooks. In the antebellum period, there are some textbooks that continue this pedagogical progression, while others do not contain the same types of pedagogical tools. During the antebellum period, applications make their first appearance, with the exception of Hutton's text, which contained minimal applications and was written before the antebellum period. Previously, the exercises found in trigonometry textbooks simply asked students to solve a triangle, but during the antebellum period, many exercises include real-life applications.

First, Lacroix's French textbook first used at USMA at West Point does not have any exercises for students. It is interesting to note that although Hassler's *Analytic Trigonometry* was written and published in the United States, he was Swiss and had difficulty with the English language. For that reason, he wrote his textbook in French and it was translated into English (Cajori, 1890). Like Lacroix, Hassler's textbook contains no exercises for students to complete. Perhaps there is a cultural difference in that regard between Continental European and American (and British) authors.

The following is a discussion on overt pedagogical techniques included in textbooks. Because they do not contain any overt pedagogical techniques, it is reasonable to exclude both of the textbooks written by French authors from this discussion. Without the French authors, pedagogical techniques become easier to examine. Peirce's 1838 textbook has the fewest types of exercises. He simply has some exercises explained fully and some exercises with answer only. Loomis' 1848 textbook has a significant improvement over Peirce's with regard to exercises. Loomis has the same types of exercises that Peirce does, but then at the end of a section, he gives six exercises with no answers given, where the exercises can pertain to any part of the preceding section.

Brewster's 1832 textbook and Davies' 1838 textbook not only have some exercises explained fully and some with answer only, but they also have real-life applications. These are given at the end of the discussion of plane trigonometry and they deal with angles of elevation and depression, surveying, and immeasurable distances (such as measuring the distance across a body of water). Pedagogically, this is similar to Loomis' technique that gives several general problems at the end of a section for students to practice. Applications, however, give the added benefit that they require interpretation

and they require students to come up with a way to use the trigonometry they have learned to solve the problem where it might not be clear how to do so.

Although the pedagogical advances during the antebellum period did not happen chronologically, they may have happened with respect to location. First, Lacroix and Hassler do not use exercises for students, and perhaps they are influenced in that respect by their European origin. Also, it is peculiar that Davies' and Brewster's textbooks both have applications while the others do not, but they were both professors at USMA at West Point, and they wrote their textbooks for use at West Point (Shell-Gellasch, 2001). Notably, Hutton, whose textbook was discussed in the previous chapter, also came from West Point and included a section of applications in his textbook, entirely angle of elevation and angle of depression problems.

Finally, Peirce's and Hassler's textbooks have pedagogy that seems typical of American textbooks when compared with Webber's and Day's textbooks that were discussed in the previous chapter.

Changes in Structure and Content that Reflect Changes in Mathematical Thinking

Two important changes in structure and content occur during the antebellum period. The first idea that grows in importance is letting the radius equal one (or occasionally another convenient value). Before the antebellum period, one author this study examined allowed the radius to equal one—Hutton. Allowing the radius to equal one simplifies formulas and calculations, but it can make trigonometric functions more confusing because although the radius is still a part of the formulas, it is no longer seen, since one is the multiplicative identity.

During the antebellum period, almost all of the authors allow the radius to equal one. Most notably, Davies allows the radius to equal one from the beginning, although he shows how to incorporate other values for the radius later, and Peirce limit's the radius to one for his entire text. On the contrary, Loomis does not allow the radius to equal one at all.

The change in allowing the radius to equal one is a change in content because more textbooks are including this innovation, and it is also a change in structure because in the textbooks that include it, the change is happening earlier and earlier. In the antebellum period, authors are giving the trigonometric formulas with the radius equal to one initially, rather than showing how allowing the radius to equal one simplifies them later.

The trigonometric ratios also represent a change both in content and structure. Previous to the antebellum period, the formulas similar to the trigonometric ratios appeared, but the ratios themselves did not appear in their finished form. In the antebellum period, almost all textbooks include the trigonometric ratios (although Loomis' textbook does not), and they gain quite a bit of prominence in some textbooks.

Whereas earlier the trigonometric ratios if included were somewhere near the end of the book, during the antebellum period, they gain so much importance in some authors' minds that they become the definitions of the trigonometric functions themselves. Those authors leave behind the notion that trigonometric functions are lines on a circle and move forward by using the trigonometric ratios to define the functions.

Debate on the Ratio System versus the Line System

The movement of trigonometry from the line system to the ratio system was not simply contained to textbooks. The teachers of trigonometry in colleges debated this issue as well. There remains the evidence of one such debate in two periodicals. First, in 1827 in the *American Quarterly Review*¹⁴, an unknown author reviewed *Elements of Analytic Trigonometry, Plane and Spherical* by Ferdinand Rudolph Hassler, which was published in 1826. Hassler, though not born in the United States, was a professor at USMA at West Point. His *Elements of Trigonometry* was the first textbook published in the United States to give the ratio definitions for the trigonometric functions, leaving out the line representations all together.

The reviewer gives Hassler the highest praise for his accomplishment. He refers to the ratio system as the “analytic method.” and the line system as the “geometric method.” He compares Hassler’s textbook to Robert Simson’s (although the reviewer misspells his name “Simpson“). Simson’s textbook was covered in detail in Chapter 4. The reviewer states that Simson’s textbook was the most commonly used for teaching trigonometry in colleges.

The reviewer finds the analytic method much easier for students and does not think the geometric method has much value at all. He says, “the usual method employed in the colleges of this country, the geometric, may be considered as either inadequate, or, when made adequate, much too laborious to the student“ (*American Quarterly Review*, 1827, p. 38).

¹⁴ *American Quarterly Review*, march 3, 1827, 1, p. 38-54.

The reviewer goes on to criticize Simson's textbook for a number of reasons, mostly because it limits the student to a fairly basic study of the subject, the Radius appears in many of the equations and makes calculations more laborious, and the language is obsolete (*American Quarterly Review*, 1827). His only praise for Simpson is that he "for clearness, precision, and close reasoning, may fairly rank with the works of the ancient mathematicians" (*American Quarterly Review*, 1827, p. 38).

Putting an end to the discussion of the geometric system versus the analytic system, the reviewer says, "It boots not to dispute whether the geometric or analytic method in trigonometry be the better; the former is no longer in use in any modern writer of reputation, and the student whose opportunities of learning have been limited to it, will have his whole labour to renew, before he can read a page of any eminent author of the present day" (*American Quarterly Review*, 1827, p. 49). This statement does not seem to be accurate if he is speaking about authors in the United States, because Hassler's *Elements of Trigonometry* was the first in the United States to do away with the line system (Cajori, 1890).

He goes on to describe the transformation from the geometric method to the analytic method. Many of his observations are in accordance with the discussion in the previous section of this chapter. He says the following:

"The elementary consideration of trigonometry in general, by the analytic method, was slowly introduced. We are informed, but have not the means of ascertaining the truth, that the Germans led the way in this respect; and in the more modern French courses of mathematics, it has, in every successive edition, been elevated to a higher degree of importance, until it has entirely superseded the ancient method. Among these works are to be named, with greatest praise, those of Legendre, Lacroix, and Biot. . . . By the joint labours of so many eminent men, the analytic method in trigonometry has at last assumed so high a rank, as to render it the only proper mode of studying the subject; and in their voluminous works,

everything that is necessary to be known, may be successfully sought.” (p. 49-50)

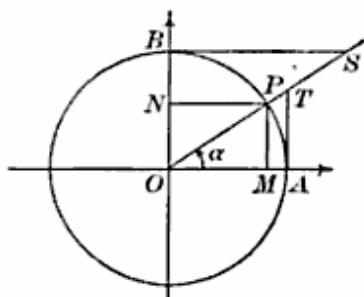
The reviewer sees the past works of trigonometry as incremental steps toward a goal, and Hassler’s *Elements of Trigonometry* is the final completion of that goal. He writes, “The time has therefore arrived, when it is possible to recast the whole subject, and reduce it to one consistent and uniform system. This is the object of the work before us [Hassler’s] and the author has been in no small degree successful in its accomplishment” (p. 50).

The reviewer makes an interesting claim that is, in fact, mathematically incorrect.

The claim is as follows, with a couple of the sentences that proceed it, to put it in the context of the review:

“In every other elementary treatise with which we are acquainted, the trigonometric functions have been considered as lines absolutely existing, in some given relation to an arc of a circle of a definite radius. In all the elementary investigations, whether geometric or analytic, this radius bears a most important part; yet, no sooner do we make a direct step into the farther analysis, than it disappears, and the functions are represented in relations to each other, that lines can never assume. Thus, for instance, if we were to seek the value of the sine of an arc, in terms of the cosine and the tangent, it would appear to be equal to their product, which, geometrically considered, would be impossible, as it is tantamount to declaring a line to be equal to a rectangle. Euler evidently conceived the angular functions to be the expressions of the ratio between lines, and not as lines themselves, and in this way they are now employed in all calculations. To render the theory consistent with the practice, it is necessary that their properties should be investigated upon a principle derived from this view of their nature. This has been at last done by Mr. Hassler.” (p. 51-52)

The reviewer here claims that the sine of an angle is equal to the product of its cosine and its tangent, which when considering the geometric system makes a line equal to a rectangle. This does seem to be nonsense, as the reviewer says. However, by returning to the line definitions of the trigonometric functions, this statement can be explained (see Figure 5.1).



$$PM = \text{sine}$$

$$NP = \text{cosine}$$

$$TA = \text{tangent}$$

$$RS = \text{cotangent}$$

$$OT = \text{secant}$$

$$OS = \text{cosecant}$$

Figure 5.1

Notice that if we add to this illustration a line segment perpendicular to \overline{TA} from T to the line \overline{RO} , at point B, as shown below, then there is a rectangle $\square OATB$. Rectangle $\square OATB$ is similar to the rectangle $\square OMPN$ because they share a diagonal, \overline{OS} . Note that TB is equal to the radius. By the similarity of these rectangles, we get $m\overline{TA}/m\overline{TB} = m\overline{PM}/m\overline{NP}$. By replacing each of these segments with their trigonometric equivalents, this becomes $\frac{\tan(\alpha)}{\text{radius}} = \frac{\sin(\alpha)}{\cos(\alpha)}$. By cross-multiplying, sine times the radius is equal to cosine times the tangent. The reviewer makes one major error. He does not remember that the radius was originally part of the formula he uses. He lets the radius

equal one and then disregards it. Now going back, he forgets that there was a radius and that he let it equal one.

It is ironic that he goes on to use his own error to praise the analytic system over the geometric system because it is the analytic system itself that has allowed him to make this error. Had he been using the geometric system, this formula would have been deduced from the trigonometric functions on the circle, and he would not have been able to make this mistake. It is only because the analytic system proves its formulas are true using algebra that the reviewer could possibly forget about the radius. Even more ironic, earlier in his review, he complains about Simson's textbook because Simson does not let the radius equal one. The reviewer does not like all the cumbersome terms in his formulas, and he much prefers Hassler's formulas, which are free from the radius terms because Hassler lets the radius equal one immediately. Following his discussion of this erroneous reason for preferring the analytic method, the reviewer argues against some of the common criticism of the analytic method.

Many people argue that the best method for teaching a subject is the method by which it was discovered. In the case of trigonometry, that would be the geometric method. About this, the reviewer says that "although perhaps most proper for discovery, [the geometric method is] neither homogenous, nor suited to elementary instruction' (p. 50). He argues that the analytic method has applied trigonometry to every possible situation, and does so in a way that is easy for beginning students to be able to use trigonometry for all its various uses.

The reviewer also praises Hassler's *Elements of Trigonometry* because it is accessible even to students who have no background knowledge of trigonometry. "...Mr.

Hassler has built a complete system of the elements of Plane Trigonometry, and of what is frequently called the Arithmetic of Sines; which system requires no further preliminary knowledge in the learner, than the first forty-seven propositions of Euclid, the four rules of arithmetic, the fundamental principle of proportion, and the solution of simple equations. We question of there be any work in modern mathematics, so simple in its basis, so clear and easy in its steps, and so full and complete in its deductions.” (p. 52)

In addition to its clarity, the reviewer believes that Hassler’s *Elements of Trigonometry* a national honor. He calls Hassler’s “a work that will afford to foreign nations a high idea of the status of knowledge in our country; and which, as the production of an adopted citizen, who, although educated in his native land, first applied him to mathematical science, as a profession, in our country, and drew it up originally for the use of an institution supported at the public expense [West Point] is unquestionably national.” (p. 53) At this time, both France and Germany favored the analytic method over the geometric method, and Britain was not far behind (Cajori, 1890). The reviewer seems to be concerned with keeping up with European countries in this way.

To conclude, the reviewer says the following: “We cannot too strongly recommend the introduction of this treatise, as a text book, into the colleges and universities of the United States. We have expressed our opinion of the geometric method, and believe it must be abandoned; a step to which has already been made in the translation of Lacroix for Harvard University. But the work before us is far simpler in its basis than that of Lacroix; more elementary and direct in its attainment of the parts applicable to ordinary calculations; and far more extensive in its views and objects.” (p. 54) Although the reviewer has one argument that does not hold up under scrutiny, he has

many reasons for highly recommending Hassler's textbook and in general for the adoption of the analytic method for teaching trigonometry. It is more straightforward, it can be applied to more topics more easily, and it is easier to understand.

This point of view, however, was not shared by all teachers of trigonometry in colleges. In particular, James Wallace argued against this reviewer. Wallace was the chair of the mathematics department at the University of South Carolina from 1820 to 1834. Previous to this he was professor of mathematics at Georgetown College. Cajori says that Wallace was a skilled and capable mathematician as well as a competent and patient teacher who loved the art of teaching (1890).

Wallace wrote in the *Southern Review* (1827) an article entitled "Geometry and Calculus,"¹⁵ in which he criticizes Hassler's *Elements of Trigonometry*, as well as the unknown reviewer who gave Hassler such praise. Wallace compares abandoning the geometric method in favor of the analytic method alone to building a house beginning with the roof. He says, "Should [the geometric] method, however, be abandoned, we have but little doubt, that we should soon become like Swift's ingenious architect at Laputa, who had contrived a new method of building houses, by beginning at the roof, and working downwards to the foundation, which he justified by the prudent practice of the bee and the spider, as others might, by the example of this well meaning critic and his author [the reviewer and Hassler]" (p. 112).

Wallace disagrees with the anonymous reviewer, who says that that the geometric method is no longer being used, saying, "Yet we challenge this critic to point out one individual writer, whatever may be his reputation, that *has* advanced, or can advance one

¹⁵ *American Quarterly Review*, March 3, 1827, 1, pp. 38-54.

step, legitimately, without it. It is the foundation, and without this foundation, the superstructure cannot exist” (p. 110).

Wallace comments that the reviewer says Hassler’s work “will afford to foreign nations, a high idea of the state of knowledge in our country.” because at this time England, France, and Italy, among other countries, were ahead of the United States in terms of giving up the line system (p. 112).

Wallace criticizes Hassler deeply for his general misunderstanding of geometry, trigonometry, and calculus, and then returns to criticizing the reviewer. At this point Wallace scrutinizes many of the references the reviewer makes:

“We should, however, probably have spared ourselves this labour, did we not find, that he had an advocate in Woodhouse, (Preface to his Trigonometry) whose opinion should not pass unnoticed. That he had, at present, the Edinburgh Review on his side (see v. xxxi, p. 377, Review of Woodhouse’s Astronomy) and that he cites the adoption of Lacroix (we suppose he means his Trigonometry) in this country, as having a tendency to banish the *geometric method*. In page 49, of the Review, he observes, that “the usual method observed in the colleges of this country, the *geometric*, may be considered as inadequate to the purpose, or when made adequate, too laborious to the student.” Again, (p. 54) “we have expressed *our opinion* of the geometric method, and *believe it must be abandoned*: a step to which, has already been made in the translation of Lacroix, for Harvard University.” (p. 114)

Wallace then shows that these references do not, in truth, support the Reviewer’s opinion.

First he tells about Lacroix’s *Trigonometry*, which begins with the geometry that is the foundation of the trigonometric functions. Lacroix also uses the radius in most of his formulas, which the Reviewer does not appreciate. Legendre’s *Trigonometry* is similar,

Wallace claims. Wallace then says, the following:

“Woodhouse does not go to the length, however, that the Reviewer does, in conscientiously believing, that *geometry* must be abandoned, although his idea of the *geometric method* appears to be the same. In his preface, speaking of trigonometry as now extended, with its collateral uses, he

remarks that to the knowledge of many of these, the geometrical method is unable to conduct us. *At some point or other of our inquiries, it must be abandoned*, and recourse be had to that, which, technically, is called the *analytic method*." Here we find, that it is only at some indefinite point, that the geometric method must be abandoned. And, if by the analytic method, he means only the application of the symbols of algebra, or the calculus, in further developing the results of geometrical investigation, he is perfectly right. By adopting the *algorithm* of the calculus, we can push out inquiries much further, and condense into a focus, as it were, the results of extensive researches. This is, undoubtedly, an advantage; but this method is not, and cannot be, essentially different from the geometrical method." (p. 117-118)

Wallace criticizes the Reviewer for misunderstanding the nuances of the shift from a geometric system to an analytic one. The Reviewer, according to Wallace, wants to abandon the geometric method all together, making trigonometry a purely analytic subject, and he claims to have many authors of textbooks on his side. Wallace digs deeper, showing that the very same authors do not intend to abandon the geometric method, but rather to use the analytic method to extend that which the geometric method alone cannot accomplish.

Wallace claims that both the Reviewer and Hassler have it wrong—although geometry can be helped by analysis, the geometry itself cannot be abandoned. "Now the *calculus* has effected this [improvement in language] in various parts of geometry; yet it is geometry still" (p. 122). The same, Wallace says, is true for trigonometry. While analytic trigonometry can go further than geometric trigonometry alone, it is not a substitute for the geometric trigonometry.

On the Field of Analytic Trigonometry

Although Wallace makes many arguments in favor of keeping the line system and ultimately he answers Hassler's anonymous reviewer very adequately, one point that

Hassler's reviewer makes has a great deal of historical legitimacy. He says, "The elementary consideration of trigonometry in general, by the analytic method, was slowly introduced. We are informed, but have not the means of ascertaining the truth, that the Germans led the way in this respect; and in the more modern French courses of mathematics, it has, in every successive edition, been elevated to a higher degree of importance, until it has entirely superseded the ancient method" (p. 49).

From the end of the sixteenth century to the eighteenth century, trigonometry moved from an entirely geometric field to an analytic field, where algebraic methods were employed. Although historians do not agree as to precisely who to credit for the shift, key players were Francois Viète, Rene Descartes, Pierre de Fermat, John Wallis, Isaac Newton, Thomas Fantel de Lagny, Jakob Bernoulli, Jakob Kresa, Freidrich Christian Mayer, Roger Cotes, Abraham DeMoivre, Georg Simon Klugel, Freiderich Willhelm Opper, and above all Leonhard Euler. It was Euler's *Introductio in Analysis Infinitorum* that made the shift to analytic trigonometry complete (Maor, 1998; Van Brummelen, 2009, Smith, "Notes on the history of trigonometry").

Out of necessity, when trigonometry became analytic and involved complex numbers, the trigonometric functions were thought of completely apart from their line representations and the circles on which they originated. Mathematically, then, the transformation of trigonometry was completed with the publication of Euler's *Introductio* in 1748 (Maor, 1998). "It is here that trigonometry comes into its own as a distinct branch of mathematics. Here is created and perfected the formal language of the science" (Smith, "Notes on the history of trigonometry").

Other important accomplishments that later affected the teaching of trigonometry were from Freiderich Willhelm Opper (c. 1746) who used algebra proved all the theorems of plane and spherical trigonometry from a few simple geometric theorems and Georg Simon Klugel who first defined the trigonometric functions as ratios (Smith, “Notes on the history of trigonometry”).

It is important to note that the developments in the teaching of trigonometry as a result of the contributions of Euler and others experienced a considerable lag in time. There were fifty-three years between the publication of Euler’s *Introductio* and the first instance of the ratio system in an elementary trigonometry textbook (Hutton, 1801). The lag in time is expected and necessary because after Euler published his *Introductio*, first it became popular, it was translated into other languages, and then it perhaps influenced those who were students at the time to write new trigonometry textbooks much later in their careers.

The first evidence that the works of Euler and others were affecting the teaching of elementary trigonometry is that trigonometry began to be considered its own field. Trigonometry first came “into its own” as a result of Euler’s *Introductio* (Smith, “Notes on the history of trigonometry”). Around the turn of the nineteenth century, trigonometry textbooks used in American colleges separated trigonometry as its own course (i.e., Playfair, 1795), and in the early part of the nineteenth century, textbooks devoted to trigonometry alone were published (i.e., Brewster, 1831).

Analytic trigonometry required the trigonometric ratios. At first, elementary trigonometry education stayed much the same, and analytic trigonometry was a separate topic, separated from elementary trigonometry both because they were each contained in

different textbooks and because typical courses of study did separated the two fields by years. Most colleges had elementary trigonometry as a standard course for first-semester sophomores and analytic trigonometry as a standard course for seniors or some for second-semester juniors. Over time, however, textbook authors began including the trigonometric ratios in works on elementary trigonometry (i.e., Davies, 1838), and some textbook authors even used the ratios to define the trigonometric functions (i.e., Hassler, 1826; Peirce, 1838).

Within trigonometry education, the idea of defining the trigonometric functions as ratios was not very popular when the first textbooks did so, as Wallace demonstrates in his review. As time went on, however, the ratio system for defining the trigonometric functions became ubiquitous in elementary trigonometry textbooks. This change was no doubt aided by Oppel who used algebra to prove all the theorems of trigonometry and Klugel who first defined the trigonometric functions as ratios. Klugel introduced the idea that trigonometric functions could be defined as ratios (rather than deducing the ratio system for right triangles after defining the trigonometric functions using the line system). Then, by proving all the trigonometric formulas and theorems using algebra, Oppel showed that it was possible to forego the trigonometric lines altogether, an option which some textbook authors took (i.e., Hassler, 1826).

The developments in the field of analytic trigonometry were also the cause of textbook authors allowing the radius to equal one to simplify trigonometric formulas. When trigonometric functions are considered to be geometric entities, it makes no sense to let the radius equal one. However, for the purpose of analytic trigonometry, it is extremely convenient to allow the radius to equal one. During middle and late nineteenth

century, most textbook authors let the radius equal one sometimes but not always, although some never did and some always did. The mathematical accomplishments of Euler and others took time to affect the teaching of elementary trigonometry, but the changes they did effect were endless.

The most important international influence during this time was French. France aided the United States greatly during the time of the American revolutionary war and following, and much of the influence it bought France was educational (Thwing, 1906). At first, French textbooks were used and then translations of French textbooks began to be used. Both the French textbooks and translations of French textbooks differed from British and British-influenced textbooks in two important ways. First, French textbooks placed a greater emphasis on algebra. In French textbooks and French translations, even though the line system defines the trigonometric functions, the ratio system is introduced early on and is used to prove some of the theorems. Second, French textbooks themselves did not include exercises for students, but the translations of French textbooks did.

Cajori (1890) says that translators altered the French textbooks so much that they resembled more closely English textbooks than the French textbooks that they translated. His was a general statement that is somewhat true when considering trigonometry textbooks. In the case of trigonometry textbooks, the French translations resembled the French textbooks more in content and approach, but they resembled British textbooks more in pedagogy.

Summary

In antebellum America, trigonometric functions were not consistently defined or understood. There was considerable discrepancy from one author to another as to how the trigonometric functions ought to be defined and understood. Some, like Loomis and perhaps Wallace, wanted to keep the trigonometric ratios away from elementary trigonometry and strictly relegated to analytic trigonometry. Although the ratios can extend trigonometry further, they cannot take over where the line definitions are already sufficient.

Others, like Peirce, Hassler, and his Reviewer, saw the ratios as the new identities of the trigonometric functions. They either included the line system as merely line representations or did not include them at all. They saw that the ratios allowed trigonometry to work just as well as the lines did, and the ratios allowed trigonometry to extend further. In that case, there is no need for the line definitions of the trigonometric functions. The antebellum period saw a great shift beginning to take place. Some authors went with it wholeheartedly while others resisted it.

The antebellum period also saw the emergence of different pedagogical schools of thought. There is a clear pedagogy coming from USMA at West Point that encouraged the use of real-life applications within trigonometry textbooks. It is also clear that, at least compared to continental Europeans, Americans were more concerned with giving students exercises and examples within their textbooks. Although continental European mathematics was clearly influential in antebellum America, textbook authors in America inserted their own pedagogical techniques, which existed in American textbooks prior to the French influence.

CHAPTER VI:

Turning Trigonometry on its Head

This chapter focuses on trigonometry education during the latter part of the nineteenth century, from 1860 to 1900. Previous to 1860, most trigonometry textbooks in the United States defined the trigonometric functions using the line system. By the last decade of the nineteenth century, all textbooks published and nearly all textbooks in use defined the trigonometric functions using the ratio system. By the time Cajori wrote his *The Teaching and History of Mathematics* (1890), the line system had all but disappeared from textbooks. Of course, some colleges and universities continued using older textbooks that defined trigonometric functions using the line system, but by and large, trigonometry education had changed.

By this time, clear influences from other countries faded away as did translations of others' textbooks. Now, textbook authors wrote their own textbooks, although many authors mention outside influences. During the late nineteenth century, many textbooks from the antebellum era were still in use, especially Davies' *Legendre* and Loomis' *Trigonometry* (Cajori, 1890). Since the previous chapter discusses those textbooks thoroughly, there is no further discussion here.

A Note on Terminology

The terminology used during the nineteenth century in trigonometry textbooks was very different from that which is used today. So that it can be most comprehensible today, this study uses modern terminology. In addition to the terminology defined in Chapters 4 and 5, the following terminology is used:

1. The *sexagesimal system* is the traditional British and American degree system. The circle is divided into 360 equal parts, and the angles (or arcs) in each of these parts are called degrees. One-sixtieth of a degree is called a minute, and one-sixtieth of a minute is called a second.
2. The *centesimal system* is similar to the degree system and was popular in continental Europe for a short time. In this system, one right angle is equal to 100 grades, one grade equals 100 minutes, and one minute equals 100 seconds.
3. The *radian system* measures an angle or an arc by comparing the length of the arc to the length of the radius. An arc (or an angle whose corresponding arc is) of equal length to the radius is one radian. A full turn around the circle is 2π radians, and so on.
4. *Trigonometric equations* are equations involving trigonometric functions, where the unknown variable is usually an angle. Trigonometric equations must usually be solved in general. For example the equation $\sin(x) = 1$ has the following solution: $x = 90^\circ + 360^\circ k$, where k is an integer.
5. Two additional trigonometric functions are introduced in one of the textbooks considered in this chapter, *suversed sine* and *sucovered sine*. Like versed sine and covered sine, these trigonometric functions are no longer considered.

$$\text{suversed sine} = 1 + \cos(x)$$

$$\text{sucovered sine} = 1 + \sin(x)$$

6. The *ambiguous case* in solving oblique triangles refers to a situation where two sides and one of the angles opposite a given side are known. In this situation, there can be two possible lengths for the unknown leg, and two possible angles for the angle opposite the unknown leg.

Late Nineteenth Century Teaching of Trigonometry

During this time period, all diagrams were given within the textbook, and diagrams were given frequently in all textbooks. To avoid unnecessary repetition, there is no further discussion of diagrams in this chapter.

In the late nineteenth century, many colleges and universities began to require trigonometry as a prerequisite for admission. For example, at Harvard in 1871, elementary plane trigonometry became a requirement for admission.¹⁶ However, many colleges continued to teach elementary plane trigonometry through 1900. By that point, most students took elementary plane trigonometry in secondary school (Allen, 1977).

Late nineteenth century textbooks on elementary plane trigonometry were long and detailed. They covered more topics than ever before, and most textbooks largely covered the same topics. For these reasons, it is impractical to give detailed descriptions of all the topics in each textbook within the text of this chapter. This information can be found in Appendix A. Omitting the details of each textbook allows for a richer discussion of the manner in which each textbook treats the topics. It also allows a more detailed discussion of the ways in which the textbooks differ.

¹⁶ Harvard University Archives, Online Collection. The Harvard University Catalog, 1872-1873.

One way in which textbooks in the late nineteenth century differed from their predecessors is a greatly decreased emphasis on the construction of trigonometric tables. By the late nineteenth century, textbook authors used the power series for sine and cosine to find the trigonometric functions of any angle to arbitrary accuracy. Once the calculation of the trigonometric tables became very straightforward because of the power series, it ceased to be a topic of much importance in trigonometry textbooks. Prior to the late nineteenth century, a description of how the author created the trigonometric tables was given a prominent place and a lengthy discussion in most textbooks.

The textbooks discussed in this chapter all cover generally the same topics. These topics are as follows:

- The measurement of angles, all including the sexagesimal and radian measures of angles, some including the centesimal measure
- The quadrants of various angles and negative angles
- Complements and supplements of angles.
- Definitions of the trigonometric functions, all given by the ratio definitions
- Pythagorean identities
- Reciprocal identities of the trigonometric functions
- The trigonometric functions of special angles 30° , 45° , and 60°
- Reference angles
- The limits and the extreme values of the trigonometric functions
- Sum and difference formulas, double angle formulas, half-angle formulas, and multiple angle formulas

- Power series for $\sin(x)$ and $\cos(x)$ ¹⁷
- The solution of trigonometric equations, including equations where trigonometric functions are raised to powers
- The solution of right and oblique triangles
- Real-life applications of the solution of triangles
- Trigonometric formulas for the area of a triangle

American Textbooks

The first American textbook that is considered is Henry H. Ludlow's *Elements of Trigonometry with logarithmic and other tables*. Ludlow's textbook was used at USMA at West Point from 1888 to 1908.

In his preface, Ludlow references Newcomb's *Trigonometry* whose method "for deducing the developments of sine and cosine have been closely followed." Newcomb's *Trigonometry* was published in New York and was used at Princeton and Yale.¹⁸ Ludlow also consulted textbooks on trigonometry written by the following authors: Chauvenet, Church, Olney, Newcomb, Todhunter, Beasley, and Hann. Chauvenet was French; Church, Olney, and Newcomb were American; Todhunter, Beasley, and Hann were English.

Ludlow begins his textbook by defining trigonometry as follows: "Trigonometry is that branch of Mathematics which treats algebraically: First. Of the measurement, and

¹⁷ Of the late nineteenth century authors considered in this chapter, Loomis is the only author that does not include the power series and the solution of trigonometric equations.

¹⁸ Princeton University Catalog, 1873 and Yale University Catalogs, 1896 and 1990. Source: Google books.

relations, of angles and their sides. Second. Of the solution of triangles” (p. 1). His textbook generally follows the format of his definition. He first gives a thorough treatment of the trigonometric functions and their relationships and then shows how to solve triangles, including all the theorems needed to do so.

Although Ludlow defines the trigonometric functions according to their ratio definitions, he does not use algebra to the exclusion of geometry in the way that Hassler, for example, did (1826). Ludlow uses a portion of the circle (see Figure 6.1) to observe geometrically several identities and formulas. From these, he uses algebra to prove other identities and formulas.

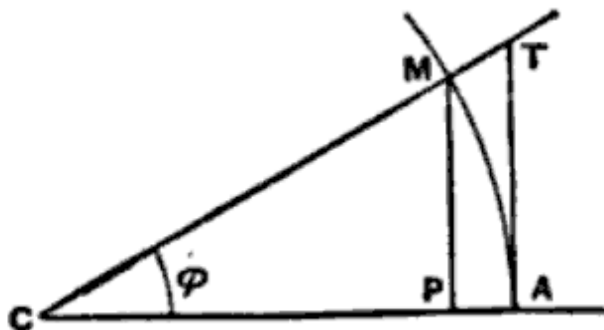


Figure 6.1

Ludlow geometrically observes the Pythagorean identities and the reciprocal identities of the trigonometric functions. He goes on to prove other theorems using these theorems with algebraic techniques.

Pedagogically, Ludlow’s textbook is quite different from its predecessors in the antebellum era. Ludlow gives many more exercises and especially many more real-life applications than antebellum textbook authors did. He also gives exercises that require more thinking on the part of the student. While most exercises from the antebellum

period and earlier required students to practice what they had previously been shown and would allow them to follow previous examples, Ludlow's textbook is full of problems that require much more independent thinking. For example, he requires students to prove trigonometric identities on their own and solve trigonometric equations. His textbook still contains a great deal of exercises that require students only to practice what previous examples have shown them. However, there is a shift away from procedural tasks in exercises and toward thinking and reasoning in exercises.

For example, many of the exercises in Ludlow's *Trigonometry* simply ask students to apply a formula that has been given to them or follow a procedure that has been given to them. One such exercise is, "Express in radians $275^{\circ} 37' 30''$ " (p. 6).

While these exercises are helpful for students, they do not require a great deal of independent thinking. In the antebellum period, simple practice exercises such as these were the only exercises in trigonometry textbooks.

Ludlow also includes exercises that require a great deal more independent thinking from students, where no formula or procedure has been given to follow when solving the problem. One such exercise is, "The length of an arc subtending an angle is equal to that of its radius. Find the angle in seconds" (p. 8).

Ludlow also gives real-life examples, which typically require students to think independently and apply the theorems they have learned. One example is, "The angle which a ship's course makes with the bearing of a light-house is $88^{\circ} 12$ minutes later the bearing of the light makes an angle of 92° with the ship's course, which remains unchanged. If the ship runs 10 knots an hour, what is the distance in miles, approximately, from the ship to the light?" (p. 8).

Ludlow also often asks thought-provoking series of questions, for example the following: “As the angle increases, which is increasing more rapidly? $\sin 30^\circ$ or $\sin 40^\circ$? $\sin 45^\circ$ or $\sin 135^\circ$? $\sin 60^\circ$ or $\sin 150^\circ$? ...” and “Construct both positive angles less than 360° which correspond to each of the following sines: $\frac{1}{2}$, ... , $-.4$, ...” (p. 18).

Ludlow asks students to solve trigonometric equations and systems of trigonometric equations. One example of a trigonometric equation that students are asked to solve is, “ $\tan 3\phi = -1$ ” (p. 50). An example of a system of trigonometric equations is:

$$r \cos v \cos u = a$$

$$r \cos v \sin u = b$$

$$r \sin v = c”$$
 (p. 75).

Finally, Ludlow asks students to prove trigonometric identities, such as:

$$“\cot^2 \phi - \cot^2 \theta = \frac{\sin(\theta + \phi) \sin(\theta - \phi)}{\sin^2 \theta \sin^3 \phi}”$$
 (p. 63). These exercises represent a dramatic shift

from antebellum-period exercises because they required students to apply the information they have learned, rather than simply practice working with a formula they were given earlier. In addition to a dramatic increase in types of exercises compared to antebellum textbooks, Ludlow’s textbook also included a great many more of each type of exercise (the numbers of exercises are detailed in Appendix A). Ludlow’s textbook contains over 200 exercises, while most antebellum textbooks contained about twenty exercises.

In terms of content, Ludlow’s textbook also has new developments when compared to antebellum textbooks. He includes lengthy sections on solving trigonometric equations and power series for sine and cosine. Both of these topics are new since the antebellum period. At the same time, Ludlow does not include trigonometric tables within the textbook, nor does he devote any time to the development of trigonometric

tables. At the same time as he published his *Trigonometry*, Ludlow did publish a separate book that contained the trigonometric tables and an explanation of how they were created. Within the textbook, however since he has the power series for sine and cosine and can find sine and cosine to arbitrary accuracy in that way, there is no need for a detailed explanation on how to create trigonometric tables.

Finally, in contrast to the antebellum period, Ludlow defines the trigonometric functions using the ratio system as functions of angles. Most textbooks in the antebellum period and earlier defined trigonometry according to the line system, and defined them as functions of arcs. Even more, Ludlow's *Trigonometry* does not contain the line representations of the trigonometric functions at all.

Compared with the antebellum period, Ludlow's textbook represents a shift in pedagogy, content, and approach.

The second textbook to consider from the late nineteenth century is Elias Loomis' revised edition of his *Elements of Plane and Spherical Trigonometry*. The textbook considered here is an 1890 reprint of the original 1886 publication. Loomis originally wrote his *Trigonometry* in 1848, but he says in the preface of the 1886 edition that after printing 60,000 copies, the stereotype plates were worn out and had to be recast, which made it possible for him to make revisions. "At the same time," he says, "I have introduced a radical change which I have had in contemplation for several years. In the former editions, in conformity with the old English mathematicians, the trigonometrical functions were regarded as lines; in this revised edition they are regarded as ratios, in conformity with the usage that has become well-nigh universal. I am of the opinion that

the old system has some important advantages in the training of students who have no decided aptitude for mathematical studies; but since the weight of the authority is decidedly against this system, I have decided to abandon it. It is hoped that the changes made in this revised edition may receive the general approval of teachers.”

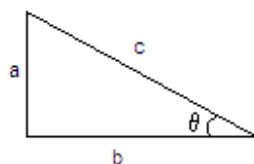
Immediately after defining the trigonometric functions according to the ratio definitions, Loomis shows that these definitions are equivalent to the old line representations. “If the radius of the circle be taken equal to unity, the trigonometric functions above defined may be represented by *straight lines*.” He goes on to say for example, “The secant of an arc is that part of the produced diameter which is intercepted between the center and the tangent.” He also defines versed sine in this section. Loomis uses the line representations to show the reciprocal relationships of the trigonometric functions.

Loomis’ 1886 textbook is an interesting case, because as the preface reveals, Loomis did not want to change his textbook to the ratio system for defining the trigonometric functions. Cajori (1890) says that Loomis was one of the last textbook authors to make the change. Loomis says in his preface that he believes the old line system “has some important advantages in the training of students who have no decided aptitude for mathematical studies.” Seemingly against his better judgment, he gives up the old line system because “the weight of the authority is decidedly against this system.”

Although Loomis has converted this textbook to the new ratio system, he presents the line representations immediately upon defining the trigonometric functions and shows that they are equivalent. By comparison with Ludlow and the other authors in this chapter, Loomis gives the line representations a great deal of consideration and

importance. He also uses them to prove some of the theorems and identities that are easily shown using the line representations. Other authors do not give them the same kind of priority.

Loomis is also unique among authors who define the trigonometric functions using the ratio system because typically authors define all six trigonometric functions as functions of one of the acute angles in a right triangle, where all six ratios are unique (see Figure 6.2).



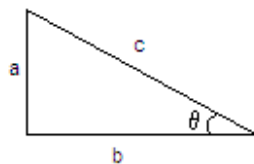
$$\sin \theta = a/c \quad \cos \theta = b/c$$

$$\sec \theta = c/b \quad \csc \theta = c/a$$

$$\tan \theta = a/b \quad \cot \theta = b/a$$

Figure 6.2

Loomis defines sine, tangent, and secant as functions of one of the acute angles in a right triangle and gives their ratios, and then defines cosine, cosecant, and cotangent as functions of the other non-right angle and defines as the same ratios as sine, secant, and tangent (see Figure 6.3).



$$\sin \theta = a/c \quad \cos \varphi = a/c$$

$$\sec \theta = c/b \quad \csc \varphi = c/b$$

$$\tan \theta = a/b \quad \cot \varphi = a/b$$

Figure 6.3

By defining cosine as the sine of the complement, cosecant as secant of the complement, and cotangent as tangent of the complement, Loomis preserves an element of the line system, even though he has converted his textbook to the ratio system.

Pedagogically, Loomis' 1886 *Trigonometry* shows considerable advancement over his 1848 *Trigonometry*. In this new version, Loomis includes more exercises, including numerous proofs and applications in addition to exercises similar to those he gave in the previous edition. He gives ten proofs of trigonometric identities, similar to those of Ludlow, and ten applications to real-life, also similar to Ludlow's. Unlike Ludlow, Loomis does not include solving trigonometric equations or systems of trigonometric equations.

Of all the textbooks this chapter considers from the late nineteenth century, Loomis' 1886 *Trigonometry*, although much more pedagogically advanced than his 1848

edition, is most like the antebellum textbooks. The clear reason is that this was not a new textbook, but rather an updated edition of an older textbook.

The next late nineteenth century textbook to consider is Emler Adelbert Lyman and Edwin Charles Goddard's *Plane and Spherical Trigonometry*. This textbook was originally written in 1890 and was reprinted in 1900. This study considered the 1900 edition, but changes between the editions were minimal. Both Lyman and Goddard were from the University of Michigan-Ann Arbor.

Like Loomis, Lyman and Goddard have substantial prefatory material that makes clear what they intended when they wrote their textbook. They acknowledge that there are an adequate number of trigonometry textbooks already in existence. However, they say, "many American text-books treat the solution of triangles quite fully; English text-books elaborate analytical trigonometry; but no book available seems to meet both needs adequately. To do that is the first aim of the present work" (p. iii).

They also have a couple of concerns about the traditional methods of presenting trigonometric theorems in the existing textbooks. They say, "For some unaccountable reason nearly all books, in proving the formulae for functions of $\alpha \pm \beta$, treat the same line as both positive and negative, thus vitiating the proof; and proofs given for acute angles are (without further discussion) supposed to apply to all angles, or it is suggested that the student can draw other figures and show that the formulae hold in all cases. As a matter of fact the average student cannot show anything of the kind; and if he could, the proof would still apply only to combinations of conditions the same as those in the figures

actually drawn” (p. iii). They remedy this by writing the proofs in such a way that they apply to any angles, avoiding figures all together.

They also say that they find it unacceptable that logarithms and inverse trigonometric functions are typically introduced at the end of a trigonometry textbook because introducing them toward the end means that students do not have much opportunity to use them, causing students not to retain this information later. To aid retention, Lyman and Goddard introduce logarithms and inverse functions earlier than most textbooks and use them more often so that students have more opportunities to practice.

Lyman and Goddard are also unique in having “oral work.” They explain the purpose of oral work as follows: “The fundamental formulae of trigonometry must be memorized. There is no substitute for this. For this purpose oral work is introduced, and there are frequent lists or review problems involving all principles and formulae previously developed. These lists serve the further purpose of throwing the student on his own resources, and compelling him to find in the problem itself, and not in any model solution, the key to its solution, thus developing power instead of ability to imitate” (p. iv).

Lyman and Goddard also object to the way that most authors divide the solution of triangles into cases, because it causes students not to think for themselves, but simply to repeat what they have been shown. About this, they say, “in the solution of triangles, divisions and subdivisions into cases are abandoned, and the student is thrown on his own judgment to determine which if the three possible sets of formulae will lead to the solution. Long experience justifies this as clearer and simpler” (p. iv).

Within the preface, Lyman and Goddard instruct teacher that sections marked with * are optional and should only be covered if time permits. They also explain that they have provided more problems and exercises than one student is expected to solve, so that teachers may assign different problems to different classes or in different years. They emphasize, “Do not assign work too fast. Make sure the student has memorized and can use each preceding formula, before taking up new ones” (p. iv).

At the end of the preface, Lyman and Goddard acknowledge some of the textbooks that have influenced the writing of their own. “The standard works of Levett and Davidson, Hobson, Henrici and Treutlein, and others have been freely consulted” (p. iv). Then they explain that most of the textbook is original, but “quality has not been knowingly sacrificed for originality” (p. iv). Levett and Davidson as well as Hobson were British, and Henrici and Treutlein were German.

Lyman and Goddard’s version of the ratio definitions for trigonometric functions is slightly different from most authors’. About the trigonometric functions, they say, “These functions are the six ratios between the sides of the triangle of reference of the given angle. The triangle of reference is formed by drawing from some point in the initial line, or the initial line produced, a perpendicular to that line meeting the terminal line of the angle.” Lyman and Goddard explain how to create a triangle of reference for any angle. No matter in which quadrant the terminal side of the angle lies, the triangle of reference is created by creating a perpendicular to the x-axis from somewhere on the terminal side of the angle (see figure 6.4).

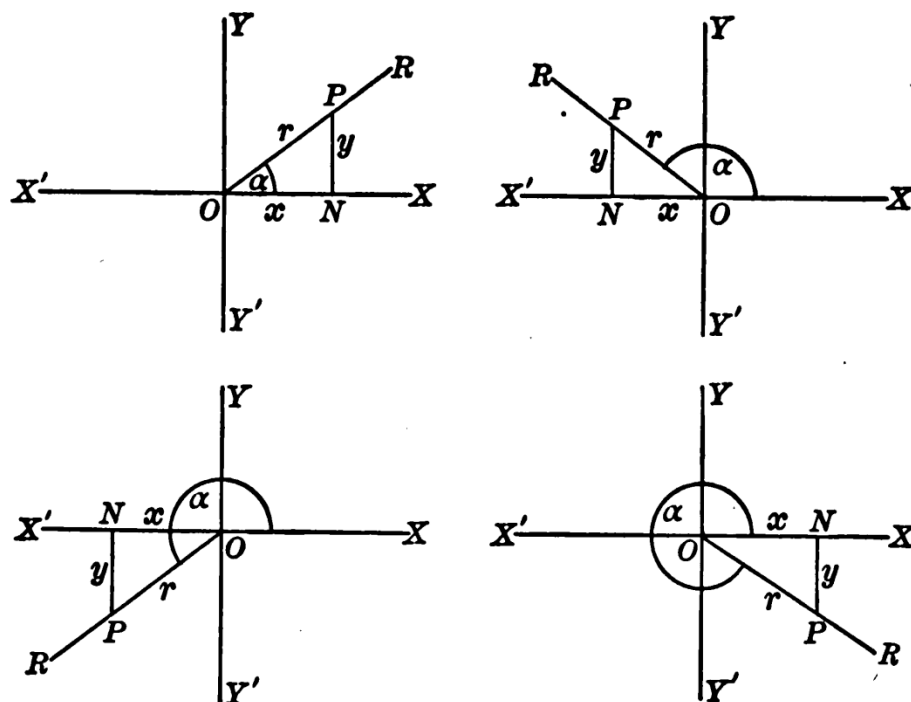


Figure 6.4

After explaining how to create a triangle of reference for any angle, Lyman and Goddard define the trigonometric functions using the ratio definitions:

$$\text{Sin } (\alpha) = y/r$$

$$\text{Cosine } (\alpha) = x/r$$

$$\text{Tangent } (\alpha) = y/x$$

$$\text{Cotangent } (\alpha) = x/y$$

$$\text{Secant } (\alpha) = r/x$$

$$\text{Cosecant } (\alpha) = r/y$$

$$\text{Versed sine} = 1 - \cos (\alpha)$$

$$\text{Covered sine} = 1 - \sin (\alpha)$$

Most authors define the trigonometric functions in the ratio system first in a right triangle, thereby making the angle that is the input of the trigonometric function acute. They generalize this definition to all angles, using the symmetry of the circle. Lyman and Goddard, however, allow any angle, and they show how to create a triangle of reference for that angle. After that, the trigonometric functions are defined on the triangle of reference. Although subtle, this way of defining trigonometric functions is significantly different from that of other authors.

Whether intended by the authors or not, Lyman and Goddard's method for defining the trigonometric functions according to the ratio system allows for easier navigation between circle trigonometry and triangle trigonometry for two important reasons. First, their definitions include a radius rather than a hypotenuse (for example, $\sin(\alpha) = y/r$). Second, the triangle of reference is located on the circle, where circle trigonometry takes place. This subtle change in the ratio system allows a much more seamless connection between triangle trigonometry and circle trigonometry.

Lyman and Goddard's textbook is also unique because it shows the periodic graphs of the trigonometric functions. This section is most likely one of the pieces of analytic trigonometry that they said in their preface was missing from most American textbooks.

Lyman and Goddard's *Trigonometry* is also the first textbook examined in this study where real-life applications are given for the measurement of angles. In other textbooks, angle measurement is not considered a topic in itself but merely a prerequisite for introducing the trigonometric functions. Angle measurement is usually treated as quickly as possible.

The greatest difference in content between Lyman and Goddard's textbook and others that came before it is that Lyman and Goddard's textbook includes a section on the periodic nature of the trigonometric functions. Other textbooks have considered how the trigonometric function changes as the angle (or arc) moves around the circle, but they have not considered trigonometric functions on their own graph, apart from the circle. Although Lyman and Goddard do not discuss it, this aspect of trigonometry has far-reaching implications, especially because it allows trigonometric functions to model real-life situations and it brings trigonometric functions into the "library of functions" (Van Brummelen, 2009, p. 284).

In terms of pedagogy, Lyman and Goddard have one unique feature—oral work. They emphasize memorizing the basic formulas of trigonometry and the values of the trigonometric functions of special angles, and they require students to be able to work with the formulas and functions in their heads on basic problems. There is oral work in nearly every section of the trigonometry textbook. Following the section that defines the trigonometric functions, the following oral work exercises are given (Figure 6.5):

ORAL WORK.

1. Which is greater, $\sin 45^\circ$ or $\frac{1}{2} \sin 90^\circ$? $\sin 60^\circ$ or $2 \sin 30^\circ$?
2. From the functions of 60° , find those of 30° ; from the functions of 90° , those of 0° . Why are the functions of 45° equal to the co-functions of 45° ?
3. Given $\sin A = \frac{1}{2}$, find $\cos A$; $\tan A$.
4. Show that $\sin B \csc B = 1$; $\cos C \sec C = 1$; $\cot x \tan x = 1$.
5. Show that $\sec^2 \theta - \tan^2 \theta = \csc^2 \theta - \cot^2 \theta = \sin^2 \theta + \cos^2 \theta$.
6. Show that $\tan 30^\circ \tan 60^\circ = \cot 60^\circ \cot 30^\circ = \tan 45^\circ$.
7. Show that $\tan 60^\circ \sin^2 45^\circ = \cos 30^\circ \sin 90^\circ$.
8. Show that $\cos \alpha \tan \alpha = \sin \alpha$; $\sin \beta \cot \beta = \cos \beta$.
9. Show that $\frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \cos 60^\circ = \frac{1}{2} \cos 0^\circ$.
10. Show that $(\tan y + \cot y) \sin y \cos y = 1$.

Figure 6.5

By the nature of the oral work exercises, it is not possible that these exercises and answers would be repeated by the teacher and class together, as in some oral exercises given for arithmetic. Rather, the authors suggest that these exercises must be done without use of a pencil and paper to help students work with the theorems by memory.

In each case where oral work is given, it is similar to that which is shown in Figure 6.5. Appendix A gives a detailed listing of where in Lyman and Goddard's *Trigonometry* oral work appears.

In addition to oral work, this textbook also contains the most exercises of any textbook analyzed in this study, but the authors say in the preface that they do not intend students to complete all of the exercises; rather, they provide them to allow teachers to choose and to have variety between classes or years.

Lyman and Goddard's *Trigonometry* has exercises of all the same types as in Ludlow's *Trigonometry*. The textbook includes simple practice exercises, exercises that require more thought (including many of the oral work exercises), proofs of trigonometric identities, solving trigonometric equations and systems of trigonometric equations, and real-life applications. Since their exercises are similar to those of Ludlow, there is no need to repeat examples of all the different types of exercises. The precise numbers of exercises are included in Appendix A.

Finally, Lyman and Goddard are the only textbook authors analyzed in this study that have pedagogical instructions and advice for teachers. They give two such pieces of advice. First, they explain that * sections are optional and that they have provided more problems than the teacher should assign. While directed at the teacher, such comments are basically just explaining how their textbook differs from others. Second, however, they emphasize, "*Do not assign work too fast. Make sure the student has memorized and can use each preceding formula, before taking up new ones*" (p. iv). That instruction is the first time in this study where a textbook author has given strong pedagogical advice to the teachers. It may indicate a general distrust as to the competence of the teachers of trigonometry. At this time, trigonometry was taught in both high schools and colleges (Cajori, 1890; Allen, 1977) so it is not possible to know whether there was distrust of college teachers of trigonometry, high school teachers of trigonometry, or both.

In terms of approach, Lyman and Goddard explain the one main difference between their textbook and others in their preface. They say that they included logarithmic trigonometry earlier in the textbook than others did so that students would have more time to practice using logarithmic trigonometric tables.

The line representations feature fairly prominently in Lyman and Goddard's *Trigonometry*, but with a great deal of emphasis that they are not the trigonometric functions and are only representations of such. Lyman and Goddard grouped them with the periodic graphs of the trigonometric functions in a section entitled, "Graphic Representation of Functions," which is an idea not seen in other textbooks of this time period.

The final textbook from the late nineteenth century to consider is *Trigonometry for Schools and Colleges*, published in 1896, written by Frederick Andregg and Edward Drake Roe, who were both professors at Oberlin College in Oberlin, Ohio. The shift toward teaching trigonometry in secondary schools is evident in the title of this textbook, because it mentions that it is both for "[Secondary] *Schools and Colleges*."

Like Lyman and Goddard, they recognize that there is a wealth of trigonometry textbooks already in print, and they justify their goal for creating another one in their preface. They say, "We have endeavored to lay down general conventions and definitions, to emphasize consistency in the use of conventions, to give general demonstrations in which we have carefully aimed at logical soundness, directness, and simplicity, and to exhibit the unity of the subject as made up of its related parts" (p. iii). They are aiming at a greater deal of cohesion and ease of understanding that they feel was lacking in contemporary textbooks.

Andregg and Roe "believe that in this way the work is simplified, the student gets the ground plan of the higher analysis, and is saved from much subsequent intellectual lameness, which results from an excessive absorption of the attention on special cases. At the same time, when it seemed pedagogically helpful, we have left up to the general

demonstration by the consideration of special cases” (p. iv). They attempt to give a simpler and more straightforward treatment of trigonometry, without sacrificing quality.

They also make reference to the other textbooks that have informed their textbook. They say, “Our views on trigonometry and related subjects have been influenced by many American, English, French, and German treatises, and especially by our respected teachers, Professors J.M. Peirce, W.E. Byerly, and B. O. Peirce, of Harvard University” (p. iv).

Andregg and Roe begin the textbook by saying “the subject matter of Trigonometry is lines, and angles” (p. 2). They define trigonometry as follows: “Trigonometry is the investigation of the relations of the sides and angles of a triangle” (p. 2).

Andregg and Roe define the trigonometric functions by saying the following: “The six ratios which can be formed by using the three sides of the triangle of reference of a given angle, two at a time, are called the six primary trigonometric functions of the angle” (p. 8). They give the ratio definitions of the trigonometric functions: sine, cosine, tangent, cotangent, secant, cosecant. They also define versed sine, covered sine, suversed sine, and sucovered sine using their related trigonometric ratios.

After giving a few basic trigonometric formulas, Andregg and Roe have a section on the line representations of trigonometric functions. About the relationship of the lines to the functions they say, “The student should remember that the line is *not* the function, but *represents* it” (p. 20). They continually reiterate throughout that section that the lines are representations, rather than the functions themselves.

In terms of content, Andregg and Roe's textbook is unique because it includes the limits of the trigonometric functions. It also includes suversed sine and sucovered sine, which are not included in any other textbook in this study. It includes the centesimal angle measurements, like Ludlow. Otherwise, this textbook is very similar to that of Lyman and Goddard. Andregg and Roe also include the periodicity of the trigonometric functions and define the trigonometric ratios by the triangles of reference, but they and Lyman and Goddard are the only in this study to do so.

In terms of approach, Andregg and Roe have set out to make a simple trigonometric textbook, and theirs is certainly simpler than some others of the day. They set out to avoid "special cases." They use cases when demonstrating the solution of triangles, but other than that, the textbook is free of cases.

Concerning pedagogy, Andregg and Roe's textbook contains many of the same pedagogical tools as the other textbooks of its time. Their textbook does not contain as many exercises as Lyman and Goddard's, but it has substantially more than earlier textbooks. The exercises include many applications, proofs, and other difficult, higher-level problems as well as simple practice problems. The types of exercises Andregg and Roe give are similar to Ludlow's and Lyman and Goddard's, so they will not be explained in detail here. Appendix A details the numbers of exercises that are given for each topic.

Final Exams in Trigonometry

In the late nineteenth century, some colleges included information about their final examinations in their course catalogs. In a couple of cases, the course catalogs

included the exams themselves. There are two such examples. The first is from Harvard, and it is the final exam for Trigonometry, given in June of 1872. During this time, Peirce's *Trigonometry* was in use as the textbook (Cajori, 1890). The final exam is as follows:

- “1. Obtain formulas by which, when the sine of an acute angle is known, the cosine, tangent, and remaining trigonometric functions can be found. Find, by these functions, the functions of an angle whose sine is $-\frac{1}{6}$.
2. Write (without proving) the formulas for the sine and cosine of the sum of two angles; and obtain from them the formulas for the sine and cosine of the *double angle*.
3. Prove the following theorems:
 - (a.) The sides of a plane triangle are proportional to the sines of the opposite angles.
 - (b.) The sum of any two sides of a plane triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.
4. Two sides of a plane oblique triangle are 672.3 and 555.9 and the included angle is $25^{\circ}16'$. Solve the triangle.
5. Two sides of a plane oblique triangle are 1.396 and .9881, and the angle opposite the second side is $32^{\circ}43'$. Solve the triangle. When are there two solutions to this problem? Why? Is the problem ever impossible? If there are two solutions in this example, give both of them.”¹⁹

From this examination comes an understanding of what was truly considered important during this time. Each problem is analyzed below to show what skills are necessary to solve it.

Problem One. The first part of problem one requires students to understand the relationship of sine to the other trigonometric functions. Since Peirce used the ratio system, this problem is made somewhat easier. The second part of problem one requires students to be able to use a trigonometric table.

¹⁹ Harvard University Archives, Online Collection. The Harvard University Catalog, 1872-1873, p. 261.

Problem Two. This problem requires students to know by memory the formula for the sine and cosine of the sum of an angle. It then requires students to realize that the formula for a double angle can be derived from the formula for the sums by making the sum $(x + x)$ instead of $(x + y)$.

Problem Three. The first part of this problem requires students to prove the law of sines, and the second part requires them to prove the law of tangents. Students would most likely have to be very familiar with the method used in the textbook for proving those theorems in order to be able to reproduce them on the final exam.

Problem Four. This problem requires students to solve a triangle. They are given two sides and the included angle, so the solution requires students to use only the law of sines.

Problem Five. This problem requires students to solve a triangle given two sides and a non-included angle (the ambiguous case). The triangle has two possible solutions, and the problem not only requires students to find both solutions but also requires students to explain the ambiguous case in general.

This final exam demonstrates that students at this time were required to memorize basic formulas, theorems, and proofs; use trigonometric tables; understand relationships between the trigonometric functions; and solve triangles, including triangles with two possible solutions.

The second examination comes from Columbia College (now Columbia University) and is given in the *Catalog* for the school year of 1874-1875. During this time, Davies' *Legendre* was in use at Columbia. Although no specific examination on trigonometry is given, the sophomore year examination papers include a mathematics

examination which has the following questions pertaining to trigonometry (questions not pertaining are omitted):

- “1. Deduce the value of $\sin(a + b)$.
2. Deduce the values of the functions of the arc $\frac{1}{2}\alpha$ in terms of the functions of the arc α .
3. Prove that the sum of the sines of two arcs is to their difference as the tangent of half the sum of the arcs is to the tangent of half their difference. ...
9. In a plane triangle, given the area, angle C, and $a + b$, find the sides.”²⁰

Again, there is a consideration of what each problem requires students to do.

Problem One. Because the problem instructs students to “deduce,” students must be able to follow the proof of the sums of sines formula. This would require that students be extremely familiar with the process for deriving that formula.

Problem Two. This problem requires students to know how to derive the half angle formulas. It is similar to Problem One because students need to be familiar with the proof of the half angle formulas in order to deduce them.

Problem Three. This problem requires students to prove the law of tangents. This would most likely require students to be very familiar with the proof given in the textbook so that they could reproduce it on the final exam.

Problem Nine. This problem requires students to have memorized the formula $Area = \frac{1}{2} ab \sin C$. By solving for ab in the aforementioned formula, students would then know the sum and the product of a and b , and using a system of equations, they could solve for the two sides. Since no numerical values are given, students were expected to solve this problem in general.

²⁰ Columbia College Archives. The Columbia College Catalog, 1874-1875, p. 81.

The important topics here are memorizing formulas, proving theorems, and understanding how many formulas can be taken apart to give new information. In contrast to Harvard's examination, there are no straightforward problems on the solution of triangles that would come from the basic four cases, although problem nine does address the solution of triangles. The examination used at Columbia College is more theoretical than that of Harvard, and the solution of triangles is not addressed at Columbia, while it is large part of the exam at Harvard.

German Influence in Mathematics Education

Although it is not something that Cajori mentions in his 1890 *History and Teaching of Mathematics*, it is well-documented that in higher education in general, after the post-revolutionary influence of the French diminished, a German influence followed (Thwing, 1906; Cohen and Kisker, 2010; Brubacher and Rudy, 1958; Lucas, 1994). The textbooks from the late nineteenth century confirm that there is at least some German influence in trigonometry education. Three textbooks out of the four in this chapter acknowledge outside influences. Two out of those three acknowledge German influences, Lyman and Goddard as well as Andregg and Roe.

The German influence is more difficult to analyze than was the French, because no German textbooks were directly used, nor were translations of German textbooks created. Instead, the influence was more subtle. American authors were reading and considering German textbooks when writing their textbooks. From the prefatory materials in textbooks, it is clear that there was a German influence, but it was much less extensive than the French influence.

Trigonometric Definitions Change Radically

Expressions of trigonometric functions as ratios grew in importance and line representations diminished in importance. Ultimately, they changed places as the “ratio system” defined trigonometry, rather than the “line system.” In the late eighteenth and early nineteenth centuries, there is no mention of the trigonometric ratios in elementary textbooks. Trigonometry consists of lines on a circle. Although the trigonometric ratios were known to mathematicians at that time (Van Brummelen, 2009), they were nowhere to be found in elementary trigonometry textbooks.

By the middle of the nineteenth century, all the textbooks contained the trigonometric ratios in some form, and the first textbooks were written that used the ratios to define the trigonometric functions. During this time, there was debate as to which was the better system for defining trigonometry. Even while the debate over the definitions took place, the ratio system grew in prominence within textbooks. In textbooks that had the line system for the definitions of trigonometric functions, the trigonometric ratios were often proved soon after and used to prove many of the other theorems of trigonometry.

Finally, by the late nineteenth century even Loomis, the textbook author who was most committed to keeping the line system as the definition of trigonometry (Cajori, 1890), conformed “with the usage that has become well-nigh universal” (Loomis, 1890). After he converted his book to define the trigonometric functions according to the ratio definitions, Loomis still gives the line representations a place of prominence in his textbook, directly after the ratio definitions, and uses them to prove theorems that are

easily shown in that way. Other authors who use the ratio system do not include the line system at all. Some, like Andregg and Roe, include it but only as a side note and do not use it for anything in particular.

By the late nineteenth century textbooks were universally using the ratio system to define trigonometric functions. However, some colleges and universities were still using textbooks that used the line system. For example, Columbia was still using Davies' *Legendre* through the turn of the twentieth century.²¹

Overall, the progression toward the ratio system was not entirely linear. First, textbooks defined trigonometry according to the line system sometimes contained the ratio system. Then, some of the first textbooks that defined trigonometry according to the ratio system did not contain the line system at all (for example, Ludlow, 1886 and Peirce, 1852). Some textbooks defined trigonometry according to the ratio system and featured the line system prominently (for example, Loomis, 1890), others mentioned it only briefly (for example, Andregg and Roe, 1896). Although there is a general trend from the line system to the ratio system, authors at the end of the nineteenth century disagree about the roles of these two systems. Authors seem to disagree especially about the continued role of the line system, even though it is no longer defining trigonometric functions. There is no consensus about what role it should play in trigonometry.

In 1908, Robert Moritz wrote an article in a journal *School Science and Mathematics*,²² where he commented on the lack of uniform order when comparing one trigonometry textbook to another. He says, "Trigonometry to-day is probably the least

²¹ Columbia College Archives. Columbia College Catalogs, 1865-1900.

²² Moritz, Robert E. "On the definition and scope of plane trigonometry." *School Science and Mathematics*. Vol. 8, no. 5, May 1908.

organized of the mathematical disciplines from arithmetic to and through the infinitesimal calculus. There appears to be no recognized order of precedence in the treatment of different topics by various authors” (p. 394).

Further, Moritz argues that the differences from author to author in the sequencing of trigonometry is caused, fundamentally, by a disagreement in the definition of trigonometry. “This lack of recognized order is due, no doubt, to an absence of unity in the conception of the subject. While most authors attempt to cover a certain number of traditional topics under the head of trigonometry, they are widely at variance as to what constitutes trigonometry proper. Scarcely two authors agree in their definition of the science” (p. 394).

Based on the definitions given by authors in this chapter, Moritz is completely correct in his judgment that authors do not tend to agree on the definition of trigonometry. For example Ludlow begins his textbook by defining trigonometry as follows: “Trigonometry is that branch of Mathematics which treats algebraically: First. Of the measurement, and relations, of angles and their sides. Second. Of the solution of triangles.”

On the other hand, Andregg and Roe define trigonometry as follows: “Trigonometry is the investigation of the relations of the sides and angles of a triangle” (p. 2). It is no wonder, as Moritz points out, that authors with such differing views on what trigonometry is have textbooks that are substantially different from one another.

Moritz suggests a solution for the fractionated nature of trigonometry. He says, “So trigonometry, which originated, it is true, in the attempt to solve triangles by numerical methods, has become vastly more than this. Its central idea is no longer

triangles, but angles and their functions. It only remains to shift the emphasis where it belongs, from the solution of triangles to the relations between angles and their functions. With trigonometry as the science of angular magnitudes, everything ordinarily treated under that head can be organized into a coherent whole” (p. 396). Further study of the teaching of trigonometry in the twentieth century would be required to find out if Moritz’s suggestion took effect.

Pedagogy Develops Further

The end of the nineteenth century saw many advances in pedagogy. First, authors began to give more exercises, and the exercises they gave covered more material, were more interesting, and were more challenging. Also, there was one pedagogical method—oral work—that appeared but was not prolific.

Before the American civil war, textbooks contained few exercises, and those exercises were often simply repetition of examples given in the text. In the late nineteenth century, however, exercises developed much more. The quantity of exercises increased ten-fold. Repetitive exercises to practice something that was exemplified in the text still existed, but they were supplemented with many exercises asking students to prove trigonometric identities, solve trigonometric equations, and apply trigonometry to real-life situations.

One pedagogical development that did not become universal was “Oral Work.” Lyman and Goddard include several sections of oral work, where students had to use formulas they memorized to solve simple trigonometric problems without writing

anything. They are the only textbook to include this pedagogical method, however, so it was not a widespread technique.

It is clear that during the late nineteenth century, textbooks took on a role in pedagogy that previously must have been filled by the teacher. The advancement of pedagogy within textbooks demonstrates the advancement of textbooks as a pedagogical tool during this time.

There are many reasons that the pedagogy in textbooks grew. In the eighteenth century, because books were expensive and difficult to print, students typically did not have their own copies of textbooks. Rather, the teacher had one textbook, and students copied their own textbooks. Since students were not using the textbooks directly, there were no overt pedagogical techniques given in the textbooks.

In the nineteenth century, textbooks could be printed more easily, so it became common for students to have their own copies. This coincides with the appearance of exercises in textbooks. Now that students had the textbook from which to study, textbook authors wanted to aid students in practicing what they were learning from the textbooks. Because exercises helped students, textbook authors included more and more exercises.

In the nineteenth century, trigonometry also became its own course of study, rather than a topic in a larger course on mathematics. Colleges began devoting typically one entire semester to trigonometry. This was reflected in textbooks, as well. In the late eighteenth century, trigonometry was included within a larger mathematics textbook. Then, in the early part of the nineteenth century, a treatise on trigonometry was typically attached to a geometry textbook. By the middle of the nineteenth century, trigonometry textbooks were completely separate textbooks. As trigonometry became a separate course

and a separate textbook, teachers had more time to devote to trigonometry and authors had more time and space to devote to exercises.

As trigonometry developed into a separate course, there was also a trend to see the trigonometry as important for students to understand thoroughly. In the late eighteenth and early nineteenth centuries, trigonometry textbooks treated trigonometry as a skill for students to master. The goal was the solution of triangles, and textbooks taught only that which was necessary for students to acquire that skill. In the middle and late nineteenth century, trigonometry began to be seen as important in and of itself and important for calculus, and that was reflected in how textbook authors presented it.

Later textbooks had students not only solving triangles, but also proving trigonometric identities, solving trigonometric equations, and understanding the periodic nature of trigonometric functions. This happened because trigonometry was no longer seen as a means to an end, but as an entire topic within the field of mathematics. It came to be a topic not taught only so that students could solve triangles, but also to exercise students' logical reasoning and to expand students' understanding of functions.

There was also a cultural difference in the way textbooks included overt pedagogical tools. American textbooks began with British textbooks and later British authors wrote textbooks for use in American colleges. Exercises included in textbooks began with British textbooks. However, when textbooks later came to the United States from France or when a French author wrote a textbook for use in the United States, exercises were not included. This represents a cultural difference between British and American textbooks and French-authored textbooks.

Through time, American textbooks came to be more and more helpful for teachers. When textbooks contained no exercises, teachers presumably created exercises for students to practice the topics they were learning. When textbooks contained a few exercises on each topic, teachers still had to create some exercises, but perhaps they used the exercises in the textbooks as a model for creating more.

By the end of the nineteenth century, some textbook authors included so many exercises that it would be impossible and unnecessary for students to complete all of them in the course, and the authors said as much in their preface (i.e. Lyman and Goddard, 1890). The authors included an excess of exercises so that teachers would be able to choose different exercises for different sections or for different years of teaching the same course.

Over time, textbook authors tried harder and harder to make the textbook as helpful as possible for the teachers who were using that textbook. By the end of the nineteenth century, textbook authors began writing pedantically to teachers in their textbooks, instructing them on how best to use the textbook (i.e., Lyman and Goddard, 1890). During this time, textbooks grew to be more and more helpful to teachers. At the same time, textbook authors grew to be less trusting of the teachers to do a good job on their own, and they wanted to make sure teachers were using their textbooks correctly.

As the college population exploded during the nineteenth century, so did the numbers of teachers that were instructing students in trigonometry. Textbook authors could no longer be certain of teachers' backgrounds. Furthermore, at the end of the nineteenth century, trigonometry began to be taught in secondary schools as well as colleges and universities (Allen, 1977). Textbook authors, no doubt, also had high school

teachers in mind when they were writing their textbooks and therefore wanted to include as many helpful tools for teachers, since in many cases high school teachers were less qualified than teachers of higher education.

One final trend can be seen in the pedagogy in textbooks in the nineteenth century—that is, the inclusion of applications of trigonometry to real life. This practice began at USMA at West Point. Beginning in the early nineteenth century with Charles Hutton's *Mathematics* (1812), every textbook published by a professor at USMA at West Point contained applications, but others did not. By the late nineteenth century, applications became a popular topic included in all textbooks. However, for several decades, applications were unique to West Point textbook authors.

Changes in the Content of Trigonometry

At the end of the eighteenth century, textbooks included trigonometric tables and explained how to use them and detailed discussions of how they were created. Textbooks also included logarithmic trigonometric tables and explanations on how to use them and some of the most basic trigonometric formulas. They explained the solution of triangles, and that was considered to be the goal and the end of trigonometry.

By the beginning of the nineteenth century, the ratio system made its first appearance in trigonometry textbooks, coming at the end of the textbook. At the nineteenth century trigonometry textbooks also grew in length and included more formulas. However, the focus was still largely geometric and computational.

Moving into the antebellum period, the topics covered by trigonometry textbooks increased somewhat. The creation of trigonometric tables and logarithmic trigonometric

tables were still very important. Many more trigonometric formulas were not only given, but also explained and proved in detail. In the antebellum period, many textbooks included the ratio system, even if they did not use it to define trigonometric functions. Also, the first textbooks appeared that used the ratio system to define trigonometric functions.

In the late nineteenth century, the number of topics included in trigonometry textbooks increased dramatically. Trigonometric tables and logarithmic trigonometric tables were given less importance because calculus allowed simple calculation of trigonometric functions to arbitrary accuracy.

The ratio system also gained importance during this time, now becoming the way that all textbooks defined the trigonometric functions. Some textbooks placed little or no importance in the line system while in some textbooks the line system remained an integral part of trigonometry. During this time, the discussion of trigonometric functions as periodically-changing functions became an important topic.

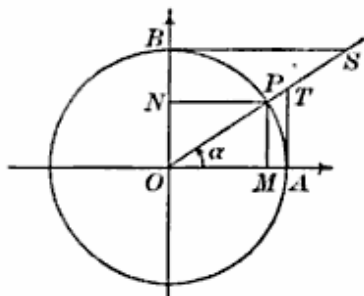
There were many complex reasons that the aforementioned changes took place in the content of trigonometry. In the beginning of the teaching of trigonometry in the United States, trigonometry was included in the curriculum because of its usefulness in surveying and navigating. Trigonometry was thought of in a geometric way and had geometric applications. It was seen as an extension of geometry—quite literally. Treatises on trigonometry at this time were often added to the end of geometry textbooks.

As calculus grew in importance, and as trigonometry was preparing more and more students for calculus, the content of trigonometry tailored itself more toward calculus. Trigonometric functions were viewed as periodically changing functions, which

are an important part of calculus. Also, the power series for sine and cosine appeared in textbooks and were used to calculate trigonometric tables. Over time, trigonometry textbooks included more and more of the topics that were essential in preparing students for calculus.

The ratio system gained importance for a number of reasons. First, there were pedagogical considerations. As more and more students attended college, many of the students did not have the elite preparation that was once associated with a college preparatory curriculum. Students struggled with trigonometry, and using the ratio system was a simple way for students to solve triangles. Solving triangles was one of the most important problems of trigonometry and one of the most applicable to real life, and the ratio system allowed textbooks and teachers to give students a straightforward way to complete these problems.

There were also pedagogical considerations for keeping the line definitions, and some textbook authors resisted the transition. Using the line system allowed students to make connections between Euclidean geometry and trigonometry, and it helped students understand where some of the trigonometric formulas originate. For example, when using the line system, the Pythagorean identities can be observed simply by looking for right triangles in the diagram that defines the trigonometric functions (see Figure 6.6).



$$\begin{array}{ll}
 PM = \sin (\alpha) & NP = \cos (\alpha) \\
 TA = \tan (\alpha) & RS = \cot (\alpha) \\
 OT = \sec (\alpha) & OS = \csc (\alpha)
 \end{array}$$

Figure 6.6

Assuming that the radius is equal to one, $\triangle OMP$ shows that $\sin^2 \alpha + \cos^2 \alpha = 1$, $\triangle OAT$ shows that $1 + \tan^2 \alpha = \sec^2 \alpha$, and $\triangle ORS$ shows that $\cot^2 \alpha + 1 = \csc^2 \alpha$. Similarly, other trigonometric formulas can be proved easily using the line system.

Proponents of the line system also argue that although the ratio system allows trigonometry to advance further, its foundation in the line definitions cannot be abandoned. Wallace (1828) likens using the ratio system to define trigonometry to building a house by the roof first.

Second, beginning as early as the seventeenth century and certainly by the middle of the eighteenth century, the trend among mathematicians was to think of the trigonometric functions as ratios and to use the ratio definitions of the trigonometric functions when proving theorems (Guralnick, 1975). There was pressure on textbook

authors and the teachers of trigonometry to place increasing importance on the ratio definitions and eventually to replace the line definitions of the trigonometric functions with the ratio definitions. Making the transition to the ratio definitions allowed students to proceed to higher levels of mathematics without having to re-learn the definitions of the trigonometric functions.

Finally, there was international pressure to convert to the ratio system. The United States was the last in the Western world to adopt the ratio definitions for teaching trigonometry. It was generally held to be true that British mathematics lagged behind French mathematics. Ironically, the United States first got its mathematics from British sources, then switched over to French sources in the beginning of the nineteenth century. By the mid nineteenth century, French and even British textbook authors had all embraced the ratio definitions for the trigonometric functions. Then, it took another several decades for the United States to make the same transition (Cajori, 1890). Understandably, some textbook authors in the United States desired to keep up with their overseas counterparts.

The transition to the ratio system was not the only shift in the content of trigonometry education during this time period. The applications of trigonometry became more extensive during the time of this study. At first, surveying and navigation were the only applications that were discussed in textbooks. Over time, many different types of applications were included, and this drove an increase in the number of trigonometric formulas and theorems that textbooks needed.

During the nineteenth century, trigonometry went through changes in content that left authors in disagreement about the content of the subject. Moritz in an article entitled

“On the definition and scope of plane trigonometry,” suggests that trigonometry textbooks are poorly organized and inconsistent from one author to another. He blames this on the lack of a single definition of the subject.

Through the nineteenth century, the content of trigonometry changed and became more broad. However, as it changed, different textbook authors and, no doubt, different teachers of trigonometry disagreed and had different conceptions of its content.

After the French influence faded, American authors began writing their own textbooks. Many credited British, French, and German textbooks as being influential, but there is no clear way to differentiate how these influences are realized within the textbooks. Hofstadter (1955) claims that after the French influence, there was a German influence in American colleges and universities. The German influence appears in trigonometry education, as well.

In addition to international political influences, there were domestic social and political influences. The USMA at West Point had a great deal of influence over other colleges and universities for a number of reasons. First, West Point professors wrote a number of textbooks. These textbooks caused the practices at West Point to proliferate the colleges and universities that used them. Furthermore, West Point graduated many students who had very strong mathematics backgrounds and went on to teach mathematics at colleges and universities across the country. With them, they brought the style and often the textbooks of West Point.

West Point’s influence was felt most clearly in the area of trigonometric applications. Including applications in trigonometry textbooks originated with West Point textbook authors, beginning with Charles Hutton. Applications were confined to West

Point textbooks for several decades before the trend caught on and became universal. Even today, applications are considered an integral part of trigonometry education, and it derived from West Point textbooks.

The pressure to be like other countries affected trigonometry education. During the eighteenth and nineteenth centuries, the United States was a follower of Western European countries rather than a world leader, and there was a clear pressure for the United States to be like the Western European countries. This pressure is especially revealed by the anonymous reviewer (*American Quarterly Review*, 1827), who calls Hassler's textbook (1823), which includes only the ratio system, "a work that will afford to foreign nations a high idea of the status of knowledge in our country." At the time the reviewer was writing, France, Germany, Switzerland, and most of Western Europe were already defining trigonometric functions using the ratio system, and soon after Britain was, too (Cajori, 1890).

In 1886, Loomis revised his original trigonometry textbook (1848), against his better judgment, to include the ratio definitions of the trigonometric functions rather than the line definitions. He does so because of external pressure since the ratio system had become "well-nigh universal." Here he is, without a doubt, referring to international as well as domestic practices. Since Loomis goes against his intuition to change the definitions of the trigonometric functions, the pressure to keep up with other countries was real.

Summary

In the late nineteenth century, the debate that took place during the antebellum period has been settled. Although not every textbook author was content with the outcome, the winner was nevertheless proclaimed—the ratio system. The runner-up, the line system, struggled to find its place within the new trigonometry.

Trigonometry was expanding during this time, most notably to include the periodic trigonometric functions and power series for sine and cosine. Periodic trigonometric functions allow the functions to be used to model real-life situations where no other type of equation can, and power series for sine and cosine revolutionized trigonometric tables. Prior to this time, the mathematics behind constructing a trigonometric table was complex and arduous, especially to do so accurately, the power series for sine and cosine allowed trigonometric functions to be computed to arbitrary accuracy very easily.

In the late nineteenth century, trigonometry was beginning to make its way into other branches of mathematics and affecting them, and other branches of mathematics were affecting and changing trigonometry. During this time, trigonometry went from the science of measuring triangles to much more than that.

CHAPTER VII:

Conclusion and Recommendations

During the time period this study encompasses, trigonometry education in the United States underwent considerable changes. In the late eighteenth century, trigonometry was taught as a topic in a larger mathematics course from a textbook that had few pedagogical tools and only the most basic of trigonometric formulas. Trigonometry was taught exclusively using the line system. By the end of the nineteenth century, the ratio system had taken over completely and trigonometry was taught as its own course that covered the topic extensively with many applications to real life. Textbooks were full of pedagogical tools.

The path that the teaching of trigonometry took through the late eighteenth and nineteenth centuries did not always move in a linear fashion. Sometimes, trigonometry education stayed the same for a long time and then was suddenly changed, but other times changes happened more gradually. There were many international influences, and there were influential Americans and influential American institutions that changed the course of trigonometry instruction in this country.

This chapter brings together all the facts of history that have been laid out in the previous chapters and explains why and how the teaching of trigonometry changed over time.

Research Questions Answered

1. How did trigonometry textbooks change from 1776-1900:

a) in content? What topics were covered during this time period, and how do the topics change over time?

During the time period of this study, trigonometry textbooks changed dramatically in their content. In the late eighteenth and early nineteenth centuries, trigonometry textbooks focused on computing trigonometric functions. Logarithms had a great deal of importance in trigonometry textbooks because they made trigonometric calculations much simpler, and the calculation of trigonometric tables was a main focus of the textbooks. Applications to trigonometry emphasized surveying and navigation. At this time, the line system was the only means of defining the trigonometric functions.

In the antebellum period, trigonometry textbooks grew in size and content. During this time, the discussion of trigonometric functions as periodically-changing functions became an important topic. Also, the ratio system appeared and grew in importance. Applications of trigonometric functions remained focused on surveying and navigation. Logarithms and trigonometric calculations were still given places of importance as well.

In the late nineteenth century, trigonometric textbooks expanded even further. The ratio system completely dominated textbooks' definitions of the trigonometric functions, although the line system was still found in some textbooks. Proving trigonometric identities and solving trigonometric equations became important

topics. The calculation of trigonometric tables decreased in importance as the Taylor series for sine and cosine were used to calculate tables to arbitrary accuracy, but logarithms remained important for calculations. Topics in trigonometry also became much more varied as trigonometry prepared students not only for surveying and navigation but also for calculus.

1. How did trigonometry textbooks change from 1776-1900:

b) in approach? Namely, in what order are the topics presented, and with what emphasis on each topic?

The major trend in the approach of trigonometry textbooks was increasing prevalence of algebraic topics and algebraic methods. In the late eighteenth and early nineteenth centuries, trigonometry was an extension of geometry. The proofs of the theorems were given by geometric methods, and the trigonometric functions were defined geometrically, as lines on a circle.

Over time, algebraic methods made an appearance. At first, the trigonometric ratios (the basis of analytic trigonometry) were given at the end of the textbook. As time went on, most textbooks began to move the trigonometric ratios earlier and earlier, and then they used the ratios to prove subsequent theorems, especially when the proofs were easier using the ratio system. Finally, the ratio system became the way to define the trigonometric functions, and many more theorems were proved using algebra rather than trigonometry.

Eventually, the ratio system and the line system changed places. At the beginning of the nineteenth century, the line system defined trigonometry and the ratio system was given at the end of the text or left out entirely. By the end of the nineteenth century, the ratio system defined trigonometry and the line system was given later or left out entirely.

1. How did trigonometry textbooks change from 1776-1900:

c) in pedagogy? Particularly, what types of questions and problems are posed to students within the textbooks, are answers and/or solutions given, and how many? What other pedagogical techniques are used?

From 1776 to 1900, the pedagogy in textbooks made great advances. In the late eighteenth century, overt pedagogical tools were basically absent from textbooks. Exercises for students to complete with either solutions given or answers only given (usually some of each) became common by the beginning of the nineteenth century. Setting off formulas and rules in special type that students needed to memorize was also common during this time. Moving forward in the nineteenth century, the numbers and different types of exercises grew substantially. Exercises that applied trigonometry to real-life situations appeared and became more prevalent throughout the nineteenth century.

By the late nineteenth century, trigonometric textbooks contained abundant exercises, about ten times more than they did in the antebellum period, ranging from converting between different types of angle measurements and solving triangles to

proving trigonometric identities and solving trigonometric equations. Sections on the applications of trigonometry to real-life situations became universal by this time. One textbook even included sections of oral work to help students memorize necessary formulas.

2. How did the contributions of Euler and others in the field of trigonometry influence the teaching of trigonometry in colleges and universities?

The contributions of Euler and others to analytic trigonometry ultimately caused three major shifts in the teaching of trigonometry in colleges and universities. First, the increased emphasis on trigonometry caused it to become its own field, with textbooks of its own, rather than being a part of a larger course on mathematics where trigonometry was simply seen as a means to an end.

Second, the work of Euler and Klugel in particular allowed the trigonometric functions to be defined by the ratio system. It was Klugel who first defined the trigonometric functions as ratios, and this way of defining the trigonometric functions eventually made its way to elementary teaching of trigonometry.

Finally, it was the body of analytic trigonometry that caused authors of elementary trigonometric textbooks to set the radius equal to one in trigonometric formulas. After analytic trigonometry popularized this practice, elementary trigonometry changed to accommodate this difference. Prior to that, the radius was seen to be a part of every trigonometric definition, formula, and theorem.

3. What were the social and political factors affecting higher education during this time period, and how did these factors affect trigonometry education?

The most significant social change affecting higher education during this time was a significant increase in numbers and diversity, both of students and colleges (Cajori, 1890; Thwing, 1906). From 1776 to 1900, the numbers of students attending higher education increased dramatically. As their numbers increased, so did the students' diversity. Students were going to college with different levels of preparedness and different goals for the future. Also, the number of colleges rose dramatically, and colleges became more diverse among themselves. At the beginning of the republic, the Eastern colleges that existed were very similar to each other, but over time state universities and other private institutions became common (Thelin, 2004). The increase in numbers of colleges and students also affected the numbers and quality of teachers, but the trend among teachers was toward a greater degree of uniformity, especially in how they were prepared (Cohen and Kisker, 2010). Still, not all teachers of mathematics in colleges were well-qualified to teach the subject (Cajori, 1890).

Higher education was also affected by international influences. The first influence was British, followed by a strong French influence, and finally there was a German influence in the late nineteenth century. The USMA at West Point was among the most powerful national influences.

Trigonometry education was affected by each of the social and political factors that affected higher education as a whole. As the numbers of students, colleges, and teachers increased, textbooks accommodated this by adding additional pedagogical tools

so that the textbook could help to teach the students, rather than relying solely on the teacher for instruction and exercises. Textbook authors also gave more specific instruction to teachers, knowing that they might not be superbly qualified to teach the subject.

Trigonometry education was affected deeply by international influences. The British influence set the standard for the type of pedagogy and exercises that American textbooks would contain. The French influence brought the ratio system, the unit circle, and the periodic trigonometric functions. The German influence brought the ratio definitions and a great deal of new pedagogy.

The USMA at West Point was the strongest national influence on trigonometry education and began many of the pedagogical practices, most especially the inclusion of real-life applications to trigonometry.

4. Who were the major players in trigonometry education during this time, and what influence did they have on the teaching of trigonometry?

Three American textbook authors were extremely influential in trigonometry education from 1776 to 1900—Charles Davies, Benjamin Peirce, and Elias Loomis. Although there were some international textbook authors that were also influential in American trigonometry education, this study will focus on those authors from within the United States.

Charles Davies' 1838 trigonometry textbook was prolific from the time it was written. It was in use until the turn of the twentieth century in certain colleges, and it was

extremely popular in a wide variety of colleges and universities (Cajori, 1890). Because this textbook was used in many colleges and universities but also because it was in use for a long time, it was very influential. This textbook was traditional in that it defined trigonometric functions according to their line definitions, but it did introduce the trigonometric ratios later in the textbook. This textbook was known for its readability and for the ease with which students were able to use it, which helps explain its enduring popularity.

Benjamin Peirce's 1838 textbook was also extremely influential, but for a different reason than Davies'. Peirce's textbook was the second textbook in the United States to define the trigonometric functions using the ratio system to define the trigonometric functions, and the first to be widely used. Although Peirce's textbook was used widely outside of Harvard because it was seen to be too difficult for some students, it was used for several decades at Harvard, and it went through several editions. Because it was used in a number of places and for a long time, Peirce's textbook was groundbreaking. It was the first textbook to define trigonometric functions as ratios and to stand the test of time as well.

Elias Loomis was another very influential textbook author. His first *Trigonometry* came out in 1848. This textbook was very popular, and it defined trigonometric functions using the line system. It was reprinted so many times that the stereotype plates became worn out and had to be recast (Loomis, 1890). At that time, in 1886, Loomis decided to change his textbook so as to define trigonometric functions according to the ratio system. In doing so, he signified the end of the era of the line system. Cajori (1890) says that Loomis was more committed to the line system than any other textbook author, and

Cajori is convinced that the line system had been put to rest ever since Loomis published his new textbook.

These three textbook authors were each extremely influential, but for different reasons. Davies wrote the most enduringly popular textbook, Peirce's textbook signaled the beginning of the era of the ratio system, and Loomis' revised textbook signaled the end of the era of the line system.

A Consideration of the Reasons for the Changes

The greatest changes in elementary trigonometry education from 1776 to 1900 were the expansion of the subject's content, the growth in its pedagogy, and the movement from geometric trigonometry to algebraic trigonometry (in particular, from the line system to the ratio system). Both the overall growth of trigonometry and the development toward algebraic trigonometry stemmed from the contributions of Euler and others in the field of analytic trigonometry. International pressure, increased numbers of students with diverse mathematical backgrounds, and alleged ease of teaching also played a role in the movement toward the ratio system. Advances in printing technology, British influence, increased numbers of students and teachers, the influence of the USMA at West Point, and several influential American textbook authors all contributed to the remarkable pedagogical progress.

Many of the pedagogical changes were mirrored in other mathematical fields at this time (Ackerberg-Hastings, 2000), but the growth of the content of trigonometry and the move toward the ratio system were unique to trigonometry. Considering how much trigonometry education changed in slightly over a century, it is not surprising that modern

authors are perplexed as to the reasons for some of the changes (Bressoud, 2010). Voices within the mathematics education community who do not agree with all of the changes, particularly the change from the line system to the ratio system are not a new phenomenon (Wallace, 1828). At some point, however, these changes became “well-nigh universal” (Loomis, 1890), and they were accepted as the standard mode of trigonometry instruction in the United States.

Limitations of the Study

This study is limited to the years from 1776 to 1900, to the United States, and to higher education. It does not consider any time before 1776 because to do so thoroughly would have required studying trigonometry education in Britain as well. Prior to 1776, America was a British colony and the educational systems of America and Great Britain were intertwined. For this reason, studying times earlier than 1776 would require also studying British trigonometry education, most especially English trigonometry education since the earliest American colleges borrowed textbooks, professors, and pedagogy from English universities.

After 1900, trigonometry was taught some colleges and universities, but it was more often taught in secondary schools. To extend the study later, it would be necessary to study secondary schools as well. Since secondary schools have their own social and political forces, influential people, teachers, and pedagogy, it was not possible for this study to extend in that direction.

Although this study includes textbooks that were used in the United States but written by foreigners, it does not study any international textbooks that were not used in

the United States. Because it is limited to those textbooks that were used in the United States, it is not possible to understand fully the foreign influences on trigonometry education. Without studying trigonometry education in all of Western Europe, there is a limited understanding of the influences coming from those countries. Unfortunately, such study is outside the scope of this dissertation.

This study is also somewhat limited by the primary sources that were available. Although many primary sources were available and were utilized, more resources would have led to a more complete study. Many of the textbooks studied here were found using Google Books, where libraries can scan and upload books that are no longer subject to copyright laws. Since the textbooks used in this study are very old, it was not possible to borrow the textbooks from distant libraries. In a few cases, the author located textbooks and the library holding the textbook was able to upload that textbook to Google Books by request. Still, there were some textbooks that would have been helpful but that could not be located.

Also, primary sources other than textbooks were difficult to locate. More of such resources (i.e., exams, course notebooks, letters or diaries of teachers or students, and so on) would have further completed the study. Especially for times as distant as the late eighteenth century, such sources are few and difficult to find. For the late nineteenth century, there were many more primary sources other than textbooks that were located and included in the study.

Recommendations for Further Study

In order to understand more fully the history of trigonometry education, study similar to this study could be conducted for the time prior to the years of this study, and it would also be helpful to study trigonometry education in Great Britain for that timeframe. A study of French, German, or British trigonometry education during the same time period as this study would also shed a great deal of light on American trigonometry education. To expand this study, it would be prudent to study trigonometry education in secondary schools in the late nineteenth and twentieth centuries.

There is also a need for further study concerning trigonometry education that is not historical study itself, but that uses history as a guide. Since teachers and mathematicians are dissatisfied with the current state of trigonometry education (i.e., Bressoud, 2010), it would be helpful to study ways in which trigonometry education could be improved by considering the history trigonometry education.

For example, there need to be studies that teach a group of students the line system in addition to or instead of the ratio system and compare the outcome using an examination with students who are taught only the ratio system.

Furthermore, where education is concerned—and, in particular, trigonometry education—it is not necessary to reinvent the wheel. Instead of coming up with new ways to teach trigonometry, it is possible to use the information gained by studying the history of trigonometry education to inform future research and future teaching practices. Extensions of this study can use it as a guide and as a basis for current trigonometry education.

Final Remarks

As Bressoud mentioned, Henri Poincare (1899) suggested that the best way to teach a science is the way that it was discovered through history. Poincare said, “The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide” (p. 159). By 1900, the teaching of trigonometry in the United States had not only divorced itself from the “old line system” (Cajori, 1890), but it had separated itself from the history of the science as well.

It is not clear whether it would be possible to go back and recover the line system at this point in trigonometry education, but it would be helpful at the very least to include the line system as part of trigonometry education in the United States. Ideally, trigonometry education would follow the path of its historical development, meaning that the line system would come first, followed by the ratio system. If that change would be too difficult to make at this point, then it is at least necessary that the line system should be included as part of the trigonometry curriculum.

Modern technology has shown geometric models to be extremely helpful for cultivating students’ understandings. For example, it is possible to create a geometric model of the line system in Geometer’s Sketchpad™ that simultaneously shows the trigonometric functions changing as the angle moves around the circle and the graphs on the x-y coordinate plane of each function. Such an addition to the current curriculum would not require any radical changes, but it would help to incorporate the line system, promote student understanding, and help to connect the origins of trigonometry with its modern day uses.

Not only would this help trigonometry to follow Poincare's recommendation, but it would also help satisfy the trigonometry teachers like Loomis and Wallace who suggested that there is benefit for the students in teaching trigonometry according to the line system. Trigonometry education in the United States has become, as Wallace warned, "a house without a foundation." If the line system was included in trigonometry education, it would help to place trigonometry on its proper foundation.

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APPENDIX A:
Late Nineteenth Century Textbooks

Henry H. Ludlow's *Elements of Trigonometry with logarithmic and other tables*.

Ludlow's first section discusses the measurement of angles. He includes the following topics:

- The measurement of angles
- Exercises on finding the measure of the interior angles of eleven regular polygons. For these exercises, the correct answers are given.
- The sexagesimal measure of angles
- The centesimal measure of angles
- The radian measure of angles
- Exercises (all with answers given) that require students to translate back and forth and find the radian measure for a variety of radii.
- The quadrant of various angles and negative angles and has
- Seventeen total exercises to practice doing so in degrees, grades, and radians including eight real-life applications.
- Complements and supplements of angles.

Exercises, sixteen finding the complement and thirteen finding the supplement using all three types of angle measure.

After his section on angle measure, Ludlow defines the trigonometric functions according to their ratio definitions. He defines the following:

Sine. Defines and discusses when sine is positive and negative and what happens to sine as the angle moves around the circle. A circular diagram accompanies the definition, and the opposite side is defined as a segment on the circle.

Exercises sixty-three in total, including to construct and determine by measuring the sine of twenty-seven different angles and a series of questions about which is the greater quantity (for example $\sin 45$ or $\frac{1}{2} \sin 90$).

Cosine. Ludlow proves as a theorem the following: the cosine of any angle is equal to the sine of its complement. He has a similar discussion and exercises as he did for sine.

Tangent. Ludlow gives a similar discussion and exercises as were given for sine and cosine.

Cotangent. Ludlow proves as a theorem: *The cotangent of any angle is equal to the tangent of its complement.* Similar discussion and exercises as given for the previous trigonometric functions, but fewer exercises.

Secant. Similar discussion and exercises as given for cotangent.

Cosecant. Ludlow proves as a theorem: *The cosecant of any angle is equal to the secant of its complement.* Similar discussion and exercises as given for cotangent and secant.

Versed sine (Ludlow calls this versine, an alternate name). Ludlow defines versed sine as the ratio of the distance from the foot of the perpendicular to the arc to the radius. Similar discussion and exercises as given for cotangent, secant, and cosecant.

Coversed sine. Ludlow defines the coversed sine as the ratio of the distance of the complementary angle's perpendicular distance to the arc to the radius. Ludlow proves as

a theorem: *The covered sine of any angle is equal to the versed-sine of its complement.*

He has a similar discussion and exercises as he had for cotangent, secant, and cosecant.

Inverse trigonometric functions. Ludlow defines them, not mentioning that they can also be called arc-functions, and he gives two exercises for students to find the inverse functions of a numerical value and then fifteen exercises for students to find the inverse function of another trigonometric function. For example, “find $\cot^{-1}(\cos 60^\circ)$ ” (p. 36).

Having given these definitions, Ludlow uses a portion of the circle (see Figure A.1) to observe geometrically several identities and formulas. From these, he uses algebra to prove other identities and formulas.

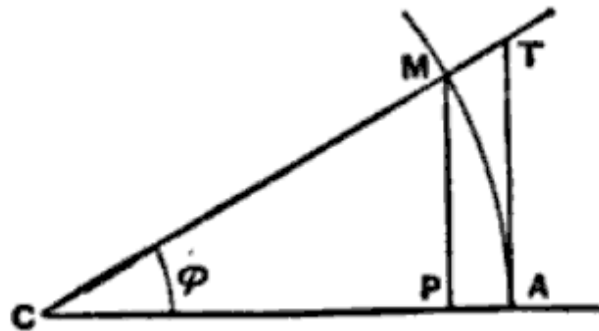


Figure A.1

Ludlow geometrically observes the following:

- Principle Pythagorean identity
- Versed sine = 1 - cosine
- $\tan(x) = \sin(x) / \cos(x)$
- $\sec(x) = 1 / \cos(x)$
- $\sec^2(x) = 1 + \tan^2(x)$ then the two other Pythagorean identities

- Reciprocal relationships of the trigonometric functions
- From these, Ludlow proves the following using algebra from the previous theorems that he observed geometrically:
 - covered sine $(x) = 1 - \sin(x)$,
 - $\cot(x) = \cos(x) / \sin(x)$
 - $\tan(x) \cot(x) = 1$
 - $\csc(x) = 1 / \sin(x)$
 - Third Pythagorean identity

After proving many identities and theorems, Ludlow poses the following problem:

“To express any trigonometric function in terms of any other of the same angle” (p. 39).

He gives two examples: he expresses sine in terms of cosine using the principle Pythagorean identity, then versed sine in terms of covered sine. He gives four exercises where students are given the value of one trigonometric function and they have to find others. Answers are given. After these exercises, Ludlow gives three exercises where students must express sine, cosine, and tangent separately in terms of each of the other trigonometric functions. For these, answers are not given.

After showing that trigonometric functions can be expressed in terms of one another, Ludlow gives a series of theorems, each followed by exercises. Theorems, as follows, are given in italics:

- *The sine of any angle less than 180° is equal to the ratio of half the chord of twice the subtending arc to the corresponding radius.*
- Two exercises, no answers given.

- *The chord of any arc is equal to twice the product of the radius by the sine of half the angle at the center.*
- Uses the aforementioned theorem to find sine of 30° , cosine of 60° , tangent of 45° .
- From the three previous trigonometric functions of special angles, Ludlow finds all the trigonometric functions of 30° , 45° , and 60° and puts them in a table.
- He gives as an exercise to verify all the trigonometric functions in the table, and to find the exact values of the trigonometric functions of 18° .

After finding the exact values for the trigonometric functions of special angles, Ludlow shows how to reduce trigonometric functions of large angles to trigonometric functions of smaller angles. First, he shows how to reduce any angle to one within 360° . He gives fourteen exercises on the topic. Ludlow then shows how to reduce any trigonometric function within 360° to one that is equivalent in the first quadrant. He gives thirty-five exercises to practice that task. Ludlow also discusses how to find equal positive angles for negative angles, with five exercises and their answers given. Finally, Ludlow concludes this section by discussing how to find all the angles that have the same trigonometric values. For this topic, he includes seventeen exercises, with answers given for first thirteen.

Ludlow goes on to prove two theorems on the boundaries and limits of trigonometric functions:

- *The radian measure of any acute angle is greater than its sine, and less than its tangent.*

- *Unity is the limit of the ratio of an angle to its sine, of an angle to its tangent, and of the tangent to the sine, as the angle approaches zero.*

Ludlow also shows some trigonometric formulas:

- *Sum and difference formulas* with eight exercises
- *Double angle formulas* with seven exercises
- *Half-angle formulas* with twelve exercises
- *Multiple angles formulas* with four exercises
- Six exercises to prove other trigonometric identities.
- In this section of trigonometric formulas, the exercises do not have answers given.

Ludlow follows these formulas with several theorems:

- *The sum of the sines of any two angles is to the difference of their sines as the tangent of half the sum of the angles is to the tangent of half their difference.*
- *The sum of the sines of any two angles is to the sine of their sum as the sine of their difference is to the difference of their sines.*

Ludlow then undertakes the task of developing sine and cosine into a power series. First he develops power series for $\sin(x)$ and $\cos(x)$, then $\sin(x+y)$, $\cos(x+y)$. Ludlow uses the power series to find sine and cosine to arbitrary accuracy. He gives four exercises to find either sine or cosine of an angle accurately to six decimal places, and for these exercises, answers are given.

After the section on power series, Ludlow shows how to reduce trigonometric functions that are raised to powers, followed by six exercises with answers given.

After that, he shows how to solve trigonometric equations using different methods as follows:

- *Literal Equations*
- Five equations for practice with solutions given
- Uses the main Pythagorean identity to solve equations
- Five equations for practice with the solution given for the first one
- Uses elimination by division
- Five equations for practice, with the solution given for first three.
- Uses sum and difference formulas to simplify and solve equations
- Five equations for practice, with answers given for all.
- Uses multiple angle formulas to simply and solve equations
- Five equations for practice, with answers given for all
- Uses inverse trigonometric functions.
- Six exercises, with answers given for five.
- Solving numerical equations.
- Three equations for practice, with the solution given for one and answers for other two
- If two equations with two unknowns are given, it might be possible to simplify this by division into one trigonometric function and solve that equation first.
- Five exercises, the solution is given for one and answers only given for other four
- Similar discussion for three equations, three unknowns.
- Three exercises, the solution is given for one and answers only are given for the other two
- Discusses how to solve when the unknown variable is within a sum or difference of angles in a trigonometric function.

- Four equations for practice, answers given for all.

After concluding his section on trigonometric equations, Ludlow has a section on solving triangles. He shows how to solve the following:

- Right plane triangles, in four cases, with a discussion and two answered exercises for each case
- Thirteen miscellaneous exercises, with answers given for all.
- Discussion of projections and other applications.
- Fourteen application exercises with answers given for all.
- Oblique plane triangles using two auxiliary right triangles.
- *Law of Sines.*
- *Law of cosines.*
- *Heron's formula for area.*
- $Area = 1/2 bc \sin A.$
- Expressions for radii of circumscribed and inscribed circles.
- Solving oblique triangles in four cases, giving between three and ten exercises for each case, one solved, all others with the answer given.
- Twenty additional exercises, with answers given.
- *Applications*, 8 exercises involving real-life applications, with no answers given.

The solution of plane triangles concludes Ludlow's textbook.

Elias Loomis' revised edition of his *Elements of Plane and Spherical Trigonometry*. The textbook considered here is a 1890 reprint of an originally 1886 publication.

At the beginning of the textbook, Loomis defines the following:

- Trigonometry
- The measurement of angles, showing that the ratios of sides of an angle is the same if the angle is the same, no matter what the lengths of the sides.
- Trigonometric functions are given as ratios of the sides of a right triangle.
- Sine
- Tangent
- Secant
- The co-functions are given as the functions of the other non-right angle of the triangle.

At this point, Loomis shows that these definitions are equivalent to the old line representations. “If the radius of the circle be taken equal to unity, the trigonometric functions above defined may be represented by *straight lines*.” He goes on to say for example, “The secant of an arc is that part of the produced diameter which is intercepted between the center and the tangent.” He also defines versed sine in this section. Loomis uses the line representations to show the reciprocal relationships of the trigonometric functions.

Loomis presents the following ideas:

- Sines and tangents of special angles
- How to use both the natural and logarithmic trigonometric tables
- Discusses the solution of right triangles in four cases.
- Each case has two exercises, one with solution given and one with answer only

Loomis says, following the first set of exercises, “The student should work this and the following examples both by natural numbers and by logarithms until he has made himself

perfectly familiar with both methods. He may then employ either method, as may appear to him most expeditious.” At the end of the section are given six exercises for practice with no answers or solutions are given. To conclude the section on right triangles, Loomis gives the Pythagorean theorem, with two exercises, one with the solution given and one with answer only.

After his section on solving right triangles, Loomis has a section on solving oblique triangles. Loomis first proves the law of sines and the law of tangents. He shows how to solve oblique triangles in four cases, with same format of exercises as the right triangle section.

Loomis moves on to a section that shows how the trigonometric functions change as the angle moves around the circle. He discusses the following:

- The signs of sine, cosine, and tangent in the four quadrants
- Sines and cosines of sums and differences
- Sines and cosines of negative angles
- Multiple and half angle formulas
- Method for computing the trigonometric tables including the law of cosines

After these discussions, Loomis gives general exercises for practice, all with answers only given. He gives a total of twenty-four exercises of which four are computational, ten are real-life applications of trigonometry, and ten are proofs. Among the applications and the proofs, many of the problems are not straightforward. These exercises conclude Loomis’ text.

Trigonometry for Schools and Colleges, published in 1896, written by Frederick Andregg and Edward Drake Roe.

Andregg and Roe define the trigonometric functions as follows: “The six ratios which can be formed by using the three sides of the triangle of reference of a given angle, two at a time, are called the six primary trigonometric functions of the angle.” They give the ratio definitions of the trigonometric functions: sine, cosine, tangent, cotangent, secant, cosecant. They also define versed sine, covered sine, suversed sine, and sucovered sine using their related trigonometric ratios. Following the definitions, they discuss the following:

- Signs of the trigonometric functions in the four quadrants
- The reciprocal relationships of the trigonometric functions
- The Pythagorean identities
- Forty-four exercises, including thirty-four proofs of trigonometric identities
- Limits of trigonometric functions

After the previous topics, Andregg and Roe have a section on the line representations of trigonometric functions. About the the relationship of the lines to the functions they say, “The student should remember that the line is *not* the function, but *represents* it.” They continually reiterate throughout that section that the lines are representations, rather than the functions themselves.

Following the section on the line representations, Andregg and Roe discuss the inverse trigonometric functions, which they call anti-functions. They do acknowledge that some authors call them arc functions and they also use inverse notation when they

are writing expressions using the inverse functions. They give twenty-three exercises, mostly concerning the ranges of values that each of the trigonometric functions achieve.

Andregg and Roe discuss the following:

- Trigonometric functions of negative angles
- Trigonometric functions when adding or subtracting 90, 180, 270
- A generalization for adding or subtracting all multiples of 90
- The periodicity of the trigonometric functions
- Trigonometric functions of special angles.
- Fifty-two exercises, including eighteen proofs and many higher-level problems, for example, obtaining general solutions to equations such as: $\cos(x) = \frac{1}{2}$, $4 \cos(x) = \sec(x)$.
- Brief discussion on projection, when line and axis are or are not coplanar
- Trigonometric functions of sums and differences
- Trigonometric functions of multiple angles
- Trigonometric functions of half angles
- Eighty-two exercises including six proofs and several real-life applications

Following these exercises, there is a section on the solution of triangles. It includes the following:

- Solving right triangles in four cases, with eight exercises
- Practical applications to problems on heights and distances (including angles of elevation and depression, immeasurable distances, etc.)
- Thirteen examples, all applications
- Solving oblique triangles

- Law of sines
- Law of tangents
- Law of cosines
- Formulas for the area of a triangle
- Formulas for radii of inscribed and circumscribed triangles.
- How to solve oblique triangles in four cases
- Twenty-six exercises, including fifteen real-life examples

The solution of triangles is the conclusion of Andregg and Roe's textbook.