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A Hybrid Spectral Element Method for Fractional Two-Point Boundary Value Problems

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. We propose a hybrid spectral element method for fractional two-point boundary value problem (FBVPs) involving both Caputo and Riemann-Liouville (RL) fractional derivatives. We first formulate these FBVPs as a second kind Volterra integral equation (VIEs) with weakly singular kernel, following a similar procedure in [16]. We then design a hybrid spectral element method with generalized Jacobi functions and Legendre polynomials as basis functions. The use of generalized Jacobi functions allow us to deal with the usual singularity of solutions at $t=0$. We establish the existence and uniqueness of the numerical solution, and derive a hp-type error estimates under $L^2(I)$-norm for the transformed VIEs. Numerical results are provided to show the effectiveness of the proposed methods.

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Key words: Boundary value problems, generalized Jacobi functions, spectral element method, fractional differential equations, error bounds.

1. Introduction

This paper is concerned with numerical solutions of the following FBVPs:

$$-{}_0^*D_t^{2-\delta}u(t)+b(t)u'(t)+c(t)u(t)=f(t),\quad t\in(0,T), \tag{1.1}$$

with Robin or Dirichlet boundary conditions

$$u(0)-\alpha_0u'(0)=\gamma_0,\qquad u(T)+\alpha_1u'(T)=\gamma_1, \tag{1.2a}$$
$$u(0)=\gamma_0,\qquad u(T)=\gamma_1, \tag{1.2b}$$

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where $\delta \in (0,1)$, and ${}^*_0D_t^{2-\delta}$ refers to either Caputo or RL fractional derivative of order $2-\delta$ (see (2.2) and (2.4), respectively). The constants α_0, α_1, γ_0, γ_1 and the functions $b(t)$, $c(t)$ and $f(t)$ are given. In the case of (1.2a), we assume that $c(t) \geq 0$ and

$$\alpha_0 \geq \frac{1}{1-\delta} \quad \text{and} \quad \alpha_1 \geq 0. \tag{1.3}$$

The conditions $c(t) \geq 0$ and (1.3) guarantee that (1.1) with (1.2a) satisfies a suitable comparison/maximum principle, from which existence and uniqueness of the solution u of (1.1) (see, Theorem 1 of [16]).

The FBVP (1.1) is motivated by the studies on anomalous diffusion processes, which model the steady state of one-dimensional superdiffusion of particle motion when convection is present see [13, 21]. Similar to the classical diffusion case, closed form solutions are usually not available, and one has to resort to numerical methods. Some recent numerical works for (1.1) include finite difference method and piecewise polynomial collocation methods for FBVPs, see [11, 13, 16, 28] and the references therein.

Two main difficulties in solving fractional PDEs such as (1.1) are: (i) fractional derivatives are non-local operators and generally lead to full matrices; and (ii) their solutions are often singular at the endpoint(s) so polynomial based approximations are not efficient.

Since spectral methods are capable of providing exceedingly accurate numerical results with less degrees of freedoms, they have been widely used for numerical approximations of PDEs, see e.g., [4, 10, 12, 24, 25]. In recent years, spectral methods have been proposed for VIEs with smooth/weakly singular kernels. We refer to [7, 8, 18] for the p version of spectral methods and [27, 29] for the hp-version of spectral collocation methods. However, these methods are based on polynomial basis functions which are not particularly suitable for FBVPs whose solutions are generally non-smooth. In some earlier work [1, 5], the authors employed non polynomial methods for weakly singular VIEs. Very recently, Shen et al. [23, 26] proposed one-step and multi-step spectral Galerkin methods using generalized Jacobi functions for weakly singular VIEs.

The main purpose of this paper is to propose and analyze an efficient hybrid spectral element methods for FBVPs. Our approach is inspired by [16] where the authors reformulated Caputo FBVPs (1.1) with Robin boundary conditions to a second kind VIEs with weakly singular kernel, and proposed a numerical scheme based on piecewise polynomial collocation. The main advantage of this approach is that, instead of solving a two-point FBVP which couples all unknowns together, one can now use a time-marching method for VIEs. The main strategies and contributions are highlighted below:

- We extend the approach in [16] to include RL and Caputo FBVPs with other admissible boundary conditions, and propose hybrid spectral element methods with basis functions that can be tuned to match the singularities of the underlying solutions.

- We analyze and characterize the hp-version error bounds of the proposed methods. The error bounds can guide us to choose parameters h and p to achieve higher accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce some basic properties of fractional calculus and generalized Jacobi functions. In Section 3, we first transform three kinds of FBVPs into weakly singular VIEs, and present hybrid spectral element methods for the transformed weakly singular VIEs. In Section 4, We establish some useful lemmas and prove the existence, uniqueness and convergence for the proposed methods. We present in Section 5 some numerical experiments, and some concluding remarks are given in the final section.

2. Preliminaries

In this section, we first review some basics of fractional itegrals/derivatives, and some properties of the shifted generalized Jacobi functions and the shifted Legendre polynomial. We then reformulate the three kinds of FBVPs as weakly singular VIEs, and propose a hybrid spectral element methods for transformed weakly singular VIEs.

2.1. Fractional calculus

We start with some definitions of fractional calculus (see, e.g., [9, 22]). To fix the idea, we restrict our attentions to the interval $(0, T)$.

For $\rho \in \mathbb{R}^+$, the left-sided and right-sided RL integrals are respectively defined as

$$ {}_0I_t^\rho u(t) = \frac{1}{\Gamma(\rho)} \int_0^t \frac{u(s)}{(t-s)^{1-\rho}} ds, \qquad t \in (0,T), \tag{2.1a} $$

$$ {}_tI_T^\rho u(t) = \frac{1}{\Gamma(\rho)} \int_t^T \frac{u(s)}{(s-t)^{1-\rho}} ds, \qquad t \in (0,T), \tag{2.1b} $$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided RL fractional derivative of order ν is defined by

$$ {}_0D_t^\nu u(t) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_0^t \frac{u(s)}{(t-s)^{\nu-m+1}} ds, \quad t \in (0,T), \tag{2.2} $$

and the right-sided RL fractional derivative of order ν is defined by

$$ {}_tD_T^\nu u(t) = \frac{(-1)^m}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_t^T \frac{u(s)}{(s-t)^{\nu-m+1}} ds, \quad t \in (0,T). \tag{2.3} $$

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided Caputo fractional derivative of order ν is defined by

$$ {}_0^CD_t^\nu u(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{u^{(m)}(s)}{(t-s)^{\nu-m+1}} ds, \quad t \in (0,T), \tag{2.4} $$

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and the right-sided Caputo fractional derivative of order ν is defined by

$$ {}_t^C D_T^\nu u(t) = \frac{(-1)^m}{\Gamma(m-\nu)} \int_t^T \frac{u^{(m)}(s)}{(s-t)^{\nu-m+1}} ds, \quad t \in (0,T). \tag{2.5} $$

It is clear that for any $m \in \mathbb{N}_0$,

$$ {}_0D_t^m = D^m, \quad {}_tD_T^m = (-1)^m D^m, \quad \text{where} \quad D^m := \frac{d^m}{dt^m}. $$

Thus, we can define the fractional derivatives as

$$ \begin{aligned} &{}_aD_t^\nu u(t) = D^m {}_0I_t^{m-\nu} u(t), && {}_tD_T^\nu u(t) = (-1)^m D^m {}_tI_T^{m-\nu} u(t), \\ &{}_0^C D_t^\nu u(t) = {}_0 I_t^{m-\nu} D^m u(t), && {}_t^C D_T^\nu u(t) = (-1)^m {}_tI_T^{m-\nu} D^m u(t). \end{aligned} $$

According to Theorem 2.14 of [9], we have that for any absolutely integrable function u, and real $\nu \geq 0$,

$$ {}_0D_t^\nu {}_0I_t^\nu u(t) = u(t), \quad {}_tD_T^\nu {}_tI_T^\nu u(t) = u(t), \qquad t \in (0,T). \tag{2.6} $$

According to Theorem 3.8 of [9], assume that $\nu \geq 0$, $m = \lceil \nu \rceil$, and $u \in A^m[a,b]$, we have

$$ {}_0I_t^\nu {}_0^C D_t^\nu u(t) = u(t) - \sum_{k=0}^{m-1} \frac{D^k u(0)}{k!} t^k, \tag{2.7} $$

where A^m denote the set of functions with an absolutely continuous $(m-1)$st derivative.

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives (see, e.g., [9, 22]).

Lemma 2.1. *For $\nu \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$$ {}_0D_t^\nu u(t) = {}_0^C D_t^\nu u(t) + \sum_{j=0}^{k-1} \frac{u^{(j)}(0)}{\Gamma(1+j-\nu)} t^{j-\nu}. \tag{2.8} $$

2.2. Properties of basic functions

We recall below properties of generalized Jacobi functions and Legendre polynomial (see, e.g., [6, 29]), which will serve as basis functions of our hybrid spectral element methods.

For $\alpha, \beta > -1$, let $P_n^{(\alpha,\beta)}(x)$, $x \in \Lambda := (-1,1)$ be the standard Jacobi polynomial of degree n, and denote the weight function $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$. The set of Jacobi polynomials is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system, i.e.,

$$ \int_{-1}^1 P_l^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \chi^{(\alpha,\beta)}(x) dx = \gamma_l^{(\alpha,\beta)} \delta_{l,m}, \tag{2.9} $$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2l+\alpha+\beta+1)} \frac{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{l!\Gamma(l+\alpha+\beta+1)}. \tag{2.10}$$

In particular, $P_0^{(\alpha,\beta)}(x) = 1$.

2.3. Shifted Jacobi polynomials

Let I_h be a mesh on the interval $I = (0, T)$,

$$I_h := \{t_n : \ 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

We denote

$$h_n := t_n - t_{n-1}, \quad h_{\max} = \max_{1\le n\le N} h_n, \quad I_n = (t_{n-1}, t_n).$$

The shifted Jacobi polynomial of degree l on I_n is defined by

$$\widetilde{P}_{n,l}^{(\alpha,\beta)}(t) = P_l^{(\alpha,\beta)}\left(\frac{2t - t_{n-1} - t_n}{h_n}\right), \quad t \in I_n, \quad l \geq 0. \tag{2.11}$$

Clearly, the set of $\{\widetilde{P}_{n,l}^{(\alpha,\beta)}(t)\}_{l\geq 0}$ is a complete $L^2_{\chi_n^{(\alpha,\beta)}}(I_n)$-orthogonal system with the weight function $\chi_n^{(\alpha,\beta)}(t) = (t_n - t)^{\alpha}(t - t_{n-1})^{\beta}$, by (2.9) and (2.11) we get that

$$\int_{I_n} \widetilde{P}_{n,l}^{(\alpha,\beta)}(t)\widetilde{P}_{n,m}^{(\alpha,\beta)}(t)\chi_n^{(\alpha,\beta)}(t)dt = \left(\frac{h_n}{2}\right)^{\alpha+\beta+1}\gamma_l^{(\alpha,\beta)}\delta_{l,m}. \tag{2.12}$$

For any integer $M_n > 0$, we denote by $\{x_{n,j}^{(\alpha,\beta)}, \omega_{n,j}^{(\alpha,\beta)}\}_{j=0}^{M_n}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation on the interval Λ. Let $\mathcal{P}_{M_n}(I_n)$ be the set of polynomials of degree at most M_n on the interval I_n, and $t_{n,j}^{(\alpha,\beta)}$ be the shifted Jacobi-Gauss quadrature nodes on the interval I_n,

$$t_{n,j}^{(\alpha,\beta)} = \frac{1}{2}(h_n x_{n,j}^{(\alpha,\beta)} + t_{n-1} + t_n), \quad 0 \leq j \leq M_n. \tag{2.13}$$

Due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi(t) \in \mathcal{P}_{2M_n+1}(I)$ (cf. [26]), we have

$$\int_{I_n} \phi(t)\chi_n^{(\alpha,\beta)}(t)dt = \left(\frac{h_n}{2}\right)^{\alpha+\beta+1}\sum_{j=0}^{M_n}\phi(t_{n,j}^{(\alpha,\beta)})\omega_{n,j}^{(\alpha,\beta)}. \tag{2.14}$$

We shall use the shifted Legendre polynomials $L_{n,l}(t) := \widetilde{P}_{n,l}^{(0,0)}(t)$ as basis functions on I_n for $n > 1$. Since solution of FBVPs (1.1) are usually non-smooth at $t = 0$, we shall use non-polynomial basis in I_1.

2.3.1. Shifted generalized Jacobi functions on I_1

For any $\alpha,\ \beta>-1$, the shifted generalized Jacobi functions on I_1 is defined by (cf. [6])

$$J_{1,l}^{(\alpha,\beta)}(t)=t^{\beta}\widetilde{P}_{1,l}^{(\alpha,\beta)}(t),\quad t\in I_1,\quad l\geq 0, \tag{2.15}$$

and the finite-dimensional fractional-polynomial space on I_1 is defined by

$$\mathcal{F}_{M_1}^{(\beta)}(I_1):=\{t^{\beta}\psi(t):\psi(t)\in\mathcal{P}_{M_1}(I_1)\}=\mathrm{span}\{J_{1,l}^{(\alpha,\beta)}:0\leq l\leq M_1\}. \tag{2.16}$$

By (2.12) and (2.15), the set of $\{J_{1,l}^{(\alpha,\beta)}(t)\}_{l\geq 0}$ is a complete $L^2_{\chi_1^{(\alpha,-\beta)}}(I_1)$-orthogonal system with the weight function $\chi_1^{(\alpha,-\beta)}(t)$, namely,

$$\begin{aligned}&\int_{I_1}J_{1,l}^{(\alpha,\beta)}(t)J_{1,m}^{(\alpha,\beta)}(t)\chi_1^{(\alpha,-\beta)}(t)dt\\ =&\int_{I_1}t^{2\beta}\widetilde{P}_{1,l}^{(\alpha,\beta)}(t)\widetilde{P}_{1,m}^{(\alpha,\beta)}(t)\chi_1^{(\alpha,-\beta)}(t)dt\\ =&\int_{I_1}\widetilde{P}_{1,l}^{(\alpha,\beta)}(t)\widetilde{P}_{1,m}^{(\alpha,\beta)}(t)\chi_1^{(\alpha,\beta)}(t)dt\\ =&\Big(\frac{h_1}{2}\Big)^{\alpha+\beta+1}\gamma_l^{(\alpha,\beta)}\delta_{l,m}.\end{aligned} \tag{2.17}$$

By (2.14), it follows that for any $\varphi(t)=t^{2\beta}\phi(t)$ and $\phi(t)\in\mathcal{P}_{2M_1+1}(I_1)$, we have

$$\begin{aligned}\int_{I_1}\varphi(t)\chi_1^{(\alpha,-\beta)}(t)dt=\int_{I_1}\phi(t)\chi_1^{(\alpha,\beta)}(t)dt&=\Big(\frac{h_1}{2}\Big)^{\alpha+\beta+1}\sum_{j=0}^{M_1}\phi(t_{1,j}^{(\alpha,\beta)})\omega_{1,j}^{(\alpha,\beta)}\\ &=\Big(\frac{h_1}{2}\Big)^{\alpha+\beta+1}\sum_{j=0}^{M_1}(t_{1,j}^{(\alpha,\beta)})^{-2\beta}\varphi(t_{1,j}^{(\alpha,\beta)})\omega_{1,j}^{(\alpha,\beta)}.\end{aligned} \tag{2.18}$$

Next, let $(u,v)_{\chi_1^{(\alpha,-\beta)}}$ and $\|v\|_{\chi_1^{(\alpha,-\beta)}}$ be the inner product and the norm of space $L^2_{\chi_1^{(\alpha,-\beta)}}(I_1)$ respectively. We also introduce the following discrete inner product on the interval I_1,

$$\langle u,v\rangle_{\chi_1^{(\alpha,-\beta)}}=\Big(\frac{h_1}{2}\Big)^{\alpha+\beta+1}\sum_{j=0}^{M_1}(t_{1,j}^{(\alpha,\beta)})^{-2\beta}u(t_{1,j}^{(\alpha,\beta)})v(t_{1,j}^{(\alpha,\beta)})\omega_{1,j}^{(\alpha,\beta)}. \tag{2.19}$$

Thanks to (2.18), for any $\phi,\psi\in\mathcal{F}_{M_1}^{(\beta)}(I_1)$,

$$(\phi,\psi)_{\chi_1^{(\alpha,-\beta)}}=\langle\phi,\psi\rangle_{\chi_1^{(\alpha,-\beta)}}. \tag{2.20}$$

3. Hybrid spectral element methods: formulation

We shall first transform (1.1) with different boundary conditions into weakly singular VIEs, then we construct efficient hybrid spectral element methods for them.

3.1. Transformation to weakly singular VIEs

We consider three admissible cases separately below:

Caputo-FBVPs (1.1) **with B. C. Robin** (1.2a).

It follows from (2.7) with $\nu = 1-\delta$, we have

$$ {}_0I_t^{1-\delta}\big({}_0^CD_t^{2-\delta}u\big)(t) = {}_0I_t^{1-\delta}\big({}_0^CD_t^{1-\delta}u'\big)(t) = u'(t) - u'(0). $$

Hence, applying ${}_0I_t^{1-\delta}$ to (1.1), we obtain that

$$ -u'(t) + u'(0) + {}_0I_t^{1-\delta}(bu' + cu)(t) = {}_0I_t^{1-\delta}(f)(t). \tag{3.1} $$

Set

$$ \mu = u'(0), \quad z(t) = u'(t) - \mu, \quad Z(t) = \int_0^t z(s)ds, \quad t \in I := [0,T]. \tag{3.2} $$

Then, we can use Rbc to conclude

$$ (cu)(t) = c(t)\Big[\int_0^t \big(u'(s) - u'(0)\big)ds + \mu t + u(0)\Big] = (cZ)(t) + \mu(t+\alpha_0)c(t) + \gamma_0 c(t). $$

Consequently, (3.1) can be rewritten as

$$ z(t) - {}_0I_t^{1-\delta}(bz + cZ)(t) = {}_0I_t^{1-\delta}(\mu g_1 + g_2)(t), \tag{3.3} $$

where

$$ g_1(t) = b(t) + (t+\alpha_0)c(t), \quad g_2 = \gamma_0 c(t) - f(t), \quad t \in I. \tag{3.4} $$

Caputo-FBVPs (1.1) **with B. C. Dirichlet** (1.2b).

By a similar argument as before, we apply ${}_0I_t^{1-\delta}$ again to (1.1) yields

$$ -u'(t) + u'(0) + {}_0I_t^{1-\delta}(bu' + cu)(t) = {}_0I_t^{1-\delta}(f)(t). \tag{3.5} $$

Meanwhile, by using the Dbc, we get that

$$ (cu)(t) = c(t)\Big[\int_0^t \big(u'(s) - u'(0)\big)ds + \mu t + u(0)\Big] = (cZ)(t) + \mu t c(t) + \gamma_0 c(t). $$

The above with (3.2) and (3.5) leads to

$$ z(t) - {}_0I_t^{1-\delta}(bz + cZ)(t) = {}_0I_t^{1-\delta}(\mu g_1 + g_2)(t), \tag{3.6} $$

where

$$ g_1(t) = b(t) + tc(t), \quad g_2 = \gamma_0 c(t) - f(t), \quad t \in I. \tag{3.7} $$

RL-FBVPs (1.1) **with B. C. Dirichlet** (1.2b).

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We now turn to following RL FBVPs with (1.2b):

$$-{}_0D_t^{2-\delta}u(t)+b(t)u'(t)+c(t)u(t)=f(t), \qquad t\in(0,T), \tag{3.8a}$$
$$u(0)=0, \qquad u(T)=\gamma_1. \tag{3.8b}$$

Note that with RL derivative, only homogeneous Dirichlet condition at $t=0$ can be considered. Clearly, we obtain from (2.8) and the above equation that

$$-{}_0^CD_t^{2-\delta}u(t)+b(t)u'(t)+c(t)u(t)=f(t)+\frac{\mu t^{\delta-1}}{\Gamma(\delta)}. \tag{3.9}$$

Analogously, applying ${}_0I_t^{1-\delta}$ to (3.9), we obtain that

$$-u'(t)+u'(0)+{}_0I_t^{1-\delta}(bu'+cu)(t)={}_0I_t^{1-\delta}(f)(t)+{}_0I_t^{1-\delta}\left(\frac{\mu t^{\delta-1}}{\Gamma(\delta)}\right). \tag{3.10}$$

Hence, we note that by $u(0)=0$, there holds

$$(cu)(t)=c(t)\Big[\int_0^t\big(u'(s)-u'(0)\big)ds+\mu t\Big]=(cZ)(t)+\mu tc(t).$$

This togerher with (3.2) and (3.10) implies that

$$z(t)-{}_0I_t^{1-\delta}(bz+cZ)(t)={}_0I_t^{1-\delta}(\mu g_1+g_2)(t), \tag{3.11}$$

where

$$g_1(t)=-\frac{t^{\delta-1}}{\Gamma(\delta)}+b(t)+tc(t), \quad g_2=-f(t), \quad t\in I. \tag{3.12}$$

In all three cases, we are led to a second kind weakly singular VIEs (3.3), (3.6) and (3.11). Hence we only need to consider the following weakly singular VIEs

$$\begin{aligned}&y(t)-C_\delta\int_0^t(t-s)^{-\delta}b(s)y(s)ds-C_\delta\int_0^t(t-s)^{-\delta}c(s)\Big[\int_0^s y(\tau)d\tau\Big]ds\\ =&C_\delta\int_0^t(t-s)^{-\delta}g(s)ds,\end{aligned} \tag{3.13}$$

where $C_\delta=1/\Gamma(1-\delta)$, g can be g_1, g_2 (see (3.4), (3.7) and (3.12)), and whose solution $y(t)$ can be v and w respectively, this imply that $y=\mu v+w$.

In order to use a spectral-element method, we shall first rewrite (3.13) as a sequence of equations in non-overlapping time intervals $\{I_n: n=1,\cdots,N\}$.

Let $y^n(t)$ the solution of (3.13) on the n-th element I_n, namely,

$$y^n(t)=y(t), \quad \forall t\in I_n, \quad 1\le n\le N.$$

From (3.13) we have that for any $t \in I_n$,

$$\begin{aligned}
&y^n(t) - C_\delta \int_0^{t_{n-1}} (t-\xi)^{-\delta} b(\xi) y^k(\xi) d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} b(s) y^n(s) ds \\
&\quad - C_\delta \int_0^{t_{n-1}} (t-\xi)^{-\delta} c(\xi) \Big[\int_0^\xi y(\sigma) d\sigma \Big] d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} c(s) \Big[\int_0^{t_{n-1}} y(\sigma) d\sigma \Big] ds \\
&\quad - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} c(s) \Big[\int_{t_{n-1}}^s y^n(\tau) d\tau \Big] ds \\
=& C_\delta \int_0^{t_{n-1}} (t-\xi)^{-\delta} g(\xi) d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} g(s) ds.
\end{aligned} \tag{3.14}$$

The Eq. (3.14) can be rewritten as

$$\begin{aligned}
&y^n(t) - C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} b(\xi) y^k(\xi) d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} b(s) y^n(s) ds \\
&\quad - C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} c(\xi) \Big[\int_0^\xi y(\tau) d\tau \Big] d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} c(s) \Big[\sum_{l=1}^{n-1} \int_{I_l} y^l(\sigma) d\sigma \Big] ds \\
&\quad - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} c(s) \Big[\int_{t_{n-1}}^s y^n(\tau) d\tau \Big] ds \\
=& C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} g(\xi) d\xi - C_\delta \int_{t_{n-1}}^t (t-s)^{-\delta} g(s) ds.
\end{aligned} \tag{3.15}$$

In order to transfer the integral intervals $(t_{n-1}, t]$ to I_n, we make the following linear transformation:

$$s = s(t,\lambda) := t_{n-1} + \frac{(\lambda - t_{n-1})(t - t_{n-1})}{h_n}, \quad \lambda \in I_n. \tag{3.16}$$

Then, the Eq. (3.15) reads

$$\begin{aligned}
&y^n(t) - C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} b(\xi) y^k(\xi) d\xi \\
&\quad - C_\delta \Big(\frac{t - t_{n-1}}{h_n} \Big)^{1-\delta} \int_{I_n} (t_n - \lambda)^{-\delta} b(s(t,\lambda)) y^n(s(t,\lambda)) d\lambda \\
&\quad - C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} c(\xi) \Big[\sum_{l=1}^{n-1} \int_{I_l} y^l(\sigma) d\sigma + \int_{t_{k-1}}^\xi y^k(\varsigma) d\varsigma \Big] d\xi \\
&\quad - C_\delta \Big(\frac{t - t_{n-1}}{h_n} \Big)^{1-\delta} \int_{I_n} (t_n - \lambda)^{-\delta} c(s(t,\lambda)) \Big[\sum_{l=1}^{n-1} \int_{I_l} y^l(\sigma) d\sigma \Big] d\lambda \\
&\quad - C_\delta \Big(\frac{t - t_{n-1}}{h_n} \Big)^{1-\delta} \int_{I_n} (t_n - \lambda)^{-\delta} c(s(t,\lambda)) \Big[\int_{t_{n-1}}^{s(t,\lambda)} y^n(\tau) d\tau \Big] d\lambda
\end{aligned}$$

$$=C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} g(\xi) d\xi - C_\delta \Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\delta} \int_{I_n} (t_n-\lambda)^{-\delta} g(s(t,\lambda)) d\lambda. \tag{3.17}$$

Finally, under following two linear transformations, which transfer the integral intervals $(t_{k-1}, \xi]$ to I_k and $(t_{n-1}, s(t,\lambda)]$ to I_n,

$$\varsigma = \varsigma(\xi, \varrho) := t_{k-1} + \frac{(\varrho - t_{k-1})(\xi - t_{k-1})}{h_k}, \qquad \varrho \in I_k, \tag{3.18a}$$

$$\tau = \tau(t, \lambda, \rho) := t_{n-1} + \frac{(\rho - t_{n-1})(s(t,\lambda) - t_{n-1})}{h_n}, \qquad \rho \in I_n. \tag{3.18b}$$

The Eq. (3.17) becomes

$$y^n(t) - \mathcal{V}_1^n y(t) - \mathcal{V}_2^n y^n(t) - \mathcal{V}_3^n y(t) - \mathcal{V}_4^n y(t) - \mathcal{V}_5^n y^n(t) = \mathcal{V}_6^n g(t) + \mathcal{V}_7^n g(t), \tag{3.19}$$

where

$$\begin{aligned}
\mathcal{V}_1^n y(t) &= C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} b(\xi) y^k(\xi) d\xi, \\
\mathcal{V}_2^n y^n(t) &= C_\delta \Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\delta} \int_{I_n} (t_n-\lambda)^{-\delta} b(s(t,\lambda)) y^n(s(t,\lambda)) d\lambda, \\
\mathcal{V}_3^n y(t) &= C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} c(\xi) \Big[\sum_{l=1}^{k-1} \int_{I_l} y^l(\sigma) d\sigma + \Big(\frac{\xi - t_{k-1}}{h_k}\Big) \int_{I_k} y^k(\varsigma(\xi,\varrho)) d\varrho \Big] d\xi, \\
\mathcal{V}_4^n y(t) &= C_\delta \Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\delta} \int_{I_n} (t_n-\lambda)^{-\delta} c(s(t,\lambda)) \Big[\sum_{l=1}^{n-1} \int_{I_l} y^l(\sigma) d\sigma \Big] d\lambda, \\
\mathcal{V}_5^n y^n(t) &= C_\delta \Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\delta} \\
&\quad \cdot \int_{I_n} (t_n-\lambda)^{-\delta} c(s(t,\lambda)) \Big[\Big(\frac{s(t,\lambda) - t_{n-1}}{h_n}\Big) \int_{I_n} y^n(\tau(t,\lambda,\rho)) d\rho \Big] d\lambda, \\
\mathcal{V}_6^n g(t) &= C_\delta \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\delta} g(\xi) d\xi, \\
\mathcal{V}_7^n g(t) &= C_\delta \Big(\frac{t-t_{n-1}}{h_n}\Big)^{1-\delta} \int_{I_n} (t_n-\lambda)^{-\delta} g(s(t,\lambda)) d\lambda.
\end{aligned}$$

3.2. Hybrid spectral element methods for weakly singular VIEs

Below we only consider Caputo FBVPs (1.1) with (1.2a), since a similar procedure can be applied to the other two cases (see, Remark 3.1).

Let $V|_{I_n} := V^n$, $W|_{I_n} := W^n$ be the numerical solution (3.19) with $g = g_1,\ g_2$ respectively for $1 \le n \le N$. It follows from Lemma 5 in [16] that

$$T + \alpha_0 + \alpha_1 + \alpha_1 V(T) + \int_0^T V \neq 0.$$

Then, according to Theorem 2 in [16], we can construct a numerical approximation of u in (1.1), denoted by U as follows:

$$U'(t) = \mu_M[V(t)+1] + W(t), \tag{3.20a}$$

$$U(t) = \gamma_0 + \mu_M\alpha_0 + \int_0^t U'(s)ds, \tag{3.20b}$$

where

$$\mu_M = \frac{\gamma_1 - \gamma_0 - \alpha_1 W(T) - \int_0^T W}{T + \alpha_0 + \alpha_1 + \alpha_1 V(T) + \int_0^T V}. \tag{3.21}$$

Therefore, we only need to solve two weakly singular VIEs (3.19) (with $g = g_1$ and $g = g_2$).

The hybrid spectral element methods for solving (3.19) is to seek $Y^1(t) \in \mathcal{F}_{M_1}^{(1-\delta)}(I_1)$ and $Y^n(t) \in \mathcal{P}_{M_n}(I_n)$ with $n \geq 2$, such that

$$\begin{cases} (Y^1,\varphi)_{\chi_1^{(-\delta,\delta-1)}} - (\mathcal{V}_2^1 Y^1 + \mathcal{V}_5^1 Y^1, \varphi)_{\chi_1^{(-\delta,\delta-1)}} & \\ \quad = (\mathcal{V}_7^1 g, \varphi)_{\chi_1^{(-\delta,\delta-1)}}, & \forall \varphi \in \mathcal{F}_{M_1}^{(1-\delta)}(I_1), \\ (Y^n,\psi)_{I_n} - (\mathcal{V}_2^n Y^n + \mathcal{V}_5^n Y^n, \psi)_{I_n} & \\ \quad = (\mathcal{V}_6^n g + \mathcal{V}_7^n g, \psi)_{I_n} + (\mathcal{V}_1^n Y + \mathcal{V}_3^n Y + \mathcal{V}_4^n Y, \psi)_{I_n}, & \forall \psi \in \mathcal{P}_{M_n}(I_n). \end{cases} \tag{3.22}$$

We now describe the numerical implementations of scheme (3.22). To this end, we set

$$\begin{cases} Y^1(t) = \sum_{p=0}^{M_1} y_p^1 J_{1,p}^{(-\delta,1-\delta)}(t), & t \in I_1, \\ Y^n(t) = \sum_{p=0}^{M_n} y_p^n L_{n,p}(t), & t \in I_n, \quad n \geq 2. \end{cases} \tag{3.23}$$

Substituting (3.23) into (3.22) and taking

$$\varphi = J_{1,q}^{(-\delta,1-\delta)}(t), \qquad 0 \leq q \leq M_1,$$
$$\psi = L_{n,q}(t), \qquad 0 \leq q \leq M_n,$$

we can obtain that

$$\begin{cases} \sum_{p=0}^{M_1} y_p^1 (J_{1,p}^{(-\delta,1-\delta)}, J_{1,q}^{(-\delta,1-\delta)})_{\chi_1^{(-\delta,\delta-1)}} \\ \quad - \sum_{p=0}^{M_1} y_p^1 \big((\mathcal{V}_2^1 + \mathcal{V}_5^1) J_{1,p}^{(-\delta,1-\delta)}, J_{1,q}^{(-\delta,1-\delta)}\big)_{\chi_1^{(-\delta,\delta-1)}} \\ \quad = (\mathcal{V}_7^1 g, J_{1,q}^{(-\delta,1-\delta)})_{\chi_1^{(-\delta,\delta-1)}}, \\ \sum_{p=0}^{M_n} y_p^n (L_{n,p}, L_{n,q})_{I_n} - \sum_{p=0}^{M_n} y_p^n \big((\mathcal{V}_2^n + \mathcal{V}_5^n) L_{n,p}, L_{n,q}\big)_{I_n} \\ \quad = \big((\mathcal{V}_6^n + \mathcal{V}_7^n) g, L_{n,q}\big)_{I_n} + \big((\mathcal{V}_1^n + \mathcal{V}_3^n + \mathcal{V}_4^n) Y, L_{n,q}\big)_{I_n}. \end{cases} \tag{3.24}$$

Set

$$\begin{aligned}
&\mathbf{y}^n=(y_0^n,\cdots,y_{M_n}^n)^T, \qquad A^n=\big(a_{pq}^n\big)_{0\le p,q\le M_n}, &\text{(3.25a)}\\
&a_{pq}^1=(J_{1,p}^{(-\delta,1-\delta)},J_{1,q}^{(-\delta,1-\delta)})_{\chi_1^{(-\delta,\delta-1)}}=\Big(\frac{h_1}{2}\Big)^{2-2\delta}\gamma_p^{(-\delta,1-\delta)}\delta_{p,q,I_n}\\
&\quad=\frac{h_n}{2p+1}\delta_{p,q}, \quad n\ge 2, &\text{(3.25b)}\\
&B^n=\big(b_{qp}^n\big)_{0\le q,p\le M_n}, &\text{(3.25c)}\\
&b_{qp}^1=\big((\mathcal{V}_2^1+\mathcal{V}_5^1)J_{1,p}^{(-\delta,1-\delta)},J_{1,q}^{(-\delta,1-\delta)}\big)_{\chi_1^{(-\delta,\delta-1)}}, &\text{(3.25d)}\\
&b_{qp}^n=\big((\mathcal{V}_2^n+\mathcal{V}_5^n)L_{n,p},L_{n,q}\big)_{I_n}, \quad n\ge 2, &\text{(3.25e)}\\
&\mathbf{c}^n=(c_0^n,\cdots,c_{M_n}^n)^T, \quad c_q^n=\big((\mathcal{V}_1^n+\mathcal{V}_3^n+\mathcal{V}_4^n)Y,L_{n,q}\big)_{I_n}, \quad n\ge 2, &\text{(3.25f)}\\
&\mathbf{g}^n=(g_0^n,\cdots,g_{M_n}^n)^T, \quad g_q^1=(\mathcal{V}_7^1 g,J_{1,q}^{(-\delta,1-\delta)})_{\chi_1^{(-\delta,\delta-1)}}, &\text{(3.25g)}\\
&g_q^n=\big((\mathcal{V}_6^n+\mathcal{V}_7^n)g,L_{n,q}\big)_{I_n}, \quad n\ge 2. &\text{(3.25h)}
\end{aligned}$$

Thus, the Eq. (3.24) is equivalent to the following linear system

$$\begin{cases} A^1\mathbf{y}^1-B^1\mathbf{y}^1=\mathbf{g}^1,\\ A^n\mathbf{y}^n-B^n\mathbf{y}^n=\mathbf{g}^n+\mathbf{c}^n, \quad n\ge 2. \end{cases} \tag{3.26}$$

In practical computation, we shall use the quadrature formulas such as (2.14) and (2.19) to approximate the terms in (3.25a).

Once we have V and W, the numerical solution U can be computed from (3.20a). Note that the integrals in (3.20a) can be computed exactly and efficiently since V and W are expressed in terms of basis functions. More precisely, we have

$$\begin{aligned}
V(T)&=\sum_{p=0}^{M_N}v_p^N L_{N,p}(T), \qquad W(T)=\sum_{p=0}^{M_N}w_p^N L_{N,p}(T),\\
\int_0^T V&=\sum_{p=0}^{M_1}v_p^1\int_{I_1}J_{1,p}^{(-\delta,1-\delta)}(t)dt+\sum_{n=2}^{N}\sum_{p=0}^{M_n}v_p^n\int_{I_n}L_{n,p}(t)dt\\
&=\sum_{p=0}^{M_1}\frac{v_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t_1)+\sum_{n=2}^{N}h_n v_0^n,\\
\int_0^T W&=\sum_{p=0}^{M_1}w_p^1\int_{I_1}J_{1,p}^{(-\delta,1-\delta)}(t)dt+\sum_{n=2}^{N}\sum_{p=0}^{M_n}w_p^n\int_{I_n}L_{n,p}(t)dt\\
&=\sum_{p=0}^{M_1}\frac{w_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t_1)+\sum_{n=2}^{N}h_n w_0^n,
\end{aligned}$$

then we can obtain μ_M by using (3.21).

Moreover, it follows from Theorem 3.1 in [6] and (3.20a) that

$$\begin{aligned}
U^1(t) =&\gamma_0+\mu_M\alpha_0+\int_0^t \frac{d}{ds}U^1(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\int_0^t \mu_M[V^1(s)+1]+W^1(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\mu_M t+\mu_M\sum_{p=0}^{M_1}v_p^1\int_0^t J_{1,p}^{(-\delta,1-\delta)}(s)ds+\sum_{p=0}^{M_1}w_p^1\int_0^t J_{1,p}^{(-\delta,1-\delta)}(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\mu_M t+\mu_M\sum_{p=0}^{M_1}\frac{v_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t)\\
&+\sum_{p=0}^{M_1}\frac{w_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t),
\end{aligned}\tag{3.27}$$

where $J_{1,p}^{(-\delta-1,2-\delta)}$ with the index $-\delta-1<-1$ is also well define in [6]. Similarly, by the derivative recurrence relations of Legendre polynomials (cf. (3.176a) in [25]), we have that for any $t\in I_N$,

$$\begin{aligned}
U^n(t) =&\gamma_0+\mu_M\alpha_0+\sum_{k=1}^{n-1}\int_{I_k}\frac{d}{ds}U^k(s)ds+\int_{t_{n-1}}^t\frac{d}{ds}U^n(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\sum_{k=1}^{n-1}\int_{I_k}\mu_M[V^k(s)+1]+W^k(s)ds\\
&+\int_{t_{n-1}}^t \mu_M[V^n(s)+1]+W^n(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\mu_M t+\mu_M\sum_{k=1}^{n-1}\int_{I_k}V^k(s)ds+\sum_{k=1}^{n-1}\int_{I_k}W^k(s)ds\\
&+\mu_M\int_{t_{n-1}}^t V^n(s)ds+\int_{t_{n-1}}^t W^n(s)ds\\
=&\gamma_0+\mu_M\alpha_0+\mu_M t+\mu_M\Big[\sum_{p=0}^{M_1}\frac{v_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t_1)+\sum_{k=2}^{n-1}h_k v_0^k\Big]\\
&+\sum_{p=0}^{M_1}\frac{w_p^1}{p+2-\delta}J_{1,p}^{(-\delta-1,2-\delta)}(t_1)+\sum_{k=2}^{n-1}h_k w_0^k\\
&+\frac{h_n}{2}\mu_M\sum_{p=1}^{M_n}\frac{v_p^n}{2p+1}\big(L_{n,p+1}(s)-L_{n,p-1}(s)\big)\Big|_{s=t_{n-1}}^{s=t}+\frac{h_n}{2}\mu_M v_0^n(t-t_{n-1})\\
&+\frac{h_n}{2}\sum_{p=1}^{M_n}\frac{w_p^n}{2p+1}\big(L_{n,p+1}(s)-L_{n,p-1}(s)\big)\Big|_{s=t_{n-1}}^{s=t}+\frac{h_n}{2}w_0^n(t-t_{n-1}).
\end{aligned}\tag{3.28}$$

In summary, we can obtain U (and U'), approximation of the solution u (and u'), through the following algorithm.

Algorithm 3.1 Algorithm for solving Caputo BVPs with B. C. Robin (1.2a).

For $n = 1, \cdots, N$ **do**

Compute the vectors $\{g_p^n\}_{p=0}^{M_n}$ (both g_1 and g_2), $\{c_q^n\}_{q=0}^{M_n}$, and the matrices A^n, B^n by (3.25a).

Compute the coefficients $\{\widetilde{v}_p^n\}_{p=0}^{M_n}$ and $\{\widetilde{w}_p^n\}_{p=0}^{M_n}$ (with g_1 and g_2) by (3.26).

end For

Compute the value μ_M by (3.21).

Compute the values $\{U^n(t_{n,j})\}_{j=0}^{M_n}$ and $\{U^n(t_n)\}_{n=1}^{N}$ by (3.27) and (3.28).

Remark 3.1. We can solve Caputo FBVPs with (1.2b) and RL FBVPs with (1.2b) in a similar fashion. Indeed, let $V(t)$ and $W(t)$ be the solutions of (3.7) (or (3.12)) with g_1 and g_2, respectively. Assume that $T + \int_0^T V \neq 0$, using (1.2b) and (3.8a) leads to

$$U'(t) = \mu_M[V(t) + 1] + W(t), \tag{3.29a}$$

$$U(t) = U(0) + \int_0^t U'(s)ds, \tag{3.29b}$$

with

$$\mu_M = \frac{\gamma_1 - \gamma_0 - \int_0^T W}{T + \int_0^T V}.$$

4. Well-posedness and error analysis

We recall some lemmas which will be used later. We denote by c a generic positive constant independent of h_k, M_k, the solutions of $y(t)$ and $Y(t)$. For any integer $m \geq 0$, we introduce the weighted Sobolev space on $(-1,1)$,

$$H^m_{\chi^{(\alpha,\beta)},A}(-1,1) = \{v : \|v\|_{H^m_{\chi^{(\alpha,\beta)},A}(-1,1)} < \infty\},$$

with the norm

$$\|v\|_{H^m_{\chi^{(\alpha,\beta)},A}(-1,1)} = \left(\sum_{k=0}^{m} \|\partial_x^k v\|^2_{L^2_{\chi^{(\alpha+k,\beta+k)}}(-1,1)}\right)^{\frac{1}{2}}.$$

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Clearly, to characterize the regularity of the solution y, we introduce the non-uniformly weighted space involving fractional derivatives in the first interval I_1:

$$\mathcal{B}^m_{\alpha,\beta}(I_1) := \{v \in L^2_{\chi_1^{(\alpha,-\beta)}}(I_1) : {}_0D_t^{\beta+r} v \in L^2_{\chi_1^{(\alpha+\beta+r,r)}}(I_1) \ \text{ for } \ 0 \le r \le m\}, \quad m \in \mathbb{N}_0,$$
$$\mathcal{H}^m_{\alpha,\beta}(I_1) := \{v \in L^2_{\chi_1^{(\alpha,-\beta)}}(I_1) : {}_0D_t^{\beta+r} v \in L^2_{\chi_1^{(\alpha,-\beta)}}(I_1) \ \text{ for } \ 0 \le r \le m\}, \qquad m \in \mathbb{N}_0.$$

We denote $\pi^{(\alpha,\beta)}_{I_1,M_1}$ is the $L^2_{\chi_1^{(\alpha,-\beta)}}(I_1)$-orthogonal projection upon $\mathcal{F}^{(\beta)}_{M_1}(I_1)$

$$\left(\pi^{(\alpha,\beta)}_{I_1,M_1} v - v, \psi\right)_{\chi_1^{(\alpha,-\beta)}} = 0, \quad \forall \psi \in \mathcal{F}^{(\beta)}_{M_1}(I_1). \tag{4.1}$$

According to Lemma 3.2 of [26], we have

Lemma 4.1. *Let $\alpha > -1$, $\beta > 0$, for any $v \in \mathcal{B}^{m_1}_{\alpha,\beta}(I_1)$, with integer $0 \le m_1 \le M_1$, we get that*

$$\|\pi^{(\alpha,\beta)}_{I_1,M_1} v - v\|_{\chi_1^{(\alpha,-\beta)}} \le c h_1^{-\beta} M_1^{-(\beta+m_1)} \|{}_0D_t^{\beta+m_1} v\|_{\chi_1^{(\alpha+\beta+m_1,m_1)}}. \tag{4.2}$$

In particular, if $v \in \mathcal{H}^{m_1}_{\alpha,\beta}(I_1)$, then

$$\|\pi^{(\alpha,\beta)}_{I_1,M_1} v - v\|_{\chi_1^{(\alpha,-\beta)}} \le c h_1^{m_1} M_1^{-(\beta+m_1)} \|{}_0D_t^{\beta+m_1} v\|_{\chi_1^{(\alpha,-\beta)}}. \tag{4.3}$$

Next, we define π_{I_n,M_n} is the standard $L^2(I_n)$-orthogonal projection upon $\mathcal{P}_{M_n}(I_n)$,

$$(\pi_{I_n,M_n} v - v, \psi)_{I_n} = 0, \quad \forall \psi \in \mathcal{P}_{M_n}(I_n). \tag{4.4}$$

According to Lemma 3.4 of [26], we have

Lemma 4.2. *For any $v \in H^m(I_n)$ with integer $1 \le m \le M_n + 1$,*

$$\|v - \pi_{I_n,M_n} v\|_{I_n} \le c M_n^{-m} \|\partial_t^m v\|_{L^2_{\chi_n^{(m,m)}}(I_n)} \le c h_n^m M_n^{-m} \|\partial_t^m v\|_{I_n}, \tag{4.5}$$

where $H^m(I_n)$ is the usual Sobolev space.

The following discrete Gronwall Lemma can be found in [27].

Lemma 4.3. *Assume that $\{k_j\}$ and $\{\rho_j\}$ $(j \ge 0)$ are given non-negative sequences, and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le \rho_0$ and*

$$\varepsilon_n \le \rho_n + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \varepsilon_j, \quad n \ge 1,$$

with $q_j \ge 0$ $(j \ge 0)$. Then

$$\varepsilon_n \le \rho_n + \sum_{j=0}^{n-1} (q_j + k_j \rho_j) \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

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4.1. Existence, uniqueness and error estimate

We first establish the existence and uniqueness of the solution of (3.22).

Since the solution of (3.13) is generally nonsmooth at $t=0$ and smooth for $t>0$, it is reasonable to assume that the solution $y|_{t\in I_1}\in\mathcal{B}^{m_1}_{-\delta,1-\delta}(I_1)$, and $y(t)|_{t\in I_n}$ belongs to the usual Sobolev space $H^{m_n}(I_n)$ with $n>1$. Let $Y(t)$ be the global numerical solution of (3.22), which is given by

$$Y(t)=Y^n(t)|_{t\in I_n},\quad 1\le n\le N.$$

Lemma 4.4. *Assume that* $b(s),c(s)\in C(0,T)$, $0<\delta<1/2$, *and* $h_{\max}$ *sufficiently small. Then the Eq.* (3.22) *possesses a unique solution.*

Proof. Consider (3.22) with $g=0$, according to the definitions (4.1) and (4.4) of the projection operator $\pi^{(-\delta,1-\delta)}_{I_1,M_1}$ and π_{I_n,M_n}, we know from (3.22) that

$$\begin{cases} Y^1=\pi^{(-\delta,1-\delta)}_{I_1,M_1}(\mathcal{V}^1_2+\mathcal{V}^1_5)Y^1,\\ Y^n=\pi_{I_n,M_n}(\mathcal{V}^n_1+\mathcal{V}^n_3+\mathcal{V}^n_4)Y+\pi_{I_n,M_n}(\mathcal{V}^n_2+\mathcal{V}^n_5)Y^n, \qquad n\ge 2. \end{cases} \tag{4.6}$$

We first proof the case $n=1$. The first formula of (4.6) along with the projection theorem, implies

$$\begin{aligned}\|Y^1\|^2_{I_1}\le& ch_1\|Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}=chi_1\|\pi^{(-\delta,1-\delta)}_{I_1,M_1}(\mathcal{V}^1_2+\mathcal{V}^1_5)Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}\\ \le& ch_1\big(\|\mathcal{V}^1_2Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}+\|\mathcal{V}^1_5Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}\big).\end{aligned} \tag{4.7}$$

Then, we derive from (4.7), the definition in (3.19) and the Cauchy-Schwarz inequality that

$$\begin{aligned}&\|\mathcal{V}^1_2Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}\\ \le& c\int_{I_1}\Big(\int_{t_0}^t(t-s)^{-\delta}b(s)Y^1(s)ds\Big)^2\chi_1^{(-\delta,\delta-1)}(t)dt\\ \le& c\int_{I_1}\Big[\int_{t_0}^t(t-s)^{-\delta}ds\int_{t_0}^t(t-s)^{-\delta}\big(Y^1(s)\big)^2ds\Big]\chi_1^{(-\delta,\delta-1)}(t)dt\\ =&c\int_{I_1}\Big(\frac{-(t-s)^{1-\delta}}{1-\delta}\Big|_{s=t_0}^{s=t}\Big)\int_{t_0}^t(t-s)^{-\delta}\big(Y^1(s)\big)^2ds\chi_1^{(-\delta,\delta-1)}(t)dt\\ \le& c\int_{I_1}(t-t_0)^{1-\delta}\int_{t_0}^t(t-s)^{-\delta}\big(Y^1(s)\big)^2ds\chi_1^{(-\delta,\delta-1)}(t)dt\\ =&c\int_{I_1}(t_1-t)^{-\delta}\int_{t_0}^t(t-s)^{-\delta}\big(Y^1(s)\big)^2dsdt\\ =&c\int_{I_1}\big(Y^1(s)\big)^2\int_s^{t_1}(t_1-t)^{-\delta}(t-s)^{-\delta}dtds\\ \le& ch_1^{1-2\delta}\|Y^1\|^2_{I_1},\end{aligned} \tag{4.8}$$

and

$$
\begin{aligned}
&\|\mathcal{V}_5^1 Y^1\|^2_{\chi_1^{(-\delta,\delta-1)}}\\
\le& c\int_{I_1}\Big[\int_{t_0}^t (t-s)^{-\delta}c(s)\Big(\int_{t_0}^s Y^1(\tau)d\tau\Big)ds\Big]^2\chi_1^{(-\delta,\delta-1)}(t)dt\\
\le& c\int_{I_1}\Big[\int_{t_0}^t (t-s)^{-\delta}ds\int_{t_0}^t (t-s)^{-\delta}\Big(\int_{t_0}^s Y^1(\tau)d\tau\Big)^2 ds\Big]\chi_1^{(-\delta,\delta-1)}(t)dt\\
\le& ch_1\int_{I_1}\Big[\int_{t_0}^t (t-s)^{-\delta}ds\int_{t_0}^t (t-s)^{-\delta}\Big(\int_{t_0}^s \big(Y^1(\tau)\big)^2 d\tau\Big)ds\Big]\chi_1^{(-\delta,\delta-1)}(t)dt\\
\le& ch_1\int_{I_1}\Big[\int_{t_0}^t (t-s)^{-\delta}ds\int_{t_0}^t (t-s)^{-\delta}ds\Big]\chi_1^{(-\delta,\delta-1)}(t)dt\int_{I_1}\big(Y^1(\tau)\big)^2 d\tau\\
=& ch_1\int_{I_1}\Big(\frac{-(t-s)^{1-\delta}}{1-\delta}\Big|_{s=t_0}^{s=t}\Big)^2\chi_1^{(-\delta,\delta-1)}(t)dt\int_{I_1}\big(Y^1(\tau)\big)^2 d\tau\\
\le& ch_1\int_{I_1}(t_1-t)^{-\delta}(t-t_0)^{1-\delta}dt\int_{I_1}\big(Y^1(\tau)\big)^2 d\tau\\
\le& ch_1^{3-2\delta}\|Y^1\|^2_{I_1}.
\end{aligned}
\tag{4.9}
$$

Therefore, a combination of the above estimates (4.7)-(4.9) leads to

$$
\|Y^1\|^2_{I_1}\le ch_1^{2-2\delta}\|Y^1\|^2_{I_1}. \tag{4.10}
$$

We now turn to show the case $n>1$, we obtain from the second formula of (4.6) and the projection theorem that

$$
\begin{aligned}
\|Y^n\|^2_{I_n}=&\|\pi_{I_n,M_n}(\mathcal{V}_1^n+\mathcal{V}_3^n+\mathcal{V}_4^n)Y+\pi_{I_n,M_n}(\mathcal{V}_2^n+\mathcal{V}_5^n)Y^n\|^2_{I_n}\\
\le& c\big(\|\mathcal{V}_1^nY\|^2_{I_n}+\|\mathcal{V}_3^nY\|^2_{I_n}+\|\mathcal{V}_4^nY\|^2_{I_n}+\|\mathcal{V}_2^nY^n\|^2_{I_n}+\|\mathcal{V}_5^nY^n\|^2_{I_n}\big).
\end{aligned}
\tag{4.11}
$$

Next, let $0<\delta<1/2$, it follows from (3.19) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\|\mathcal{V}_1^nY\|^2_{I_n}=&C_\delta\int_{I_n}\Big(\int_0^{t_{n-1}}(t-s)^{-\delta}b(s)Y(s)ds\Big)^2dt\\
\le& c\int_{I_n}\Big[\int_0^{t_{n-1}}(t-s)^{-2\delta}ds\int_0^{t_{n-1}}\big(Y(s)\big)^2ds\Big]dt\\
=& c\int_{I_n}\Big(\frac{-(t-s)^{1-2\delta}}{1-2\delta}\Big|_{s=0}^{s=t_{n-1}}\Big)\int_0^{t_{n-1}}\big(Y(s)\big)^2dsdt\\
=& c\int_{I_n}\Big((t-t_{n-1})^{1-2\delta}+t^{1-2\delta}\Big)dt\int_0^{t_{n-1}}\big(Y(s)\big)^2ds\\
\le& ch_n\sum_{k=1}^{n-1}\|Y^k\|^2_{I_k}.
\end{aligned}
\tag{4.12}
$$

Similarly, by (3.19) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\mathcal{V}_3^n Y\|_{I_n}^2 =& C_\delta \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-\delta} c(s) \Big(\int_0^s Y(\tau) d\tau \Big) ds \Big]^2 dt \\
\le& c \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-2\delta} ds \int_0^{t_{n-1}} \Big(\int_0^s Y(\tau) d\tau \Big)^2 ds \Big] dt \\
\le& cT \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-2\delta} ds \int_0^{t_{n-1}} \Big(\int_0^s \big(Y(\tau)\big)^2 d\tau \Big) ds \Big] dt \\
\le& cT^2 \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-2\delta} ds \Big] dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
=& cT^2 \int_{I_n} \Big(\frac{-(t-s)^{1-2\delta}}{1-2\delta} \Big|_{s=0}^{s=t_{n-1}} \Big) dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
\le& cT^2 \int_{I_n} \Big((t-t_{n-1})^{1-2\delta} + t^{1-2\delta} \Big) dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
\le& cT^2 h_n \sum_{k=1}^{n-1} \|Y^k\|_{I_k}^2,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
\|\mathcal{V}_4^n Y\|_{I_n}^2 =& C_\delta \int_{I_n} \Big[\int_{t_{n-1}}^t (t-s)^{-\delta} c(s) \Big(\int_0^{t_{n-1}} Y(\tau) d\tau \Big) ds \Big]^2 dt \\
\le& c \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-2\delta} ds \int_0^{t_{n-1}} \Big(\int_0^{t_{n-1}} Y(\tau) d\tau \Big)^2 ds \Big] dt \\
\le& cT^2 \int_{I_n} \Big[\int_0^{t_{n-1}} (t-s)^{-2\delta} ds \Big] dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
=& cT^2 \int_{I_n} \Big(\frac{-(t-s)^{1-2\delta}}{1-2\delta} \Big|_{s=0}^{s=t_{n-1}} \Big) dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
\le& cT^2 \int_{I_n} \Big((t-t_{n-1})^{1-2\delta} + t^{1-2\delta} \Big) dt \int_0^{t_{n-1}} \big(Y(\tau)\big)^2 d\tau \\
\le& cT^2 h_n \sum_{k=1}^{n-1} \|Y^k\|_{I_k}^2.
\end{aligned} \tag{4.14}$$

Furthermore, by (3.14) and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
\|\mathcal{V}_2^n Y^n\|_{I_n}^2 =& C_\delta \int_{I_n} \Big(\int_{t_{n-1}}^t (t-s)^{-\delta} b(s) Y^n(s) ds \Big)^2 dt \\
\le& c \int_{I_n} \int_{t_{n-1}}^t (t-s)^{-\delta} ds \int_{t_{n-1}}^t (t-s)^{-\delta} \big(Y^n(s)\big)^2 ds dt \\
=& c \int_{I_n} \Big[\frac{-(t-s)^{1-\delta}}{1-\delta} \Big|_{s=t_{n-1}}^{s=t} \Big] \int_{t_{n-1}}^t (t-s)^{-\delta} \big(Y^n(s)\big)^2 ds dt
\end{aligned}$$

$$
\begin{aligned}
\leq & c h_n^{1-\delta} \int_{I_n} \int_{t_{n-1}}^{t} (t-s)^{-\delta} \big(Y^n(s)\big)^2 ds dt \\
\leq & c h_n^{1-\delta} \int_{I_n} \big(Y^n(s)\big)^2 \int_{s}^{t_n} (t-s)^{-\delta} dt ds \\
\leq & c h_n^{2-2\delta} \|Y^n\|_{I_n}^2 .
\end{aligned} \tag{4.15}
$$

Similarly,

$$
\begin{aligned}
\|\mathcal{V}_5^n Y^n\|_{I_n}^2 = & C_\delta \int_{I_n} \Big[\int_{t_{n-1}}^{t} (t-s)^{-\delta} c(s) \int_{t_{n-1}}^{s} Y^n(\tau) d\tau ds \Big]^2 dt \\
\leq & c \int_{I_n} \Big[\int_{t_{n-1}}^{t} (t-s)^{-\delta} ds \int_{t_{n-1}}^{t} (t-s)^{-\delta} \Big(\int_{t_{n-1}}^{s} Y^n(\tau) d\tau \Big)^2 ds \Big] dt \\
\leq & c h_n \int_{I_n} \Big[\int_{t_{n-1}}^{t} (t-s)^{-\delta} ds \int_{t_{n-1}}^{t} (t-s)^{-\delta} \int_{t_{n-1}}^{s} \big(Y^n(\tau)\big)^2 d\tau ds \Big] dt \\
\leq & c h_n \int_{I_n} \Big[\int_{t_{n-1}}^{t} (t-s)^{-\delta} ds \int_{t_{n-1}}^{t} (t-s)^{-\delta} ds \Big] dt \int_{I_n} \big(Y^n(\tau)\big)^2 d\tau \\
= & c h_n \int_{I_n} \Big(\frac{-(t-s)^{1-\delta}}{1-\delta} \Big|_{s=t_{n-1}}^{s=t} \Big)^2 dt \int_{I_n} \big(Y^n(\tau)\big)^2 d\tau \\
\leq & c h_n^{3-2\delta} \int_{I_n} \big(Y^n(\tau)\big)^2 d\tau \leq c h_n^{2-2\delta} \|Y^n\|_{I_n}^2 .
\end{aligned} \tag{4.16}
$$

Again, a combination of the above estimates (4.11) - (4.16), leads to

$$
\|Y^n\|_{I_n}^2 \leq c \sum_{k=1}^{n-1} h_n \|Y^k\|_{I_k}^2 + c h_n^{2-2\delta} \|Y^n\|_{I_n}^2 . \tag{4.17}
$$

Finally, by using Lemma 4.3 we have

$$
\|Y^n\|_{I_n}^2 \leq c h_n^{2-2\delta} \|Y^n\|_{I_n}^2 . \tag{4.18}
$$

Thus, if $h_{\max}$ is sufficiently small such that $c h_{\max}^{2-2\delta} \leq \beta < 1$, we find from above that $\|Y^n\|_{I_n} = 0,\ n \geq 1$. This implies that (3.22) admits a unique solution since $\mathcal{F}_{M_1}^{(1-\delta)}(I_1)$ and $\mathcal{P}_{M_n}(I_n)$ are finite-dimensional. □

Remark 4.1. The condition $0 < \delta < 1/2$ is technically required for the above analysis, but it may not be necessary. Our numerical experiments show that the scheme is still well posed for $\delta \in [1/2, 1)$.

Next, we carry out an error analysis. Hereafter, let $e(t) = e_n(t)|_{t \in I_n} = y^n(t) - Y^n(t)|_{t \in I_n},\ 1 \leq n \leq N$, and $M_{\min} = \min_{1 \leq n \leq N} M_n$.

Lemma 4.5. *Let y^n be the solution of* (3.19) *and Y^n be the solution of* (3.22)*. Assume that $b(t), c(t) \in C(0,T)$, $y|_{t\in I_1} \in \mathcal{B}^{m_1}_{-\delta,1-\delta}(I_1)$, $y|_{t\in I_n} \in H^{m_n}(I_n)$ with $n \geq 2$, and integer $1 \leq m_n \leq M_n + 1$, and $0 < \delta < 1/2$. Then, for $h_{\max}$ is sufficiently small and $2 \leq n \leq N$, we have*

$$\|e_1\|_{I_1} \leq c h_1^{\delta-1/2} M_1^{-(1-\delta+m_1)} \|{}_0D_t^{1-\delta+m_1} y\|_{\chi_1^{(1-2\delta+m_1,m_1)}}, \tag{4.19a}$$

$$\|e_n\|_{I_n} \leq c h_n^{m_n} M_n^{-m_n} \|\partial_t^{m_n} y\|_{I_n}, \quad n \geq 2. \tag{4.19b}$$

In particular, if $y|_{t\in I_1} \in \mathcal{H}^{m_1}_{-\delta,1-\delta}(I_1)$, then

$$\|e_1\|_{I_1} \leq c h_1^{m_1+1/2} M_1^{-(1-\delta+m_1)} \|{}_0D_t^{1-\delta+m_1} y\|_{\chi_1^{-\delta,\delta-1}}, \tag{4.20a}$$

$$\|e_n\|_{I_n} \leq c h_n^{m_n} M_n^{-m_n} \|\partial_t^{m_n} y\|_{I_n}, \quad n \geq 2. \tag{4.20b}$$

Proof. According to the definitions (4.1) and (4.4) of the projection operator $\pi^{(-\delta,1-\delta)}_{I_1,M_1}$ and π_{I_n,M_n}, we know from (3.22) that

$$\begin{cases} Y^1 = \pi^{(-\delta,1-\delta)}_{I_1,M_1} \mathcal{V}_7^1 g + \pi^{(-\delta,1-\delta)}_{I_1,M_1} (\mathcal{V}_2^1 + \mathcal{V}_5^1) Y^1, \\ Y^n = \pi_{I_n,M_n} (\mathcal{V}_6^n + \mathcal{V}_7^n) g + \pi_{I_n,M_n} (\mathcal{V}_1^n + \mathcal{V}_3^n + \mathcal{V}_4^n) Y \\ \qquad +\pi_{I_n,M_n} (\mathcal{V}_2^n + \mathcal{V}_5^n) Y^n, \quad n \geq 2. \end{cases} \tag{4.21}$$

By subtracting (4.21) from (3.19), we deduce that

$$\begin{cases} e_1 = \mathcal{V}_7^1 g - \pi^{(-\delta,1-\delta)}_{I_1,M_1} \mathcal{V}_7^1 g + \big(\mathcal{I} - \pi^{(-\delta,1-\delta)}_{I_1,M_1}\big)(\mathcal{V}_2^1 + \mathcal{V}_5^1) Y^1, \\ e_n = (\mathcal{V}_6^n + \mathcal{V}_7^n) g - \pi_{I_n,M_n} (\mathcal{V}_6^n + \mathcal{V}_7^n) g + \big(\mathcal{I} - \pi_{I_n,M_n}\big)(\mathcal{V}_1^n + \mathcal{V}_3^n + \mathcal{V}_4^n) Y \\ \qquad +\big(\mathcal{I} - \pi_{I_n,M_n}\big)(\mathcal{V}_2^n + \mathcal{V}_5^n) Y^n, \qquad n > 1. \end{cases}$$

Then, by (3.19) there holds

$$\begin{cases} \mathcal{V}_7^1 g - \pi^{(-\delta,1-\delta)}_{I_1,M_1} \mathcal{V}_7^1 g = (\mathcal{I} - \pi^{(-\delta,1-\delta)}_{I_1,M_1}) y^1 + \big(\pi^{(-\delta,1-\delta)}_{I_1,M_1} - \mathcal{I}\big)(\mathcal{V}_2^1 + \mathcal{V}_5^1) y^1, \\ (\mathcal{V}_6^n + \mathcal{V}_7^n) g - \pi_{I_n,M_n} (\mathcal{V}_6^n + \mathcal{V}_7^n) g \\ \quad = (\mathcal{I} - \pi_{I_n,M_n}) y^n + \big(\pi_{I_n,M_n} - \mathcal{I}\big)(\mathcal{V}_1^n + \mathcal{V}_3^n + \mathcal{V}_4^n) y \\ \qquad +\big(\pi_{I_n,M_n} - \mathcal{I}\big)(\mathcal{V}_2^n + \mathcal{V}_5^n) y^n, \qquad n > 1, \end{cases}$$

with $\mathcal{I}$ the identity operator. A combination of the previous two equalities, we get that

$$\begin{cases} e_1 = (\mathcal{I} - \pi^{(-\delta,1-\delta)}_{I_1,M_1}) y^1 + \pi^{(-\delta,1-\delta)}_{I_1,M_1} (\mathcal{V}_2^1 + \mathcal{V}_5^1) e_1, \\ e_n = (\mathcal{I} - \pi_{I_n,M_n}) y^n + \pi_{I_n,M_n} (\mathcal{V}_1^n + \mathcal{V}_3^n + \mathcal{V}_4^n) e \\ \qquad +\pi_{I_n,M_n} (\mathcal{V}_2^n + \mathcal{V}_5^n) e_n, \quad n > 1. \end{cases} \tag{4.22}$$

Moreover, by (4.22) and the projection theorem, we obtain that

$$\begin{aligned} \|e_1\|^2_{I_1} \leq & 2\|(\mathcal{I} - \pi^{(-\delta,1-\delta)}_{I_1,M_1}) y^1\|^2_{I_1} + 2\|\pi^{(-\delta,1-\delta)}_{I_1,M_1} (\mathcal{V}_2^1 + \mathcal{V}_5^1) e_1\|^2_{I_1} \\ \leq & c h_1 \left(\|(\mathcal{I} - \pi^{(-\delta,1-\delta)}_{I_1,M_1}) y^1\|^2_{\chi_1^{(-\delta,\delta-1)}} + \|\pi^{(-\delta,1-\delta)}_{I_1,M_1} (\mathcal{V}_2^1 + \mathcal{V}_5^1) e_1\|^2_{\chi_1^{(-\delta,\delta-1)}} \right) \\ \leq & c h_1 \left(\|D_1\|^2_{\chi_1^{(-\delta,\delta-1)}} + \|D_2\|^2_{\chi_1^{(-\delta,\delta-1)}} + \|D_3\|^2_{\chi_1^{(-\delta,\delta-1)}} \right), \end{aligned} \tag{4.23}$$

where

$$D_1=(\mathcal{I}-\pi^{(-\delta,1-\delta)}_{I_1,M_1})y^1,\quad D_2=\mathcal{V}^1_2e_1,\quad D_3=\mathcal{V}^1_5e_1.$$

Next, we estimate

$$\|D_j\|^2_{\chi_1^{(-\delta,\delta-1)}},\quad j=1,2,3.$$

By using Lemma 4.1, we find that for integer $1\le m_1\le M_1+1$,

$$\begin{aligned}\|D_1\|^2_{\chi_1^{(-\delta,\delta-1)}}\le& ch_1^{2\delta-2}M_1^{-2(1-\delta+m_1)}\|_0D_t^{1-\delta+m_1}y\|^2_{\chi_1^{(1-2\delta+m_1,m_1)}}\\ \le& ch_1^{2m_1}M_1^{-2(1-\delta+m_1)}\|_0D_t^{1-\delta+m_1}y\|^2_{\chi_1^{(-\delta,\delta-1)}}.\end{aligned}\tag{4.24}$$

By using similar arguments as in (4.8) and (4.9), we have

$$\|D_2\|^2_{\chi_1^{(-\delta,\delta-1)}}\le ch_1^{1-2\delta}\|e_1\|^2_{I_1},\tag{4.25a}$$

$$\|D_3\|^2_{\chi_1^{(-\delta,\delta-1)}}\le ch_1^{3-2\delta}\|e_1\|^2_{I_1}.\tag{4.25b}$$

Therefore, for h_{max} sufficiently small, we derive from (4.23)-(4.25b) that

$$\|e_1\|^2_{I_1}\le ch_1^{2m_1+1}M_1^{-2(1-\delta+m_1)}\|_0D_t^{1-\delta+m_1}y\|^2_{\chi_1^{(-\delta,\delta-1)}}.\tag{4.26}$$

In addition, we derive from the second formula of (4.22) and the projection theorem that

$$\begin{aligned}\|e_n\|^2_{I_n}\le&3\|(\mathcal{I}-\pi_{I_n,M_n})y^n\|^2_{I_n}+3\|\pi_{I_n,M_n}(\mathcal{V}^n_1+\mathcal{V}^n_3+\mathcal{V}^n_4)e\|^2_{I_n}\\ &+3\|\pi_{I_n,M_n}(\mathcal{V}^n_2+\mathcal{V}^n_5)e_n\|^2_{I_n}\\ \le& c\left(\|D_4\|^2_{I_n}+\|D_5\|^2_{I_n}+\|D_6\|^2_{I_n}+\|D_7\|^2_{I_n}+\|D_8\|^2_{I_n}+\|D_9\|^2_{I_n}\right),\end{aligned}\tag{4.27}$$

where

$$\begin{aligned}&D_4=(\mathcal{I}-\pi_{I_n,M_n})y^n, && D_5=\mathcal{V}^n_1e, && D_6=\mathcal{V}^n_3e,\\ &D_7=\mathcal{V}^n_4e, && D_8=\mathcal{V}^n_2e_n, && D_9=\mathcal{V}^n_5e_n.\end{aligned}$$

By Lemma 4.2, we get that for integer $1\le m_n\le M_n+1$,

$$\|D_4\|^2_{I_n}\le ch_n^{2m_n}M_n^{-2m_n}\|\partial_t^{m_n}y\|^2_{I_n}.\tag{4.28}$$

Let $0<\delta<1/2$, in the same fashion as (4.12)-(4.14), we get that

$$\|D_5\|^2_{I_n}\le ch_n\sum_{k=1}^{n-1}\|e_k\|^2_{I_k},\tag{4.29a}$$

$$\|D_6\|^2_{I_n}\le cT^2h_n\sum_{k=1}^{n-1}\|e_k\|^2_{I_k},\tag{4.29b}$$

$$\|D_7\|^2_{I_n}\le cT^2h_n\sum_{k=1}^{n-1}\|e_k\|^2_{I_k}.\tag{4.29c}$$

Furthermore, by using similar arguments as (4.15) and (4.16), we have

$$\|D_8\|^2_{I_n} \leq ch_n^{2-2\delta} \int_{I_n} e_n^2(s)ds \leq ch_n^{2-2\delta}\|e_n\|^2_{I_n}, \tag{4.30a}$$

$$\|D_9\|^2_{I_n} \leq ch_n^{3-2\delta} \int_{I_n} e_n^2(\tau)d\tau \leq ch_n^{2-2\delta}\|e_n\|^2_{I_n}. \tag{4.30b}$$

A combination of the above estimates (4.27)-(4.30a) leads to

$$\|e\|^2_{I_n} \leq c\sum_{k=1}^{n-1} h_n\|e_k\|^2_{I_k} + ch_n^{2m_n}M_n^{-2m_n}\|\partial_t^{m_n}y\|^2_{I_n}. \tag{4.31}$$

Finally, it follows from Lemma 4.3 that

$$\|e_n\|^2_{I_n} \leq ch_n^{2m_n}M_n^{-2m_n}\|\partial_t^{m_n}y\|^2_{I_n}. \tag{4.32}$$

This ends the proof. □

A direct consequence of Lemma 4.5 is the following Theorem.

Theorem 4.1. *Let y be the solution of* (3.13) *and Y be the global numerical solution* (3.13). *Assume that $b(t), c(t) \in C(0,T)$, $y|_{t\in I_1} \in \mathcal{B}^m_{-\delta,1-\delta}(I_1)$, $y|_{t\in I_n} \in H^m(I_n)$ with $n > 1$, and integer $1 \leq m \leq M_{\min}+1$. Then, for h_{max} sufficiently small and $0 < \delta < 1/2$, there holds*

$$\begin{aligned}&\|y-Y\|_{L^2(I)}\\ \leq& ch_1^{\delta-1/2}M_1^{-(1-\delta+m)}\|{}_0D_t^{1-\delta+m}y\|_{\chi_1^{(1-2\delta+m_1,m_1)}} + \sum_{n=2}^N h_n^m M_n^{-m}\|\partial_t^m y\|_{L^2(I)}.\end{aligned} \tag{4.33}$$

In particular, if $y|_{t\in I_1} \in \mathcal{H}^m_{-\delta,1-\delta}(I_1)$, then

$$\begin{aligned}\|y-Y\|_{L^2(I)} \leq& ch_1^{m+1/2}M_1^{-(1-\delta+m)}\|{}_0D_t^{1-\delta+m}y\|_{\chi_1^{(-\delta,\delta-1)}}\\ &+\sum_{n=2}^N h_n^m M_n^{-m}\|\partial_t^m y\|_{L^2(I)}.\end{aligned} \tag{4.34}$$

5. Numerical results

In this section, we present some numerical results to illustrate the efficiency of the hybrid spectral element methods. To quantify the numerical results, we set

$$E_1(T)=\Big(\sum_{k=1}^{N}\frac{h_k}{2}\sum_{j=0}^{M_k}\Big(\frac{d}{dt}u^k(t_{k,j})-\frac{d}{dt}U^k(t_{k,j})\Big)^2\omega_{k,j}\Big)^{\frac{1}{2}}$$
$$\approx\Big(\int_0^T\Big(\frac{d}{dt}u(t)-\frac{d}{dt}U(t)\Big)^2dt\Big)^{\frac{1}{2}}, \tag{5.1a}$$
$$E_2(T)=\Big(\sum_{k=1}^{N}\frac{h_k}{2}\sum_{j=0}^{M_k}(u^k(t_{k,j})-U^k(t_{k,j}))^2\omega_{k,j}\Big)^{\frac{1}{2}}$$
$$\approx\Big(\int_0^T(u(t)-U(t))^2dt\Big)^{\frac{1}{2}}. \tag{5.1b}$$

Since solutions of (1.1) are in general non-smooth at $t=0$, we use a graded mesh $t_n=\big(n/N\big)^r$ for $n=0,1,\cdots,N$ with $r=M/(1-\delta)$ in all examples below. At each interval, we use polynomials/GJFs of degree M.

5.1. Accuracy test

We consider the following Caputo FBVP:

$$-{}_0^CD_t^{2-\delta}u(t)+\lambda u'(t)=f(t),\quad t\in(0,T), \tag{5.2}$$

with either B. C. Robin

$$u(0)-\frac{\delta}{1+\delta}u'(0)=\gamma_0, \tag{5.3a}$$
$$u(T)+\frac{1}{2}u'(T)=\gamma_1, \tag{5.3b}$$

or B. C. Dirichlet $u(0)=\gamma_0$ and $u(T)=\gamma_1$.

We choose smooth data $f\equiv 1$, and γ_0, γ_1 such that the equation has the (non smooth exact) solution

$$u(t)=2+\frac{t}{\lambda}+\frac{\lambda-1}{\lambda}\int_0^t E_{1-\delta}(\lambda s^{1-\delta})ds, \tag{5.4}$$

where

$$E_{1-\delta}(\lambda t^{1-\delta})=\sum_{k=0}^{\infty}\frac{(\lambda t^{1-\delta})^k}{\Gamma(1+k(1-\delta))}.$$

In order to test accuracy, we use 800-point Legendre-Gauss quadrature to approximate the integral in $u(t)$ in (5.4).

We first consider the case with B. C. Robin Figs. 1-2 show the numerical errors vs. the numbers of intervals N with fixed uniform mode $M=4,5,6,7$, in log-log scale with $T=1$, $\lambda=1/2$, $\delta=1/3$. They indicate that the numerical errors decay algebraically as N increases. In Figs. 3-4, we plot the numerical errors vs. the number of modes in each interval M with fixed interval number $N=30,40,50,60$, in log-log scale with $T=1$,

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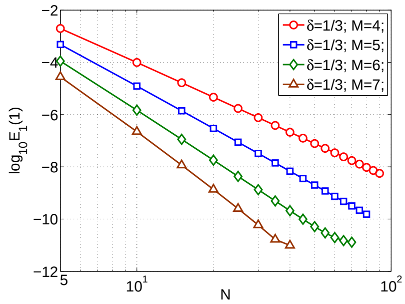

Figure 1: The numerical errors $E_1(1)$.

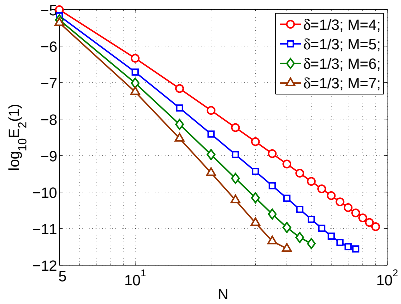

Figure 2: The numerical errors $E_2(1)$.

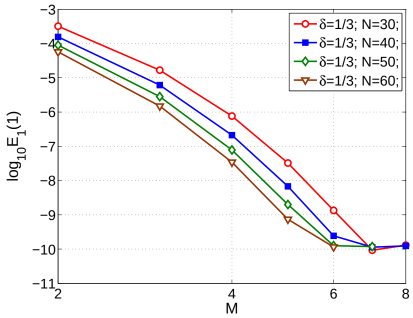

Figure 3: The numerical errors $E_1(1)$.

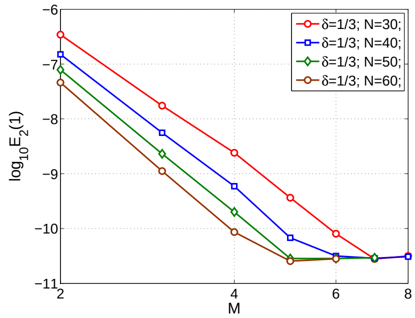

Figure 4: The numerical errors $E_2(1)$.

$\lambda = 1/2$, $\delta = 1/3$. They also indicate that the numerical errors decay algebraically as M increases, although Fig. 4 shows better than algebraic rate but this is probably due to the fact that we are still in pre-asymptotic range. Since the solution is not smooth, we do not expect exponential convergence w.r.t. M. However, one observe that with a fixed total numbers of unknowns, increasing M leads to more accurate results than increasing N. Despite the non-smooth solution, the method still provides very accurate results for both $u'(t)$ and $u(t)$, thanks to the graded mesh.

Next we consider (5.2) with B. C. Dirichlet. In Figs. 5-6, we plot, in log-log scale, the numerical errors vs. the numbers of intervals N with fixed uniform mode $M = 4, 5, 6, 7$ with $T = 1$, $\lambda = 1/2$, $\delta = 2/3$. And Figs. 7-8 show, in log-log scale, the numerical errors vs. the number of modes in each interval M with $T = 1$, $\lambda = 1/2$, $\delta = 2/3$.. These results are similar to the above case with B. C.Robin

5.2. Comparison with the collocation method in [16]

We now provide a comparison with the collocation method in [16]. Consider the FBVPs with $b(t) = \cos(t) - t^2$, $c(t) \equiv 0$, $\alpha_0 = 1/(1-\delta)$ and $\alpha_1 = 3/5$. We choose f, g_0 and g_1 such that the equation has the exact solution

$$u(t) = 2t^{2-\delta} - t^{3-2\delta} + 1 + 2t - 3t^3 + \frac{1}{2}t^4,$$

where regularity is typical of solutions of (1.1). This example was considered in [16].

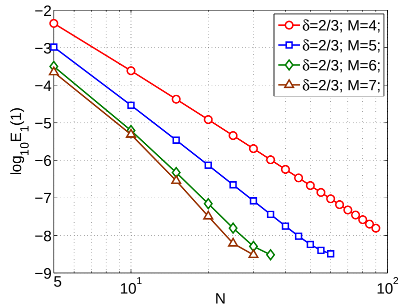

Figure 5: The numerical errors $E_1(1)$.

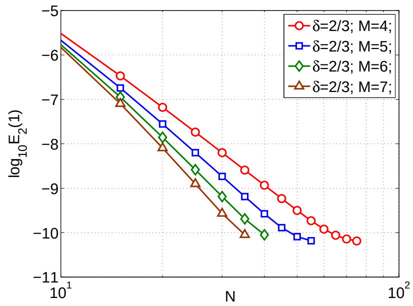

Figure 6: The numerical errors $E_2(1)$.

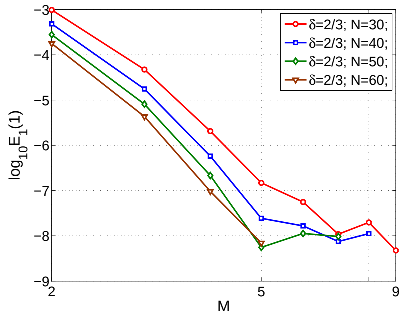

Figure 7: The numerical errors $E_1(1)$.

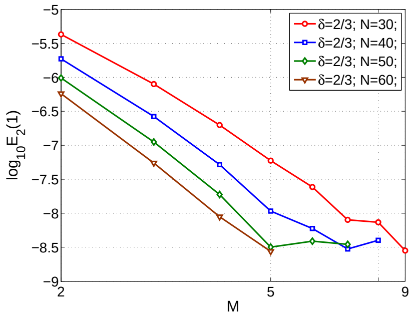

Figure 8: The numerical errors $E_2(1)$.

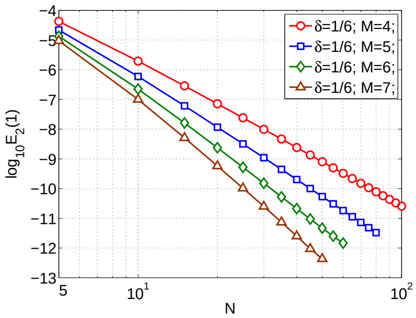

Figure 9: The numerical errors $E_1(1)$.

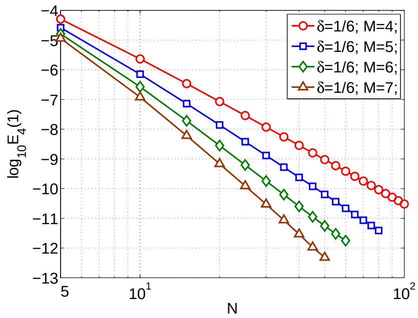

Figure 10: The numerical errors $E_2(1)$.

In Figs. 9-10, we plot the numerical errors vs. the numbers of intervals N with fixed uniform mode $M = 4, 5, 6, 7$, in log-log scale, of hybrid spectral element methods with $T = 1$, $\delta = 1/6$. As in previous examples, we observe an algebraic convergence.

In Tables 1-2, we compare the maximum errors at $T = 1$ of our algorithm, with $M = 7$ and $M = 11$ respectively, and of the collocation method (see, Table 3 of [16]). We observe that our method provides much more accurate numerical results with the same degree of freedom (see the second - fifth columns of Tables 1-2). We also observe that for a fixed number of total unknowns, $M = 11$ provides better accuracy than $M = 7$.

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Table 1: A comparison of the numerical errors (with $M = 7$ in our algorithm).

	DOF$= 128$		DOF$= 256$		DOF$= 4096$
	Ref. [16]	Our	Ref. [16]	Our	Ref. [16]
$\delta = 0.1$	3.787e-04	9.291e-10	9.476e-05	4.154e-12	3.704e-07
$\delta = 0.2$	4.051e-04	4.762e-09	1.015e-04	2.191e-11	3.976e-07
$\delta = 0.3$	4.277e-04	1.941e-08	1.076e-04	9.234e-11	4.232e-07
$\delta = 0.4$	4.361e-04	7.636e-08	1.104e-04	3.793e-10	4.392e-07
$\delta = 0.5$	6.530e-04	3.147e-07	1.619e-04	1.672e-09	6.244e-07
$\delta = 0.6$	1.337e-03	1.409e-06	3.346e-04	8.621e-09	1.313e-06
$\delta = 0.7$	2.992e-03	7.941e-06	7.619e-04	5.858e-08	3.111e-06
$\delta = 0.8$	7.721e-03	5.285e-05	2.015e-03	6.519e-07	8.774e-06
$\delta = 0.9$	2.963e-02	5.038e-04	7.953e-03	1.702e-05	3.724e-05

Table 2: A comparison of the numerical errors (with $M = 11$ in our algorithm).

	DOF$= 192$		DOF$= 768$		DOF$= 6144$
	Ref. [16]	Our	Ref. [16]	Our	Ref. [16]
$\delta = 0.1$	4.885e-07	1.681e-11	7.690e-09	3.444e-13	1.504e-11
$\delta = 0.2$	1.346e-06	1.407e-10	2.111e-08	4.138e-13	4.116e-11
$\delta = 0.3$	3.005e-06	9.837e-10	4.696e-08	3.526e-13	9.110e-11
$\delta = 0.4$	6.449e-06	7.005e-09	1.008e-07	4.214e-13	1.944e-10
$\delta = 0.5$	1.409e-05	5.552e-08	2.230e-07	3.900e-13	4.287e-10
$\delta = 0.6$	3.251e-05	5.200e-07	5.319e-07	4.236e-13	1.031e-09
$\delta = 0.7$	8.312e-05	5.860e-06	1.445e-06	2.773e-12	2.888e-09
$\delta = 0.8$	2.604e-04	6.451e-05	4.998e-06	6.519e-10	1.064e-08
$\delta = 0.9$	1.508e-03	6.566e-04	3.303e-05	9.867e-08	7.659e-08

6. Concluding Remarks

We proposed in this paper a hybrid spectral element methods for Caputo- and RL-FBVPs with two-point Robin and Dirichlet conditions. To avoid solving linear system which couples all unknowns, we first reformulated the FBVPs as weakly singular VIEs, and then we designed an efficient hybrid spectral element method, which use GJFs in the first interval to deal with the solution singularity at $t = 0$ and Legendre polynomials as basis functions. We established the existence and uniqueness of the numerical solution, and derived the hp-type error estimates for the hybrid spectral element methods. Numerical experiments demonstrated that the proposed method is capable of proving very accurate results despite the solution singularity.

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