Complex Systems **8** (1994) 407–417

A Markov Chain Analysis of Genetic Algorithms with a State Dependent Fitness Function*

Herbert Dawid†
Department of Operations Research and Systems Theory, Vienna University of Technology, Argentinierstr. 8/119, A-1040 Vienna, Austria

Abstract. We analyze the behavior of a Simple Genetic Algorithm (GA) in systems where the fitness of a string is determined by a function depending on the state of the whole population. The GA is modeled by a Markov chain and we conclude that for small mutation probabilities the limit distribution will put almost all the weight to the homogeneous states. We derive conditions under which a homogeneous state will be stable for the dynamics representing the expected behavior of the GA. Interpreting the state dependent fitness function as an economic system we prove that any strict economic equilibrium will be asymptotically stable with respect to the expected behavior of the GA.

1. Introduction

Inspired by the exemplar of natural evolution John Holland [17] developed a model of evolutionary processes called Genetic Algorithms (GAs). GAs were designed as a tool to find solutions of static optimization problems in poorly understood large spaces. Since then GAs have been used with great success to calculate almost optimal solutions to a large number of problems like function optimizing or various combinatorial problems (e.g., Goldberg [14], Goldberg, Milman, and Tidd [15] or Davis [5]). Although there is large empirical evidence of the optimizing properties of GAs, for a long time there have been few theoretical explanations for the success of GAs. A first attempt was Hollands' Schema Theorem [17], but in spite of the huge impact this theorem had on GA research, several researchers have pointed out that the Schema Theorem is not able to give a full explanation of what happens in genetic search (e.g., Mühlenbein [24, 23]). Only recently has more attention been paid to mathematical considerations regarding the behavior of a GA

*This research was supported by the Austrian Science Foundation under contract No. P9112-SOZ. Helpful comments from an anonymous referee are gratefully acknowledged.
†Electronic mail address: `dawid@e119ws1.tuwien.ac.at`

(e.g., Mühlenbein [22, 23], Vose and Liepins [28], Nix and Vose [25], Fogel [13], Hartl [16], Suzuki [27], Rudolph [26], and Davis and Principe [7]). The approach to model the evolution of a GA as a Markov process has proved especially useful. Vose and Liepins [28] and Nix and Vose [25] calculate the exact transition matrix of a Simple Genetic Algorithm and also derive some local stability results for static problems. Suzuki [27] and Rudolph [26] show that a GA using some kind of elitistic strategy (i.e., always keep the fittest string in the population) will always find the optimum. Also Eiben, Aarts, and Van Hee [12] derive conditions under which the optimum will always be found, by using Markov theory.

In the last few years GAs have become more and more important in economics. In economic or game theoretic situations a GA is no longer seen as a technical tool to find optima, but as a model of an interacting population. In a well known work Axelrod [3] used GAs to learn optimal strategies in the iterated prisoner's dilemma. He shows that GAs are able to learn strategies that are even more successful than Rappoport's "tit for tat" strategy. Mühlenbein [21] extended Axelrod's work by addressing the question of whether or not a spatial population structure will improve the performance of the GA. (GAs with spatial population structures have also been analyzed by Deb and Goldberg [11], Davidor [4], or Horn [19].) Relying on the *continent cycle theory* of Darwin, Mühlenbein does simulations where the population is periodically split into islands and the small island populations evolve separately for some time until the population is reunited. Miller [20] was the first to use GAs to learn equilibria in economic systems. Recently, Arifovic [1, 2] analyzed the learning behavior of GAs in several economic models. By using the example of a cobweb model, she demonstrates that the behavior of a GA differs significantly less from experimental data than standard econometric learning rules. In recent papers Dawid and Mehlmann [9, 10] and Dawid [8] analyze the ability of a GA to learn equilibria in strategic games. Holland and Miller [18] give a larger review of applications of GAs to economic systems.

Contrary to static optimization problems, in these economic models the fitness of a string is a function of the state of the population. Further on we will call such a system with State Depending Fitness an SDF-system. Typical questions concerning SDF-systems will be whether the population always converges to a homogeneous state and how the limit set of the system can be characterized. However, up to now no one has investigated these problems from an analytic point of view. In this paper we try to fill this vacuum and to provide some theoretical insights on the behavior of a GA with a state dependent fitness function. We use the exact representation of GAs by Vose and Liepins [28] to gain several stability results which enable us to answer the questions stated above, at least partly by the means of analytical arguments. It should be clear that any static optimization problem can be interpreted as an SDF-system with a constant fitness function f, and therefore all our results hold also for the "classical" applications of GAs.

2. The Markov model of the Genetic Algorithm

We consider a simple GA incorporating the three standard operators: selection, crossover, and mutation. We assume that proportional selection is used and denote the crossover probability with χ, the mutation probability with μ. Let P be the population consisting of a finite number of binary strings of length l. The number of strings in the population will be denoted by n. Further let Ω be the set of all binary strings of length l. Obviously the cardinality of Ω is given by $|\Omega| = r = 2^l$. We will denote each element k of Ω by an integer given by $\sum_{i=1}^{l} k(i)2^{i-1}$, where $k(i)$ is the value of the ith bit of k. A state of the population P is given by a vector ϕ of length r, where ϕ_k, $k = 0, \ldots, r-1$ denotes the relative frequency of the string $k \in \Omega$ in the population. We call the set of all possible population states S. Simple calculations show that the number of possible states is given by $|S| = N = \binom{n+r-1}{r-1}$ (see Nix and Vose [25]). Further we assume that the fitness function of an arbitrary SDF-system is given by a function $f : S \to \mathbb{R}^r$ where $f_k(\phi)$ denotes the fitness the string k receives if the population is in state ϕ. We assume that the fitness function is positive, continuous, and continuously differentiable.

Our formal description of the GA relies heavily on the work of Nix and Vose [25] and Vose and Liepins [28]. We interpret the behavior of the GA as a homogeneous Markov chain on the state space S. Let Q be the transition matrix. Calculations show that $q_{\phi\phi'}$ is given by

$$q_{\phi\phi'} = n! \prod_{k=0}^{r-1} \frac{p_k(\phi)^{n\phi'_k}}{(n\phi'_k)!}, \tag{1}$$

where $p_k(\phi)$ is the probability that an arbitrary offspring produced from a population in state ϕ will be a string k. To calculate $p_k(\phi)$ we define by $m_{i,j}(k)$, $i, j, k \in \Omega$ the probability that string k results from the recombination process consisting of crossover and mutation with parents i and j. We denote by M a $r \times r$ matrix whose elements are given by $m_{i,j}(0)$, $i, j \in \Omega$. This implies that the probability that the string 0 results from the reproduction procedure in a population in state ϕ is given by $\phi^T M \phi$. Let $\oplus$ be the exclusive or, so that $j \oplus k$ is the bitwise difference vector between j and k. Using this notation, symmetry arguments show that

$$m_{k\oplus i, k\oplus j}(k) = m_{i,j}(0).$$

This implies that the probability that a string k is produced by crossover and mutation in state ϕ is given by $(\sigma_k\phi)^T M \sigma_k \phi$, where σ_k is a permutation matrix incorporating a permutation of the basis of S where the basis vector $e_{j\oplus k}$ is transformed to the basis vector e_j:

$$\sigma_k \langle \phi_0, \ldots, \phi_{r-1} \rangle^T = \langle \phi_{k\oplus 0}, \ldots, \phi_{k\oplus(r-1)} \rangle^T.$$

We define the operator $\mathcal{M} : S \to \Lambda = \{x \in \mathbb{R}^r : x_i \geq 0, \sum_{i=0}^{r-1} x_i = 1\}$ by setting $\mathcal{M}_k(\phi)$, $k \in \Omega$, $\phi \in S$, equal to the probability that string k is generated by reproduction in a population in state ϕ. This means

$$\mathcal{M}(\phi) = \langle (\sigma_0\phi)^T M(\sigma_0\phi), \ldots, (\sigma_{r-1}\phi)^T M(\sigma_{r-1}\phi) \rangle^T.$$

For a formal proof of these facts see Vose and Liepins [28] and Nix and Vose [25].

In order to represent the selection process we define the matrix $F(\phi) = \text{diag}(f(\phi))$ as a diagonal matrix with $F_{kk}(\phi) = f_k(\phi)$. As we assume that proportional selection is used, the probability of a string k being selected if the population is in state ϕ is given by $\frac{f_k(\phi)\phi_k}{f^T(\phi)\phi}$. We define the selection operator as

$$\mathcal{S}(\phi) = \frac{F(\phi)\phi}{f^T(\phi)\phi} \tag{2}$$

and get finally

$$p_k(\phi) = \mathcal{M}(\mathcal{S}(\phi))_k. \tag{3}$$

It is easy to see that for $\mu > 0$ the transition matrix Q is strictly positive and the Markov chain has therefore an unique limit distribution. With zero mutation, the steady state distribution is concentrated at the absorbing states, namely the vertices of Λ. Moreover, it is simply a left eigenvector which depends continuously on the matrix Q which depends continuously on μ. Thus by continuity as $\mu \to 0$, the steady state distribution becomes concentrated near the vertices of Λ. Davis [5] shows further that the limit of the steady state distribution for $\mu \to 0$ attaches a strictly positive probability to every homogeneous state. We do not know however how much weight the limit distribution will assign to the different homogeneous states. To gain some insight into this problem we derive some results in the next section concerning the stability of the homogeneous states.

3. Stability of the homogeneous states

From (1) and (3) it is obvious that the expected evolving behavior of the GA is given by the difference equation

$$\phi^{t+1} = \mathcal{M}(\mathcal{S}(\phi^t)), \tag{4}$$

where ϕ^t is the state of the population at time t. As for large n the probability of deviation from this expected path is small, the stability of a state with respect to (4) may be used to analyze whether the Markov chain will stay near a state or will be driven away. We will analyze the case where $\mu = 0$, but due to the fact that the right-hand side of (4) depends continuously on μ, we state that the stability properties derived for $\mu = 0$ hold also for small $\mu > 0$. Note that for $\mu = 0$ every homogeneous state e_k, $k = 0, \ldots, r-1$ is a fixed point of (4). In Proposition 1 the spectrum of the linearization of (4) at a fixed point $e_k \in S$ is calculated.

Proposition 1. For $\mu = 0$ the spectrum of the linearization of (4) at $e_k \in S$ is given by

$$\Lambda(e_k) = \{0\} \cup \{2m_{0,j\oplus k}\frac{f_j(e_k)}{f_k(e_k)}, \quad j \in \Omega,\ j \neq k\}. \tag{5}$$

Proof. We calculate explicitly the $r \times r$ matrix $\mathcal{D}_{\mathcal{M}\circ\mathcal{S}}(e_k) = \mathcal{D}_{\mathcal{M}}(e_k)\mathcal{D}_{\mathcal{S}}(e_k)$. Direct calculation yields

$$\mathcal{D}_{\mathcal{M}}(\phi) = 2\sum_{j=0}^{r-1} \sigma_j^{-1} M^* \sigma_j \phi_j$$

with $M^* = [m_{i\oplus j,i}]_{i,j\in\Omega}$ (see Vose and Liepins [28]), and therefore $\mathcal{D}_{\mathcal{M}}(e_k) = 2\sigma_k^{-1}M^*\sigma_k$. Note further that M^* is upper triangular for $\mu = 0$. This can be seen by considering $m^*{}_{i,j} = m_{i\oplus j,i}$ for $i > j$. As $i > j$ there has to be a bit position p with $i(p) = 1$ and $j(p) = 0$, and therefore $(i \oplus j)(p) = 1 = i(p)$. Because of this string 0 can never emerge from crossover between $i \oplus j$ and i, which implies $m^*{}_{i,j} = m_{i\oplus j,i} = 0\ \forall i, j \in \Omega$, with $i > j$. Next we calculate the linearization of the selection operator given by (2):

$$\begin{aligned}
\mathcal{D}_{\mathcal{S}}(e_k) &= \tfrac{1}{(e_k^T f(e_k))^2}[(\mathrm{diag}(f(e_k)) + \mathrm{diag}(e_k)\mathcal{D}_f(e_k))(e_k^T f(e_k)) \\
&\quad -\mathrm{diag}(f(e_k))e_k(f^T(e_k) + e_k^T\mathcal{D}_f(e_k))] \\
&= \tfrac{1}{f_k{}^2(e_k)}[f_k(e_k)\mathrm{diag}(f(e_k)) + f_k(e_k)\mathrm{diag}(e_k)\mathcal{D}_f(e_k) \\
&\quad - \underbrace{\mathrm{diag}(f(e_k))e_k}_{=f_k(e_k)e_k} f_k^T(e_k) - \underbrace{\mathrm{diag}(f(e_k))e_k e_k^T}_{=f_k(e_k)\mathrm{diag}(e_k)} \mathcal{D}_f(e_k)] \\
&= \tfrac{1}{f_k(e_k)}[\mathrm{diag}(f(e_k)) - e_k f^T(e_k)].
\end{aligned}$$

Let $D^*{}_k$ be given by

$$\begin{aligned}
D^*{}_k &= \sigma_k\mathcal{D}_{\mathcal{S}}(e_k)\sigma_k = \tfrac{1}{f_k(e_k)}[\underbrace{\sigma_k\mathrm{diag}(f(e_k))\sigma_k}_{=\mathrm{diag}(\sigma_k f(e_k))} - \underbrace{\sigma_k e_k}_{=e_0} f^T(e_k)\sigma_k] \\
&= \tfrac{1}{f_k(e_k)}[\mathrm{diag}(\sigma_k f(e_k)) - e_0 f^T(e_k)\sigma_k].
\end{aligned}$$

This means that $D^*{}_k$ is the difference between a diagonal matrix and a matrix which has positive values in the first row only. Therefore $D^*{}_k$ is upper triangular, and it is easy to see that the elements on the diagonal are given by

$$\begin{aligned}
d^*{}_{jj} &= 0, & j &= 0, \\
d^*{}_{jj} &= \tfrac{f_{j\oplus k}(e_k)}{f_k(e_k)}, & j &= 1, \ldots, r-1.
\end{aligned}$$

Using $\sigma_k{}^{-1} = \sigma_k$ we get

$$\begin{aligned}
\mathcal{D}_{\mathcal{M}\circ\mathcal{S}}(e_k) &= 2\sigma_k M^*\sigma_k\mathcal{D}_{\mathcal{S}}(e_k) = 2\sigma_k M^*\sigma_k\sigma_k D^*{}_k\sigma_k \\
&= 2\sigma_k M^* D^*{}_k\sigma_k.
\end{aligned}$$

As a permutation of the basis does not change the eigenvalues of a matrix, the spectrum of $\mathcal{D}_{\mathcal{M}\circ\mathcal{S}}(e_k)$ coincides with the spectrum of $2M^*D^*{}_k$. As M^* and $D^*{}_k$ are both upper triangular matrices, the eigenvalues of the product are given by the product of the diagonal elements of the two matrices, which yields

$$\Lambda(e_k) = \{0\} \cup \{2m_{0,i}\frac{f_{i\oplus k}(e_k)}{f_k(e_k)},\ i = 1, \ldots, r-1\}.$$

Substituting $i \oplus k$ by j establishes the claim of Proposition 1. ■

Proposition 1 enables us to derive stability results for different crossover operators by just calculating the term $m_{0,j\oplus k}$. We will do so for two of the most popular crossover operators, namely one point crossover and uniform crossover. To formulate the stability properties for one point crossover, we denote by $d(j) = \max\{|x-y| \, |j(x) = j(y) = 1\}$ the longest distance between two bits with value 1 in string j.

Proposition 2. A homogeneous state e_k is local asymptotically stable for the expected dynamics of a GA (4) with $\mu = 0$ and one point crossover with probability $\chi \in (0, 1]$ if and only if

$$\frac{d(j \oplus k)}{l-1} > \frac{1}{\chi}\left(1 - \frac{f_k(e_k)}{f_j(e_k)}\right)$$

holds for all $j \in \Omega, j \neq k$.

Proof. It is well known that a fixed point of a discrete time dynamical system is stable if and only if all eigenvalues of the linearization of the system at the fixed point lie in the interior of the unit circle. Therefore e_k is stable if and only if

$$2m_{0,j\oplus k}\frac{f_j(e_k)}{f_k(e_k)} < 1$$

holds for all $j \in \Omega,\ j \neq k$. Remember that $m_{0,j\oplus k}$ is the probability that an arbitrary offspring, say the first of the two offsprings, is 0, when the parents are 0 and $j \oplus k$. One of the two offsprings will be 0 if no crossover takes place, or if the crossover point is left of the leftmost 1 in $j \oplus k$ or right of the rightmost 1 in $j \oplus k$. This yields

$$m_{0,j\oplus k} = \frac{1}{2}\left(1 - \frac{\chi d(j \oplus k)}{l-1}\right).$$

Inserting this expression into the inequality above proves the claim of Proposition 2. ■

Note that $d(j \oplus k)$ is the length between the two outmost bits where j and k differ in value. Proposition 2 claims therefore, that a state consisting almost only of strings k will converge to the homogeneous state e_k if the strings receiving a higher payoff in the current state differ from k in bits positioned far apart. To understand this result consider a situation where few strings j appear in a population consisting mainly of strings k. On one hand the number of strings j will increase due to the effects of selection but on the other hand they will be destryoed with high probability whenever they are mated for crossover with a string k because any crossover point position between the two outmost differing bits will destroy j. As long as the fitness of j is not far too much larger than that of k the second effect will be stronger and the strings j will disappear. An important implication of Proposition 2 is that the coding of a problem may determine the stability or instability of certain homogeneous states.

Proposition 3 deals with uniform crossover, where for any bit position x an offspring attains the values of the parents $i(x)$ and $j(x)$ with probability 0.5.

Proposition 3. A homogeneous state e_k is local asymptotically stable for the expected dynamics of a GA (4) with $\mu = 0$ and uniform crossover with probability $\chi \in (0, 1]$ if and only if

$$\frac{f_k(e_k)}{f_j(e_k)} > (1-\chi) + \chi \left(\frac{1}{2}\right)^{|j \oplus k|-1}$$

holds for all $j \in \Omega, j \neq k$.

Proof. The proof is analogous to the proof of Proposition 2, but $m_{0,j\oplus k}$ is now given by

$$m_{0,j\oplus k} = \frac{1}{2}(1-\chi) + \chi \left(\frac{1}{2}\right)^{|j \oplus k|} . \blacksquare$$

After having completed this paper the author was informed that Vose and Wright [29] have independently at the same time derived results similar to our Propositions 1, 2, and 3 for state independent fitness functions. As state independent fitness functions are a special case of the fitness functions considered here this can be seen as a confirmation of our results.

As pointed out in the introduction, the main reason why we analyze GAs with a state dependent fitness function is that we would like to have some analytical insight into the behavior of GAs in economic systems. The main question arising in the context of learning in economic systems is the question of stability of the economic equilibria with respect to the learning dynamics. With the help of Proposition 1 we are able to answer this question for genetic learning. Let us now interpret the population P as a population of economic agents, where each string codes the current action of one agent. The fitness function $f_k(\phi)$ is interpreted as the payoff an agent receives when he executes the action coded by string k and the rest of the population behaves according to ϕ. An economic equilibrium of this system is a state ϕ where every agent acts in a way to maximize his payoff under the assumption that the rest of the population acts according to ϕ. This means that ϕ is an economic equilibrium if for all $k \in \Omega$ with $\phi_k > 0$

$$f_k(\phi) \geq f_j(\phi) \qquad \forall\, j \in \Omega. \tag{6}$$

We call the equilibrium strict if every agent has a unique best reply to the population strategy. In the case of a homogeneous strict equilibrium state the condition is therefore $f_k(e_k) > f_j(e_k)$, $\forall j \neq k$. This means that in the surrounding of a population in state e_k the string k will have a strictly higher fitness than any other string j would have. Of course this does however not imply that the string k will be optimal in the surrounding of population states $\phi \neq e_k$. In Proposition 4 it is shown that at least every homogeneous strict equilibrium state e_k, is local asymptotically stable with respect to the expected behavior of the GA.

Proposition 4. Let e_k be a homogeneous strict equilibrium state of an SDF-system, then e_k is local asymptotically stable with respect to (4) for $\mu = 0$ and for any kind of crossover operator.

Proof. Due to our assumption, $f_k(e_k) > f_j(e_k)$ has to hold for any $j \neq k$. On the other hand $m_{0,j} \leq \frac{1}{2}$, $\forall j \neq 0$ holds for any crossover operator, as the crossover between the string 0 and any string not equal to 0 can never create two offspring 0. With (5) this yields that all eigenvalues of the linearization of (4) at e_k are strictly smaller than 1. ■

Although the main purpose of this paper is the analysis of GAs with a state dependent fitness function we will close this section with a result concerning GAs with a "traditional" state independent fitness function. We do this because the result is in our opinion quite interesting and can be easily obtained as a direct corollary of Proposition 4. If the fitness function is state independent (i.e., a vector $[f_j]_{j\in\Omega}$), the condition $f_j(e_k) < f_k(e_k)$, $\forall j \neq k$ reduces to $f_j < f_k$, $\forall j \neq k$, which means that the string k has maximum fitness.

Corollary 1. Let a static optimization problem be given by the fitness vector $[f_j]_{j\in\Omega}$, and let k be the unique optimal solution to the problem. Then the state e_k is local asymptotically stable with regard to (4) for $\mu = 0$ and any crossover operator.

4. Conclusion

The aim of this paper is to gain some initial insights into the characterization of the limit set of a GA in the general setting of systems where the fitness of a string depends on the state of the population. We have used the result that for small mutation probabilities the limit distribution of the corresponding Markov chain will be concentrated mostly on the homogeneous states and have stated conditions saying which of these homogeneous states will be asymptotically stable under the expected behavior of the GA. The stability properties are only local but we may conclude that unstable points will have only small weight in the limit distributions as the mutations will disrupt the homogeneous state to some point off of the stable manifold and the state of the system will be driven away from this homogeneous state (although we have derived our stability results for $\mu = 0$ they will still hold for very small $\mu > 0$). We may therefore conjecture that the limit distribution will be concentrated on the homogeneous states satisfying the stability conditions in Proposition 2 or Proposition 3. Unfortunately, the highly nonlinear structure of the system describing the expected behavior of the GA does not allow us to derive any global results, but at least in cases where there is only one locally stable state our results indicate that the system will with high probability be in this state after some transient time. The result of Proposition 4 that a homogeneous strict economic equilibrium state will be locally stable with respect to the dynamics of the GA can be seen as a mathematical foundation of the different empirical results indicating that in various economic or game theoretic models a GA will converge to an equilibrium state.

References

[1] J. Arifovic, "Adaptation of Genetic Algorithm in Environments with Changing Parameters," unpublished manuscript (McGill University, Montreal, 1992).

[2] J. Arifovic, "Genetic Algorithm Learning and the Cobweb Model," *Journal of Economic Dynamics and Control*, **18** (1994) 3–28.

[3] R. Axelrod, "The Evolution of Strategies in the Iterated Prisoner's Dilemma," in *Genetic Algorithms and Simulated Annealing*, edited by L. Davis (Morgan Kaufmann, Los Altos, 1987), 32–41.

[4] Y. Davidor, "A Naturally Occurring Niche and Species Phenomenon: The Model and First Results," in *Proceedings of the Fourth International Conference on Genetic Algorithms*, edited by R. K. Belew and L. B. Booker (Morgan Kaufmann, 1991), 257–263.

[5] L. Davis (editor), *Handbook of Genetic Algorithms* (Van Nostrand Reinhold, New York, 1991).

[6] T. E. Davis, *Toward an Extrapolation of the Simulated Annealing Convergence Theory Onto the Simple Genetic Algorithm*, Ph.D. thesis (University of Florida, Gainesville, 1991).

[7] T. E. Davis and J. C. Principe, "A Markov Chain Framework for the Simple Genetic Algorithm," *Evolutionary Computation*, **1** (1993).

[8] H. Dawid, "Learning by Genetic Algorithms in Evolutionary Games," in *Operations Research Proceedings 1994*, edited by U. Derigs *et al.* (Springer, Berlin, 1995), 261–266.

[9] H. Dawid and A. Mehlmann, "Two Population Contests and the Language of Genetics," to be published in *Mathematical Linguistics and Related Topics*, edited by G. Paun (Editura Academiei, Bucarest, 1995).

[10] H. Dawid and A. Mehlmann, "Genetic Learning in Strategic Form Games," unpublished manuscript (Vienna University of Technology, 1994).

[11] K. Deb and D. E. Goldberg, "An Investigation of Niche and Species Formation in Genetic Function Optimization," in *Proceedings of the Third International Conference on Genetic Algorithms*, edited by J. D. Schaffer (Morgan Kaufmann, 1989), 42–50.

[12] A. E. Eiben, E. H. L. Aarts, and K. M. Van Hee, "Global Convergence of Genetic Algorithms: A Markov Chain Analysis," in *Parallel Problem Solving from Nature*, edited by H.-P. Schwefel and R. Männer (Springer, Berlin, 1991), 4–12.

[13] D. B. Fogel, "Asymptotic Convergence Properties of Genetic Algorithms and Evolutionary Programming: Analysis and Experiments," *Cybernetics and Systems*, **25** (1994) 389–407.

[14] D. E. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning* (Addison-Wesley, Reading, MA, 1989).

[15] D. E. Goldberg, K. Milman, and C. Tidd, "Genetic Algorithms: A Bibliography," IlliGAL Report No. 92008 (University of Illinois, Urbana-Champaign, 1992).

[16] R. F. Hartl, "A Global Convergence Proof for a Class of Genetic Algorithms," unpublished manuscript (Vienna University of Technology, 1991).

[17] J. H. Holland, *Adaptation in Natural and Artificial Systems* (University of Michigan, Ann Arbor, 1975).

[18] J. H. Holland and J. H. Miller, "Artificial Adaptive Agents in Economic Theory," in *American Economic Review: Papers and Proceedings of the 103rd Annual Meeting of the American Economic Association*, 1991, 335–350.

[19] J. Horn, "Finite Markov Chain Analysis of Genetic Algorithms with Niching," in *Proceedings of the Fifth International Conference on Genetic Algorithms*, edited by S. Forrest (Morgan Kaufmann, 1993), 110–117.

[20] J. H. Miller, "A Genetic Model of Economic Adaptive Behavior," Working Paper (University of Michigan, Ann Arbor, 1986).

[21] H. Mühlenbein, "Darwin's Continent Cycle Theory and Its Simulation by the Prisoner's Dilemma," *Complex Systems*, **5** (1991) 459–478.

[22] H. Mühlenbein, "How Genetic Algorithms Really Work: Mutation and Hill-climbing," in *Parallel Problem Solving from Nature 2*, edited by R. Männer and B. Manderick (North Holland, Amsterdam, 1992), 15–26.

[23] H. Mühlenbein, "Evolutionary Algorithms: Theory and Applications," in *Local Search in Combinatorial Optimization*, edited by E. H. L Aarts and J. K. Lenstra (Wiley, 1993).

[24] H. Mühlenbein, "Evolution in Time and Space, the Parallel Genetic Algorithm," in *Foundation of Genetic Algorithms*, edited by G. Rawlins (Morgan Kaufmann, 1991), 316–337.

[25] A. E. Nix and M. D. Vose, "Modeling Genetic Algorithms with Markov Chains," *Annals of Mathematics and Artificial Intelligence*, **5** (1992) 79–88.

[26] G. Rudolph, "Convergence Analysis of Canonical Genetic Algorithms," *IEEE Transactions on Neural Networks*, **5** (1994) 96–101.

[27] J. Suzuki, "A Markov Chain Analysis on a Genetic Algorithm," in *Proceedings of the Fifth International Conference on Genetic Algorithms*, edited by S. Forrest (Morgan Kaufmann, 1993), 146–153.

[28] M. D. Vose and G. E. Liepins, "Punctuated Equilibria in Genetic Search," *Complex Systems*, **5** (1991) 31–44.

[29] M. D. Vose and A. H. Wright, "Stability of Vertex Fixed Points and Applications," to be published in: *Foundations of Genetic Algorithms III*.