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Abstract

This article introduces a certain class of stochastic processes, which we suggest to call mild Itô processes, and a new - somehow mild - Itô type formula for such processes. Examples of mild Itô processes are mild solutions of stochastic partial differential equations (SPDEs) and their numerical approximation processes.

Contents

1 Introduction

The following setting is considered in this introductory section. Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ and $(V, \|\cdot\|_V, \langle\cdot,\cdot\rangle_V)$ be separable real Hilbert spaces, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $(W_t)_{t\in[0,\infty)}$ be a cylindrical standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Wiener process on U, let $A\colon D(A) \subset H \to H$ be a generator of an analytic semigroup and let $\alpha \in (-1, 0]$, $\beta \in (-\frac{1}{2}, 0]$, $\eta \in [0, \infty)$ be real numbers such that $\eta - A$ is bijective and positive. Moreover, to simplify the notation define $\|v\|_{H_r} := \|(\eta - A)^r v\|_H$ for all $v \in H_r := D((\eta - A)^r)$ and all $r \in \mathbb{R}$ and let $F\colon H \to H_\alpha$ and $B\colon H \to HS(U, H_\beta)$ be globally Lipschitz continuous functions and let $X\colon [0,\infty) \times \Omega \to H$ be an adapted stochastic process with continuous sample paths satisfying $\sup_{s\in[0,t]} \mathbb{E}[\|X_s\|_H^p] < \infty$ and

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s \tag{1}$$

$\mathbb{P}$-a.s. for all $t, p \in [0, \infty)$. The stochastic process $X: [0, \infty) \times \Omega \to H$ is thus a *mild solution* of the stochastic partial differential equation (SPDE) (1) (SPDEs have been extremely intensively studied in the last decades; see, e.g., the books [95, 21, 35, 22, 12, 19, 96, 14, 89, 61, 47] and lecture notes [105, 57, 67, 90, 1, 24, 40] and the references therein). A simple example of this framework is the following setting: If $H = U = L^2((0,1), \mathbb{R})$ is the Hilbert space of equivalence classes of Lebesgue square integrable functions, if $A: D(A) \subset H \to H$ is the Laplacian with Dirichlet boundary conditions on $(0,1)$ and if $\big(F(v)\big)(x) = f(x, v(x))$ and $\big(B(v)u\big)(x) = f(x, v(x)) \cdot u(x)$ for all $x \in (0,1)$, $u, v \in H$ where $f, b: (0,1) \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions with globally bounded derivatives, then the above framework is fulfilled with $\alpha = \eta = 0$ and $\beta \in (-\frac{1}{2}, -\frac{1}{4})$ and (1) reduces to the SPDE

$$dX_t(x) = \Big[\tfrac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x))\Big]\, dt + b(x, X_t(x))\, dW_t(x) \tag{2}$$

with $X_t(0) = X_t(1) = 0$ for $x \in (0,1)$ and $t \in [0, \infty)$. Further examples of the above described framework and existence and uniqueness results for (1) can, e.g., be found in Da Prato & Zabczyk [21, 22], Brzeźniak [8] (see Theorem 4.3 in [8]), Van Neerven, Veraar & Weis [103] (see Theorem 6.2 and Section 10 in [103]) and in the references therein.

Our aim is to derive an Itô type formula for the solution process X of the SPDE (1). Let us briefly review some related Itô type formula results from the literature. First, note that if $\alpha = \beta = 0$ and if the mild solution process X of the SPDE (1) is also a $D(A)$-valued strong solution of the SPDE (1), then the standard Itô formula (see Itô [48]) in infinite dimensions can be applied to X. More precisely, in that case, Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9] implies

$$\begin{aligned} \varphi(X_t) = \varphi(X_{t_0}) &+ \int_{t_0}^{t} \varphi'(X_s)\,[AX_s + F(X_s)]\, ds + \int_{t_0}^{t} \varphi'(X_s) B(X_s)\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi''(X_s)\big(B(X_s) g_j, B(X_s) g_j\big)\, ds \end{aligned} \tag{3}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0, \infty)$ with $t_0 \leq t$ and all twice continuously Fréchet differentiable functions $\varphi \in C^2(H, V)$ where $\mathcal{J}$ is a set and $(g_j)_{j \in \mathcal{J}} \subset U$ is an arbitrary orthonormal basis of U. The case where X is not $D(A)$-valued and thus not a strong solution of (1) is more subtle. There are a few results in the literature in this direction. First, in the case $\alpha \geq -\frac{1}{2}$ and $\beta = 0$ (i.e., B maps from H to $HS(U, H)$), in the case where $A: D(A) \subset H \to H$ is self-adjoined and in the case of the special test function $\varphi(v) = \|v\|_H^2$ for all $v \in H$, (3) can be generalized and then reads as

$$\begin{aligned} \|X_t\|_H^2 = \|X_{t_0}\|_H^2 &+ 2 \int_{t_0}^{t} \langle X_s, AX_s \rangle_H\, ds + 2 \int_{t_0}^{t} \langle X_s, F(X_s) \rangle_H\, ds + 2 \int_{t_0}^{t} \langle X_s, B(X_s)\, dW_s \rangle_H \\ &+ \int_{t_0}^{t} \|B(X_s)\|_{HS(U,H)}^2\, ds \end{aligned} \tag{4}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0, \infty)$ with $t_0 \leq t$; see Pardoux's pioneering work [85, 86, 87] and see, e.g., also [68, 38, 39, 88, 83, 91, 90] for generalizations and reviews of this Itô formula for the squared norm (the above mentioned results from the literature consider slightly different frameworks and, in particular, often allow A to be nonlinear too). Note that in that case X enjoys values in $H_{1/2} = D\big((\eta - A)^{1/2}\big)$ (see Theorem 4.2 in Kruse & Larsson [66]) and therefore, the integral $\int_0^t \langle X_s, AX_s \rangle_H\, ds := \eta \int_0^t \|X_s\|_H^2\, ds - \int_0^t \|(\eta - A)^{\frac{1}{2}} X_s\|_H^2\, ds$ in (4) is well defined. Formula (4) is a crucial ingredient in the *variational approach* for SPDEs (see the monographs [87, 68, 95, 90]). Formula (4) is an Itô formula for possibly non-strong solutions of SPDEs in the case of the special test function $\|\cdot\|_H^2$. There are also a few results in the literature which establish Itô type formulas for possible non-strong solutions of SPDEs for more general test functions than the squared norm $\|\cdot\|_H^2$; see [88, 110, 33, 69, 72, 70]. In Theorem 5.1 in Pardoux [88], formula (4) is generalized to a special class of test functions which have similar topological properties as the function $\|\cdot\|_H^2$. In Zambotti [110], the standard Itô formula is applied to regularized versions of the solution process of the stochastic heat equation with additive noise and then the limit of these regularized Itô formulas is made sense through a suitable renormalization term that appears in the resulting formula. In Gradinaru, Nourdin & Tindel [33], Malliavin calculus and a Skorokhod integral is used to prove an Itô type formula for the solution of the stochastic heat equation with additive noise (see also Leon & Tindel [72] for a related Itô formula result for the stochastic heat equation with additive fractional noise). In Lanconelli [69], a Wick product is used to formulate an Itô type formula for the solution process of the stochastic heat equation with additive noise and the relation between the formulas in [110, 33] is analyzed (see Section 3 in [69] and see also Lanconelli [70] for some consequences of this Wick produckt Itô type formula for the stochastic heat equation with additive noise).

In general it is not clear how and whether (3) can be generalized to the case where $X\colon [0,\infty)\times\Omega\to H$ is not a $D(A)$-valued strong solution of (1). This article suggests a different approach for deriving an Itô formula for solutions of (1). We do not aim for a suitable generalization of (3) to the case of non-strong solutions but instead we suggest a somehow different Itô type formula for (1) which naturally holds for (1) in its full generality for all smooth test functions. More precisely, we establish in Corollary 2 in Subsection 3.2 below the identity

$$\begin{aligned}\varphi(X_t) = \varphi(e^{A(t-t_0)}X_{t_0}) + \int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\, e^{A(t-s)}F(X_s)\, ds + \int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\, e^{A(t-s)}B(X_s)\, dW_s \\ + \frac{1}{2}\sum_{j\in\mathcal{J}}\int_{t_0}^t \varphi''(e^{A(t-s)}X_s)\Big(e^{A(t-s)}B(X_s)g_j, e^{A(t-s)}B(X_s)g_j\Big)\, ds\end{aligned} \tag{5}$$

$\mathbb{P}$-a.s. for all $t_0, t\in[0,\infty)$ with $t_0\le t$ and all $\varphi\in\cup_{r<\min(\alpha+1,\beta+1/2)}C^2(H_r,V)\supset C^2(H,V)$. Corollary 2 also ensures that all terms in (5) are well defined (see (50)–(52) in Subsection 3.2). In the case of (2), natural examples for the test functions $\varphi\in\cup_{r<\min(\alpha+1,\beta+1/2)}C^2(H_r,V)$ in (5) are Nemytskii operators and nonlinear integral operators such as $H_r\ni v\mapsto\int_0^1\psi(x,v(x))\,dx\in\mathbb{R}$ for any $0<r<\min(\alpha+1,\beta+1/2)$ and any sufficiently regular function $\psi\colon(0,1)\times\mathbb{R}\to\mathbb{R}$. In the special case $\varphi=id_H\colon H\ni v\mapsto v\in H=V$, equation (5) reduces to the variant of constants formula (1) and in that sense, (5) is somehow a *mild Itô formula*. In the deterministic case $B\equiv 0$, equation (5) is somehow a *mild chain rule*; see Example 2 in Section 2.2 below for more details. The identity (5) can be generalized to a much larger class of stochastic processes than solution processes of the SPDE (1). To be more precise, in Definition 1 in Subsection 2.1 a class of stochastic processes which exhibit a similar algebraic structure as (1) is introduced and referred as *mild Itô processes*. Examples of mild Itô processes are solution processes of SPDEs such as (1) (see Subsection 3.2) as well as their numerical approximation processes (see Subsection 3.3). The identity (5) is then a special case of equation (22) in Theorem 1 below in which a mild Itô formula for mild Itô processes is established.

Let us outline how (5) and Theorem 1 respectively are established. A central idea in the proof of (5) is to consider a suitable transformation of the solution process $X\colon[0,\infty)\times\Omega\to H$ of the SPDE (1). The transformed stochastic process is then a standard Itô process to which the standard Itô formula (see (3)) can be applied. Relating then the transformed stochastic process in an appropriate way to the original solution process $X\colon[0,\infty)\times\Omega\to H$ of the SPDE (1) finally results in the mild Itô formula (5). Two types of transformations are well suited for this job. One possibility is, roughly speaking, to multiply the solution process X of the SPDE (1) by e^{-At}, $t\in[0,\infty)$, where e^{-At}, $t\in[0,\infty)$, has to be understood in an appropriate large Hilbert space (see Subsection 2.2 below for details). In that sense the transformed stochastic process becomes rougher than the solution process X of the SPDE (1). This transformation has been suggested in Teichmann [101] and Filipović, Tappe & Teichmann [31] (see also Hausenblas & Seidler [45, 46]). The other possible transformation goes into the other direction and, roughly speaking, consists of multiplying the solution process X of the SPDE (1) by $e^{A(T-t)}$, $t\in[0,T]$, for some large value $T\in(0,\infty)$. This transformation is based on an idea in Conus & Dalang [16] and Conus [15] (see also Debussche & Printems [27], Lindner & Schilling [73] and Kovács, Larsson & Lindgren [62]). The second transformation, which makes the transformed process smoother than the solution process X of the SPDE (1), turns out to be more flexible and allows us to prove Theorem 1 in its full generality. For more details on the proofs of (5) and Theorem 1 respectively, the reader is referred to Subsection 2.2 below.

In the remainder of this introductory section, a few consequences of the mild Ito formula (5) and its generalization in Theorem 1 are illustrated. For this let $X^x\colon[0,\infty)\times\Omega\to H$, $x\in H$, be a family of adapted stochastic processes with continuous sample paths satisfying $X_t=e^{At}x+\int_0^t e^{A(t-s)}F(X_s^x)\,ds+\int_0^t B(X_s^x)\,dW_s$ $\mathbb{P}$-a.s. for all $t\in[0,\infty)$ and all $x\in H$ (see, e.g., Theorem 4.3 in Brzeźniak [8] or Theorem 6.2 in Van Neerven, Veraar & Weis [103] for the up to indistinguishability unqiue existence of such processes). Then for every $r\in\big(-\infty,\min(\alpha+1,\beta+1/2)\big)$ and every at most polynomially growing continuous function $\varphi\in C(H_r,V)$ define the continuous function $u_\varphi\colon[0,\infty)\times H_r\to V$ through $u_\varphi(t,x):=\mathbb{E}\big[\varphi(X_t^x)\big]$ for all $(t,x)\in[0,\infty)\times H_r$. Under the assumption that $\alpha=\beta=0$ and that F and B are three times continuously Fréchet differentiable with globally bounded derivatives, the functions $u_\varphi\colon[0,\infty)\times H\to V$, $\varphi\colon H\to V$ twice continuous Fréchet differentiable with globally bounded derivatives, are strict solutions of the infinite dimensional Kolmogorov partial differential equation (PDE) $\frac{\partial}{\partial t}u_\varphi(t,x)=(Lu_\varphi)(t,x)$ with $u_\varphi(0,x)=\varphi(x)$ for $(t,x)\in(0,\infty)\times H_1$ where $L\colon C^2(H,V)\to C(H_1,V)$ is defined through

$$(L\varphi)(x):=\frac{1}{2}\mathrm{Tr}\Big(\big(B(x)\big)^*\varphi''(x)\,B(x)\Big)+\varphi'(x)\big[Ax+F(x)\big] \tag{6}$$

for all $x\in H_1=D(A)$ and all $\varphi\in C^2(H,V)$ (see Theorem 7.5.1 in Da Prato & Zabczyk [30]). Infinite dimensional Kolmogorov equation have been intensively investigated in the last two decades (see, e.g., the monographs [78, 13, 23, 19] and articles [109, 93, 94, 20] and the references mentioned therein). We prove here that the functions $u_\varphi\colon [0,\infty)\times H \to V$, φ sufficiently smooth, also solve another kind of Kolmogorov equation. More precisely, from (5) we derive in Subsection 3.2.2 below (see (70)) the identity

$$u_\varphi(t,x) = \varphi(e^{At}x) + \int_0^t u_{L_{t-s}(\varphi)}(s,x)\,ds \tag{7}$$

for all $(t,x)\in(0,\infty)\times H$ and all $\varphi\in\cup_{r<\min(\alpha+1,\beta+1/2)}C^2(H_r,V)$ with at most polynomially growing derivatives where $L_t\colon \cup_{r<\min(\alpha+1,\beta+1/2)}C^2(H_r,V)\to C(H,V)$, $t\in(0,\infty)$, is a family of bounded linear operators defined through

$$\big(L_t\varphi\big)(x) := \frac{1}{2}\mathrm{Tr}\Big(\big(e^{At}B(x)\big)^{*}\varphi''(e^{At}x)\,e^{At}B(x)\Big) + \varphi'(e^{At}x)\,e^{At}F(x) \tag{8}$$

for all $x\in H$, $\varphi\in\cup_{r<\min(\alpha+1,\beta+1/2)}C^2(H_r,V)$, $t\in(0,\infty)$. Equation (7) is somehow a *mild Kolmogorov backward equation*. From (7) we derive new regularity properties of solutions of second-order PDEs in Hilbert spaces. More precisely, using (7) we establish in Theorem 2 below the existence of real numbers $c_{\delta,\rho,q,T}\in[0,\infty)$, $\delta,\rho,q,T\in[0,\infty)$, such that the *regularity estimate*

$$\sup_{x\in H_\delta}\left(\frac{\|u_\varphi(t,x)\|_V}{(1+\|x\|_{H_\delta})^{(q+2)}}\right) \leq c_{\delta,\rho,q,T}\cdot\|\varphi\|^{\delta,\rho}_{t,q} \tag{9}$$

holds for all $t\in(0,T]$, $\varphi\in C^2(H_\rho,V)$ with $\sup_{x\in H_\rho}\frac{\|\varphi''(x)\|_{L^{(2)}(H_\rho,V)}}{(1+\|x\|_{H_\rho})^q}<\infty$, $q,\delta,T\in[0,\infty)$, $\rho\in[0,\min(\alpha+1,\beta+\frac{1}{2}))$ where

$$\|\varphi\|^{\delta,\rho}_{t,q} := \sup_{x\in H_\delta}\left[\frac{\big\|\varphi(e^{At}x)\big\|_V}{(1+\|x\|_{H_\delta})^{(q+2)}}\right] + \int_0^t (t-s)^{\min(\delta-\rho,0)}\sup_{x\in H_\rho}\left[\frac{\|(K_t\varphi)'(x)\|_{L(H_\alpha,V)}}{\big(1+\|x\|_{H_\rho}\big)^{(q+1)}} + \frac{\|(K_t\varphi)''(x)\|_{L^{(2)}(H_\beta,V)}}{\big(1+\|x\|_{H_\rho}\big)^{q}}\right]ds \tag{10}$$

for all $\varphi\in C^2(H_\rho,V)$ with $\sup_{x\in H_\rho}\frac{\|\varphi''(x)\|_{L^{(2)}(H_\rho,V)}}{(1+\|x\|_{H_\rho})^q}<\infty$, $t,q,\delta\in[0,\infty)$, $\rho\in[0,\min(\alpha+1,\beta+\frac{1}{2}))$ and where $K_t\colon C(H_1,V)\to C(H,V)$, $t\in(0,\infty)$, is defined through $(K_t\varphi)(x)=\varphi(e^{At}x)$ for all $x\in H$, $t\in(0,\infty)$. The constants $c_{\delta,\rho,q,T}$, $\delta,\rho,q,T\in[0,\infty)$, appearing in (9) are described explicitly in Theorem 2 below. Next a direct consequence of the regularity estimate (9) is presented. For this let $(C^2_{Lip}(H,\mathbb{R}),\|\cdot\|_{C^2_{Lip}(H,\mathbb{R})})$ be the real Banach space of all twice continuously differentiable globally Lipschitz continuous real valued functions on H with globally Lipschitz continuous derivatives (see (42) in Subsection 3.1 for details). Moreover, for every $t\in(0,\infty)$ let $(\mathcal{G}_t(H,\mathbb{R}),\|\cdot\|_{\mathcal{G}_t(H,\mathbb{R})})$ be the completion of the normed real vector space $(C^2_{Lip}(H,\mathbb{R}),\|\cdot\|^{0,0}_{t,0})$. Then consider the mapping $\mathcal{I}\colon\{\mu\colon\mathcal{B}(H)\to[0,1]$ probability measure$\colon\int\|x\|_H\,\mu(dx)<\infty\}\to(C^2_{Lip}(H,\mathbb{R}))'$ given by $\big(\mathcal{I}(\mu)\big)(\varphi)=\int\varphi(x)\,\mu(dx)$ for all $\varphi\in C^2_{Lip}(H,\mathbb{R})$ and all probability measures $\mu\colon\mathcal{B}(H)\to[0,1]$ with $\int\|x\|_H\,\mu(dx)<\infty$. Lemma 6 below proves that $\mathcal{I}$ is injective, that is, $\mathcal{I}$ embeds the probability measures with finite first absolute moments into linear forms on $C^2_{Lip}(H,\mathbb{R})$. Next note that $\mathcal{I}(\mathbb{P}_{X_t})\in(C^2_{Lip}(H,\mathbb{R}))'$ for all $t\in[0,\infty)$ where $\mathbb{P}_{X_t}[A]=\mathbb{P}\big[X_t\in A\big]$ for all $A\in\mathcal{B}(H)$, $t\in[0,\infty)$ are the probability measures of the solution process X_t, $t\in[0,\infty)$, of the SPDE (1). From (9) we then infer for every $t\in(0,\infty)$ that $\mathcal{I}(\mathbb{P}_{X_t})\in(C^2_{Lip}(H,\mathbb{R}))'$ is not only in $(C^2_{Lip}(H,\mathbb{R}))'$ but in the smaller space $(\mathcal{G}_t(H,\mathbb{R}))'$ too (the embedding $(\mathcal{G}_t(H,\mathbb{R}))'\subset(C^2_{Lip}(H,\mathbb{R}))'$ continuously is proved in Lemma 5 below). We thus have established more *regularity of the probability measures* $\mathbb{P}_{X_t}$, $t\in(0,\infty)$, of the solution process of the SPDE (1).

Another application of the regularity estimate (9) and the mild Kolmogorov backward equation (7) is the analysis of continuity properties of solutions of second-order PDEs in Hilbert spaces (see, e.g., the books [78, 13, 23, 19]). More precisely, Corollary 7 in Section 3.2.3 below proves that there exist real number $c_{r,\delta,\rho,T}\in[0,\infty)$, $r,\delta,\rho,T\in[0,\infty)$, such that

$$\|u_\varphi(t_1,x_1)-u_\varphi(t_2,x_2)\|_V \leq \left[\frac{c_{r,\delta,\rho,T}\,(1+\|x_1\|^3_{H_\delta}+\|x_2\|^3_{H_\delta})\,\|\varphi\|_{C^2_{Lip}(H_\rho,V)}}{|\min(t_1,t_2)|^{\max(r+\rho-\delta,0)}}\right]\Big[\,|t_1-t_2|^r+\|x_1-x_2\|_{H_\delta}\,\Big] \tag{11}$$

for all $t_1,t_2\in(0,T]$, $x_1,x_2\in D((-A)^\delta)$, $\varphi\in C^2_{Lip}(H_\rho,V)$, $r\in[0,1+\alpha-\rho)\cap[0,1+2\beta-2\rho)$, $\delta,T\in[0,\infty)$, $\rho\in[0,\min(\alpha+1,\beta+\frac{1}{2}))$. Inequality (11) thus proves Hölder continuity of the solutions $u_\varphi\colon[0,\infty)\times H\to V$,

$\varphi \in C^2_{Lip}(H,V)$, of second-order Kolmogorov PDEs in infinite dimensions. In particular, in the case of the example SPDE (2), inequality (11) ensures that

$$\sup_{\substack{t_1,t_2\in[0,T]\\ t_1<t_2}} \left(\frac{|t_1|^r \left\| \mathbb{E}\big[\varphi(X_{t_1})\big] - \mathbb{E}\big[\varphi(X_{t_2})\big] \right\|_V}{|t_1-t_2|^r} \right) < \infty \tag{12}$$

for all $\varphi \in C^2_{Lip}(H,V)$, $T \in (0,\infty)$ and all $r \in [0, \frac{1}{2})$. Results in the literature imply that (12) holds for all $r \in [0, \frac{1}{4}]$. More formally, in the case of the SPDE (2), we get from the global Lipschitz continuity of φ that

$$\sup_{\substack{t_1,t_2\in[0,T]\\ t_1<t_2}} \frac{|t_1|^r \left\| \mathbb{E}\big[\varphi(X_{t_1})\big] - \mathbb{E}\big[\varphi(X_{t_2})\big] \right\|_V}{|t_1-t_2|^r} \leq \left[\sup_{\substack{x,y\in H\\ x\neq y}} \tfrac{\|\varphi(x)-\varphi(y)\|_V}{\|x-y\|_H} \right] \left[\sup_{\substack{t_1,t_2\in[0,T]\\ t_1<t_2}} \frac{|t_1|^r\, \mathbb{E}\big[\|X_{t_1}-X_{t_2}\|_H\big]}{|t_1-t_2|^r} \right] < \infty \tag{13}$$

for all $\varphi \in C^2_{Lip}(H,V)$, $T \in (0,\infty)$ and all $r \in [0, \frac{1}{4}]$ where the second factor on the right hand side of (13) is finite due to Theorem 6.3 in Van Neerven, Veraar & Weiss [103] for the case $r \in [0, \frac{1}{4})$ and due to Corollaries A.16 and A.35 in [51] for the case $r = \frac{1}{4}$ (see also Brzeźniak [8], Kruse & Larsson [66], Van Neerven, Veraar & Weiss [104] for related results). This shows that regularity results in the literature ensure that (12) holds for all $r \in [0, \frac{1}{4}]$. Up to our best knowledge, this is the first result in the literature which establishes that (12) also holds in the regime $r \in (\frac{1}{4}, \frac{1}{2})$.

A further application of the regularity estimate (9) and the mild Kolmogorov backward equation (7) is the weak approximation of SPDEs. Let us illstrate this in the case of spectral Galerkin projections for the example SPDE (2). More precisely, in the case of the SPDE (2), Corollary 8 in Section 3.2.3 implies that there exist real numbers $C_{r,T} \in [0,\infty)$, $r, T \in [0,\infty)$, such that

$$\left\| \mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi\big(P_N(X_T)\big)\big] \right\|_V \leq \frac{C_{r,T}\, \|\varphi\|_{C^2_{Lip}(H,V)}}{N^r} \tag{14}$$

for all $N \in \mathbb{N}$, $T \in (0,\infty)$, $\varphi \in C^2_{Lip}(H,V)$ and all $r \in [0,1)$ where $P_N \in L(H)$, $N \in \mathbb{N}$, are spectral Galerkin projections defined by $(P_N v)(x) := \sum_{n=1}^N 2\sin(n\pi x) \int_0^1 v(y)\, \sin(n\pi y)\, dy$ for all $x \in (0,1)$, $v \in H = L^2((0,1),\mathbb{R})$ and all $N \in \mathbb{N}$. Inequality (14) and Corollary 8 respectively are a straightforward consequence of the regularity estimate (9) (see Section 3.2.3 for details). In the case of the stochastic heat equation with additive noise $f(x,y) = 0$ and $b(x,y) = 1$ for all $x \in (0,1)$, $y \in \mathbb{R}$ in (1), inequality (14) follows for all $r \in [0,1)$ from the results in [27, 73, 62, 63] (see also [43, 25, 32, 44, 26, 7, 65] for further numerical weak approximation results for SPDEs). In addition, in the general setting of the SPDE (1), it is well know that inequality (14) holds for all $r \in [0, \frac{1}{2}]$. Indeed, in that case, we get from the global Lipschitz continuity of φ that

$$\begin{aligned} &\left\| \mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi\big(P_N(X_T)\big)\big] \right\|_V \leq \|\varphi\|_{C^2_{Lip}(H,V)}\, \mathbb{E}\big[\|(I-P_N)X_T\|_H\big] \\ &\leq \|\varphi\|_{C^2_{Lip}(H,V)}\, \mathbb{E}\big[\|X_T\|_{H_{1/4}}\big] \|(I-P_N)(\eta-A)^{-1/4}\|_{L(H)} \leq \frac{\|\varphi\|_{C^2_{Lip}(H,V)}\, \mathbb{E}\big[\|X_T\|_{H_{1/4}}\big]}{\sqrt{N\pi}} < \infty \end{aligned} \tag{15}$$

for all $N \in \mathbb{N}$, $T \in (0,\infty)$ and all $\varphi \in C^2_{Lip}(H,V)$ where finiteness of $\mathbb{E}\big[\|X_T\|_{H_{1/4}}\big]$ for all $T \in (0,\infty)$ follows, e.g., from Lemma A.23 and Corollary A.37 in [51] (see also Kruse & Larsson [66] and Van Neerven, Veraar & Weiss [104] and the references therein for similar results). This shows that regularity results from the literature ensure that (14) holds for all $r \in [0, \frac{1}{2}]$. The present article proves that (14) also holds for all $r \in [0,1)$. Observe that (14) estimates the weak approximation error of spatial spectral Galerkin projections only instead of more complicated spatial approximations (see also Corollary 8 in Section 3.2.3 below for a generalization of (14)) and also the time interval and the noise are not discretized in (14). We believe that the mild Kolmogorov backward equation (7) can also be used to solve these more complicated weak numerical approximation problems for SPDEs and plan to develop these results in a future publication.

Another application of the mild Itô formula (5) and the mild Kolmogorov backward equation (7) respectively are the derivation of strong and weak stochastic Taylor expansions of solutions of SPDEs. Details can be found in Subsection 3.2.4 below. These Taylor expansions can then be used to derive higher order numerical schemes for SPDEs. In Subsection 3.3.2 below this is illustrated in the case of Milstein scheme for SPDEs.

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2 Mild stochastic calculus

Throughout this section assume that the following setting is fulfilled. Let $\mathbb{I} \subset [0,\infty)$ be a closed and convex subset of $[0,\infty)$ with nonempty interior, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in\mathbb{I}}$ and let $\big(\check{H}, \langle\cdot,\cdot\rangle_{\check{H}}, \|\cdot\|_{\check{H}}\big)$, $\big(\tilde{H}, \langle\cdot,\cdot\rangle_{\tilde{H}}, \|\cdot\|_{\tilde{H}}\big)$, $\big(\hat{H}, \langle\cdot,\cdot\rangle_{\hat{H}}, \|\cdot\|_{\hat{H}}\big)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\check{H} \subset \tilde{H} \subset \hat{H}$ continuously and densely. In addition, let $Q\colon U \to U$ be a bounded nonnegative symmetric linear operator and let $(W_t)_{t\in\mathbb{I}}$ be a cylindrical Q-Wiener process with respect to $(\mathcal{F}_t)_{t\in\mathbb{I}}$. Moreover, by $\big(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0}\big)$ the $\mathbb{R}$-Hilbert space with $U_0 = Q^{1/2}(U)$ and $\|u\|_{U_0} = \|Q^{-1/2}(u)\|_U$ for all $u \in U_0$ is denoted (see, for example, Proposition 2.5.2 in Prévôt & Röckner [90]). Here and below $S^{-1}\colon \mathrm{im}(S) \subset U \to U$ denotes the pseudo inverse of a bounded linear operator $S\colon U \to U \in L(U)$ (see, e.g., Appendix C in [90]). In addition, let $i_v\colon L(\hat{H}, \check{H}) \to \check{H} \in L\big(L(\hat{H}, \check{H}), \check{H}\big)$, $v \in \hat{H}$, be a family of bounded linear operators defined through $i_v A = Av$ for all $A \in L(\hat{H}, \check{H})$ and all $v \in \hat{H}$. Then by $\mathcal{S}(\hat{H}, \check{H}) := \sigma(\cup_{v\in\hat{H}} i_v^{-1}(\mathcal{B}(\check{H}))) = \sigma(\{i_v^{-1}(\mathcal{A}) \subset L(\hat{H}, \check{H})\colon v \in \hat{H}, \mathcal{A} \in \mathcal{B}(\check{H})\})$ the strong sigma algebra on $L(\hat{H}, \check{H})$ is denoted (see, for instance, Section 1.2 in Da Prato & Zabczyk [21]). Finally, let $\angle \subset \mathbb{I}^2$ be a set defined through $\angle := \big\{(t_1, t_2) \in \mathbb{I}^2 \colon t_1 < t_2\big\}$ and let $\tau \in \mathbb{I}$ be defined throught $\tau := \inf(\mathbb{I})$.

2.1 Mild stochastic processes

Definition 1 (Mild Itô process)**.** *Let $S\colon \angle \to L(\hat{H}, \check{H})$ be a $\mathcal{B}(\angle)/\mathcal{S}(\hat{H}, \check{H})$-measurable mapping satisfying $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$ for all $t_1, t_2, t_3 \in \mathbb{I}$ with $t_1 < t_2 < t_3$. Additionally, let $Y\colon \mathbb{I}\times\Omega \to \hat{H}$ and $Z\colon \mathbb{I}\times\Omega \to HS(U_0, \hat{H})$ be two predictable stochastic processes with $\int_\tau^t \|S_{s,t} Y_s\|_{\check{H}}\, ds < \infty$ $\mathbb{P}$-a.s. and $\int_\tau^t \|S_{s,t} Z_s\|^2_{HS(U_0,\check{H})}\, ds < \infty$ $\mathbb{P}$-a.s. for all $t \in \mathbb{I}$. Then a predictable stochastic process $X\colon \mathbb{I}\times\Omega \to \tilde{H}$ satisfying*

$$X_t = S_{\tau,t}\, X_\tau + \int_\tau^t S_{s,t}\, Y_s\, ds + \int_\tau^t S_{s,t}\, Z_s\, dW_s \tag{16}$$

$\mathbb{P}$-a.s. for all $t \in \mathbb{I} \cap (\tau, \infty)$ is called a mild Itô process (with semigroup S, mild drift Y and mild diffusion Z).

Note that if $(\check{H}, \langle\cdot,\cdot\rangle_{\check{H}}, \|\cdot\|_{\check{H}}) = (\hat{H}, \langle\cdot,\cdot\rangle_{\hat{H}}, \|\cdot\|_{\hat{H}})$ and if the semigroup $S\colon \angle \to L(\hat{H})$ satisfies $S_{t_1,t_2} = I$ for all $(t_1, t_2) \in \angle$, then the mild Itô process (16) is nothing else but a standard Itô process. (Throughout this article the terminology "standard Itô process" instead of simply "Itô process" is used in order to distinguish more clearly from the above notion of a mild Itô process.) Every standard Itô process is thus a mild Itô process too. However, a mild Itô process is, in general, not a standard Itô process (see Section 3 for some examples). The concept of a mild Itô process in Definition 1 thus generalizes the concept of a standard Itô process. In concrete examples of mild Itô processes it will be crucial that the semigroup $S\colon \angle \to L(\hat{H}, \check{H})$ in Definition 1 depends explicitly on both variables t_1 and t_2 with $(t_1, t_2) \in \angle$ instead of on $t_2 - t_1$ only (see Subsection 3.3 for details). The assumptions $\int_\tau^t \|S_{s,t} Y_s\|_{\check{H}}\, ds < \infty$ $\mathbb{P}$-a.s. and $\int_\tau^t \|S_{s,t} Z_s\|^2_{HS(U_0,\check{H})}\, ds < \infty$ $\mathbb{P}$-a.s. for all $t \in \mathbb{I}$ in Definition 1 ensure that both the deterministic and the stochastic integral in (16) are well defined. In the next step an immediate consequence of Definition 1 is presented.

Proposition 1. *Let $X\colon \mathbb{I}\times\Omega \to \tilde{H}$ be a mild Itô process with semigroup $S\colon \angle \to L(\hat{H}, \check{H})$, mild drift $Y\colon \mathbb{I}\times\Omega \to \hat{H}$ and mild diffusion $Z\colon \mathbb{I}\times\Omega \to HS(U_0, \hat{H})$. Then*

$$X_{t_2} = S_{t_1,t_2}\, X_{t_1} + \int_{t_1}^{t_2} S_{s,t_2}\, Y_s\, ds + \int_{t_1}^{t_2} S_{s,t_2}\, Z_s\, dW_s \tag{17}$$

$\mathbb{P}$-a.s. for all $t_1, t_2 \in \mathbb{I}$ with $t_1 < t_2$.

Proposition 1 follows directly from Theorem 1 below. Obviously, equation (17) in Proposition 1 generalizes equation (16) in the definition of a mild Itô process. Let us complete this subsection on mild Itô processes with the notion of a certain subclass of mild Itô processes.

Definition 2 (Mild martingale). *A mild Itô process $X\colon \mathbb{I}\times\Omega\to\tilde{H}$ with semigroup $S\colon \angle\to L(\hat{H},\check{H})$, mild drift $Y\colon \mathbb{I}\times\Omega\to\hat{H}$ and mild diffusion $Z\colon \mathbb{I}\times\Omega\to HS(U_0,\hat{H})$ satisfying $\mathbb{E}\big[\|X_t\|_{\tilde{H}}\big]<\infty$ for all $t\in\mathbb{I}$ is called a mild martingale (with respect to the filtration $\mathcal{F}_t, t\in\mathbb{I}$, and with respect to the semigroup S) if*

$$\mathbb{E}\big[X_{t_2}\big|\mathcal{F}_{t_1}\big]=S_{t_1,t_2}\,X_{t_1} \tag{18}$$

$\mathbb{P}$-a.s. for all $t_1,t_2\in\mathbb{I}$ with $t_1<t_2$.

Proposition 2 (Stochastic convolutions). *Let $X\colon \mathbb{I}\times\Omega\to\tilde{H}$ be a mild Itô process with semigroup $S\colon\angle\to L(\hat{H},\check{H})$, mild drift $Y\colon \mathbb{I}\times\Omega\to\hat{H}$ and mild diffusion $Z\colon \mathbb{I}\times\Omega\to HS(U_0,\hat{H})$ satisfying $\mathbb{E}\big[\|X_t\|^2_{\tilde{H}}\big]<\infty$ for all $t\in\mathbb{I}$. If $\mathbb{P}\big[Y_t=0\big]=1$ for all $t\in\mathbb{I}$, then X is a mild martingale with respect to the filtration $\mathcal{F}_t$, $t\in\mathbb{I}$, and with respect to the semigroup S.*

Proof of Propostion 2. Propostion 1 yields

$$X_{t_2}=S_{t_1,t_2}\,X_{t_1}+\int_{t_1}^{t_2}S_{s,t_2}\,Z_s\,dW_s \tag{19}$$

$\mathbb{P}$-a.s. for all $t_1,t_2\in\mathbb{I}$ with $t_1<t_2$. Equation (19) and the assumption $\mathbb{E}\big[\|X_t\|^2_{\tilde{H}}\big]<\infty$ for all $t\in\mathbb{I}$ imply equation (18). The proof of Proposition 2 is thus completed. □

2.2 Mild Itô formula

Let $\mathcal{J}$ be a set and let $g_j\in U_0$, $j\in\mathcal{J}$, be an arbitrary orthonormal basis of the $\mathbb{R}$-Hilbert space $\big(U_0,\langle\cdot,\cdot\rangle_{U_0},\|\cdot\|_{U_0}\big)$. For an $\mathbb{R}$-vector space $(V,\|\cdot\|_V)$ and a function $\varphi\in C^{1,2}(\mathbb{I}\times\check{H},V)$ we denote by $\partial_1\varphi\in C(\mathbb{I}\times\check{H},V)$, $\partial_2\varphi\in C(\mathbb{I}\times\check{H},L(\check{H},V))$ and $\partial_2^2\varphi\in C(\mathbb{I}\times\check{H},L^{(2)}(\check{H},V))$ the partial Fréchet derivatives of φ given by $(\partial_1\varphi)(t,x)=\big(\frac{\partial\varphi}{\partial t}\big)(t,x)$, $(\partial_2\varphi)(t,x)=\big(\frac{\partial\varphi}{\partial x}\big)(t,x)$ and $(\partial_2^2\varphi)(t,x)=\big(\frac{\partial^2\varphi}{\partial x^2}\big)(t,x)$ for all $t\in\mathbb{I}$ and all $x\in\check{H}$.

Theorem 1 (Mild Itô formula). *Let $X\colon \mathbb{I}\times\Omega\to\tilde{H}$ be a mild Itô process with semigroup $S\colon\angle\to L(\hat{H},\check{H})$, mild drift $Y\colon \mathbb{I}\times\Omega\to\hat{H}$ and mild diffusion $Z\colon \mathbb{I}\times\Omega\to HS(U_0,\hat{H})$ and let $(V,\langle\cdot,\cdot\rangle_V,\|\cdot\|_V)$ be a separable $\mathbb{R}$-Hilbert space. Then*

$$\mathbb{P}\bigg[\int_{t_0}^t\|(\partial_2\varphi)(s,S_{s,t}X_s)S_{s,t}Y_s\|_V+\|(\partial_2\varphi)(s,S_{s,t}X_s)S_{s,t}Z_s\|^2_{HS(U_0,V)}\,ds<\infty\bigg]=1, \tag{20}$$

$$\mathbb{P}\bigg[\int_{t_0}^t\|(\partial_1\varphi)(s,X_s)\|_V+\|(\partial_2^2\varphi)(s,S_{s,t}X_s)\|_{L^{(2)}(\check{H},V)}\,\|S_{s,t}Z_s\|^2_{HS(U_0,\check{H})}\,ds<\infty\bigg]=1 \tag{21}$$

and

$$\begin{aligned}\varphi(t,X_t)&=\varphi(t_0,S_{t_0,t}X_{t_0})+\int_{t_0}^t(\partial_1\varphi)(s,S_{s,t}X_s)\,ds+\int_{t_0}^t(\partial_2\varphi)(s,S_{s,t}X_s)\,S_{s,t}\,Y_s\,ds\\&\quad+\int_{t_0}^t(\partial_2\varphi)(s,S_{s,t}X_s)\,S_{s,t}\,Z_s\,dW_s+\frac{1}{2}\sum_{j\in\mathcal{J}}\int_{t_0}^t(\partial_2^2\varphi)(s,S_{s,t}X_s)\,(S_{s,t}Z_sg_j,S_{s,t}Z_sg_j)\,ds\end{aligned} \tag{22}$$

$\mathbb{P}$-a.s. for all $t_0,t\in\mathbb{I}$ with $t_0<t$ and all $\varphi\in C^{1,2}(\mathbb{I}\times\check{H},V)$.

Note that (20) and (21) ensure that the possibly infinite sum and all integrals in (22) are well defined. Indeed, equation (21) implies

$$\begin{aligned}&\sum_{j\in\mathcal{J}}\int_{t_0}^t\big\|(\partial_2^2\varphi)(s,S_{s,t}X_s)(S_{s,t}Z_sg_j,S_{s,t}Z_sg_j)\big\|_V\,ds\\&\leq\int_{t_0}^t\big\|(\partial_2^2\varphi)(s,S_{s,t}X_s)\big\|_{L^{(2)}(\check{H},V)}\Big(\textstyle\sum_{j\in\mathcal{J}}\|S_{s,t}Z_sg_j\|^2_{\check{H}}\Big)\,ds\\&=\int_{t_0}^t\big\|(\partial_2^2\varphi)(s,S_{s,t}X_s)\big\|_{L^{(2)}(\check{H},V)}\,\|S_{s,t}Z_s\|^2_{HS(U_0,\check{H})}\,ds<\infty\end{aligned} \tag{23}$$

$\mathbb{P}$-a.s. for all $t_0, t \in \mathbb{I}$ with $t_0 < t$ and all $\varphi \in C^{1,2}(\mathbb{I} \times \check{H}, V)$. Moreover, note that the mild Itô formula (22) is independent of the particular choice of the orthonormal basis $g_j \in U_0$, $j \in \mathcal{J}$, of U_0.

In the next step a certain flow property of the mild Itô formula (22) is observed. To be more precise, the mild Itô formula (22) on the time interval $[\hat{t}, t]$ applied to the test function $\varphi(s, v)$, $s \in [\hat{t}, t]$, $v \in \check{H}$, reads as

$$\begin{aligned} \varphi(t, X_t) &= \varphi(\hat{t}, S_{\hat{t},t} X_{\hat{t}}) + \int_{\hat{t}}^{t} (\partial_1 \varphi)(s, S_{s,t} X_s)\, ds + \int_{\hat{t}}^{t} (\partial_2 \varphi)(s, S_{s,t} X_s)\, S_{s,t}\, Y_s\, ds \\ &+ \int_{\hat{t}}^{t} (\partial_2 \varphi)(s, S_{s,t} X_s)\, S_{s,t}\, Z_s\, dW_s + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{\hat{t}}^{t} (\partial_2^2 \varphi)(s, S_{s,t} X_s)\, (S_{s,t} Z_s g_j, S_{s,t} Z_s g_j)\, ds \end{aligned} \tag{24}$$

$\mathbb{P}$-a.s. for all $\hat{t}, t \in \mathbb{I}$ with $\hat{t} < t$ and all $\varphi \in C^{1,2}(\mathbb{I} \times \check{H}, V)$. Moreover, observe that the mild Itô formula (22) on the time interval $[t_0, \hat{t}]$ applied to the test function $\varphi(s, S_{\hat{t},t} v)$, $s \in [t_0, \hat{t}]$, $v \in \check{H}$, reads as

$$\begin{aligned} \varphi(\hat{t}, S_{\hat{t},t} X_{\hat{t}}) &= \varphi(t_0, S_{t_0,t} X_{t_0}) + \int_{t_0}^{\hat{t}} (\partial_1 \varphi)(s, S_{s,t} X_s)\, ds + \int_{t_0}^{\hat{t}} (\partial_2 \varphi)(s, S_{s,t} X_s)\, S_{s,t}\, Y_s\, ds \\ &+ \int_{t_0}^{\hat{t}} (\partial_2 \varphi)(s, S_{s,t} X_s)\, S_{s,t}\, Z_s\, dW_s + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{\hat{t}} (\partial_2^2 \varphi)(s, S_{s,t} X_s)\, (S_{s,t} Z_s g_j, S_{s,t} Z_s g_j)\, ds \end{aligned} \tag{25}$$

$\mathbb{P}$-a.s. for all $t_0, \hat{t}, t \in \mathbb{I}$ with $t_0 < \hat{t} < t$ and all $\varphi \in C^{1,2}(\mathbb{I} \times \check{H}, V)$. Putting (25) into (24) then results in the mild Itô formula (22) on the time interval $[t_0, t]$ for $t_0, t \in \mathbb{I}$ with $t_0 < t$. If the test function $(\varphi(t,x))_{t \in \mathbb{I}, x \in \check{H}} \in C^{1,2}(\mathbb{I} \times \check{H}, V)$ in the mild Itô formula (22) does not depend on $t \in \mathbb{I}$, then the mild Itô formula in Theorem 1 reads as follows.

Corollary 1. *Let $X \colon \mathbb{I} \times \Omega \to \tilde{H}$ be a mild Itô process with semigroup $S \colon \angle \to L(\hat{H}, \check{H})$, mild drift $Y \colon \mathbb{I} \times \Omega \to \hat{H}$ and mild diffusion $Z \colon \mathbb{I} \times \Omega \to HS(U_0, \hat{H})$ and let $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ be a separable $\mathbb{R}$-Hilbert space. Then*

$$\mathbb{P}\Big[\int_{t_0}^{t} \|\varphi'(S_{s,t} X_s) S_{s,t} Y_s\|_V + \|\varphi'(S_{s,t} X_s) S_{s,t} Z_s\|^2_{HS(U_0,V)}\, ds < \infty \Big] = 1, \tag{26}$$

$$\mathbb{P}\Big[\int_{t_0}^{t} \|\varphi''(S_{s,t} X_s)\|_{L^{(2)}(\check{H},V)}\, \|S_{s,t} Z_s\|^2_{HS(U_0,\check{H})}\, ds < \infty \Big] = 1 \tag{27}$$

and

$$\begin{aligned} \varphi(X_t) &= \varphi(S_{t_0,t} X_{t_0}) + \int_{t_0}^{t} \varphi'(S_{s,t} X_s)\, S_{s,t}\, Y_s\, ds + \int_{t_0}^{t} \varphi'(S_{s,t} X_s)\, S_{s,t}\, Z_s\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi''(S_{s,t} X_s)\, (S_{s,t} Z_s g_j, S_{s,t} Z_s g_j)\, ds \end{aligned} \tag{28}$$

$\mathbb{P}$-a.s. for all $t_0, t \in \mathbb{I}$ with $t_0 < t$ and all $\varphi \in C^2(\check{H}, V)$.

Let us illustrate Theorem 1 and Corollary 1 by two simple examples. The first one is a mild version of the stochastic integration by parts formula (see, e.g., Corollary 2.6 in [9]).

Example 1 (Mild stochastic integration by parts)**.** *Let $(\check{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\check{\mathcal{H}}}, \|\cdot\|_{\check{\mathcal{H}}})$, $(\tilde{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}}, \|\cdot\|_{\tilde{\mathcal{H}}})$, $(\hat{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\hat{\mathcal{H}}}, \|\cdot\|_{\hat{\mathcal{H}}})$, $(\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}})$ and $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ be separable $\mathbb{R}$-Hilbert spaces with $\check{\mathcal{H}} \subset \tilde{\mathcal{H}} \subset \hat{\mathcal{H}}$ continuously and densely, let $X \colon \mathbb{I} \times \Omega \to \tilde{H}$ be a mild Itô process with semigroup $S \colon \angle \to L(\hat{H}, \check{H})$, mild drift $Y \colon \mathbb{I} \times \Omega \to \hat{H}$ and mild diffusion $Z \colon \mathbb{I} \times \Omega \to HS(U_0, \hat{H})$ and let $\mathcal{X} \colon \mathbb{I} \times \Omega \to \tilde{\mathcal{H}}$ be a mild Itô process with semigroup $\mathcal{S} \colon \angle \to L(\hat{\mathcal{H}}, \check{\mathcal{H}})$, mild drift $\mathcal{Y} \colon \mathbb{I} \times \Omega \to \hat{\mathcal{H}}$ and mild diffusion $\mathcal{Z} \colon \mathbb{I} \times \Omega \to HS(U_0, \hat{\mathcal{H}})$. Corollary 1 then shows*

$$\begin{aligned} \varphi(X_t, \mathcal{X}_t) &= \varphi\big(S_{t_0,t} X_t, \mathcal{S}_{t_0,t} \mathcal{X}_t\big) + \int_{t_0}^{t} \varphi\big(S_{s,t} Y_s, \mathcal{S}_{s,t} \mathcal{X}_s\big)\, ds + \int_{t_0}^{t} \varphi\big(S_{s,t} X_s, \mathcal{S}_{s,t} \mathcal{Y}_s\big)\, ds \\ &+ \int_{t_0}^{t} \varphi\big(S_{s,t} Z_s(\cdot), \mathcal{S}_{s,t} \mathcal{X}_s\big)\, dW_s + \int_{t_0}^{t} \varphi\big(S_{s,t} X_s, \mathcal{S}_{s,t} \mathcal{Z}_s(\cdot)\big)\, dW_s + \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi\big(S_{s,t} Z_s g_j, \mathcal{S}_{s,t} \mathcal{Z}_s g_j\big)\, ds \end{aligned} \tag{29}$$

$\mathbb{P}$-a.s. for all $t_0, t \in \mathbb{I}$ with $t_0 < t$ and all bounded bilinear mappings $\varphi \colon \check{H} \times \check{\mathcal{H}} \to V$.

Example 2 (Mild chain rule)**.** *Let $S\colon \angle \to L(\hat{H}, \check{H})$ be an $\mathcal{B}(\angle)/\mathcal{S}(\hat{H}, \check{H})$-measurable mapping satisfying $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$ for all $t_1, t_2, t_3 \in \mathbb{I}$ with $t_1 < t_2 < t_3$ and let $x\colon \mathbb{I} \to \tilde{H}$ and $y\colon \mathbb{I} \to \hat{H}$ be two Borel measurable functions with $\int_\tau^t \|S_{s,t} y_s\|_{\tilde{H}}\, ds < \infty$ and $x_t = S_{\tau,t}\, x_\tau + \int_\tau^t S_{s,t}\, y_s\, ds$ for all $t \in \mathbb{I}$. Corollary 1 then shows*

$$\varphi(x_t) = \varphi(S_{t_0,t} x_{t_0}) + \int_{t_0}^t \varphi'(S_{s,t} x_s)\, S_{s,t}\, y_s\, ds \tag{30}$$

for all $t_0, t \in \mathbb{I}$ with $t_0 < t$ and all $\varphi \in C^2(\check{H}, V)$. Equation (30) is somehow a mild chain rule for the mild process $x\colon \mathbb{I} \to \tilde{H}$.

Let us now concentrate on proofs of the mild Itô formula (22). A central difficulty in order to establish an Itô formula for the stochastic process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ is that this stochastic process is, in general, not a standard Itô process to which the standard Itô formula (see, e.g., Theorem 4.17 in Section 4.5 in Da Prato & Zabczyk [21]) could be applied. (Here and below the terminology "standard Itô formula" instead of simply "Itô formula" is used in order to distinguish more clearly from the above mild Itô formula.) The stochastic process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ is, in general, not a standard Itô process since it satisfies the Itô-Volterra type equation (16) in which the integrand processes $S_{s,t}\, Y_s$, $s \in [\tau, t]$, and $S_{s,t}\, Z_s$, $s \in [\tau, t]$, depend explicitly on $t \in \mathbb{I}$ too (this was the reason for introducing the notion of a mild Itô process; see Definition 1). Below we present two proofs which overcome this difficulty and which establish the mild Itô formula (22). Both proofs consider appropriate transformations of the mild Itô process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$. The transformed stochastic processes are then standard Itô processes to which the standard Itô formula can be applied. Relating then the transformed stochastic processes in a suitable way to the original mild Itô process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ finally results in the mild Itô formula (22). The main difference of the two proofs is the type of transformation applied to the mild Itô process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$.

The first proof makes use of a transformation in Teichmann [101] and Filipović, Tappe & Teichmann [31] (see equations (1.3) and (1.4) in Teichmann [101] and Section 8 in Filipović, Tappe & Teichmann [31] and see also Hausenblas & Seidler [45, 46]). The first proof does not show Theorem 1 in the general case but in the case in which the semigroup of the mild Itô process is pseudo-contractive (see below for the precise description of the used assumptions). Under this additional assumption, the semigroup $(S_{t_1,t_2})_{(t_1,t_2) \in \angle}$ on the Hilbert space $\hat{H}$ can be dilated to a group $(\mathcal{U}_t)_{t \in \mathbb{R}}$ on a larger Hilbert space (see, e.g., Szökefalvi-Nagy [98, 99] and Theorem I.81 in Szökefalvi-Nagy & Foiaş [100] for the so-called dilations of the unitary theorem). On this larger Hilbert space, the mild Itô process (16) can be transformed into a standard Itô process by – roughly speaking – multiplying with $\mathcal{U}_{-t}$ for $t \in \mathbb{I}$. Next the standard Itô formula can be applied to the transformed standard Itô process. Relating this transformed standard Itô process then in a suitable way to the original mild Itô process finally results in the mild Itô formula (22).

The second proof establishes Theorem 1 in the general case. It makes use of an idea in Conus & Dalang [16] and Conus [15] (see Section 6 in Conus & Dalang [16] and equations (1.7) and (7.6) in Conus [15] and see also Section 3 in Debussche & Printems [27], Theorem 4 in Lindner & Schilling [73] and Theorem 3.1 in Kovács, Larsson & Lindgren [62]) and exploits a more elementary transformation. Roughly speaking, the mild Itô process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ is transformed in the second proof by multiplying with $S_{t,T}$ for $t \in [\tau, T)$ with a fixed $T \in \mathbb{I}$ (compare that the transformation in first proof is based on multiplying with the group at the negative time value $-t$). Since $T - t > 0$, the transformed process of the $\tilde{H}$-valued process $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ enjoys values in $\tilde{H}$ too (this is in contrast to the first proof where the transformed process of $X\colon \mathbb{I} \times \Omega \to \tilde{H}$ takes values in a larger Hilbert space in which $\tilde{H}$ is continuously embedded). Nonetheless, as in the first proof, the transformed stochastic process is a standard Itô process to which the standard Itô formula can be applied. Relating the transformed standard Itô process in a suitable way to the original mild Itô process then again results in the mild Itô formula (22).

Both proofs thus essentially consist of three steps: a *transformation*, an *application of the standard Itô formula* and the use of a suitable *relation* of the transformed standard Itô process and the original mild Itô process. The second proof also uses the following simple result.

Lemma 1. *Let $Y, Z\colon \mathbb{I} \times \Omega \to [0, \infty)$ be two product measurable stochastic processes with $\mathbb{P}\big[Y_t = Z_t\big] = 1$ for all $t \in \mathbb{I}$ and with $\mathbb{P}\big[\int_{\mathbb{I}} Y_s\, ds < \infty\big] = 1$. Then $\mathbb{P}\big[\int_{\mathbb{I}} Z_s\, ds < \infty\big] = 1$.*

The proof of Lemma 1 is clear and therefore omitted. Instead the first proof of Theorem 1 in the special case of a pseudo-contractive semigroup is now presented.

Proof. Proof of Theorem 1 in the case where the partial Fréchet derivatives $\partial_1 \varphi$, $\partial_2 \varphi$ and $\partial_2^2 \varphi$ of φ are globally bounded, where $Y\colon \mathbb{I} \times \Omega \to \hat{H}$ and $Z\colon \mathbb{I} \times \Omega \to HS(U_0, \hat{H})$ have continuous sample paths, where $(\check{H}, \langle \cdot, \cdot \rangle_{\check{H}}, \|\cdot\|_{\check{H}}) = (\tilde{H}, \langle \cdot, \cdot \rangle_{\tilde{H}}, \|\cdot\|_{\tilde{H}}) = (\hat{H}, \langle \cdot, \cdot \rangle_{\hat{H}}, \|\cdot\|_{\hat{H}})$, where $U_t \in L(\tilde{H})$, $t \in [0, \infty)$, is a strongly continuous *pseudo-contractive semigroup on* $\tilde{H}$ *and where* $S_{t_1,t_2} = U_{(t_2-t_1)} \in L(\tilde{H})$ *for all* $(t_1, t_2) \in \angle$. First, observe that, under these additional assumptions, (20) and (21) are obviously fulfilled. Moreover, due to Proposition 8.7 in [31], there exists a separable $\mathbb{R}$-Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}, \|\cdot\|_{\mathcal{H}})$ with $\tilde{H} \subset \mathcal{H}$ and $\|v\|_{\tilde{H}} = \|v\|_{\mathcal{H}}$ for all $v \in \tilde{H}$ and a strongly continuous group $\mathcal{U}_t \in L(\mathcal{H})$, $t \in \mathbb{R}$, such that

$$U_t(v) = P\big(\mathcal{U}_t(v)\big) \tag{31}$$

for all $v \in \tilde{H} \subset \mathcal{H}$ and all $t \in [0, \infty)$ where $P \colon \mathcal{H} \to \tilde{H}$ is the orthogonal projection from $\mathcal{H}$ to $\tilde{H}$. In this first proof the mild Itô process $X \colon \mathbb{I} \times \Omega \to \tilde{H}$ is now transformed into a standard Itô process by - roughly speaking - multiplying with $\mathcal{U}_{-t}$ for $t \in \mathbb{I}$. In a more concrete setting this transformation has been proposed in Teichmann [101] and Filipović, Tappe & Teichmann [31]; see equations (1.3) and (1.4) in Teichmann [101] and Section 8 in Filipović, Tappe & Teichmann [31] and see also Hausenblas & Seidler [45, 46]. Let us now go into details. Let $\bar{X} \colon \mathbb{I} \times \Omega \to \mathcal{H}$ be the up to indistinguishability unique adapted stochastic process with continuous sample paths satisfying

$$\bar{X}_t = X_\tau + \int_\tau^t \mathcal{U}_{-s}\, Y_s\, ds + \int_\tau^t \mathcal{U}_{-s}\, Z_s\, dW_s \tag{32}$$

$\mathbb{P}$-a.s. for all $t \in \mathbb{I}$ (*Transformation*; see also equation (1.4) in [101] and equation (8.6) in [31]). Next observe that the identity $X_t = \mathcal{P}\big(\mathcal{U}_t(\bar{X}_t)\big)$ $\mathbb{P}$-a.s. (see also Theorem 8.8 in [31]) and the standard Itô formula in infinite dimensions (see Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9]) applied to the test function $\varphi\big(s, P\big(\mathcal{U}_t(v)\big)\big)$, $s \in [t_0, t]$, $v \in \mathcal{H}$, give

$$\begin{aligned} \varphi(t, X_t) &= \varphi\big(t, P\big(\mathcal{U}_t(\bar{X}_t)\big)\big) = \varphi\big(t_0, P\big(\mathcal{U}_t(\bar{X}_{t_0})\big)\big) + \int_{t_0}^t (\partial_1 \varphi)\big(s, P\big(\mathcal{U}_t(\bar{X}_s)\big)\big)\, ds \\ &+ \int_{t_0}^t (\partial_2 \varphi)\big(s, P\big(\mathcal{U}_t(\bar{X}_s)\big)\big)\, P\, \mathcal{U}_{(t-s)}\, Y_s\, ds + \int_{t_0}^t (\partial_2 \varphi)\big(s, P\big(\mathcal{U}_t(\bar{X}_s)\big)\big)\, P\, \mathcal{U}_{(t-s)}\, Z_s\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^t (\partial_2^2 \varphi)\big(s, P\big(\mathcal{U}_t(\bar{X}_s)\big)\big)\big(P\, \mathcal{U}_{(t-s)} Z_s g_j, P\, \mathcal{U}_{(t-s)} Z_s g_j\big)\, ds \end{aligned} \tag{33}$$

$\mathbb{P}$-a.s. for all $t_0, t \in \mathbb{I}$ with $t_0 \le t$ and all $\varphi \in C^{1,2}(\mathbb{I} \times \tilde{H}, V)$ (*Application of the standard Itô formula*). Next note that equation (31) gives

$$\begin{aligned} P\big(\mathcal{U}_t(\bar{X}_s)\big) &= P\big(\mathcal{U}_t(X_\tau)\big) + \int_\tau^s P\, \mathcal{U}_{(t-u)}\, Y_u\, du + \int_\tau^s P\, \mathcal{U}_{(t-u)}\, Z_u\, dW_u \\ &= S_{\tau,t}\, X_\tau + \int_\tau^s S_{u,t}\, Y_u\, du + \int_\tau^s S_{u,t}\, Z_u\, dW_u \\ &= S_{s,t} \left(S_{\tau,s}\, X_\tau + \int_\tau^s S_{u,s}\, Y_u\, du + \int_\tau^s S_{u,s}\, Z_u\, dW_u \right) = S_{s,t}\, X_s \end{aligned} \tag{34}$$

$\mathbb{P}$-a.s. for all $s, t \in \mathbb{I}$ with $s \le t$ (*Relation* of the transformed standard Itô process $\bar{X} \colon \mathbb{I} \times \Omega \to \mathcal{H}$ and the original mild Itô process $X \colon \mathbb{I} \times \Omega \to \tilde{H}$). Using (31) and (34) in (33) finally shows (22). The proof is thus completed. □

In the next step the proof of Theorem 1 in the general case is given. Above an outline of this second proof is given.

Proof of Theorem 1. In this second proof the time variable $t \in \mathbb{I}$ within the integrand processes in (16) is fixed and then, the standard Itô formula is applied to the resulting standard Itô process. In a more concrete setting this trick has been proposed in Conus & Dalang [16] and Conus [15]; see Section 6 in Conus & Dalang [16] and equations (1.7) and (7.6) in Conus [15] and see also Section 5 in Lindner & Schilling [73] and Section 3 in Kovacs, Larsson & Lindgren [62]. Another related result can be found in Section 3 in Debussche & Printemps [27]. Let us now go into details. Let $\bar{X}^t \colon [\tau, t] \times \Omega \to \check{H}$, $t \in \mathbb{I} \cap (\tau, \infty)$, be a family of adapted stochastic processes with continuous sample paths given by

$$\bar{X}_u^t = S_{\tau,t}\, X_\tau + \int_\tau^u S_{s,t}\, Y_s\, ds + \int_\tau^u S_{s,t}\, Z_s\, dW_s \tag{35}$$

$\mathbb{P}$-a.s. for all $u \in [\tau, t]$ and all $t \in \mathbb{I} \cap (\tau, \infty)$ (*Transformation*; see also Section 6 in [16], Section 7 in [15], Section 5 in [73] and Section 3 in [62]). Note that the assumptions $\mathbb{P}\big[\int_\tau^t \|S_{s,t} Y_s\|_{\check{H}}\, ds < \infty\big] = 1$ and

$\mathbb{P}\big[\int_\tau^t \|S_{s,t} Z_s\|^2_{HS(U_0,\check{H})}\,ds < \infty\big] = 1$ for all $t \in \mathbb{I}$ (see Definition 1) ensure that $\bar{X}^t\colon [\tau,t]\times\Omega \to \check{H}$, $t \in \mathbb{I}\cap(\tau,\infty)$, in (35) are indeed well defined adapted stochastic processes with continuous sample paths. In the next step the continuity of the partial derivatives of $\varphi\colon \mathbb{I}\times\check{H}\to V$, the continuity of the sample paths of $\bar{X}^t\colon [\tau,t]\times\Omega\to\check{H}$ and again the assumptions $\mathbb{P}\big[\int_\tau^t \|S_{s,t}Y_s\|_{\check{H}}\,ds < \infty\big] = 1$ and $\mathbb{P}\big[\int_\tau^t \|S_{s,t}Z_s\|^2_{HS(U_0,\check{H})}\,ds < \infty\big] = 1$ in Definition 1 imply

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|(\partial_2\varphi)(s,\bar{X}^t_s)S_{s,t}Y_s\big\|_V + \big\|(\partial_2\varphi)(s,\bar{X}^t_s)S_{s,t}Z_s\big\|^2_{HS(U_0,V)}\,ds < \infty\bigg] = 1 \tag{36}$$

and

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|(\partial_1\varphi)(s,\bar{X}^t_s)\big\|_V + \big\|(\partial_2^2\varphi)(s,\bar{X}^t_s)\big\|_{L^{(2)}(\check{H},V)}\,\|S_{s,t}Z_s\|^2_{HS(U_0,\check{H})}\,ds < \infty\bigg] = 1 \tag{37}$$

for all $t_0 \in [\tau,t]$, $t \in \mathbb{I}\cap(\tau,\infty)$ and all $\varphi \in C^{1,2}(\mathbb{I}\times\check{H},V)$. Moreover, the identity $X_t = \bar{X}^t_t$ $\mathbb{P}$-a.s. and the standard Itô formula (see Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9]) give

$$\begin{aligned}\varphi(t,X_t) = \varphi(t,\bar{X}^t_t) = \varphi(t_0,\bar{X}^t_{t_0}) &+ \int_{t_0}^t (\partial_1\varphi)(s,\bar{X}^t_s)\,ds + \int_{t_0}^t (\partial_2\varphi)(s,\bar{X}^t_s)\,S_{s,t}\,Y_s\,ds\\ + \int_{t_0}^t (\partial_2\varphi)(s,\bar{X}^t_s)\,S_{s,t}\,Z_s\,dW_s &+ \frac{1}{2}\sum_{j\in\mathcal{J}}\int_{t_0}^t (\partial_2^2\varphi)(s,\bar{X}^t_s)\,(S_{s,t}\,Z_s\,g_j, S_{s,t}\,Z_s\,g_j)\,ds\end{aligned} \tag{38}$$

$\mathbb{P}$-a.s. for all $t_0, t \in \mathbb{I}$ with $t_0 < t$ and all $\varphi \in C^{1,2}(\mathbb{I}\times\check{H},V)$ (*Application of the standard Itô formula*; see also Section 6 in [16], equations (1.7) and (7.6) in [15], Theorem 4 in [73] and Theorem 3.1 in [62]). Equation (38) is an expansion formula for the stochastic processes $\varphi(t,X_t)$, $t\in\mathbb{I}\cap(\tau,\infty)$, for $\varphi\in C^{1,2}(\mathbb{I}\times\check{H},V)$. Nevertheless, this formula seems to be of limited use since the integrands in (38) contain the transformed stochastic processes $\bar{X}^t_s$, $s\in[t_0,t]$, $t_0,t\in\mathbb{I}$, $t_0<t$, instead of the mild Itô process X_s, the mild drift Y_s and the mild diffusion Z_s for $s\in[t_0,t]$, $t_0,t\in\mathbb{I}$, $t_0<t$, only. However, a key observation here is to exploit the elementary identity

$$\begin{aligned}\bar{X}^t_s &= S_{\tau,t}\,X_\tau + \int_\tau^s S_{u,t}\,Y_u\,du + \int_\tau^s S_{u,t}\,Z_u\,dW_u\\ &= S_{s,t}\left(S_{\tau,s}\,X_\tau + \int_\tau^s S_{u,s}\,Y_u\,du + \int_\tau^s S_{u,s}\,Z_u\,dW_u\right) = S_{s,t}\,X_s\end{aligned} \tag{39}$$

$\mathbb{P}$-a.s. for all $s,t\in\mathbb{I}$ with $s<t$ in equation (38) (*Relation* of the transformed standard Itô processes $\bar{X}^t\colon[\tau,t]\times\Omega\to\mathcal{H}$, $t\in\mathbb{I}\cap(\tau,\infty)$, and the original mild Itô process $X\colon\mathbb{I}\times\Omega\to\tilde{H}$). This enables us to obtain a closed formula for the stochastic processes $\varphi(t,X_t)$, $t\in\mathbb{I}\cap(\tau,\infty)$, for $\varphi\in C^{1,2}(\mathbb{I}\times\check{H},V)$. More precisely, (39), (36), (37) and Lemma 1 imply (20) and (21). Putting (39) into (38) then gives (22). The proof of Theorem 1 is thus completed. □

Let us close this section on mild stochastic calculus with a remark on possible generalizations.

Remark 1. *Note that here mild Itô processes, mild drifts and mild diffusions with values in separable Hilbert spaces are considered. Instead one could develop the above notions and the above mild Itô formula for stochastic processes with values in an appropriate class of possibly non-separable Banach spaces too. Indeed, the standard Itô formula also holds for stochastic processes with values in UMD Banach spaces (see Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9]). Details on the stochastic integration in UMD Banach spaces can be found in Van Neerven, Veraar & Weis [102, 103] and in the references therein. Another possible generalization is to consider more general integrators than the cylindrical Wiener process $(W_t)_{t\in\mathbb{I}}$. This leads to the concept of a mild semimartingale instead of a mild Itô process in Definition 1. In particular, the fourth integral in the mild Itô formula* (22) *then needs to be replaced by an integral involving the quadratic variation of the driving noise process.*

3 Stochastic partial differential equations (SPDEs)

3.1 Setting and assumptions

Throughout this section suppose that the following setting and the following assumptions are fulfilled. Let $T\in(0,\infty)$ be a real number, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$, $(U,\langle\cdot,\cdot\rangle_U,\|\cdot\|_U)$ and $(V,\langle\cdot,\cdot\rangle_V,\|\cdot\|_V)$ be separable $\mathbb{R}$-Hilbert spaces. In addition, let $Q\colon U\to U$ be a bounded nonnegative symmetric linear operator and let $(W_t)_{t\in[0,\infty)}$ be a cylindrical Q-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$. Moreover, by $\left(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0}\right)$ the separable $\mathbb{R}$-Hilbert space with $U_0 = Q^{1/2}(U)$ and $\|u\|_{U_0} = \|Q^{-1/2}(u)\|_U$ for all $u \in U_0$ is denoted.

Assumption 1 (Linear operator A)**.** *Let $A\colon D(A) \subset H \to H$ be a generator of a strongly continuous analytic semigroup $e^{At} \in L(H)$, $t \in [0,\infty)$.*

Let $\eta \in [0,\infty)$ be a nonnegative real number such that $\sigma(A) \subset \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) < \eta\}$ where $\sigma(A) \subset \mathbb{C}$ denotes as usual the spectrum of the linear operator $A\colon D(A) \subset H \to H$. Such a real number exists since A is assumed to be a generator of a strongly continuous semigroup (see Assumption 1). By $H_r := D((\eta - A)^r)$ equipped with the norm $\|v\|_{H_r} := \|(\eta - A)^r v\|_H$ for all $v \in H_r$ and all $r \in \mathbb{R}$, the $\mathbb{R}$-Hilbert spaces of domains of fractional powers of the linear operator $\eta - A\colon D(A) \subset H \to H$ are denoted (see, e.g., Subsection 11.4.2 in Renardy & Roggers [92]).

Assumption 2 (Drift term F)**.** *Let $\alpha, \gamma \in \mathbb{R}$ be real numbers with $\gamma - \alpha < 1$ and let $F\colon H_\gamma \to H_\alpha$ be globally Lipschitz continuous.*

Assumption 3 (Diffusion term B)**.** *Let $\beta \in \mathbb{R}$ be a real number with $\gamma - \beta < \frac{1}{2}$ and let $B\colon H_\gamma \to HS(U_0, H_\beta)$ be globally Lipschitz continuous.*

Assumption 4 (Initial value ξ)**.** *Let $p \in [2,\infty)$ be a real number and let $\xi\colon \Omega \to H_\gamma$ be an $\mathcal{F}_0/\mathcal{B}(H_\gamma)$-measurable mapping with $\mathbb{E}\big[\|\xi\|^p_{H_\gamma}\big] < \infty$.*

Furthermore, similar as in Section 2, let $\angle \subset [0,T]^2$ be defined through $\angle := \big\{(t_1,t_2) \in [0,T]^2\colon t_1 < t_2\big\}$. In addition to the above assumptions, the following notations will be used in the remainder of this article. For two $\mathbb{R}$-Banach spaces $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ and real numbers $n \in \{0,1,2,\dots\}$ and $q \in [0,\infty)$ define $\|v\|_{L^{(0)}(V_1,V_2)} := \|v\|_{V_2}$ for every $v \in V_1$, define

$$\|\varphi\|_{G^n_q(V_1,V_2)} := \sum_{i=0}^{n-1} \|\varphi^{(i)}(0)\|_{L^{(i)}(V_1,V_2)} + \sup_{v\in V_1}\left(\frac{\|\varphi^{(n)}(v)\|_{L^{(n)}(V_1,V_2)}}{(1+\|v\|_{V_1})^q}\right) \in [0,\infty], \tag{40}$$

$$\|\varphi\|_{\mathrm{Lip}^{n+1}_q(V_1,V_2)} := \sum_{i=0}^{n} \|\varphi^{(i)}(0)\|_{L^{(i)}(V_1,V_2)} + \sup_{\substack{v,w\in V_1\\ v\neq w}}\left(\frac{\|\varphi^{(n)}(v)-\varphi^{(n)}(w)\|_{L^{(n)}(V_1,V_2)}}{(1+\max(\|v\|_{V_1},\|w\|_{V_1}))^q\,\|v-w\|_{V_1}}\right) \in [0,\infty] \tag{41}$$

and

$$\|\varphi\|_{C^n_{Lip}(V_1,V_2)} := \|\varphi(0)\|_{V_2} + \sum_{i=1}^{n} \|\varphi^{(i)}\|_{L^\infty(V_1,L^{(i)}(V_1,V_2))} + \sup_{\substack{v,w\in V_1\\ v\neq w}}\left(\frac{\|\varphi^{(n)}(v)-\varphi^{(n)}(w)\|_{L^{(n)}(V_1,V_2)}}{\|v-w\|_{V_1}}\right) \in [0,\infty] \tag{42}$$

for every $\varphi \in C^n(V_1,V_2)$, define $G^n_q(V_1,V_2) := \{\varphi \in C^n(V_1,V_2)\colon \|\varphi\|_{G^n_q(V_1,V_2)} < \infty\}$, define $\mathrm{Lip}^{n+1}_q(V_1,V_2) := \{\varphi \in C^n(V_1,V_2)\colon \|\varphi\|_{\mathrm{Lip}^{n+1}_q(V_1,V_2)} < \infty\}$ and define $C^n_{Lip}(V_1,V_2) := \{\varphi \in C^n(V_1,V_2)\colon \|\varphi\|_{C^n_{Lip}(V_1,V_2)} < \infty\}$ and note that $\|\varphi\|_{G^m_q(V_1,V_2)} = \|\varphi\|_{\mathrm{Lip}^m_q(V_1,V_2)}$ for every $\varphi \in C^m(V_1,V_2)$ and every $m \in \mathbb{N}$. Let us collect a few simple properties of the defined objects. More precisely, observe that

$$\|\varphi\|_{G^n_q(V_1,V_2)} = \sum\nolimits_{i=0}^{n-1} \|\varphi^{(i)}(0)\|_{L^{(i)}(V_1,V_2)} + \|\varphi^{(n)}\|_{G^0_q(V_1,V_2)} \tag{43}$$

$$\|\varphi\|_{G^0_{q+n}(V_1,V_2)} \leq \|\varphi\|_{G^{n-k}_{q+k}(V_1,V_2)} \leq \|\varphi\|_{G^n_q(V_1,V_2)}, \tag{44}$$

$$\|\varphi^{(k)}(v)\|_{L^{(k)}(V_1,V_2)} \leq \|\varphi\|_{G^n_q(V_1,V_2)}\,(1+\|v\|_{V_1})^{(q+n-k)} \tag{45}$$

$$\|\varphi^{(k)}(v)-\varphi^{(k)}(w)\|_{L^{(k)}(V_1,V_2)} \leq \|\varphi\|_{\mathrm{Lip}^{n+1}_q(V_1,V_2)}\,\big(1+\max(\|v\|_{V_1},\|w\|_{V_1})\big)^{(q+n-k)}\,\|v-w\|_{V_1} \tag{46}$$

for all $v,w \in V_1$, $\varphi \in C^n(V_1,V_2)$, $k \in \{0,1,\dots,n\}$, $n \in \{0,1,\dots\}$, $q \in [0,\infty)$ and all $\mathbb{R}$-Banach spaces $(V_1,\|\cdot\|_{V_1})$ and $(V_2,\|\cdot\|_{V_2})$. Moreover, note for all $n \in \{0,1,2,\dots\}$, $q \in [0,\infty)$ and all $\mathbb{R}$-Banach spaces $(V_1,\|\cdot\|_{V_1})$ and $(V_2,\|\cdot\|_{V_2})$ that the pairs $(G^n_q(V_1,V_2),\|\cdot\|_{G^n_q(V_1,V_2)})$, $(\mathrm{Lip}^{n+1}_q(V_1,V_2),\|\cdot\|_{\mathrm{Lip}^{n+1}_q(V_1,V_2)})$ and $(C^n_{Lip}(V_1,V_2), \|\cdot\|_{C^n_{Lip}(V_1,V_2)})$ are $\mathbb{R}$-Banach spaces with $G^{n+1}_q(V_1,V_2) \subset \mathrm{Lip}^{n+1}_q(V_1,V_2) \subset G^n_{q+1}(V_1,V_2)$ continuously. More function spaces of similar type can be found in Dörsek & Teichmann [29].

3.2 Solution processes of SPDEs

The following proposition shows that the setting in Section 3.1 ensures that the SPDE (47) below admits an up to modifications unique mild solution process. It is similar to special cases of Theorem 4.3 in Brzeźniak [8] and Theorem 6.2 in Van Neerven, Veraar & Weis [103]. Its proof is clear and therefore omitted.

Proposition 3. *Assume that the setting in Section 3.1 is fulfilled. Then there exists an up to modifications unique predictable stochastic process* $X\colon [0,T]\times\Omega\to H_\gamma$ *which fulfills* $\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t\|^p_{H_\gamma}\big]<\infty$ *and*

$$X_t = e^{At}\,\xi + \int_0^t e^{A(t-s)} F(X_s)\,ds + \int_0^t e^{A(t-s)} B(X_s)\,dW_s \tag{47}$$

$\mathbb{P}$*-a.s. for all* $t\in[0,T]$*. In addition, we have* $X\in\cap_{r\in(-\infty,\gamma]} C^{\min(\gamma-r,1/2)}([0,T],L^p(\Omega;H_r))$*.*

Proposition 3, in particular, ensures that the mild solution process $X\colon[0,T]\times\Omega\to H_\gamma$ of the SPDE (47) is a mild Itô process with semigroup $e^{A(t_2-t_1)}\in L(H_{\min(\alpha,\beta.\gamma)},H_\gamma)$, $(t_1,t_2)\in\angle$, with mild drift

$$F(X_t),\; t\in[0,T], \tag{48}$$

and with mild diffusion

$$B(X_t),\; t\in[0,T]. \tag{49}$$

This fact now enables us to apply the mild Itô formula (22) to the solution process X of the SPDE (47). To this end let $\mathcal{J}$ be a set and let $g_j\in U_0$, $j\in\mathcal{J}$, be an arbitrary orthonormal basis of the $\mathbb{R}$-Hilbert space $(U_0,\langle\cdot,\cdot\rangle_{U_0},\|\cdot\|_{U_0})$. A direct consequence of Theorem 1 and Corollary 1 is the next corollary.

Corollary 2 (A new - somehow mild - Itô formula for solutions of SPDEs)**.** *Assume that the setting in Section 3.1 is fulfilled. Then*

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi'(e^{A(t-s)}X_s)\,e^{A(t-s)}F(X_s)\big\|_V\,ds<\infty\bigg]=1, \tag{50}$$

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi'(e^{A(t-s)}X_s)\,e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,V)}\,ds<\infty\bigg]=1, \tag{51}$$

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi''(e^{A(t-s)}X_s)\big\|_{L^{(2)}(H_r,V)}\,\big\|e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H_r)}\,ds<\infty\bigg]=1 \tag{52}$$

and

$$\begin{aligned}\varphi(X_t) &= \varphi(e^{A(t-t_0)}X_{t_0}) + \int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\,e^{A(t-s)}F(X_s)\,ds\\ &\quad+\int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\,e^{A(t-s)}B(X_s)\,dW_s\\ &\quad+\frac{1}{2}\sum_{j\in\mathcal{J}}\int_{t_0}^t \varphi''(e^{A(t-s)}X_s)\Big(e^{A(t-s)}B(X_s)g_j,e^{A(t-s)}B(X_s)g_j\Big)\,ds\end{aligned} \tag{53}$$

$\mathbb{P}$*-a.s. for all* $t_0,t\in[0,T]$ *with* $t_0<t$*, all* $\varphi\in C^2(H_r,V)$ *and all* $r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$*.*

First, observe that the possibly infinite sum and all integrals in (53) are well defined due to (50)–(52). Next define mappings $K_0\colon\cup_{r\in\mathbb{R}}C(H_r,V)\to\cup_{r\in\mathbb{R}}C(H_r,V)$ and $K_t\colon\cup_{r\in\mathbb{R}}C(H_r,V)\to C(H_{\min(\alpha,\beta,\gamma)},V)$, $t\in(0,\infty)$, through $K_0(\varphi):=\varphi$ and

$$\big(K_t\varphi\big)(x):=\varphi(e^{At}x) \tag{54}$$

for all $x\in H_{\min(\alpha,\beta,\gamma)}$, $\varphi\in\cup_{r\in\mathbb{R}}C(H_r,V)$ and all $t\in(0,\infty)$. Note that $K_{t_1}\circ K_{t_2}=K_{t_1+t_2}$ for all $t_1,t_2\in[0,\infty)$. In addition, define linear operators $L^{(0)}\colon C^2(H_{\min(\alpha,\beta,\gamma)},V)\to C(H_\gamma,V)$ and $L^{(1)}\colon C^1(H_{\min(\beta,\gamma)},V)\to C(H_\gamma,V)$ through

$$\begin{aligned}(L^{(0)}\varphi)(x) &:= \varphi'(x)F(x)+\frac{1}{2}\sum_{j\in\mathcal{J}}\varphi''(x)\big(B(x)g_j,B(x)g_j\big)\\ &= \varphi'(x)F(x)+\frac{1}{2}\mathrm{Tr}\Big(\big(B(x)\big)^*\varphi''(x)\,B(x)\Big)\end{aligned} \tag{55}$$

for all $x \in H_\gamma$, $\varphi \in C^2(H_{\min(\alpha,\beta,\gamma)}, V)$ and through $(L^{(1)}\varphi)(x) := \varphi'(x)B(x)$ for all $x \in H_\gamma$, $\varphi \in C^1(H_{\min(\beta,\gamma)}, V)$. Furthermore, define mappings $L^{(0)}_t \colon \cup_{r\in\mathbb{R}} C^2(H_r, V) \to C(H_\gamma, V)$, $t \in (0,\infty)$, and $L^{(1)}_t \colon \cup_{r\in\mathbb{R}} C^1(H_r, V) \to C(H_\gamma, HS(U_0, V))$, $t \in (0,\infty)$, through $L^{(0)}_t(\varphi) := L^{(0)}(K_t(\varphi))$ for all $\varphi \in \cup_{r\in\mathbb{R}} C^2(H_r, V)$ and through $L^{(1)}_t(\varphi) := L^{(1)}(K_t(\varphi))$ for all $\varphi \in \cup_{r\in\mathbb{R}} C^1(H_r, V)$. Note that these definitions imply

$$\begin{aligned}\big(L^{(0)}_t\varphi\big)(x) &= \varphi'(e^{At}x)\,e^{At}F(x) + \frac{1}{2}\sum_{j\in\mathcal{J}} \varphi''(e^{At}x)\big(e^{At}B(x)g_j, e^{At}B(x)g_j\big) \\ &= \varphi'(e^{At}x)\,e^{At}F(x) + \frac{1}{2}\mathrm{Tr}\Big(\big(e^{At}B(x)\big)^*\varphi''(e^{At}x)\,e^{At}B(x)\Big)\end{aligned} \tag{56}$$

for all $x \in H_\gamma$, $\varphi \in \cup_{r\in\mathbb{R}} C^2(H_r, V)$, $t \in (0,\infty)$ and

$$\big(L^{(1)}_t\varphi\big)(x) = \varphi'(e^{At}x)\,e^{At}B(x) \tag{57}$$

for all $x \in H_\gamma$, $\varphi \in \cup_{r\in\mathbb{R}} C^1(H_r, V)$, $t \in (0,\infty)$. The mild Itô formula (53) can thus be written as

$$\begin{aligned}\varphi(X_t) &= \varphi(e^{A(t-t_0)}X_{t_0}) + \int_{t_0}^t \big(L^{(0)}_{(t-s)}\varphi\big)(X_s)\,ds + \int_{t_0}^t \big(L^{(1)}_{(t-s)}\varphi\big)(X_s)\,dW_s \\ &= \big(K_{(t-t_0)}\varphi\big)(X_{t_0}) + \int_{t_0}^t \big(L^{(0)}_{(t-s)}\varphi\big)(X_s)\,ds + \int_{t_0}^t \big(L^{(1)}_{(t-s)}\varphi\big)(X_s)\,dW_s \\ &= \big(K_{(t-t_0)}\varphi\big)(X_{t_0}) + \int_{t_0}^t \big(L^{(0)}K_{(t-s)}\varphi\big)(X_s)\,ds + \int_{t_0}^t \big(L^{(1)}K_{(t-s)}\varphi\big)(X_s)\,dW_s\end{aligned} \tag{58}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 < t$ and all $\varphi \in \cup_{r<\min(\alpha+1,\beta+\frac{1}{2})} C^2(H_r, V)$. Moreover, taking expectations on both side of (58) gives

$$\begin{aligned}\mathbb{E}\Big[\varphi(X_t)\Big] &= \mathbb{E}\Big[\big(K_{(t-t_0)}\varphi\big)(X_{t_0})\Big] + \int_{t_0}^t \mathbb{E}\Big[\big(L^{(0)}_{(t-s)}\varphi\big)(X_s)\Big]\,ds \\ &= \mathbb{E}\Big[\big(K_{(t-t_0)}\varphi\big)(X_{t_0})\Big] + \int_{t_0}^t \mathbb{E}\Big[\big(L^{(0)}K_{(t-s)}\varphi\big)(X_s)\Big]\,ds\end{aligned} \tag{59}$$

for all $t_0, t \in [0,T]$ with $t_0 < t$ and all $\varphi \in \cup_{r<\min(\alpha+1,\beta+\frac{1}{2})} G^2_0(H_r, V)$. Based on (59) a mild Kolmogorov backward equation is derived in Subsection 3.2.2 below. Other kinds of Itô type formulas for solutions of SPDEs can be found in [14, 21, 33, 38, 39, 68, 69, 70, 72, 83, 85, 88, 90, 91, 110]. In the next step Corollary 2 is illustrated by two simple examples.

Example 3 (Identity)**.** *Assume that the setting in Section 3.1 is fulfilled, let* $V = H_\gamma$*, let* $\|v\|_V = \|v\|_{H_\gamma}$ *for all* $v \in H_\gamma$ *and let* $\varphi \colon H_\gamma \to H_\gamma$ *be the identity on* H_γ*, i.e.,* $\varphi(v) = v$ *for all* $v \in H_\gamma$*. The mild Itô formula* (53) *in Corollary 2 then reduces to*

$$X_t = e^{A(t-t_0)}X_{t_0} + \int_{t_0}^t e^{A(t-s)}F(X_s)\,ds + \int_{t_0}^t e^{A(t-s)}B(X_s)\,dW_s \tag{60}$$

$\mathbb{P}$*-a.s. for all* $t_0, t \in [0,T]$ *with* $t_0 \leq t$*. This is nothing else but the mild formulation of the SPDE* (47)*. In this sense, the formula* (53) *is somehow a mild Itô formula for SPDEs.*

Example 4 (Squared norm)**.** *Assume that the setting in Section 3.1 is fulfilled, let* $V = \mathbb{R}$*,* $\|v\|_V = |v|$ *for all* $v \in V = \mathbb{R}$*, assume* $\min(\alpha+1, \beta+\frac{1}{2}) > 0$ *and let* $\varphi \colon H \to V$ *be given by* $\varphi(v) = \|v\|^2_H$ *for all* $v \in H$*. The mild Itô formula* (53) *in Corollary 2 then reduces to*

$$\begin{aligned}\|X_t\|^2_H &= \big\|e^{A(t-t_0)}X_{t_0}\big\|^2_H + 2\int_{t_0}^t \Big\langle e^{A(t-s)}X_s, e^{A(t-s)}F(X_s)\Big\rangle_H\,ds \\ &\quad + 2\int_{t_0}^t \Big\langle e^{A(t-s)}X_s, e^{A(t-s)}B(X_s)\,dW_s\Big\rangle_H + \int_{t_0}^t \big\|e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H)}\,ds\end{aligned} \tag{61}$$

$\mathbb{P}$*-a.s. for all* $t_0, t \in [0,T]$ *with* $t_0 < t$ *(see also Example 1 above). We refer to [38, 39, 68, 83, 85, 90, 91] for other Itô type formulas with the particular test function* $\varphi(v) = \|v\|^2_H$*,* $v \in H$*. If* $X_0 = 0$ *and* $F(v) = 0$ *for all* $v \in H$ *in addition to the above assumptions, then* (61) *simplifies to*

$$\|X_t\|^2_H = 2\int_0^t \Big\langle e^{A(t-s)}X_s, e^{A(t-s)}B(X_s)\,dW_s\Big\rangle_H + \int_0^t \big\|e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H)}\,ds \tag{62}$$

$\mathbb{P}$*-a.s. for all* $t \in [0,T]$ *and this, in particular, gives*

$$\mathbb{E}\Big[\|X_t\|_H^2\Big] = \int_0^t \mathbb{E}\Big[\big\|e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H)}\Big]\,ds \tag{63}$$

for all $t \in [0,T]$*. Clearly, equation* (63) *is nothing else but a special case of Itô's isometry.*

3.2.1 SPDEs with time dependent coefficients

In addition to the setting in Section 3.1 assume in this subsection that $\tilde{F}\colon [0,\infty)\times H_\gamma \to H_\alpha$ and $\tilde{B}\colon [0,\infty)\times H_\gamma \to HS(U_0,H_\beta)$ are two globally Lipschitz continuous mappings. Then there exists an up to modifications unique predictable stochastic process $\tilde{X}\colon [0,\infty)\times\Omega \to H_\gamma \in C([0,\infty), L^p(\Omega;H_\gamma))$ which fulfills

$$\tilde{X}_t = e^{At}\xi + \int_0^t e^{A(t-s)}\tilde{F}(s,\tilde{X}_s)\,ds + \int_0^t e^{A(t-s)}\tilde{B}(s,\tilde{X}_s)\,dW_s \tag{64}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$. Next define mappings $L^{(0)}_{s,t}\colon \cup_{r\in\mathbb{R}} C^2(H_r,V) \to C(H_\gamma,V)$, $s,t\in[0,\infty)$, $s<t$, and $L^{(1)}_{s,t}\colon \cup_{r\in\mathbb{R}} C^1(H_r,V)\to C(H_\gamma,V)$, $s,t\in[0,\infty)$, $s<t$, through

$$\big(L^{(0)}_{s,t}\varphi\big)(x) := \varphi'(e^{A(t-s)}x)\,e^{A(t-s)}\tilde{F}(s,x) + \frac{1}{2}\sum_{j\in\mathcal{J}}\varphi''(e^{A(t-s)}x)\big(e^{A(t-s)}\tilde{B}(s,x)g_j, e^{A(t-s)}\tilde{B}(s,x)g_j\big) \tag{65}$$

for all $x\in H_\gamma$, $\varphi \in \cup_{r\in\mathbb{R}}C^2(H_r,V)$ and all $s,t\in[0,\infty)$ with $s<t$ and through

$$\big(L^{(1)}_{s,t}\varphi\big)(x) := \varphi'(e^{A(t-s)}x)\,e^{A(t-s)}\tilde{B}(s,x) \tag{66}$$

for all $x\in H_\gamma$, $\varphi\in\cup_{r\in\mathbb{R}}C^1(H_r,V)$ and all $s,t\in[0,\infty)$ with $s<t$. Corollary 1 then implies

$$\varphi(\tilde{X}_t) = \big(K_{(t-t_0)}\varphi\big)(\tilde{X}_{t_0}) + \int_{t_0}^t \big(L^{(0)}_{s,t}\varphi\big)(\tilde{X}_s)\,ds + \int_{t_0}^t \big(L^{(1)}_{s,t}\varphi\big)(\tilde{X}_s)\,dW_s \tag{67}$$

for all $t_0,t\in[0,\infty)$ with $t_0<t$ and all $\varphi\in\cup_{r<\min(\alpha+1,\beta+\frac{1}{2})}C^2(H_r,V)$. The mild Itô formula (67) is nothing else but the counterpart of (53) for SPDEs with time dependent coefficients.

3.2.2 Mild Kolmogorov backward equation for SPDEs

Based on (59) a mild Kolmogorov backward equation is derived in this subsection. Proposition 3 implies the existence of predictable stochastic processes $X^x\colon[0,\infty)\times\Omega\to H_\gamma\in\cap_{q\in[1,\infty)}C([0,\infty),L^q(\Omega;H_\gamma))$, $x\in H_\gamma$, such that

$$X^x_t = e^{At}x + \int_0^t e^{A(t-s)}F(X^x_s)\,ds + \int_0^t e^{A(t-s)}B(X^x_s)\,dW_s \tag{68}$$

$\mathbb{P}$-a.s. for all $t\in[0,\infty)$ and all $x\in H_\gamma$. Proposition 3 also implies that $\mathbb{P}\big[X^x_t\in H_r\big]=1$ for all $t\in(0,\infty)$ and all $r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$. Then define mappings $P_t\colon \cup_{q\in[0,\infty)}\cup_{r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))}G^0_q(H_r,V)\to \cup_{q\in[0,\infty)}\cup_{r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))}G^0_q(H_r,V)$, $t\in[0,\infty)$, through $P_0(\varphi):=\varphi$ for all $\varphi\in\cup_{q\in[0,\infty)}\cup_{r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))}$ and through $P_t(\varphi)\in\cup_{q\in[0,\infty)}G^0_q(H_\gamma,V)$ and

$$(P_t\varphi)(x) := \mathbb{E}\big[\varphi(X^x_t)\big] \tag{69}$$

for all $x\in H_\gamma$, $\varphi\in\cup_{q\in[0,\infty)}\cup_{r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))}$ and all $t\in(0,\infty)$. Note that $P_t\big(G^0_q(H_r,V)\big)\subset G^0_q(H_\gamma,V)$ for all $t\in(0,\infty)$, $r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$ and all $q\in[0,\infty)$. The next lemma collect a few simple properties of the linear operators P_t, $t\in[0,\infty)$.

Lemma 2 (Properties of P_t, $t\in[0,\infty)$)**.** *Assume that the setting in Section 3.1 is fulfilled. Then* $P_t\in L(G^0_q(H_\gamma,V))$ *and* $\sup_{s\in[0,t]}\|P_s\|_{L(G^0_q(H_\gamma,V))}<\infty$ *for all* $t,q\in[0,\infty)$ *and it holds for every* $q\in[0,\infty)$ *that the function* $[0,\infty)\ni t\mapsto P_t\in L(Lip^1_q(H_\gamma,V),G^0_{q+1}(H_\gamma,V))$ *is locally* $\frac{1}{2}$*-Hölder continuous.*

The proof of Lemma 2 is a straightforward consequence of inequality (46) and is therefore omitted. Next the mild Itô formula in (58) implies

$$\begin{aligned}\big(P_t\varphi\big)(x) &= \mathbb{E}\Big[\big(K_{(t-t_0)}\varphi\big)(X^x_{t_0})\Big] + \int_{t_0}^t \mathbb{E}\Big[\big(L^{(0)}_{(t-s)}\varphi\big)(X^x_s)\Big]\,ds \\ &= \big(P_{t_0}K_{(t-t_0)}\varphi\big)(x) + \int_{t_0}^t \big(P_s L^{(0)}_{(t-s)}\varphi\big)(x)\,ds \\ &= \big(P_{t_0}K_{(t-t_0)}\varphi\big)(x) + \int_{t_0}^t \big(P_s\, L^{(0)} K_{(t-s)}\varphi\big)(x)\,ds\end{aligned} \tag{70}$$

for all $t_0, t \in [0,\infty)$ with $t_0 < t$, $x \in H_\gamma$ and all $\varphi \in \cup_{q\in[0,\infty)}\cup_{r\in(-\infty,\min(\alpha+1,\beta+\frac{1}{2}))}$. In the next subsection we will use (70) to study regularity properties of solutions of SPDEs. In this regularity analysis we also use the following two lemmas.

Lemma 3 (Estimates for K_t, $t \in (0,\infty)$)**.** *Assume that the setting in Section 3.1 is fulfilled. Then the function $(0,\infty) \ni t \mapsto K_t \in L\big(Lip^{n+1}_q(H_{r_1},V), G^n_{q+1}(H_{r_2},V)\big)$ is continuous for every $r_1, r_2 \in \mathbb{R}$, $q \in [0,\infty)$, $n \in \{0,1,\dots\}$ and it holds that $K_t \in L(Lip^n_q(H_{r_1},V), Lip^n_q(H_{r_2},V))$ and that*

$$\|K_t(\varphi_0)\|_{G^0_q(H_{r_1},V)} \leq \max\big(1, \|e^{At}\|^q_{L(H_{r_2},H_{r_1})}\big)\,\|\varphi_0\|_{G^0_q(H_{r_2},V)}, \tag{71}$$

$$\|(K_t\varphi_1)'\|_{G^0_q(H_\gamma,L(H_\alpha,V))} \leq \max\big(1, \|e^{At}\|^q_{L(H)}\big)\,\|e^{At}\|_{L(H_\alpha,H_\gamma)}\,\|\varphi_1'\|_{G^0_q(H_\gamma,L(H_\gamma,V))}, \tag{72}$$

$$\|(K_t\varphi_2)''\|_{G^0_q(H_\gamma,L^{(2)}(H_\beta,V))} \leq \max\big(1, \|e^{At}\|^q_{L(H)}\big)\,\|e^{At}\|^2_{L(H_\beta,H_\gamma)}\,\|\varphi_2''\|_{G^0_q(H_\gamma,L^{(2)}(H_\gamma,V))} \tag{73}$$

for all $\varphi_0 \in C(H_{r_2},V)$, $\varphi_1 \in C^1(H_\gamma,V)$, $\varphi_2 \in C^2(H_\gamma,V)$, $q \in [0,\infty)$, $n \in \mathbb{N}$, $r_1, r_2 \in \mathbb{R}$ and all $t \in (0,\infty)$.

Lemma 4 (Estimates for $L^{(0)}$)**.** *Assume that the setting in Section 3.1 is fulfilled. Then it holds that $L^{(0)} \in L(G^2_q(H_{\min(\alpha,\beta,\gamma)},V), G^0_{q+2}(H_\gamma,V))$, that $L^{(0)} \in L(Lip^3_q(H_{\min(\alpha,\beta,\gamma)},V), Lip^1_{q+2}(H_\gamma,V))$ and that*

$$\begin{aligned}&\|L^{(0)}(\varphi)\|_{G^0_{q+2}(H_r,V)} \\ &\leq \|F\|_{G^0_1(H_r,H_\alpha)}\,\|\varphi'\|_{G^0_{q+1}(H_r,L(H_\alpha,V))} + \tfrac{1}{2}\,\|B\|^2_{G^0_1(H_r,HS(U_0,H_\beta))}\,\|\varphi''\|_{G^0_q(H_r,L^{(2)}(H_\beta,V))}, \\ &\leq \max\big(\|F\|_{G^0_1(H_r,H_\alpha)}, \|B\|^2_{G^0_1(H_r,HS(U_0,H_\beta))}\big)\big(\|\varphi'\|_{G^0_{q+1}(H_r,L(H_\alpha,V))} + \|\varphi''\|_{G^0_q(H_r,L^{(2)}(H_\beta,V))}\big)\end{aligned} \tag{74}$$

for all $r \in [\gamma,\infty)$, $\varphi \in C^2(H_{\min(\alpha,\beta,\gamma)},V)$ and all $q \in [0,\infty)$.

The proofs of Lemma 3 and Lemma 4 are straightforward and therefore omitted. The proof of Lemma 3 makes use of inequality (46) above. The next corollary follows from Lemmas 2–4.

Corollary 3. *Assume that the setting in Section 3.1 is fulfilled. Then the function $(t_0,t) \ni s \mapsto P_s\, L^{(0)} K_{t-s} \in L(Lip^3_q(H_\gamma,V), G^0_{q+3}(H_\gamma,V))$ is continuous and satisfies $\int_{t_0}^t \|P_s\, L^{(0)} K_{t-s}\|_{L(Lip^3_q(H_\gamma,V),G^0_{q+3}(H_\gamma,V))}\,ds < \infty$ for every $t_0, t \in [0,\infty)$ with $t_0 < t$ and every $q \in [0,\infty)$.*

Proof of Corollary 3. The triangle inequality implies that

$$\begin{aligned}
&\big\|P_s L^{(0)} K_{t-s} - P_{s_0} L^{(0)} K_{t-s_0}\big\|_{L(Lip^3_q(H_\gamma,V),G^0_{q+3}(H_\gamma,V))} \\
&\leq \big\|P_s L^{(0)} K_{t-s} - P_s L^{(0)} K_{t-s_0}\big\|_{L(Lip^3_q(H_\gamma,V),G^0_{q+3}(H_\gamma,V))} \\
&+ \big\|P_s L^{(0)} K_{t-s_0} - P_{s_0} L^{(0)} K_{t-s_0}\big\|_{L(Lip^3_q(H_\gamma,V),G^0_{q+3}(H_\gamma,V))} \\
&\leq \|P_s\|_{L(G^0_{q+3}(H_\gamma,V))} \|L^{(0)}\|_{L(G^2_{q+1}(H_{\min(\alpha,\beta,\gamma)},V),G^0_{q+3}(H_\gamma,V))} \\
&\quad \cdot \|K_{t-s} - K_{t-s_0}\|_{L(Lip^3_q(H_\gamma,V),G^2_{q+1}(H_{\min(\alpha,\beta,\gamma)},V))} \\
&+ \big\|P_s - P_{s_0}\big\|_{L(Lip^1_{q+2}(H_\gamma,V),G^0_{q+3}(H_\gamma,V))} \|L^{(0)}\|_{L(Lip^3_q(H_{\min(\alpha,\beta,\gamma)},V),Lip^1_{q+2}(H_\gamma,V))} \\
&\quad \cdot \|K_{t-s_0}\|_{L(Lip^3_q(H_\gamma,V),Lip^3_q(H_{\min(\alpha,\beta,\gamma)},V))} \\
&\leq \underbrace{\left[\|L^{(0)}\|_{L(G^2_{q+1}(H_{\min(\alpha,\beta,\gamma)},V),G^0_{q+3}(H_\gamma,V))} \cdot \sup_{u\in[t_0,t]} \|P_u\|_{L(G^0_{q+3}(H_\gamma,V))}\right]}_{<\infty \text{ due to Lemmas 4 and 2}} \\
&\quad \cdot \|K_{t-s} - K_{t-s_0}\|_{L(Lip^3_q(H_\gamma,V),G^2_{q+1}(H_{\min(\alpha,\beta,\gamma)},V))} \\
&+ \underbrace{\left[\|L^{(0)}\|_{L(Lip^3_q(H_{\min(\alpha,\beta,\gamma)},V),Lip^1_{q+2}(H_\gamma,V))} \cdot \|K_{t-s_0}\|_{L(Lip^3_q(H_\gamma,V),Lip^3_q(H_{\min(\alpha,\beta,\gamma)},V))}\right]}_{<\infty \text{ due to Lemmas 4 and 3}} \\
&\quad \cdot \|P_s - P_{s_0}\|_{L(Lip^1_{q+2}(H_\gamma,V),G^0_{q+3}(H_\gamma,V))}
\end{aligned} \tag{75}$$

for all $s_0, s \in (t_0, t)$, $q \in [0,\infty)$ and $t_0, t \in [0,\infty)$ with $t_0 \leq t$. Combining (75) with Lemma 2 and Lemma 3 shows that the function $(t_0,t) \ni s \mapsto P_s\, L^{(0)} K_{t-s} \in L(Lip^3_q(H_\gamma,V), G^0_{q+3}(H_\gamma,V))$ is continuous for every $t_0, t \in [0,\infty)$ with $t_0 < t$ and every $q \in [0,\infty)$. Combining this with Lemmas 2–4 completes the proof of Corollary 3. ☐

In the following we reformulate (70) in a suitable abstract way by using Corollary 3. More precisely, combining Corollary 3 with equation (70) shows that

$$\begin{aligned}
P_t(\varphi) &= P_{t_0}\big(K_{(t-t_0)}(\varphi)\big) + \int_{t_0}^t P_s\big(L^{(0)}_{(t-s)}(\varphi)\big)\, ds \\
&= P_{t_0}\big(K_{(t-t_0)}(\varphi)\big) + \int_{t_0}^t P_s\big(L^{(0)}\big(K_{(t-s)}(\varphi)\big)\big)\, ds
\end{aligned} \tag{76}$$

in $G^0_{q+3}(H_\gamma,V)$ for all $t_0, t \in [0,\infty)$ with $t_0 \leq t$, $\varphi \in Lip^3_q(H_\gamma,V)$ and all $q \in [0,\infty)$ where the integrals in (76) are Bochner integrals in $\mathbb{R}$-Banach space $G^0_{q+3}(H_\gamma,V)$. According to Corollary 3, these Bochner integrals are indeed well defined. We would like to add to the mild Kolmogorov backward equation (76) that the mild Kolmogorov operators $L^{(0)}_t$, $t \in (0,\infty)$, appearing in (76) do, in general, not commute with the semigroup operators P_t, $t \in [0,\infty)$, i.e, we do, in general, not have that $(P_t L^{(0)}_s)(\varphi) = (L^{(0)}_s P_t)(\varphi)$ for all $s,t \in (0,\infty)$ and all $\varphi \in G^2_0(H_\gamma,V)$. This is in contrast to the standard Kolmogorov backward equation where the semigroup and the Kolmogorov operator do commute (see, e.g., Section 8.1 in Oeksendal [84]). In the next step let $\mathcal{K}_t\colon L(G^2_1(H_{\min(\alpha,\beta,\gamma)},V), G^0_3(H_\gamma,V)) \to L(Lip^3_0(H_\gamma,V), G^0_3(H_\gamma,V))$, $t \in (0,\infty)$, let $\mathcal{K}_0\colon L(G^0_3(H_\gamma,V)) \to L(Lip^3_0(H_\gamma,V), G^0_3(H_\gamma,V))$ and let $\mathcal{L}\colon L(G^0_3(H_\gamma,V)) \to L(G^2_1(H_{\min(\alpha,\beta,\gamma)},V), G^0_3(H_\gamma,V))$ be bounded linear operators defined through

$$(\mathcal{K}_t\Phi)(\varphi) := \Phi(K_t(\varphi)) \tag{77}$$

for all $t \in (0,\infty)$, $\varphi \in Lip^3_0(H_\gamma,V)$ and all $\Phi \in L(G^2_1(H_{\min(\alpha,\beta,\gamma)},V), G^0_3(H_\gamma,V))$, through $(\mathcal{K}_0\Phi)(\varphi) := \Phi(\varphi)$ for all $\varphi \in Lip^3_0(H_\gamma,V)$ and all $\Phi \in L(G^0_3(H_\gamma,V))$ and through

$$(\mathcal{L}\Phi)(\varphi) := \Phi\big(L^{(0)}(\varphi)\big) \tag{78}$$

for all $\varphi \in G^2_1(H_{\min(\alpha,\beta,\gamma)}$ and all $\Phi \in L(G^0_3(H_\gamma,V))$. Lemmas 3 and 4 ensure that $\mathcal{K}_t$, $t \in [0,\infty)$, and $\mathcal{L}$ are indeed well defined bounded linear operators. The next corollary follows from Lemmas 2–4.

Corollary 4. *Assume that the setting in Section 3.1 is fulfilled. Then the function $(t_0,t) \ni s \mapsto \mathcal{K}_{t-s}(\mathcal{L}(P_s)) \in L(Lip^3_0(H_\gamma,V), G^0_3(H_\gamma,V))$ is continuous and satisfies $\int_{t_0}^t \|\mathcal{K}_{t-s}(\mathcal{L}(P_s))\|_{L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))}\, ds < \infty$ for every $t_0, t \in [0,\infty)$ with $t_0 < t$.*

Proof of Corollary 4. Note that the triangle inequality implies that

$$
\begin{aligned}
&\left\|\mathcal{K}_{t-s}(\mathcal{L}(P_s)) - \mathcal{K}_{t-s_0}(\mathcal{L}(P_{s_0}))\right\|_{L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))} \\
&\leq \left\|\mathcal{K}_{t-s}(\mathcal{L}(P_s)) - \mathcal{K}_{t-s_0}(\mathcal{L}(P_s))\right\|_{L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))} \\
&+ \left\|\mathcal{K}_{t-s_0}(\mathcal{L}(P_s)) - \mathcal{K}_{t-s_0}(\mathcal{L}(P_{s_0}))\right\|_{L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))} \\
&\leq \|\mathcal{K}_{t-s} - \mathcal{K}_{t-s_0}\|_{L\left(L(G^2_1(H_{\min(\alpha,\beta,\gamma)}),V),G^0_3(H_\gamma,V)),L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))\right)} \\
&\cdot \underbrace{\|\mathcal{L}\|_{L\left(L(G^0_3(H_\gamma,V)),L(G^2_1(H_{\min(\alpha,\beta,\gamma)}),V),G^0_3(H_\gamma,V))\right)}}_{<\infty \text{ due to Lemma 4}} \cdot \underbrace{\left[\sup_{u\in[t_0,t]} \|P_u\|_{L(G^0_3(H_\gamma,V))}\right]}_{<\infty \text{ due to Lemma 2}} \qquad (79) \\
&+ \underbrace{\left[\sup_{\substack{\Phi\in L(G^0_3(H_\gamma,V)) \\ \|\Phi\|_{L(Lip^1_2(H_\gamma,V),G^0_3(H_\gamma,V))}\leq 1}} \|\mathcal{K}_{t-s_0}(\mathcal{L}(\Phi))\|_{L(Lip^3_0(H_\gamma,V),G^0_3(H_\gamma,V))}\right]}_{<\infty \text{ due to Lemmas 3 and 4}} \cdot \|P_s - P_{s_0}\|_{L(Lip^1_2(H_\gamma,V),G^0_3(H_\gamma,V))}
\end{aligned}
$$

for all $s_0, s \in (t_0, t)$ and all $t_0, t \in [0,\infty)$ with $t_0 \leq t$. Combining (79) with Lemma 3 and Lemma 2 completes the proof of Corollary 4. □

We now use Corollary 4 to reformulate equation (76). More precisely, combining Corollary 4, equation (76), definition (78) and definition (77) shows that

$$P_t = \mathcal{K}_{(t-t_0)}(P_{t_0}) + \int_{t_0}^t \mathcal{K}_{(t-s)}\big(\mathcal{L}(P_s)\big)\, ds \qquad (80)$$

in $L(Lip^3_0(H_\gamma, V), G^0_3(H_\gamma, V))$ for all $t_0, t \in [0,\infty)$ with $t_0 \leq t$ and this, in particular, implies that

$$P_t = \mathcal{K}_t(P_0) + \int_0^t \mathcal{K}_{(t-s)}\big(\mathcal{L}(P_s)\big)\, ds \qquad (81)$$

in $L(Lip^3_0(H_\gamma, V), G^0_3(H_\gamma, V))$ for all $t \in [0,\infty)$ where the integrals in (80) and (81) are understood to be Bochner integrals in $L(G^3_0(H_\gamma, V), G^0_3(H_\gamma, V))$. According to Corollary 4, these Bochner integrals are indeed well defined. Equation (81) and equation (80) are somehow *mild Kolmogorov backward equations* for the P_t, $t \in [0,\infty)$, (see (69)) associated to the SPDE (68).

3.2.3 Weak regularity for solutions of SPDEs

Another consequence of the mild Itô formula (53) is to study weak regularity of solutions of SPDEs. To be more precise, in this subsection regularity of the probability measures $\mathbb{P}_{X_t}$, $t \in (0,T]$, of the solution process X_t, $t \in [0,T]$, of the SPDE (47) are studied by using the mild Kolmogorov backward equation (70) above. Below (see the illustrations below Lemma 6) we also describe in more detail what we understand by regularity of a probability measure. While strong regularity of solutions of SPDEs have been intensively analyzed in the literature (see, e.g., Da Prato & Zabczyk [21, 22], Brzeźniak [8], Brzeźniak, Van Neerven, Veraar & Weis [9], Van Neerven, Veraar & Weis [102, 103, 104], Jentzen & Röckner [56], Kruse & Larsson [66] and the references therein), weak regularity for solutions of SPDEs seem to be much less investigated.

Let us now go into details. An important ingredient in our analysis on weak regularity of solutions of SPDEs are the following mappings. Let $\|\cdot\|^{\delta,\rho}_{t,q} : G^2_q(H_\rho, V) \to [0,\infty)$, $t \in (0,\infty)$, $q \in [0,\infty)$, $\delta \in (\rho-1,\infty)$, $\rho \in \mathbb{R}$, be a family of functions defined through

$$
\begin{aligned}
&\|\varphi\|^{\delta,\rho}_{t,q} \qquad (82) \\
&:= \|K_t(\varphi)\|_{G^0_{q+2}(H_\delta,V)} + \int_0^t (t-s)^{\min(\delta-\rho,0)} \Big(\|(K_s\varphi)'\|_{G^0_{q+1}(H_\rho,L(H_\alpha,V))} + \|(K_s\varphi)''\|_{G^0_q(H_\rho,L^{(2)}(H_\beta,V))}\Big)\, ds
\end{aligned}
$$

for all $t \in (0,\infty)$, $\varphi \in G^2_q(H_\rho, V)$, $\delta \in (\rho-1,\infty)$, $q \in [0,\infty)$ and all $\rho \in \mathbb{R}$. Please note that the integrand in (82) is indeed Borel measurable in $s \in [0,\infty)$ since H_ρ is separable for every $\rho \in \mathbb{R}$. The next lemma collects some properties of the functions $\|\cdot\|^{\delta,\rho}_{t,q} : G^2_q(H_\rho, V) \to [0,\infty)$, $t \in (0,\infty)$, $q \in [0,\infty)$, $\delta \in (\rho-1,\infty)$, $\rho \in \mathbb{R}$.

Lemma 5 (Properties of $\|\cdot\|_{t,q}^{\delta,\rho}$, $t \in (0,\infty)$, $q \in [0,\infty)$, $\delta \in (\rho-1,\infty)$, $\rho \in \mathbb{R}$)**.** *Assume that the setting in Section 3.1 is fulfilled and let* $\|\cdot\|_{t,q}^{\delta,\rho}$*,* $t \in (0,\infty)$*,* $q \in [0,\infty)$*,* $\delta \in (\rho-1,\infty)$*,* $\rho \in \mathbb{R}$*, be defined through* (82)*. Then*

$$\|\varphi\|_{t,q}^{\delta,\rho} \leq \|\varphi\|_{G_q^2(H_\rho,V)} \Bigg(\max\big(1, \|e^{At}\|_{L(H_\rho,H_\delta)}^{(q+2)}\big) + \max\big(1, \|e^{At}\|_{L(H)}^{(q+1)}\big) \int_0^t (t-s)^{\min(\delta-\rho,0)} \Big[\|e^{As}\|_{L(H_\alpha,H_\rho)} + \|e^{As}\|_{L(H_\beta,H_\rho)}^2\Big]\,ds \Bigg) < \infty \tag{83}$$

for all $\varphi \in G_q^2(H_\rho,V)$*,* $q \in [0,\infty)$*,* $t \in (0,\infty)$*,* $\delta \in (\rho-1,\infty)$*,* $\rho \in (-\infty, \min(\alpha+1,\beta+\frac{1}{2}))$ *and it holds for every* $t \in (0,\infty)$*,* $q \in [0,\infty)$*,* $\rho \in (-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$*,* $\delta \in (\rho-1,\infty)$ *that the mapping* $\|\cdot\|_{t,q}^{\delta,\rho} : G_q^2(H_\rho,V) \to [0,\infty)$ *is a norm on* $G_q^2(H_\rho,V)$*.*

Proof of Lemma 5. Combining (71)–(73), (43) and (44) shows (83). Next observe for every $t \in (0,\infty)$, $q \in [0,\infty)$, $\rho \in (-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$ and every $\delta \in (\rho-1,\infty)$ that $\|\cdot\|_{t,q}^{\delta,\rho} : G_q^2(H_\rho,V) \to [0,\infty)$ is a semi-norm on $G_q^2(H_\rho,V)$. In addition, note for every $\rho,\delta \in \mathbb{R}$, $q \in [0,\infty)$, $t \in (0,\infty)$ and every $\varphi \in G_q^0(H_\rho,V)$ that if $\|K_t(\varphi)\|_{G_q^0(H_\delta,V)} = 0$, then $\sup_{x \in e^{At}(H_\delta)} \|\varphi(x)\|_V = 0$. The fact that for every $\rho,\delta \in \mathbb{R}$ the set $e^{At}(H_\delta)$ is dense in H_ρ therefore shows for every $t \in (0,\infty)$, $q \in [0,\infty)$, $\rho \in (-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$ and every $\delta \in (\rho-1,\infty)$ that $\|\cdot\|_{t,q}^{\delta,\rho} : G_q^2(H_\rho,V) \to [0,\infty)$ is indeed a norm on $G_q^2(H_\rho,V)$. The proof of Lemma 5 is thus completed. □

In the next step we denote for every $t \in (0,\infty)$, $q \in [0,\infty)$, $\rho \in (-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$, $\delta \in (\rho-1,\infty)$ by $(\mathcal{G}_{t,q}^{2,\delta}(H_\rho,V), \|\cdot\|_{t,q}^{\delta,\rho})$ the completion of the normed $\mathbb{R}$-vector space $(G_q^2(H_\rho,V), \|\cdot\|_{t,q}^{\delta,\rho})$. The pairs $(\mathcal{G}_{t,q}^{2,\delta}(H_\rho,V), \|\cdot\|_{t,q}^{\delta,\rho})$ for $t \in (0,\infty)$, $q \in [0,\infty)$, $\delta \in (\rho-1,\infty)$ and $\rho \in (-\infty,\min(\alpha+1,\beta+\frac{1}{2}))$ are thus $\mathbb{R}$-Banach spaces.

Theorem 2 (Weak regularity for P_t, $t \in (0,\infty)$)**.** *Assume that the setting in Section 3.1 is fulfilled. Then* $P_t \in L(\mathcal{G}_{t,q-2}^{2,\delta}(H_\rho,V), G_q^0(H_\delta,V))$ *and*

$$\begin{aligned} &\|P_t\|_{L(\mathcal{G}_{t,q-2}^{2,\delta}(H_\rho,V),G_q^0(H_\delta,V))} \\ &\leq \max\Big(1, \|F\|_{G_1^0(H_\rho,H_\alpha)}, \|B\|_{G_1^0(H_\rho,HS(U_0,H_\beta))}^2\Big) \max\Bigg(1, \sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G_q^0(H_\rho,V),G_q^0(H_\delta,V))}\Big]\Bigg) < \infty \end{aligned} \tag{84}$$

for all $t \in (0,\infty)$*,* $\delta \in [\gamma,\infty) \cap (\rho-1,\infty)$*,* $q \in [2,\infty)$ *and all* $\rho \in [\gamma,\min(\alpha+1,\beta+\frac{1}{2}))$*.*

Proof of Theorem 2. Equation (70) implies

$$\begin{aligned} &\|P_t(\varphi)\|_{G_q^0(H_\delta,V)} \\ &\leq \|K_t(\varphi)\|_{G_q^0(H_\delta,V)} \\ &\quad + \Bigg(\sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G_q^0(H_\rho,V),G_q^0(H_\delta,V))}\Big]\Bigg) \left(\int_0^t (t-s)^{-\max(\rho-\delta,0)} \|L_s^{(0)}(\varphi)\|_{G_q^0(H_\rho,V)}\,ds\right) \\ &\leq \max\Bigg(1, \sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G_q^0(H_\rho,V),G_q^0(H_\delta,V))}\Big]\Bigg) \\ &\quad \cdot \left(\|K_t(\varphi)\|_{G_q^0(H_\delta,V)} + \int_0^t (t-s)^{\min(\delta-\rho,0)} \|L^{(0)}(K_s(\varphi))\|_{G_q^0(H_\rho,V)}\,ds\right) \end{aligned} \tag{85}$$

for all $t \in (0,\infty)$, $\varphi \in G_q^2(H_\rho,V)$, $\delta \in [\gamma,\infty) \cap (\rho-1,\infty)$, $q \in [2,\infty)$ and all $\rho \in [\gamma,\min(\alpha+1,\beta+\frac{1}{2}))$. Inequality (85) and Lemma 4 then complete the proof of Theorem 2. □

Below we illustrate Theorem 2 through some consequences. To do so, we need the following elementary lemma for probability measures on separable Hilbert spaces.

Lemma 6 (An embedding for probability measures)**.** *Let* $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ *and* $(V, \|\cdot\|_V, \langle\cdot,\cdot\rangle_V)$ *be separable real Hilbert spaces with* $V \neq \{0\}$ *and let* $\mu_1,\mu_2 \colon \mathcal{B}(H) \to [0,1]$ *be two probability measures with* $\int_H \varphi(x)\,\mu_1(dx) = \int_H \varphi(x)\,\mu_2(dx)$ *for all infinitely often Fréchet differentiable functions* $\varphi \colon H \to V$ *with compact support. Then* $\mu_1 = \mu_2$*.*

Proof of Lemma 6. First of all, we denote throughout this proof for every $x \in H$ and every $r \in (0,\infty)$ by $B_r(x) := \{y \in H\colon \|x-y\|_H \leq r\}$ the ball in H on x with radius r. Next let $v_0 \in V$ be a vector which satisfies $\|v_0\|_V = 1$. Such a vector does indeed since we assumed that $V \neq \{0\}$. Furthermore, let $\psi_k\colon \mathbb{R} \to [0,1]$, $k \in \mathbb{N}$, be a sequence of infinitely often differentiable functions with $\psi_k(x) = 1$ for all $x \in [-1,1]$ and all $k \in \mathbb{N}$ and with $\psi_k(x) = 0$ for all $x \in (-\infty, -1-\frac{1}{k}) \cup (1+\frac{1}{k}, \infty)$ and all $k \in \mathbb{N}$. Moreover, let $N \in \mathbb{N}$, let $x_1, \ldots, x_N \in H$ and let $r_1, \ldots, r_N \in (0,\infty)$. In the next step we define a sequence $\varphi_k\colon H \to V$, $k \in \mathbb{N}$, of functions by

$$\varphi_k(x) := v_0 \cdot \prod_{n=1}^{N} \psi_k\Big(\tfrac{\|x-x_n\|_H^2}{(r_n)^2}\Big) \tag{86}$$

for all $x \in H$ and all $k \in \mathbb{N}$. Note for every $k \in \mathbb{N}$ that φ_k is infinitely often Fréchet differentiable with a compact support. Therefore, we obtain

$$\int_H \varphi_k(x)\,\mu_1(dx) = \int_H \varphi_k(x)\,\mu_2(dx) \tag{87}$$

for all $k \in \mathbb{N}$. In the next step observe that $\varphi_k(x) = v_0$ for all $x \in \cap_{n=1}^N B_{r_n}(v_n)$ and all $k \in \mathbb{N}$, that $\sup_{k\in\mathbb{N}} \sup_{x\in H} \|\varphi_k(x)\|_V \leq 1$ and that

$$\lim_{k\to\infty} \varphi_k(x) = \begin{cases} 1 & : x \in \cap_{n=1}^N B_{r_n}(v_n) \\ 0 & : x \in H \backslash \big(\cap_{n=1}^N B_{r_n}(v_n)\big) \end{cases} \tag{88}$$

for all $x \in H$. Combining this and (87) with Lebesgue's theorem on dominated convergence then proves that $\mu_1\big(\cap_{n=1}^N B_{r_n}(x_n)\big) = \mu_2\big(\cap_{n=1}^N B_{r_n}(x_n)\big)$. Combining this, the fact that the set

$$\cup_{M\in\mathbb{N}} \big\{\cap_{m=1}^M B_{s_m}(y_m) \subset H\colon s_1, \ldots, s_M \in (0,\infty), y_1, \ldots, y_M \in H\big\} \tag{89}$$

is a $\cap$-stable generator of the Borel sigma-algebra $\mathcal{B}(H)$ and the uniqueness theorem for measures (see, e.g., Lemma 1.42 in Klenke [58]) then completes the proof of Lemma 6. $\square$

Let us now illustrate Theorem 2 by a simple application. First, we denote by $G_0^2(H_\gamma,\mathbb{R})' := L(G_0^2(H_\gamma,\mathbb{R}),\mathbb{R})$ and $\mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,\mathbb{R})' := L(\mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,\mathbb{R}),\mathbb{R})$ for $t \in (0,\infty)$ the topological dual spaces of $G_0^2(H_\gamma,\mathbb{R})$ and $\mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,\mathbb{R})$ for $t \in (0,\infty)$ respectively. Moreover, we denote by $\mathcal{M}_2(H_\gamma)$ the set of all probability measures $\mu\colon \mathcal{B}(H_\gamma) \to [0,1]$ which satisfy $\int_{H_\gamma} \|x\|_{H_\gamma}^2\,\mu(dx) < \infty$ and we consider the mapping $\mathcal{I}\colon \mathcal{M}_2(H_\gamma) \to G_0^2(H_\gamma,\mathbb{R})'$ given by $(\mathcal{I}\mu)(\varphi) = \int_{H_\gamma} \varphi(x)\,\mu(dx)$ for all $\varphi \in G_0^2(H_\gamma,\mathbb{R})$ and all $\mu \in \mathcal{M}_2(H_\gamma)$. Lemma 6 then proves that $\mathcal{I}$ is injective and through $\mathcal{I}$ we can thus identify the probability measures $\mathcal{M}_2(H_\gamma)$ with finite second moment as a subset of linear forms in $G_0^2(H_\gamma,\mathbb{R})'$. Next note that Proposition 3 proves that the probability measure $\mathbb{P}_{X_t}$ of the solution process of the SPDE (47) at every time $t \in (0,T]$ has a finite second moment and is thus in $\mathcal{M}_2(H_\gamma)$. Hence, the linear form $\mathcal{I}(\mathbb{P}_{X_t}) = \int_{H_\gamma} (\cdot)\,d\mathbb{P}_{X_t} \in G_0^2(H_\gamma,\mathbb{R})'$ corresponding to the probability measure $\mathbb{P}_{X_t}$ of the solution of the SPDE (47) at time $t \in (0,T]$ is in $G_0^2(H_\gamma,V)'$. In addition, observe that Theorem 2, in particular, implies that $\int_{H_\gamma} (\cdot)\,d\mathbb{P}_{X_t} \in \mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,V)'$ for all $t \in (0,T]$. Moreover, note that Lemma 5 implies that $\mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,V)' \subset G_0^2(H_\gamma,V)'$ continuously for all $t \in (0,\infty)$. Theorem 2 thus proves for every $t \in (0,T]$ that $\mathcal{I}(\mathbb{P}_{X_t}) = \int_{H_\gamma} (\cdot)\,d\mathbb{P}_{X_t}$ does not only lie in $G_0^2(H_\gamma,V)'$ but also in the smaller space $\mathcal{G}_{t,0}^{2,\gamma}(H_\gamma,V)'$ too. In this sense Theorem 2 proves more regularity of the probability measures $\mathbb{P}_{X_t}$, $t \in (0,T]$, of the solution of the SPDE (47). It thus establishes "weak regularity" for the solution of the SPDE (47). In the remainder of this subsection some further consequences of Theorem 2 are derived.

Corollary 5. *Assume that the setting in Section 3.1 is fulfilled, assume $\alpha \leq \gamma$, assume $\beta \leq \gamma$, let $\rho \in [\gamma, \min(\alpha+1, \beta+\frac{1}{2}))$ be a real number, let $(\tilde{H}, \langle\cdot,\cdot\rangle_{\tilde{H}}, \|\cdot\|_{\tilde{H}})$ be a separable $\mathbb{R}$-Hilbert space, let $R, \tilde{R} \in L(H_\rho, \tilde{H})$, let $\varphi \in C_{Lip}^2(\tilde{H}, V)$ and let $\psi\colon H_\rho \to \tilde{H}$ be given by $\psi(x) = \varphi(Rx) - \varphi(\tilde{R}x)$ for all $x \in H_\rho$. Then*

$$\begin{aligned}
&\|P_t(\psi)\|_{G_q^0(H_\delta,V)} \\
&\leq \max\Big(1, \|F\|_{G_1^0(H_\rho,H_\alpha)}, \|B\|_{G_1^0(H_\rho,HS(U_0,H_\beta))}^2\Big) \max\Bigg(1, \sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G_q^0(H_\rho,V),G_q^0(H_\delta,V))}\Big]\Bigg) \\
&\quad\cdot \|\varphi\|_{C_{Lip}^2(\tilde{H},V)} \frac{\max(t,1)}{t^{\max(r+\rho-\delta,0)}} \|R-\tilde{R}\|_{L(H_{\rho+r},\tilde{H})} \Bigg[\sup_{u\in[\rho-\delta,1]\cup[0,1]} \sup_{s\in(0,t]} \Big(s^{\max(u,0)} \|e^{As}\|_{L(H,H_u)}\Big)\Bigg]^3 \\
&\quad\cdot \big[1 + \|R\|_{L(H_\rho,\tilde{H})} + \|\tilde{R}\|_{L(H_\rho,\tilde{H})}\big]^2 \left(1 + \int_0^1 (1-s)^{\min(\delta-\rho,0)}\, s^{[\min(\alpha-\rho,2\beta-2\rho)-r]}\,ds\right) < \infty
\end{aligned} \tag{90}$$

for all $t \in (0,\infty)$, $q \in [3,\infty)$, $\delta \in [\gamma,\infty)$ and all $r \in [0, \min(1+\alpha-\rho, 1+2\beta-2\rho))$. In particular, we have

$$\sup_{\Phi \in C^2_{Lip}(H_\rho,V)\setminus\{0\}} \sup_{S \in L(H_\rho)} \sup_{t \in (0,T]} \left(\frac{t^{\max(r+\rho-\delta,0)} \, \|P_t(\Phi) - P_t\big(\Phi(S(\cdot))\big)\|_{G^0_3(H_\delta,V)}}{\|I-S\|_{L(H_{\rho+r},H_\rho)} \, \big(1+\|S\|^2_{L(H_\rho)}\big) \, \|\varphi\|_{C^2_{Lip}(H_\rho,V)}} \right) < \infty \tag{91}$$

for all $\delta \in [\gamma,\infty)$ and all $r \in [0, \min(1+\alpha-\rho, 1+2\beta-2\rho))$.

Proof of Corollary 5. Throughout this the proof the real numbers $\kappa_{r,t} \in [1,\infty)$, $r \in [0,\infty)$, $t \in (0,\infty)$, defined by

$$\kappa_{r,t} := \sup_{u \in [-r,1]} \sup_{s \in (0,t]} \left(s^{\max(u,0)} \|e^{As}\|_{L(H,H_u)} \right) < \infty \tag{92}$$

for all $r \in [0,\infty)$ are used. The quantities $\kappa_{r,t}$, $r \in [0,\infty)$, $t \in (0,\infty)$, are indeed finite since e^{At}, $t \in [0,\infty)$, is an analytic semigroup. The estimate

$$\|e^{At}\|_{L(H_a,H_b)} = \|e^{At}\|_{L(H,H_{(b-a)})} \le \kappa_{\max(a-b,0),t} \, t^{\min(a-b,0)} \tag{93}$$

for all $t \in (0,\infty)$ and all $a, b \in \mathbb{R}$ with $b - a \le 1$ then shows

$$\begin{aligned} \|K_t(\psi)\|_{G^0_q(H_\delta,V)} &\le \|\varphi'\|_{L^\infty(\tilde{H},L(\tilde{H},V))} \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \|e^{At}\|_{L(H_\delta,H_r)} \\ &\le \|\varphi\|_{C^2_{Lip}(\tilde{H},V)} \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \kappa_{\max(\delta-r,0),t} \, t^{\min(\delta-r,0)} \end{aligned} \tag{94}$$

and

$$\begin{aligned} &\|(K_t\psi)'\|_{G^0_{q-1}(H_\rho,L(H_\alpha,V))} \\ &\le \sup_{x\in H_\rho} \frac{\|(\varphi'(Re^{At}x) - \varphi'(\tilde{R}e^{At}x))Re^{At}\|_{L(H_\alpha,V)}}{\big(1+\|x\|_{H_\rho}\big)^{(q-1)}} \\ &\quad + \sup_{x\in H_\rho} \frac{\|\varphi'(\tilde{R}e^{At}x)(R-\tilde{R})e^{At}\|_{L(H_\alpha,V)}}{\big(1+\|x\|_{H_\rho}\big)^{(q-1)}} \\ &\le \|\varphi''\|_{L^\infty(\tilde{H},L^{(2)}(\tilde{H},V))} \, \|R\|_{L(H_\rho,\tilde{H})} \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \underbrace{\|e^{At}\|_{L(H_\alpha,H_\rho)} \, \|e^{At}\|_{L(H_\rho,H_r)}}_{\le(\kappa_{0,t})^2 t^{(\alpha-r)}} \\ &\quad + \|\varphi'\|_{L^\infty(\tilde{H},L(\tilde{H},V))} \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \underbrace{\|e^{At}\|_{L(H_\alpha,H_r)}}_{\le\kappa_{0,t}\, t^{(\alpha-r)}} \end{aligned} \tag{95}$$

and

$$\begin{aligned} &\|(K_t\psi)''\|_{G^0_{q-2}(H_\rho,L^{(2)}(H_\beta,V))} \\ &\le \sup_{x\in H_\rho} \sup_{\|v\|_{H_\beta}=\|w\|_{H_\beta}=1} \frac{\big\|\big(\varphi''(Re^{At}x) - \varphi''(\tilde{R}e^{At}x)\big)(Re^{At}v, Re^{At}w)\big\|_V}{\big(1+\|x\|_{H_\rho}\big)^{(q-2)}} \\ &\quad + \sup_{x\in H_\rho} \sup_{\|v\|_{H_\beta}=\|w\|_{H_\beta}=1} \frac{\big\|\varphi''(\tilde{R}e^{At}x)\big((R-\tilde{R})e^{At}v, Re^{At}w\big)\big\|_V}{\big(1+\|x\|_{H_\rho}\big)^{(q-2)}} \\ &\quad + \sup_{x\in H_\rho} \sup_{\|v\|_{H_\beta}=\|w\|_{H_\beta}=1} \frac{\big\|\varphi''(\tilde{R}e^{At}x)\big(\tilde{R}e^{At}v, (R-\tilde{R})e^{At}w\big)\big\|_V}{\big(1+\|x\|_{H_\rho}\big)^{(q-2)}} \\ &\le \left[\sup_{\substack{x,y\in\tilde{H} \\ x\neq y}} \frac{\|\varphi''(x)-\varphi''(y)\|_{L^{(2)}(\tilde{H},V)}}{\|x-y\|_{\tilde{H}}} \right] \|R\|^2_{L(H_\rho,\tilde{H})} \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \underbrace{\|e^{At}\|^2_{L(H_\beta,H_\rho)} \, \|e^{At}\|_{L(H_\rho,H_r)}}_{\le(\kappa_{0,t})^3 t^{(2\beta-\rho-r)}} \\ &\quad + \|\varphi''\|_{L^\infty(\tilde{H},L^{(2)}(\tilde{H},V))} \Big[\|R\|_{L(H_\rho,\tilde{H})} + \|\tilde{R}\|_{L(H_\rho,\tilde{H})}\Big] \, \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \, \underbrace{\|e^{At}\|_{L(H_\beta,H_\rho)} \, \|e^{At}\|_{L(H_\beta,H_r)}}_{\le(\kappa_{0,t})^2 t^{(2\beta-\rho-r)}} \end{aligned} \tag{96}$$

for all $q \in [3,\infty)$, $t \in (0,T]$, $\delta \in [\gamma,\infty)$ and all $r \in [\rho, \min(1+\alpha, 1+2\beta-\rho))$. Combining (94)–(96) implies

$$\begin{aligned}
\|\psi\|_{t,q-2}^{\delta,\rho} &\leq \|\varphi\|_{C^2_{Lip}(\tilde{H},V)} \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \left[\kappa_{\max(\delta-r,0),t}\right]^3 \left[1+\|R\|_{L(H_\rho,\tilde{H})}+\|\tilde{R}\|_{L(H_\rho,\tilde{H})}\right]^2 \\
&\quad\cdot \left(t^{\min(\delta-r,0)} + \int_0^t (t-s)^{\min(\delta-\rho,0)} \max\Big(s^{(\alpha-r)}, s^{(2\beta-\rho-r)}\Big)\, ds\right) \\
&\leq \|\varphi\|_{C^2_{Lip}(\tilde{H},V)} \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \left[\kappa_{\max(\delta-r,0),t}\right]^3 \left[1+\|R\|_{L(H_\rho,\tilde{H})}+\|\tilde{R}\|_{L(H_\rho,\tilde{H})}\right]^2 \max\big(1, t^{|2\beta-\rho-\alpha|}\big) \\
&\quad\cdot \left(t^{\min(\delta-r,0)} + t^{[\min(\delta-\rho,0)+\min(\alpha,2\beta-\rho)+1-r]} \int_0^1 (1-s)^{\min(\delta-\rho,0)}\, s^{\min(\alpha-r,2\beta-\rho-r)}\, ds\right) \\
&\leq \|\varphi\|_{C^2_{Lip}(\tilde{H},V)} \|R-\tilde{R}\|_{L(H_r,\tilde{H})} \left[\kappa_{\max(\delta-r,0)}\right]^3 \left[1+\|R\|_{L(H_\rho,\tilde{H})}+\|\tilde{R}\|_{L(H_\rho,\tilde{H})}\right]^2 \max(t,1) \\
&\quad\cdot t^{\min(\delta-r,0)} \left(1 + \int_0^1 (1-s)^{\min(\delta-\rho,0)}\, s^{\min(\alpha-r,2\beta-\rho-r)}\, ds\right)
\end{aligned} \tag{97}$$

for all $t \in (0,T]$, $q \in [3,\infty)$, $\delta \in [\gamma,\infty)$ and all $r \in [\rho, \min(1+\alpha, 1+2\beta-\rho))$. Next observe that Theorem 2 implies that

$$\begin{aligned}
\|P_t(\psi)\|_{G^0_q(H_\delta,V)} &\leq \max\Big(1, \|F\|_{G^0_1(H_\rho,H_\alpha)}, \|B\|^2_{G^0_1(H_\rho,HS(U_0,H_\beta))}\Big) \\
&\qquad \cdot \max\left(1, \sup_{s\in(0,t)} \left[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G^0_q(H_\rho,V),G^0_q(H_\delta,V))}\right]\right) \cdot \|\psi\|_{t,q-2}^{\delta,\rho}
\end{aligned} \tag{98}$$

for all $t \in (0,T]$, $q \in [3,\infty)$ and all $\delta \in [\gamma,\infty)$. Combining (97) and (98) then shows (90). Inequality (90) implies (91). This completes the proof of Corollary 5. □

In the remainder of this subsection, Corollary 5 is illustrated by three simple consequences (Corollary 6, Corollary 7 and Corollary 8). Corollary 6 follows immediately from inequality (91) in Corollary 5 and its proof is therefore omitted.

Corollary 6 (Spatial weak semigroup regularity)**.** *Assume that the setting in Section 3.1 is fulfilled and assume $\alpha \leq \gamma$ and $\beta \leq \gamma$. Then*

$$\sup_{\varphi \in C^2_{Lip}(H_\rho,V)\setminus\{0\}} \sup_{t\in(0,T]} \sup_{h\in(0,T]} \left(\frac{t^{\max(\rho-\delta+r,0)} \|P_t(K_h(\varphi)) - P_t(\varphi)\|_{G^0_3(H_\delta,V)}}{h^r \|\varphi\|_{C^2_{Lip}(H_\rho,V)}}\right) < \infty \tag{99}$$

for all $\delta \in [\gamma,\infty)$, $r \in [0, 1+\alpha-\rho) \cap [0, 1+2\beta-2\rho)$ and all $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$. In particular, if the real number $p \in [2,\infty)$ in Assumption 4 satisfies $p \geq 3$, then

$$\sup_{\varphi \in C^2_{Lip}(H_\gamma,V)\setminus\{0\}} \sup_{t\in[0,T]} \sup_{h\in(0,T]} \left(\frac{t^r \left\|\mathbb{E}\big[\varphi(e^{Ah}X_t)\big] - \mathbb{E}\big[\varphi(X_t)\big]\right\|_V}{h^r \|\varphi\|_{C^2_{Lip}(H_\gamma,V)}}\right) < \infty \tag{100}$$

for all $r \in [0, 1+\alpha-\gamma) \cap [0, 1+2\beta-2\gamma)$.

Corollary 7 (Temporal weak regularity)**.** *Assume that the setting in Section 3.1 is fulfilled and assume $\alpha \leq \gamma$ and $\beta \leq \gamma$. Then*

$$\sup_{\substack{t_1,t_2\in(0,T] \\ t_1<t_2}} \left(\frac{|t_1|^{\max(\rho-\delta+r,0)} \|P_{t_2} - P_{t_1}\|_{L(C^2_{Lip}(H_\rho,V),G^0_3(H_\delta,V))}}{|t_2-t_1|^r}\right) < \infty \tag{101}$$

for all $\delta \in [\gamma,\infty)$, $r \in [0, 1+\alpha-\rho) \cap [0, 1+2\beta-2\rho)$ and all $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$. In particular, if the real number $p \in [2,\infty)$ in Assumption 4 satisfies $p \geq 3$, then

$$\sup_{\varphi \in C^2_{Lip}(H_\gamma,V)\setminus\{0\}} \sup_{\substack{t_1,t_2\in[0,T] \\ t_1\neq t_2}} \left(\frac{|t_1|^r \left\|\mathbb{E}\big[\varphi(X_{t_2})\big] - \mathbb{E}\big[\varphi(X_{t_1})\big]\right\|_V}{|t_2-t_1|^r \|\varphi\|_{C^2_{Lip}(H_\gamma,V)}}\right) < \infty \tag{102}$$

for all $r \in [0, 1+\alpha-\gamma) \cap [0, 1+2\beta-2\gamma)$.

Proof of Corollary 7. First, define real numbers $c_{\rho,\delta,r} \in [0,\infty)$, $r \in [0, 1+\alpha-\rho) \cap [0, 1+2\beta-2\rho)$, $\delta \in [\gamma,\infty)$, $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$, through

$$c_{\rho,\delta,r} := \sup_{\varphi \in C^2_{Lip}(H_\rho,V)\setminus\{0\}} \sup_{t\in(0,T]} \sup_{h\in(0,T]} \left(\frac{t^{\max(\rho-\delta+r,0)} \, \|P_t(K_h(\varphi)) - P_t(\varphi)\|_{G^0_3(H_\delta,V)}}{h^r \, \|\varphi\|_{C^2_{Lip}(H_\rho,V)}} \right) < \infty \tag{103}$$

for all $r \in [0, 1+\alpha-\rho) \cap [0, 1+2\beta-2\rho)$, $\delta \in [\gamma,\infty)$ and all $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$. Corollary 6 shows that these real numbers are indeed finite. In the next step we combine (70) and the definition of $c_{\rho,\delta,r} \in [0,\infty)$ to obtain that

$$\begin{aligned}
&\frac{|t_1|^{\max(\rho-\delta+r,0)} \, \|(P_{t_2}\varphi)(x) - (P_{t_1}\varphi)(x)\|_V}{|t_2-t_1|^r} \\
&\leq \frac{|t_1|^{\max(\rho-\delta+r,0)} \, \big\|(P_{t_1}K_{(t_2-t_1)}\varphi)(x) - (P_{t_1}\varphi)(x)\big\|_V}{|t_2-t_1|^r} + \int_{t_1}^{t_2} \left[\frac{s^{\max(\rho-\delta+r,0)} \, \|(P_s \, L^{(0)}_{(t_2-s)}\varphi)(x)\|_V}{|t_2-t_1|^r} \right] ds \\
&\leq c_{\rho,\delta,r} \left[1 + \|x\|_{H_\delta}\right]^3 \|\varphi\|_{C^2_{Lip}(H_\rho,V)} + (1+T) \int_{t_1}^{t_2} \left[\frac{s^{\max(\rho-\delta,0)} \, \|(P_s \, L^{(0)}_{(t_2-s)}\varphi)(x)\|_V}{|t_2-t_1|^{\min(1+\alpha-\rho,1+2\beta-2\rho)}} \right] ds \\
&\leq c_{\rho,\delta,r} \left[1 + \|x\|_{H_\delta}\right]^3 \|\varphi\|_{C^2_{Lip}(H_\rho,V)} \\
&\quad + (1+T) \left[\sup_{t\in(0,T]} \sup_{s\in(0,t)} \Big(s^{\max(\rho-\delta,0)} \, |t-s|^{\max(\rho-\alpha,2\rho-2\beta)} \, \|(P_s \, L^{(0)}_{t-s}\varphi)(x)\|_V \Big) \right]
\end{aligned} \tag{104}$$

for all $t_1, t_2 \in (0,T]$ with $t_1 < t_2$, $x \in H_\delta$, $r \in [0, 1+\alpha-\rho) \cap [0, 1+2\beta-2\rho)$, $\delta \in [\gamma,\infty)$, $\varphi \in C^2_{Lip}(H_\rho, V)$ and all $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$. Furthermore, observe that Lemma 4 shows that

$$\begin{aligned}
&\sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \, (t-s)^{\max(\rho-\alpha,2\rho-2\beta)} \, \|P_s(L^{(0)}_{t-s}(\varphi))\|_{G^0_3(H_\delta,V)} \Big] \\
&\leq \sup_{s\in(0,t)} \Big[s^{\max(\rho-\delta,0)} \|P_s\|_{L(G^0_3(H_\rho,V),G^0_3(H_\delta,V))} \, (t-s)^{\max(\rho-\alpha,2\rho-2\beta)} \, \|L^{(0)}_{t-s}(\varphi))\|_{G^0_3(H_\delta,V)} \Big] \\
&\leq \left[\sup_{s\in(0,t)} s^{\max(\rho-\delta,0)} \|P_s\|_{L(G^0_3(H_\rho,V),G^0_3(H_\delta,V))} \right] \max\big(\|F\|_{G^0_1(H_\rho,H_\alpha)}, \|B\|^2_{G^0_1(H_\rho,HS(U_0,H_\beta))}\big) \\
&\quad \cdot \left[\sup_{s\in(0,t)} s^{\max(\rho-\alpha,2\rho-2\beta)} \Big(\|(K_s\varphi)'\|_{G^0_2(H_\rho,L(H_\alpha,V))} + \|(K_s\varphi)''\|_{G^0_1(H_\rho,L^{(2)}(H_\beta,V))} \Big) \right] \\
&\leq \left[\sup_{s\in(0,T]} s^{\max(\rho-\delta,0)} \|P_s\|_{L(G^0_3(H_\rho,V),G^0_3(H_\delta,V))} \right] \max\big(\|F\|_{G^0_1(H_\rho,H_\alpha)}, \|B\|^2_{G^0_1(H_\rho,HS(U_0,H_\beta))}\big) \\
&\quad \cdot \max\left(1, \sup_{s\in[0,T]} \|e^{As}\|^2_{L(H)}\right) \\
&\quad \cdot \left[\sup_{s\in(0,T]} s^{\max(\rho-\alpha,2\rho-2\beta)} \Big(\|e^{As}\|_{L(H_\alpha,H_\rho)} \, \|\varphi'\|_{G^0_2(H_\rho,L(H_\rho,V))} + \|e^{As}\|^2_{L(H_\beta,H_\rho)} \, \|\varphi''\|_{G^0_1(H_\rho,L^{(2)}(H_\rho,V))} \Big) \right] \\
&\leq \left[\sup_{s\in(0,T]} s^{\max(\rho-\delta,0)} \, \|P_s\|_{L(G^0_3(H_\rho,V),G^0_3(H_\delta,V))} \right] \max\big(\|F\|_{G^0_1(H_\rho,H_\alpha)}, \|B\|^2_{G^0_1(H_\rho,HS(U_0,H_\beta))}\big) \\
&\quad \cdot \max\left(1, \sup_{s\in[0,T]} \|e^{As}\|^2_{L(H)}\right) \Big(\|\varphi'\|_{G^0_2(H_\rho,L(H_\rho,V))} + \|\varphi''\|_{G^0_1(H_\rho,L^{(2)}(H_\rho,V))} \Big) \\
&\quad \cdot \left[\sup_{s\in(0,T]} s^{\max(\rho-\alpha,2\rho-2\beta)} \max\big(\|e^{As}\|_{L(H_\alpha,H_\rho)}, \|e^{As}\|^2_{L(H_\beta,H_\rho)}\big) \right]
\end{aligned} \tag{105}$$

for all $t \in (0,T]$, $\delta \in [\gamma,\infty)$, $\varphi \in G^2_1(H_\rho, V)$ and all $\rho \in [\gamma, \alpha+1) \cap [\gamma, \beta+\frac{1}{2})$. Combining (104) and (105) completes the proof of Corollary 7. □

Corollary 8 (Galerkin approximations)**.** *Assume that the setting in Section 3.1 is fulfilled, assume* $\alpha \leq \gamma$ *and* $\beta \leq \gamma$*, let* $\rho \in [\gamma, \alpha + 1) \cap [\gamma, \beta + \frac{1}{2})$ *and* $\delta \in [\gamma, \infty)$ *be real numbers and let* $P_N \in L(H_\rho)$*,* $N \in \mathbb{N}$*, be a sequence of bounded linear operators with* $\sup_{N\in\mathbb{N}} \|P_N\|_{L(H_\rho)} < \infty$*. Then*

$$\sup_{\varphi \in C^2_{Lip}(H_\rho,V)\backslash\{0\}} \sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(\frac{t^{\max(\rho-\delta+r,0)} \, \|P_t(\varphi) - P_t(\varphi(P_N(\cdot)))\|_{G^0_3(H_\delta,V)}}{\|I - P_N\|_{L(H_{\rho+r},H_\rho)} \, \|\varphi\|_{C^2_{Lip}(H_\rho,V)}} \right) < \infty \tag{106}$$

for all $r \in [0, 1 + \alpha - \rho) \cap [0, 1 + 2\beta - 2\rho)$*. In particular, if* $(\lambda_n)_{n\in\mathbb{N}} \subset (0,\infty)$ *is a non-decreasing sequence of real numbers, if* $(e_n)_{n\in\mathbb{N}} \subset H$ *is an orthonormal basis of* H *with* $D(A) = \{v \in H \colon \sum_{n=1}^\infty |\lambda_n|^2 |\left<e_n, v\right>_H|^2 < \infty\}$ *and* $Av = \sum_{n=1}^\infty -\lambda_n \left<e_n, v\right>_H e_n$ *for all* $v \in D(A)$*, if* $\rho = \gamma = 0$ *and* $p \geq 3$ *and if* $P_N(v) = \sum_{n=1}^N \left<e_n, v\right>_H e_n$ *for all* $v \in H$*,* $N \in \mathbb{N}$*, then*

$$\sup_{\varphi \in C^2_{Lip}(H,V)\backslash\{0\}} \sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(\frac{t^{\max(r-\delta,0)} \, \|P_t(\varphi) - P_t(\varphi(P_N(\cdot)))\|_{G^0_3(H_\delta,V)}}{(\lambda_N)^{-r} \, \|\varphi\|_{C^2_{Lip}(H,V)}} \right) < \infty \tag{107}$$

and

$$\sup_{\varphi \in C^2_{Lip}(H,V)\backslash\{0\}} \sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(\frac{t^r \, \big\|\mathbb{E}\big[\varphi(X_t)\big] - \mathbb{E}\big[\varphi(P_N(X_t))\big]\big\|_V}{(\lambda_N)^{-r} \, \|\varphi\|_{C^2_{Lip}(H,V)}} \right) < \infty \tag{108}$$

for all $r \in [0, 1 + \alpha) \cap [0, 1 + 2\beta)$*.*

Corollary 8 follows directly from Corollary 5 and its proof is therefore omitted. Corollary 8 is a certain spatial weak numerical approximation result for SPDEs. Further weak numerical approximation results for SPDEs can be found in [43, 25, 27, 32, 44, 73, 26, 62, 63, 7, 65].

3.2.4 Stochastic Taylor expansions for solutions of SPDEs

A further application of the mild Itô formula (53) is the derivation of stochastic Taylor expansions for solutions of stochastic partial differential equations. In Kloeden & Platen [59] stochastic Taylor expansions are derived for solutions of finite dimensional stochastic ordinary differential equations by an iterated application of the standard Itô formula. Clearly, this strategy can not be accomplished in the infinite dimensional SPDE setting since the standard Itô formula can, in general, not be applied to the solution process of an SPDE. However, by using the mild Itô formula (53) instead of the standard Itô formula, this approach can be generalized to solutions of SPDEs in a straightforward way. The main difference to the finite dimensional setting in Kloeden & Platen [59] is that the linear operators $L^{(0)}_t$, $t \in (0,T]$, and $L^{(1)}_t$, $t \in (0,T]$, in (55) and (57) here depend explicitly on the time variable $t \in (0,T]$ too (compare (55) and (57) here with (1.13) and (1.14) in Chapter 5 in [59]; see also Theorem 3 and Theorem 4 below for more details). Similar and related stochastic Taylor expansions for SPDEs can be found in Buckdahn & Ma [11], Bayer & Teichmann [4], Conus [15], Jentzen & Kloeden [54], Buckdahn, Bulla & Ma [10] and Jentzen [49].

For formulating the stochastic Taylor expansions below some notations are introduced (see also Chapter 5 in [59]). By $\mathcal{M} := \{\emptyset\} \cup (\cup_{n=1}^\infty \{0,1\}^n)$ the set of multi-indices is denoted. Moreover, define two functions $|\cdot| : \mathcal{M} \to \{0,1,2,\dots\}$ and $-(\cdot) \colon \mathcal{M}\backslash\{\emptyset\} \to \mathcal{M}$ by $|\emptyset| := 0$, by $|(\alpha_1, \alpha_2, \dots, \alpha_n)| := n$ and by $-(\alpha_1, \alpha_2, \dots, \alpha_n) := (\alpha_2, \alpha_3, \dots, \alpha_n)$ for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \{0,1\}$ and all $n \in \mathbb{N}$. Thus note that $|\alpha| \geq 1$ and $\alpha_1, \dots, \alpha_{|\alpha|} \in \{0,1\}$ for all $\alpha \in \mathcal{A}\backslash\{\emptyset\}$. Furthermore, a finite nonempty subset $\mathcal{A} \subset \mathcal{M}$ of $\mathcal{M}$ is called *hierarchical set* if $-\alpha \in \mathcal{A}$ for all $\alpha \in \mathcal{A}\backslash\{\emptyset\}$. Next define a function $\mathbb{B} \colon \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ by $\mathbb{B}(\mathcal{A}) := \{\alpha \in \mathcal{M}\backslash\mathcal{A} \colon -\alpha \in \mathcal{A}\}$ for all $\mathcal{A} \subset \mathcal{M}$. Finally, let $W^0 \colon [0,T] \to \mathbb{R}$ be a function and let $(W^1_t)_{t\in[0,T]}$ be a cylindrical Q-Wiener process defined by $W^0(t) := t$ and $W^1_t := W_t$ for all $t \in [0,T]$. Using this notation the mild Itô formula (58) can be written as

$$\varphi(X_t) = \varphi(e^{A(t-t_0)} X_{t_0}) + \sum_{i=0}^1 \int_{t_0}^t \big(L^{(i)}_{(t-s)} \varphi\big)(X_s) \, dW^i_s \tag{109}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \leq t$ and all $\varphi \in C^2(H_\gamma, V)$. Moreover, for two normed $\mathbb{R}$-vector spaces and $n \in \{0,1,2,\dots\}$ we define $C^n_b(V_1, V_2) := \big\{\varphi \in C^n(V_1,V_2) \colon \|\varphi\|_{L^\infty(V_1,V_2)} + \sum_{k=1}^n \|\varphi^{(k)}\|_{L^\infty(V_1, L^{(k)}(V_1,V_2))} < \infty\big\}$, $C_b(V_1,V_2) := C^0_b(V_1,V_2)$ and $C^\infty_b(V_1,V_2) := \cap_{k\in\mathbb{N}} C^k_b(V_1,V_2)$. We are now ready to present the stochastic Taylor expansions based on the mild Itô formula (109).

Theorem 3 (Strong stochastic Taylor expansions)**.** *Assume that the setting in Section 3.1 is fulfilled, assume* $F \in C_b^\infty(H_\gamma, H_\alpha)$*, assume* $B \in C_b^\infty(H_\gamma, HS(U_0, H_\beta))$ *and let* $\varphi \in C_b^\infty(H_\gamma, V)$*. Then*

$$\begin{aligned}\varphi(X_t) &= \varphi\big(e^{A(t-t_0)} X_{t_0}\big) \\ &+ \sum_{\substack{\alpha\in\mathcal{A}\\ \alpha\neq\emptyset}} \int_{t_0}^{t}\int_{t_0}^{s_{|\alpha|}} \dots \int_{t_0}^{s_2} \Big(L^{(\alpha_1)}_{(s_2-s_1)} \dots L^{(\alpha_{|\alpha|-1})}_{(s_{|\alpha|}-s_{|\alpha|-1})} L^{(\alpha_{|\alpha|})}_{(t-s_{|\alpha|})} \varphi\Big)\big(e^{A(s_1-t_0)} X_{t_0}\big)\, dW^{\alpha_1}_{s_1}\, dW^{\alpha_2}_{s_2} \dots dW^{\alpha_{|\alpha|}}_{s_{|\alpha|}} \\ &+ \sum_{\alpha\in\mathbb{B}(\mathcal{A})} \int_{t_0}^{t}\int_{t_0}^{s_{|\alpha|}} \dots \int_{t_0}^{s_2} \Big(L^{(\alpha_1)}_{(s_2-s_1)} \dots L^{(\alpha_{|\alpha|-1})}_{(s_{|\alpha|}-s_{|\alpha|-1})} L^{(\alpha_{|\alpha|})}_{(t-s_{|\alpha|})} \varphi\Big)\big(X_{s_1}\big)\, dW^{\alpha_1}_{s_1}\, dW^{\alpha_2}_{s_2} \dots dW^{\alpha_{|\alpha|}}_{s_{|\alpha|}}\end{aligned} \tag{110}$$

for all $t_0, t \in [0,T]$ *with* $t_0 \le t$ *and all hierarchical sets* $\mathcal{A} \subset \mathcal{M}$*.*

Proof of Theorem 3. Theorem 3 immediately follows from an iterated application of the mild Itô formula (109). □

The term $\varphi\big(e^{A(t-t_0)}X_{t_0}\big) + \sum_{\alpha\in\mathcal{A},\alpha\neq\emptyset} \dots$, $t \in [t_0,T]$, on the left hand side of (110) is referred as *strong stochastic Taylor approximation* (or truncated strong stochastic Taylor expansion) of $\varphi(X_t)$, $t \in [t_0,T]$, corresponding to the hierarchical set $\mathcal{A} \subset \mathcal{M}$ for $t_0 \in [0,T]$. The expression $\sum_{\alpha\in\mathbb{B}(\mathcal{A})} \dots$, $t \in [t_0,T]$, on the left hand side of (110) is called *remainder term* of the strong stochastic Taylor expansions of $\varphi(X_t)$, $t \in [t_0,T]$, corresponding to the hierarchical set $\mathcal{A} \subset \mathcal{M}$ for $t_0 \in [0,T]$. Next observe that, in the case $H = \mathbb{R}^d$ with $d \in \mathbb{N}$ and $A = 0$, Theorem 3 essentially reduces to Theorem 5.5.1 in Kloeden & Platen [59]. Let us also add the following remark on possible generalizations of Theorem 3.

Remark 2. *The assumption in Theorem 3 that* F*,* B *and* φ *are infinitely often Fréchet differentiable can be relaxed. To be more precise, to obtain* (110) *for a given hierarchical set* $\mathcal{A} \subset \mathcal{M}$*, it is sufficient to assume that* $F \in C_b(H_\gamma, H_\alpha)$ *is* $\max_{\alpha\in\mathbb{B}(\mathcal{A}),\alpha_1=0} \min\big\{2k-2-\sum_{i=1}^{k-1}\alpha_i \colon k \in \{1,\dots,|\alpha|\}, \alpha_{k+1} = \ldots = \alpha_{|\alpha|} = 1\big\}$*–times, that* $B \in C_b(H_\gamma, HS(U_0,H_\beta))$ *is* $\max_{\alpha\in\mathbb{B}(\mathcal{A})}(2|\alpha|-2-\sum_{i=1}^{|\alpha|-1}\alpha_i)$*–times and that* $\varphi \in C_b(H_\gamma, V)$ *is* $\max_{\alpha\in\mathbb{B}(\mathcal{A})}(2|\alpha|-\sum_{i=1}^{|\alpha|}\alpha_i)$*–times continuously Fréchet differentiable with globally bounded Fréchet derivatives. Moreover, the boundedness assumptions on* F, B *and* φ *and its derivatives can be reduced if* $p \in [2,\infty)$ *in Assumption 4 is assumed to be sufficiently large.*

In the next step Theorem 3 is illustrated with two possible examples. First, in the case of the hierarchical set $\mathcal{A} = \{\emptyset\}$, equation (110) reduces to (109), i.e., we have

$$\varphi(X_t) = \underbrace{\varphi(e^{A(t-t_0)}X_{t_0})}_{\substack{\text{strong stochastic Taylor}\\ \text{approximation corresponding}\\ \text{to the hierarchical set } \mathcal{A}=\{\emptyset\}}} + \underbrace{\sum_{i=0}^{1}\int_{t_0}^{t}\big(L^{(i)}_{(t-s)}\varphi\big)(X_s)\, dW^i_s}_{\substack{\text{remainder term corresponding}\\ \text{to the hierarchical set } \mathcal{A}=\{\emptyset\}}} \tag{111}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \le t$ and all $\varphi \in C_b^\infty(H_\gamma, V)$. Second, in the case of the hierarchical set $\mathcal{A} = \{\emptyset, (1)\}$, equation (110) simplifies to

$$\begin{aligned}\varphi(X_t) = &\underbrace{\varphi(e^{A(t-t_0)}X_{t_0}) + \int_{t_0}^{t}\varphi'(e^{A(t-t_0)}X_{t_0})\, e^{A(t-s)}B(e^{A(s-t_0)}X_{t_0})\, dW_s}_{\text{strong stochastic Taylor approximation corresponding to } \mathcal{A}=\{\emptyset,(1)\}} \\ &\underbrace{\begin{aligned}&+ \int_{t_0}^{t}\big(L^{(0)}_{(t-s)}\varphi\big)(X_s)\, ds + \int_{t_0}^{t}\int_{t_0}^{s}\big(L^{(0)}_{(s-u)}L^{(1)}_{(t-s)}\varphi\big)(X_u)\, du\, dW_s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s}\big(L^{(1)}_{(s-u)}L^{(1)}_{(t-s)}\varphi\big)(X_u)\, dW_u\, dW_s\end{aligned}}_{\text{remainder term corresponding to } \mathcal{A}=\{\emptyset,(1)\}}\end{aligned} \tag{112}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \le t$ and all $\varphi \in C_b^\infty(H_\gamma, V)$. After having presented strong stochastic Taylor expansions in Theorem 3, we now formula the corresponding weak stochastic Taylor expansions based on the mild Itô formula (109).

Theorem 4 (Weak stochastic Taylor expansions)**.** *Assume that the setting in Section 3.1 is fulfilled, let* $n \in \mathbb{N}$*, assume* $F \in C_b^{(2n-2)}(H_\gamma, H_\alpha)$*, assume* $B \in C_b^{(2n-2)}(H_\gamma, HS(U_0, H_\beta))$ *and let* $\varphi \in C_b^{2n}(H_\gamma, V)$*. Then*

$$\begin{aligned}
&\mathbb{E}\big[\varphi(X_t)\big] = \mathbb{E}\big[\varphi\big(e^{A(t-t_0)} X_{t_0}\big)\big] \\
&+ \sum_{k=1}^{n-1} \int_{t_0}^{t} \int_{t_0}^{s_k} \dots \int_{t_0}^{s_2} \mathbb{E}\Big[\big(L^{(0)}_{(s_2-s_1)} \dots L^{(0)}_{(s_k-s_{k-1})} L^{(0)}_{(t-s_k)} \varphi\big)\Big(e^{A(s_1-t_0)} X_{t_0}\Big)\Big] \, ds_1 \, ds_2 \dots ds_k \\
&+ \int_{t_0}^{t} \int_{t_0}^{s_n} \dots \int_{t_0}^{s_2} \mathbb{E}\Big[\big(L^{(0)}_{(s_2-s_1)} \dots L^{(0)}_{(s_n-s_{n-1})} L^{(0)}_{(t-s_n)} \varphi\big)(X_{s_1})\Big] \, ds_1 \, ds_2 \dots ds_n
\end{aligned} \tag{113}$$

for all $t_0, t \in [0, T]$ *with* $t_0 \leq t$*.*

Proof of Theorem 4. Equation (113) immediately follows by taking expectations on both sides of equation (110) with the hierarchical set $\mathcal{A} = \{\alpha \in \mathcal{M} \colon |\alpha| \leq n, \sum_{i=1}^{|\alpha|} \alpha_i = 0\}$. □

Using definition (69), the weak stochastic Taylor expansions in Theorem 4 can also be written in the following form.

Corollary 9. *Assume that the setting in Section 3.1 is fulfilled, let* $n \in \mathbb{N}$*, assume* $F \in C_b^{(2n-2)}(H_\gamma, H_\alpha)$*, assume* $B \in C_b^{(2n-2)}(H_\gamma, HS(U_0, H_\beta))$ *and let* $\varphi \in C_b^{2n}(H_\gamma, V)$*. Then*

$$\begin{aligned}
\big(P_t \varphi\big)(x) &= \big(K_t \varphi\big)(x) \\
&+ \sum_{k=1}^{n-1} \int_0^t \int_0^{s_k} \dots \int_0^{s_2} \big(K_{s_1} L^{(0)}_{(s_2-s_1)} \dots L^{(0)}_{(s_k-s_{k-1})} L^{(0)}_{(t-s_k)} \varphi\big)(x) \, ds_1 \, ds_2 \dots ds_k \\
&+ \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \big(P_{s_1} L^{(0)}_{(s_2-s_1)} \dots L^{(0)}_{(s_n-s_{n-1})} L^{(0)}_{(t-s_n)} \varphi\big)(x) \, ds_1 \, ds_2 \dots ds_n
\end{aligned} \tag{114}$$

for all $x \in H_\gamma$ *and all* $t \in [0, \infty)$*.*

3.2.5 Further mild Itô formulas for solutions of SPDEs

This subsection presents two slightly different variants (Corollary 10 and Proposition 4) of the mild Itô formula in Corollary 2. Both variants assume that the test function φ in Corollary 2 fulfills additional regularity. The first variant (see Corollary 10 below) is a direct consequence of Corollary 2.

Corollary 10 (Another - somehow mild - Itô type formula for solutions of SPDEs)**.** *Assume that the setting in Section 3.1 is fulfilled. Then*

$$\mathbb{P}\bigg[\int_{t_0}^{t} \big\|\varphi'\big((I + e^{A(t-t_0)} - e^{A(t-s)}) X_{t_0}\big)\big\|_{L(H_r, V)} \big\|A \, e^{A(t-s)} \, X_{t_0}\big\|_{H_r} \, ds < \infty\bigg] = 1 \tag{115}$$

and

$$\begin{aligned}
\varphi(X_t) = \varphi(X_{t_0}) &+ \int_{t_0}^{t} \varphi'\big((I + e^{A(t-t_0)} - e^{A(t-s)}) X_{t_0}\big) A \, e^{A(t-s)} \, X_{t_0} \, ds \\
&+ \int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s) \, e^{A(t-s)} F(X_s) \, ds + \int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s) \, e^{A(t-s)} B(X_s) \, dW_s \\
&+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi''(e^{A(t-s)} X_s)\big(e^{A(t-s)} B(X_s) g_j, e^{A(t-s)} B(X_s) g_j\big) \, ds
\end{aligned} \tag{116}$$

$\mathbb{P}$*-a.s. for all* $t_0, t \in [0, T]$ *with* $t_0 \leq t$*,* $\varphi \in C^2(H_r, V)$ *and all* $r \in (-\infty, \gamma)$*.*

Proof of Corollary 10. Let $r \in (-\infty, \gamma)$ be a fixed real number and define a family $\bar{X}^{t_0,t} \colon [t_0, t] \times \Omega \to H_r$, $(t_0, t) \in \angle$, of adapted stochastic processes with continuous sample paths by

$$\begin{aligned}
\bar{X}^{t_0,t}_u &:= X_{t_0} + \int_{t_0}^{u} A \, e^{A(t-s)} X_{t_0} \, ds = X_{t_0} + e^{A(t-u)} \Big(e^{A(u-t_0)} - I\Big) X_{t_0} \\
&= \Big(I + e^{A(t-t_0)} - e^{A(t-u)}\Big) X_{t_0}
\end{aligned} \tag{117}$$

for all $u \in [t_0, t]$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. The fundamental theorem of calculus then implies

$$\begin{aligned} \varphi(e^{A(t-t_0)} X_{t_0}) &= \varphi\bigg(X_{t_0} + \int_{t_0}^t A\, e^{A(t-s)} X_{t_0}\, ds\bigg) = \varphi(\bar{X}_t^{t_0,t}) \\ &= \varphi(X_{t_0}) + \int_{t_0}^t \varphi'(\bar{X}_s^{t_0,t})\, A\, e^{A(t-s)}\, X_{t_0}\, ds \end{aligned} \tag{118}$$

for all $t_0, t \in [0, T]$ with $t_0 \leq t$ and all $\varphi \in C^2(H_r, V)$. Combining (118) and Corollary 10 then completes the proof of Corollary 2. □

Observe that equations (50)–(52) and equation (115) ensure that all deterministic and stochastic integrals in (116) are well defined.

Proposition 4 (A further - somehow mild - Itô type formula for solutions of SPDEs)**.** *Assume that the setting in Section 3.1 is fulfilled. Then*

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi'\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big)\big\|_{L(H_r,V)} \big\|A\, e^{A(t-s)} X_{t_0}\big\|_{H_r}\, ds < \infty\bigg] = 1, \tag{119}$$

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi'\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big)\big\|_{L(H_r,V)} \big\|e^{A(t-s)} F(X_s)\big\|_{H_r}\, ds < \infty\bigg] = 1, \tag{120}$$

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi'\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big)\, e^{A(t-s)} B(X_s)\big\|^2_{HS(U_0,V)}\, ds < \infty\bigg] = 1, \tag{121}$$

$$\mathbb{P}\bigg[\int_{t_0}^t \big\|\varphi''\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big)\big\|_{L^{(2)}(H_r,V)} \big\|e^{A(t-s)} B(X_s)\big\|^2_{HS(U_0,H_r)}\, ds < \infty\bigg] = 1 \tag{122}$$

and

$$\begin{aligned} \varphi(X_t) = \varphi(X_{t_0}) &+ \int_{t_0}^t \varphi'\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big) \big[A\, e^{A(t-s)} X_{t_0} + e^{A(t-s)} F(X_s)\big]\, ds \\ &+ \int_{t_0}^t \varphi'\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big)\, e^{A(t-s)} B(X_s)\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^t \varphi''\big(X_{t_0} + e^{A(t-s)}(X_s - X_{t_0})\big) \big(e^{A(t-s)} B(X_s) g_j, e^{A(t-s)} B(X_s) g_j\big)\, ds \end{aligned} \tag{123}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0, T]$ with $t_0 \leq t$, $\varphi \in C^2(H_r, V)$ and all $r \in (-\infty, \gamma)$.

Proof of Proposition 4. First, observe that the well known identity $e^{At} v = v + \int_0^t A\, e^{As} v\, ds$ for all $v \in H_\gamma$ and all $t \in [0, \infty)$ shows

$$X_t = X_{t_0} + \int_{t_0}^t \Big[A\, e^{A(t-s)} X_{t_0} + e^{A(t-s)} F(X_s)\Big]\, ds + \int_{t_0}^t e^{A(t-s)} B(X_s)\, dW_s \tag{124}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0, T]$ with $t_0 \leq t$. In the next step let $r \in (-\infty, \gamma)$ be a fixed real number and let $\bar{X}^{t_0,t} \colon [t_0, t] \times \Omega \to H_r$, $(t_0, t) \in \angle$, be a family of adapted stochastic processes with continuous sample paths given by

$$\begin{aligned} \bar{X}_u^{t_0,t} &= X_{t_0} + \int_{t_0}^u \Big[A\, e^{A(t-s)} X_{t_0} + e^{A(t-s)} F(X_s)\Big]\, ds + \int_{t_0}^u e^{A(t-s)} B(X_s)\, dW_s \\ &= X_{t_0} + e^{A(t-t_0)} X_{t_0} - e^{A(t-u)} X_{t_0} + \int_{t_0}^u e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^u e^{A(t-s)} B(X_s)\, dW_s \end{aligned} \tag{125}$$

$\mathbb{P}$-a.s. for all $u \in [t_0, t]$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$ (see also (35) above). The standard Itô formula in infinite dimensions (see Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9]) then gives

$$\begin{aligned} \varphi(\bar{X}_u^{t_0,t}) = \varphi(\bar{X}_{t_0}^{t_0,t}) &+ \int_{t_0}^u \varphi'(\bar{X}_s^{t_0,t}) \Big[A\, e^{A(t-s)}\, X_{t_0} + e^{A(t-s)} F(X_s)\Big]\, ds \\ &+ \int_{t_0}^u \varphi'(\bar{X}_s^{t_0,t})\, e^{A(t-s)} B(X_s)\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^u \varphi''(\bar{X}_s^{t_0,t}) \Big(e^{A(t-s)} B(X_s) g_j, e^{A(t-s)} B(X_s) g_j\Big)\, ds \end{aligned} \tag{126}$$

$\mathbb{P}$-a.s. for all $u \in [t_0, t]$, $t_0, t \in [0, T]$ with $t_0 \le t$ and all $\varphi \in C^2(H_r, V)$. This, in particular, shows

$$\begin{aligned} \varphi(\bar{X}_t^{t_0,t}) = \varphi(X_t) = \varphi(X_{t_0}) &+ \int_{t_0}^t \varphi'(\bar{X}_s^{t_0,t}) \left[A\, e^{A(t-s)}\, X_{t_0} + e^{A(t-s)} F(X_s) \right] ds \\ &+ \int_{t_0}^t \varphi'(\bar{X}_s^{t_0,t})\, e^{A(t-s)} B(X_s)\, dW_s \\ &+ \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^t \varphi''(\bar{X}_s^{t_0,t}) \left(e^{A(t-s)} B(X_s) g_j, e^{A(t-s)} B(X_s) g_j \right) ds \end{aligned} \tag{127}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0, T]$ with $t_0 \le t$ and all $\varphi \in C^2(H_r, V)$ (see also (38) above). Putting the identity

$$\begin{aligned} \bar{X}_s^{t_0,t} &= X_{t_0} + \int_{t_0}^s \left[A\, e^{A(t-u)} X_{t_0} + e^{A(t-u)} F(X_u) \right] ds + \int_{t_0}^s e^{A(t-u)} B(X_u)\, dW_u \\ &= X_{t_0} + e^{A(t-s)} \int_{t_0}^s \left[A\, e^{A(s-u)} X_{t_0} + e^{A(s-u)} F(X_u) \right] ds \\ &\quad + e^{A(t-s)} \int_{t_0}^s e^{A(s-u)} B(X_u)\, dW_u = X_{t_0} + e^{A(t-s)} \left(X_s - X_{t_0} \right) \end{aligned} \tag{128}$$

$\mathbb{P}$-a.s. for all $s \in [t_0, T]$ and all $t_0, t \in [0, T]$ with $t_0 \le t$ (see also (39) above) into (127) finally shows (123). The proof of Proposition 4 is thus completed. □

Note that equations (119)–(122) imply that all deterministic and stochastic integrals in (123) are well defined. Finally, observe that the Itô type formulas in Corollary 10 and Proposition 4 can be generalized to the more general case of mild Itô processes (or mild semimartingales, cf. Remark 1) if additional assumptions on the semigroup are fulfilled.

3.3 Numerical approximations processes for SPDEs

This subsection demonstrates how different types of numerical approximation processes for SPDEs can be formulated as mild Itô processes. To this end the following notation is used. Let $\lfloor \cdot \rfloor_N \colon [0, T] \to [0, T]$, $N \in \mathbb{N}$, be a sequence of mappings given by

$$\lfloor t \rfloor_N := \max\!\Big(s \in \Big\{ 0, \tfrac{T}{N}, \tfrac{2T}{N}, \dots, \tfrac{(N-1)T}{N}, T \Big\} : s \le t \Big) \tag{129}$$

for all $t \in [0, T]$ and all $N \in \mathbb{N}$. Both Euler (see Subsection 3.3.1) and Milstein (see Subsection 3.3.2) type approximations for SPDEs are formulated as mild Itô processes. We begin with Euler type approximations for SPDEs in Subsection 3.3.1.

3.3.1 Euler type approximations for SPDEs

It is illustrated here how exponential Euler approximations, accelerated exponential Euler approximations, linear implicit Euler approximations and linear implicit Crank-Nicolson approximations can be formulated as mild Itô processes.

Exponential Euler approximations for SPDEs Let $Y^N \colon [0, T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by

$$\begin{aligned} Y_t^N &= e^{At}\, \xi + \int_0^t e^{A(t - \lfloor s \rfloor_N)}\, F\big(Y_{\lfloor s \rfloor_N}^N\big)\, ds + \int_0^t e^{A(t - \lfloor s \rfloor_N)}\, B\big(Y_{\lfloor s \rfloor_N}^N\big)\, dW_s \\ &= e^{At}\, \xi + \int_0^t e^{A(t-s)}\, e^{A(s - \lfloor s \rfloor_N)}\, F\big(Y_{\lfloor s \rfloor_N}^N\big)\, ds + \int_0^t e^{A(t-s)}\, e^{A(s - \lfloor s \rfloor_N)}\, B\big(Y_{\lfloor s \rfloor_N}^N\big)\, dW_s \end{aligned} \tag{130}$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$ and all $N \in \mathbb{N}$. Observe that for each $N \in \mathbb{N}$ the stochastic process $Y^N \colon [0, T] \times \Omega \to H_\gamma$ is a mild Itô process with semigroup $e^{A(t_2 - t_1)} \in L(H_{\min(\alpha,\beta,\gamma)}, H_\gamma)$, $(t_1, t_2) \in \angle$, with mild drift

$$e^{A(t - \lfloor t \rfloor_N)}\, F\big(Y_{\lfloor t \rfloor_N}^N\big), \qquad t \in [0, T], \tag{131}$$

and with mild diffusion

$$e^{A(t - \lfloor t \rfloor_N)}\, B\big(Y_{\lfloor t \rfloor_N}^N\big), \qquad t \in [0, T]. \tag{132}$$

Proposition 1 hence shows

$$Y_t^N = e^{A(t-t_0)}\, Y_{t_0}^N + \int_{t_0}^t e^{A(t-\lfloor s\rfloor_N)}\, F\big(Y_{\lfloor s\rfloor_N}^N\big)\, ds + \int_{t_0}^t e^{A(t-\lfloor s\rfloor_N)}\, B\big(Y_{\lfloor s\rfloor_N}^N\big)\, dW_s \tag{133}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \leq t$ and from (133) we conclude

$$\begin{aligned} Y_{\frac{(n+1)T}{N}}^N &= e^{A\frac{T}{N}}\, Y_{\frac{nT}{N}}^N + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} e^{A\frac{T}{N}}\, F\big(Y_{\frac{nT}{N}}^N\big)\, ds + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} e^{A\frac{T}{N}}\, B\big(Y_{\frac{nT}{N}}^N\big)\, dW_s \\ &= e^{A\frac{T}{N}} \left(Y_{\frac{nT}{N}}^N + \frac{T}{N} \cdot F\big(Y_{\frac{nT}{N}}^N\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(Y_{\frac{nT}{N}}^N\big)\, dW_s \right) \end{aligned} \tag{134}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. The mild Itô processes Y^N, $N \in \mathbb{N}$, are thus nothing else but appropriate time continuous interpolations of exponential Euler approximations (a.k.a. splitting-up approximations or exponential integrator approximations; see, e.g., [37, 74, 17, 77] and the references therein). Note that the mild drift (131) and the mild diffusion (132) of the exponential Euler approximations (134) contain the *correction term* $e^{A(t-\lfloor t\rfloor_N)}$, $t \in [0,T]$, when compared to the mild drift (48) and the mild diffusion (49) of the exact solution of the SPDE (47).

Accelerated exponential Euler approximations for SPDEs This paragraph demonstrates that accelerated exponential Euler approximations can be written as mild Itô processes. Let $\tilde{Y}^N \colon [0,T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by

$$\tilde{Y}_t^N = e^{At}\, \xi + \int_0^t e^{A(t-s)}\, F\big(\tilde{Y}_{\lfloor s\rfloor_N}^N\big)\, ds + \int_0^t e^{A(t-s)}\, B\big(\tilde{Y}_{\lfloor s\rfloor_N}^N\big)\, dW_s \tag{135}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and all $N \in \mathbb{N}$. Note that for each $N \in \mathbb{N}$ the stochastic process $\tilde{Y}^N \colon [0,T] \times \Omega \to H_\gamma$ is a mild Itô process with semigroup $e^{A(t_2-t_1)} \in L(H_{\min(\alpha,\beta,\gamma)}, H_\gamma)$, $(t_1,t_2) \in \angle$, with mild drift

$$F\big(\tilde{Y}_{\lfloor t\rfloor_N}^N\big), \qquad t \in [0,T], \tag{136}$$

and with mild diffusion

$$B\big(\tilde{Y}_{\lfloor t\rfloor_N}^N\big), \qquad t \in [0,T]. \tag{137}$$

Proposition 1 therefore gives

$$\tilde{Y}_t^N = e^{A(t-t_0)}\, \tilde{Y}_{t_0}^N + \int_{t_0}^t e^{A(t-s)}\, F\big(\tilde{Y}_{\lfloor s\rfloor_N}^N\big)\, ds + \int_{t_0}^t e^{A(t-s)}\, B\big(\tilde{Y}_{\lfloor s\rfloor_N}^N\big)\, dW_s \tag{138}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \leq t$ and this implies

$$\tilde{Y}_{\frac{(n+1)T}{N}}^N = e^{A\frac{T}{N}}\, \tilde{Y}_{\frac{nT}{N}}^N + \left(\int_0^{\frac{T}{N}} e^{As}\, ds \right) F\big(\tilde{Y}_{\frac{nT}{N}}^N\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} e^{A\left(\frac{(n+1)T}{N}-s\right)}\, B\big(\tilde{Y}_{\frac{nT}{N}}^N\big)\, dW_s \tag{139}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. In particular, in the case of additive noise, i.e., $B(v) = B(0)$ for all $v \in H_\gamma$, equation (139) reduces to

$$\tilde{Y}_{\frac{(n+1)T}{N}}^N = e^{A\frac{T}{N}}\, \tilde{Y}_{\frac{nT}{N}}^N + \left(\int_0^{\frac{T}{N}} e^{As}\, ds \right) F\big(\tilde{Y}_{\frac{nT}{N}}^N\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} e^{A\left(\frac{(n+1)T}{N}-s\right)}\, B(0)\, dW_s \tag{140}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. The mild Itô processes $\tilde{Y}^N$, $N \in \mathbb{N}$, are thus nothing else but appropriate time continuous interpolations of the numerical approximations in [53] in the case of additive noise (see (3.3) in [53]) and in [49] in the general case (see (21) and (50) in [49]).

Note that the mild drift (136) and the diffusion (137) of the approximation processes (135) do not contain the correction term $e^{A(t-\lfloor t\rfloor_N)}$, $t \in [0,T]$, in mild drift (131) and the mild diffusion (132) of the exponential Euler approximations (130). The approximation processes (135) thus seem to be more close to the exact solution (47) than the exponential Euler approximations (130) (compare the mild drifts (136), (131), (48) and the mild diffusions (137), (132), (49)). Indeed, under suitable assumptions, it has been shown (see [53, 55] for details) that $\tilde{Y}^N$, $N \in \mathbb{N}$, converges to X significantly faster than Y^N, $N \in \mathbb{N}$. This motivates to call approximations of the form (139) and (140) as *accelerated exponential Euler approximations.* The crucial point in the accelerated exponential Euler approximations is that they contain the stochastic integrals

$$\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} e^{A\left(\frac{(n+1)T}{N}-s\right)} B\big(\tilde{Y}^N_{\frac{nT}{N}}\big)\, dW_s \tag{141}$$

for $n \in \{0, 1, \ldots, N-1\}$ and $N \in \mathbb{N}$ in the scheme instead of simply increments of driving noise process. This enables them to converge, under suitable assumptions, significantly faster to X than schemes using only increments of the driving noise process such as (134). In addition, in the case of additive noise, the stochastic integrals (141) in (140) depends linearly on the cylindrical Wiener process W_t, $t \in [0, T]$ and are easy to simulate. Therefore, the accelerated exponential Euler approximations (140) can in the case of additive noise be simulated quite efficiently (see Section 3 in [53] and, particularly, see Figure 2 in [55] for details). Further investigations and related results on approximation methods that make use of stochastic integrals of the form (141) can, e.g., be found in [53, 52, 75, 76, 28, 55, 50, 79, 108].

Linear implicit Euler approximations for SPDEs Next it is shown that linear implicit Euler approximations can be formulated as mild Itô processes. For this we assume $\eta = 0$ in the following in order to avoid trivial complications. Moreover, let $\bar{S}^N \colon \angle \to L(H_\gamma)$, $N \in \mathbb{N}$, be a sequence of mappings given by

$$\bar{S}^N_{t_1,t_2} := \Big(I - A\,(t_1 - \lfloor t_1 \rfloor_N)\Big)\Big(I - A\,(t_2 - \lfloor t_2 \rfloor_N)\Big)^{-1}\Big(I - \tfrac{T}{N}A\Big)^{(\lfloor t_1 \rfloor_N - \lfloor t_2 \rfloor_N)\frac{N}{T}} \tag{142}$$

for all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and all $N \in \mathbb{N}$. Moreover, let $\bar{Y}^N \colon [0, T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by $\bar{Y}^N_0 = \xi$ and

$$\begin{aligned} \bar{Y}^N_t &= \bar{S}^N_{0,t}\, \xi + \int_0^t \bar{S}^N_{\lfloor s \rfloor_N, t}\, F\big(\bar{Y}^N_{\lfloor s \rfloor_N}\big)\, ds + \int_0^t \bar{S}^N_{\lfloor s \rfloor_N, t}\, B\big(\bar{Y}^N_{\lfloor s \rfloor_N}\big)\, dW_s \\ &= \bar{S}^N_{0,t}\, \xi + \int_0^t \bar{S}^N_{s,t}\, \big(I - A\,(s - \lfloor s \rfloor_N)\big)^{-1} F\big(\bar{Y}^N_{\lfloor s \rfloor_N}\big)\, ds \\ &\quad + \int_0^t \bar{S}^N_{s,t}\, \big(I - A\,(s - \lfloor s \rfloor_N)\big)^{-1} B\big(\bar{Y}^N_{\lfloor s \rfloor_N}\big)\, dW_s \end{aligned} \tag{143}$$

$\mathbb{P}$-a.s. for all $t \in (0, T]$ and all $N \in \mathbb{N}$. Observe that for each $N \in \mathbb{N}$ the stochastic process $\bar{Y}^N \colon [0, T] \times \Omega \to H_\gamma$ is a mild Itô process with semigroup $\bar{S}^N$, with mild drift

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \big(I - A\,(t - \lfloor t \rfloor_N)\big)^{-1} F\big(\bar{Y}^N_{\lfloor t \rfloor_N}\big), \qquad t \in [0, T], \tag{144}$$

and with mild diffusion

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \big(I - A\,(t - \lfloor t \rfloor_N)\big)^{-1} B\big(\bar{Y}^N_{\lfloor t \rfloor_N}\big), \qquad t \in [0, T]. \tag{145}$$

Proposition 1 therefore implies

$$\bar{Y}^N_{\frac{(n+1)T}{N}} = \Big(I - \tfrac{T}{N}A\Big)^{-1}\left(\bar{Y}^N_{\frac{nT}{N}} + \frac{T}{N} \cdot F\big(\bar{Y}^N_{\frac{nT}{N}}\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(\bar{Y}^N_{\frac{nT}{N}}\big)\, dW_s\right) \tag{146}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. This shows that the stochastic processes $\bar{Y}^N$, $N \in \mathbb{N}$, are nothing else but appropriate time continuous interpolations of linear implicit Euler approximations (see, e.g., [36, 41, 42, 106, 82, 81, 64, 18] and the references therein) for the SPDE (47). Note that the semigroups (142) of the linear implicit Euler approximations (143) depend explicitly on both variables t_1 and t_2 with $0 \le t_1 \le t_2 \le T$ instead of on the difference $t_2 - t_1$ only although the semigroup e^{At}, $t \in [0, T]$, of the underlying SPDE (47) depends on one variable only.

Linear implicit Crank-Nicolson approximations for SPDEs Finally, in this paragraph it is demonstrated that linear implicit Crank-Nicolson approximations can be formulated as mild Itô processes too. As in the case of the linear implicit Euler approximations we assume $\eta = 0$ in the following in order to avoid trivial complications. Let $\hat{S}^N \colon \angle \to L(H_\gamma)$, $N \in \mathbb{N}$, be a sequence of mappings given by

$$\hat{S}^N_{t_1,t_2} := \Big(I - A\tfrac{(t_1 - \lfloor t_1 \rfloor_N)}{2}\Big)\Big(I - A\tfrac{(t_2 - \lfloor t_2 \rfloor_N)}{2}\Big)^{-1}\Big(I - \tfrac{T}{2N}A\Big)^{(\lfloor t_1 \rfloor_N - \lfloor t_2 \rfloor_N)\frac{N}{T}} \tag{147}$$

for all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and all $N \in \mathbb{N}$. Moreover, let $\hat{Y}^N\colon [0,T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by $\hat{Y}_0^N = \xi$ and

$$\begin{aligned} \hat{Y}_t^N &= \hat{S}_{0,t}^N\, \xi + \int_0^t \hat{S}_{\lfloor s \rfloor_N, t}^N \left(\tfrac{1}{2} A\, \hat{Y}_{\lfloor s \rfloor_N}^N + F\big(\hat{Y}_{\lfloor s \rfloor_N}^N\big) \right) ds + \int_0^t \hat{S}_{\lfloor s \rfloor_N, t}^N\, B\big(\hat{Y}_{\lfloor s \rfloor_N}^N\big)\, dW_s \\ &= \hat{S}_{0,t}^N\, \xi + \int_0^t \hat{S}_{s,t}^N \left(I - A \tfrac{(s - \lfloor s \rfloor_N)}{2} \right)^{-1} \left(\tfrac{1}{2} A\, \hat{Y}_{\lfloor s \rfloor_N}^N + F\big(\hat{Y}_{\lfloor s \rfloor_N}^N\big) \right) ds \\ &\quad + \int_0^t \hat{S}_{s,t}^N \left(I - A \tfrac{(s - \lfloor s \rfloor_N)}{2} \right)^{-1} B\big(\hat{Y}_{\lfloor s \rfloor_N}^N\big)\, dW_s \end{aligned} \tag{148}$$

$\mathbb{P}$-a.s. for all $t \in (0,T]$ and all $N \in \mathbb{N}$. Observe for every $N \in \mathbb{N}$ that the stochastic process $\hat{Y}^N\colon [0,T] \times \Omega \to H_\gamma$ is a mild Itô process with semigroup $\hat{S}^N$, with mild drift

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N) \left(I - A \tfrac{(t - \lfloor t \rfloor_N)}{2} \right)^{-1} \left(\tfrac{1}{2} A\, \hat{Y}_{\lfloor t \rfloor_N}^N + F\big(\hat{Y}_{\lfloor t \rfloor_N}^N\big) \right),\ t \in [0,T], \tag{149}$$

and with mild diffusion

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N) \left(I - A \tfrac{(t - \lfloor t \rfloor_N)}{2} \right)^{-1} B\big(\hat{Y}_{\lfloor t \rfloor_N}^N\big),\ t \in [0,T]. \tag{150}$$

Proposition 1 hence gives

$$\hat{Y}_{\frac{(n+1)T}{N}}^N = \left(I - \tfrac{T}{2N} A \right)^{-1} \left(\left(I + \tfrac{T}{2N} A \right) \hat{Y}_{\frac{nT}{N}}^N + \frac{T}{N} \cdot F\big(\hat{Y}_{\frac{nT}{N}}^N\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(\hat{Y}_{\frac{nT}{N}}^N\big)\, dW_s \right) \tag{151}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. This shows that the stochastic processes $\hat{Y}^N$, $N \in \mathbb{N}$, are nothing else but appropriate time continuous interpolations of linear implicit Crank-Nicolson approximations (see, e.g., [97, 106, 6] and the references therein) for the SPDE (47).

3.3.2 Milstein type approximations for SPDEs

The stochastic Taylor expansions in Subsection 3.2.4 can be used to derive higher order numerical approximation methods for SPDEs. In the sequel this is illustrated for Milstein type approximations for SPDEs (see [34, 60, 80, 71, 56, 3, 2, 107, 5]). For these derivations we assume that the diffusion term $B\colon H_\gamma \to HS(U_0, H_\beta)$ is twice continuously Fréchet differentiable with globally bounded derivatives and that $\beta = \gamma$. First note, in view of the mild Itô formula (58), that equation (47) can also be written as

$$X_t = e^{A(t-t_0)} X_{t_0} + \int_{t_0}^t \big(L_{(t-s)}^{(0)} \mathrm{id}\big)(X_s)\, ds + \int_{t_0}^t \big(L_{(t-s)}^{(1)} \mathrm{id}\big)(X_s)\, dW_s \tag{152}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \leq t$ where $\mathrm{id} = \mathrm{id}_{H_\gamma}\colon H_\gamma \to H_\gamma$ is the identity on H_γ. Next the mild Itô formula (58) is applied to the test function $\big(L_{(t-s)}^{(1)} \mathrm{id}\big)(x) \in HS(U_0, H_\beta)$, $x \in H_\gamma$, to obtain

$$\begin{aligned} &\big(L_{(t-s)}^{(1)} \mathrm{id}\big)(X_s) \\ &= \big(L_{(t-s)}^{(1)} \mathrm{id}\big)\big(e^{A(s-t_0)} X_{t_0}\big) + \int_{t_0}^s \big(L_{(s-u)}^{(0)} L_{(t-s)}^{(1)} \mathrm{id}\big)(X_u)\, du + \int_{t_0}^s \big(L_{(s-u)}^{(1)} L_{(t-s)}^{(1)} \mathrm{id}\big)(X_u)\, dW_u \end{aligned} \tag{153}$$

$\mathbb{P}$-a.s. for all $t_0, s, t \in [0,T]$ with $t_0 \leq s \leq t$. Putting (153) into (152) then gives

$$\begin{aligned} X_t &= e^{A(t-t_0)} X_{t_0} + \int_{t_0}^t \big(L_{(t-s)}^{(0)} \mathrm{id}\big)(X_s)\, ds + \int_{t_0}^t \big(L_{(t-s)}^{(1)} \mathrm{id}\big)\big(e^{A(s-t_0)} X_{t_0}\big)\, dW_s \\ &\quad + \int_{t_0}^t \int_{t_0}^s \big(L_{(s-u)}^{(0)} L_{(t-s)}^{(1)} \mathrm{id}\big)(X_u)\, du\, dW_s + \int_{t_0}^t \int_{t_0}^s \big(L_{(s-u)}^{(1)} L_{(t-s)}^{(1)} \mathrm{id}\big)(X_u)\, dW_u\, dW_s \\ &= e^{A(t-t_0)} X_{t_0} + \int_{t_0}^t e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^t e^{A(t-s)} B\big(e^{A(s-t_0)} X_{t_0}\big)\, dW_s \\ &\quad + \int_{t_0}^t \int_{t_0}^s \big(L_{(s-u)}^{(0)} L_{(t-s)}^{(1)} \mathrm{id}\big)(X_u)\, du\, dW_s + \int_{t_0}^t \int_{t_0}^s e^{A(t-s)} B'\big(e^{A(s-u)} X_u\big)\, e^{A(s-u)} B(X_u)\, dW_u\, dW_s \end{aligned} \tag{154}$$

$\mathbb{P}$-a.s. for all $t_0, t \in [0,T]$ with $t_0 \leq t$. The identity (154) corresponds to the strong stochastic Taylor expansion in Theorem 3 which is described by the hierarchical set $\mathcal{A} = \{\emptyset, (1)\}$; see Subsection 3.2.4 for details. Using the approximations $e^{Ah} \approx I$ and $X_{t_0+h} \approx X_{t_0}$ for small $h \in [0,T]$ and omitting the integral $\int_{t_0}^{t}\int_{t_0}^{s}\big(L^{(0)}_{(s-u)}L^{(1)}_{(t-s)}\mathrm{id}\big)(X_u)\,du\,dW_s$ in (154) then results in the approximation

$$\begin{aligned} X_t &\approx e^{A(t-t_0)}X_{t_0} + \int_{t_0}^{t} e^{A(t-s)}F(X_{t_0})\,ds + \int_{t_0}^{t} e^{A(t-s)}B(X_{t_0})\,dW_s \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} e^{A(t-s)}B'(X_{t_0})\,e^{A(s-u)}B(X_{t_0})\,dW_u\,dW_s \\ &\approx e^{A(t-t_0)}\left(X_{t_0} + F(X_{t_0})\cdot(t-t_0) + \int_{t_0}^{t} B(X_{t_0})\,dW_s + \int_{t_0}^{t}\int_{t_0}^{s} B'(X_{t_0})\,B(X_{t_0})\,dW_u\,dW_s\right) \end{aligned} \tag{155}$$

for $t_0, t \in [0,T]$ with small $t - t_0 \geq 0$. The approximation (155) can then be used to define exponential Milstein approximations for SPDEs (see equations (156) and (159) below for details).

In the following it is demonstrated how different types of Milstein approximations for SPDEs can be formulated as mild Itô processes. To this end we assume in the remainder of this subsection that $B\colon H_\gamma \to HS(U_0, H_\beta)$ in Assumption 3 is once continuously Fréchet differentiable and that $\beta = \gamma$.

Exponential Milstein approximations for SPDEs This paragraph formulates exponential Milstein approximations as mild Itô processes. To be more precise, let $Z^N\colon [0,T]\times\Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by

$$\begin{aligned} Z^N_t &= e^{At}\,\xi + \int_0^t e^{A(t-\lfloor s\rfloor_N)}\,F\big(Z^N_{\lfloor s\rfloor_N}\big)\,ds + \int_0^t e^{A(t-\lfloor s\rfloor_N)}\,B\big(Z^N_{\lfloor s\rfloor_N}\big)\,dW_s \\ &\quad + \int_0^t e^{A(t-\lfloor s\rfloor_N)}\,B'\big(Z^N_{\lfloor s\rfloor_N}\big)\left(\int_{\lfloor s\rfloor_N}^{s} B\big(Z^N_{\lfloor s\rfloor_N}\big)\,dW_u\right)dW_s \end{aligned} \tag{156}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and all $N \in \mathbb{N}$. Note for every $N \in \mathbb{N}$ that the stochastic process $Z^N\colon [0,T]\times\Omega \to H_\gamma$ is a mild Itô processes with semigroup $e^{A(t_2-t_1)} \in L(H_{\min(\alpha,\beta,\gamma)}, H_\gamma)$, $(t_1,t_2) \in \angle$, with mild drift

$$e^{A(t-\lfloor t\rfloor_N)}\,F\big(Z^N_{\lfloor t\rfloor_N}\big),\; t \in [0,T], \tag{157}$$

and with mild diffusion

$$e^{A(t-\lfloor t\rfloor_N)}\left(B\big(Z^N_{\lfloor t\rfloor_N}\big) + B'\big(Z^N_{\lfloor t\rfloor_N}\big)\left(\int_{\lfloor t\rfloor_N}^{t} B\big(Z^N_{\lfloor t\rfloor_N}\big)\,dW_s\right)\right),\; t \in [0,T]. \tag{158}$$

Proposition 1 hence yields

$$\begin{aligned} Z^N_{\frac{(n+1)T}{N}} &= e^{A\frac{T}{N}}\left(Z^N_{\frac{nT}{N}} + \frac{T}{N}\cdot F\big(Z^N_{\frac{nT}{N}}\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(Z^N_{\frac{nT}{N}}\big)\,dW_s \right.\\ &\quad \left. + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B'\big(Z^N_{\frac{nT}{N}}\big)\left(\int_{\frac{nT}{N}}^{s} B\big(Z^N_{\frac{nT}{N}}\big)\,dW_u\right)dW_s\right) \end{aligned} \tag{159}$$

$\mathbb{P}$-a.s. for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}$. The mild Itô processes Z^N, $N \in \mathbb{N}$, are thus nothing else but appropriate time continuous interpolations of exponential Milstein approximations (see [80, 56, 2]).

Linear implicit Euler-Milstein approximations for SPDEs In this paragraph linear implicit Euler-Milstein approximations are formulated as mild Itô processes. Let $\bar{Z}^N\colon [0,T]\times\Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by $\bar{Z}^N_0 = \xi$ and

$$\begin{aligned} \bar{Z}^N_t &= \bar{S}^N_{0,t}\,\xi + \int_0^t \bar{S}^N_{\lfloor s\rfloor_N,t}\,F\big(\bar{Z}^N_{\lfloor s\rfloor_N}\big)\,ds + \int_0^t \bar{S}^N_{\lfloor s\rfloor_N,t}\,B\big(\bar{Z}^N_{\lfloor s\rfloor_N}\big)\,dW_s \\ &\quad + \int_0^t \bar{S}^N_{\lfloor s\rfloor_N,t}\,B'\big(\bar{Z}^N_{\lfloor s\rfloor_N}\big)\left(\int_{\lfloor s\rfloor_N}^{s} B\big(\bar{Z}^N_{\lfloor s\rfloor_N}\big)\,dW_u\right)dW_s \end{aligned} \tag{160}$$

$\mathbb{P}$-a.s. for all $t \in (0,T]$ and all $N \in \mathbb{N}$ (see (142) for the definition of the mappings $\bar{S}^N \colon \angle \to L(H_\gamma)$, $N \in \mathbb{N}$). Observe for every $N \in \mathbb{N}$ that the stochastic process $\bar{Z}^N \colon [0,T] \times \Omega \to H_\gamma$ is a mild Itô process with semigroup $\bar{S}^N$, with mild drift

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \bar{S}^N_{\lfloor t \rfloor_N, t}\, F\big(\bar{Z}^N_{\lfloor t \rfloor_N}\big), \quad t \in [0,T], \tag{161}$$

and with mild diffusion

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \bar{S}^N_{\lfloor t \rfloor_N, t} \left(B\big(\bar{Z}^N_{\lfloor t \rfloor_N}\big) + B'\big(\bar{Z}^N_{\lfloor t \rfloor_N}\big)\bigg(\int_{\lfloor t \rfloor_N}^{t} B\big(\bar{Z}^N_{\lfloor t \rfloor_N}\big)\, dW_s \bigg) \right), \quad t \in [0,T]. \tag{162}$$

Proposition 1 hence gives

$$\begin{aligned} \bar{Z}^N_{\frac{(n+1)T}{N}} = \Big(I - \tfrac{T}{N} A\Big)^{-1} \Bigg(& \bar{Z}^N_{\frac{nT}{N}} + \frac{T}{N} \cdot F\big(\bar{Z}^N_{\frac{nT}{N}}\big) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(\bar{Z}^N_{\frac{nT}{N}}\big)\, dW_s \\ & + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B'\big(\bar{Z}^N_{\frac{nT}{N}}\big)\bigg(\int_{\frac{nT}{N}}^{s} B\big(\bar{Z}^N_{\frac{nT}{N}}\big)\, dW_u \bigg)\, dW_s \Bigg) \end{aligned} \tag{163}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. The mild Itô processes $\bar{Z}^N$, $N \in \mathbb{N}$, are thus nothing else but appropriate time continuous interpolations of linear implicit Euler-Milstein approximations (see [60, 80, 2, 5]).

Linear implicit Crank-Nicolson-Milstein Milstein approximations for SPDEs Finally, let $\hat{Z}^N \colon [0,T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}$, be a sequence of predictable stochastic processes given by $\hat{Z}^N_0 = \xi$ and

$$\begin{aligned} \hat{Z}^N_t = \hat{S}^N_{0,t}\, \xi + \int_0^t \hat{S}^N_{\lfloor s \rfloor_N, t} \Big(\tfrac{1}{2} A\, \hat{Z}^N_{\lfloor s \rfloor_N} + F\big(\hat{Z}^N_{\lfloor s \rfloor_N}\big) \Big)\, ds + \int_0^t \hat{S}^N_{\lfloor s \rfloor_N, t}\, B\big(\hat{Z}^N_{\lfloor s \rfloor_N}\big)\, dW_s \\ + \int_0^t \hat{S}^N_{\lfloor s \rfloor_N, t}\, B'\big(\hat{Z}^N_{\lfloor s \rfloor_N}\big)\bigg(\int_{\lfloor s \rfloor_N}^{s} B\big(\hat{Z}^N_{\lfloor s \rfloor_N}\big)\, dW_u \bigg)\, dW_s \end{aligned} \tag{164}$$

$\mathbb{P}$-a.s. for all $t \in (0,T]$ and all $N \in \mathbb{N}$ (see (147) for the definition the mappings $\hat{S}^N \colon \angle \to L(H_\gamma)$, $N \in \mathbb{N}$) and note for each $N \in \mathbb{N}$ that the stochastic process $\hat{Z}^N \colon [0,T] \times \Omega \to H_\gamma$ is a mild Itô processes with semigroup $\hat{S}^N$, with mild drift

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \hat{S}^N_{\lfloor t \rfloor_N, t} \Big(\tfrac{1}{2} A\, \hat{Y}^N_{\lfloor t \rfloor_N} + F\big(\hat{Y}^N_{\lfloor t \rfloor_N}\big) \Big), \quad t \in [0,T], \tag{165}$$

and with mild diffusion

$$\mathbb{1}_{(0,\infty)}(t - \lfloor t \rfloor_N)\, \hat{S}^N_{\lfloor t \rfloor_N, t} \left(B\big(\hat{Z}^N_{\lfloor t \rfloor_N}\big) + B'\big(\hat{Z}^N_{\lfloor t \rfloor_N}\big)\bigg(\int_{\lfloor t \rfloor_N}^{t} B\big(\hat{Z}^N_{\lfloor t \rfloor_N}\big)\, dW_s \bigg) \right), \quad t \in [0,T]. \tag{166}$$

Proposition 1 therefore shows

$$\begin{aligned} \hat{Z}^N_{\frac{(n+1)T}{N}} = \Big(I - \tfrac{T}{2N} A\Big)^{-1} \Bigg(& \Big(I + \tfrac{T}{2N} A\Big)\, \hat{Z}^N_{\frac{nT}{N}} + \frac{T}{N} \cdot F\big(\hat{Z}^N_{\frac{nT}{N}}\big) \\ & + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B\big(\hat{Z}^N_{\frac{nT}{N}}\big)\, dW_s + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B'\big(\hat{Z}^N_{\frac{nT}{N}}\big)\bigg(\int_{\frac{nT}{N}}^{s} B\big(\hat{Z}^N_{\frac{nT}{N}}\big)\, dW_u \bigg)\, dW_s \Bigg) \end{aligned} \tag{167}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. The mild Itô processes $\hat{Z}^N$, $N \in \mathbb{N}$, are thus nothing else but appropriate time continuous interpolations of linear implicit Crank-Nicolson-Milstein approximations for SPDEs (see [80, 2]).

References

[1] Albeverio, S., Flandoli, F., and Sinai, Y. G. *SPDE in hydrodynamic: recent progress and prospects*, vol. 1942 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008. Lectures given at the C.I.M.E. Summer School held in Cetraro, August 29–September 3, 2005, Edited by Giuseppe Da Prato and Michael Röckner.

[2] Barth, A., and Lang, A. Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises. *SAM-Report 2011-36* (2011), 25 pages.

[3] Barth, A., and Lang, A. Simulation of stochastic partial differential equations using finite element methods. *Stochastics* (2011), 15 pages. DOI:10.1080/17442508.2010.523466.

[4] Bayer, C., and Teichmann, J. Cubature on Wiener space in infinite dimension. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 464*, 2097 (2008), 2493–2516.

[5] Becker, S. Analysis of the linear implicit Euler-Milstein scheme. *Work in progress* (2012).

[6] Beskos, A., Roberts, G., Stuart, A., and Voss, J. MCMC methods for diffusion bridges. *Stoch. Dyn. 8*, 3 (2008), 319–350.

[7] Bréhier, C.-E. Approximation of the invariant measure with an euler scheme for stochastic PDE's driven by space-time white noise. *arXiv:1202.2707v1* (2012), 31 pages.

[8] Brzeźniak, Z. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep. 61*, 3-4 (1997), 245–295.

[9] Brzeźniak, Z., van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Itô's formula in UMD Banach spaces and regularity of solutions of the Zakai equation. *J. Differential Equations 245*, 1 (2008), 30–58.

[10] Buckdahn, R., Bulla, I., and Ma, J. Pathwise Taylor Expansions for Itô Random Fields. *arXiv:1008.3221v1* (2010), 38 pages.

[11] Buckdahn, R., and Ma, J. Pathwise stochastic Taylor expansions and stochastic viscosity solutions for fully nonlinear stochastic PDEs. *Ann. Probab. 30*, 3 (2002), 1131–1171.

[12] Carmona, R. A., and Rozovskii, B., Eds. *Stochastic partial differential equations: six perspectives*, vol. 64 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[13] Cerrai, S. *Second order PDE's in finite and infinite dimension*, vol. 1762 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. A probabilistic approach.

[14] Chow, P.-L. *Stochastic partial differential equations*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[15] Conus, D. *The Non-linear Stochastic Wave Equation in High Dimensions: Existence, Hölder-continuity and Itô-Taylor Expansion*. École Polytechnique Fédéderale De Lausanne, 2008. Dissertation.

[16] Conus, D., and Dalang, R. C. The non-linear stochastic wave equation in high dimensions. electronic journal of probability. *Electronic Journal of Probability 13* (2008), 629–670.

[17] Cox, S., and van Neerven, J. Convergence rates of the splitting scheme for parabolic linear stochastic Cauchy problems. *SIAM J. Numer. Anal. 48*, 2 (2010), 428–451.

[18] Cox, S. G., and van Neerven, J. M. A. M. Pathwise Holder convergence of the implicit Euler scheme for semi-linear SPDEs with multiplicative noise. *arXiv:1201.4465v1* (2012), 67 pages.

[19] Da Prato, G. *Kolmogorov equations for stochastic PDEs*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2004.

[20] Da Prato, G. Kolmogorov equations for stochastic PDEs with multiplicative noise. *Stochastic Anal. Appl. 2* (2007), 235–263.

[21] Da Prato, G., and Zabczyk, J. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[22] Da Prato, G., and Zabczyk, J. *Ergodicity for infinite-dimensional systems*, vol. 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[23] Da Prato, G., and Zabczyk, J. *Second order partial differential equations in Hilbert spaces*, vol. 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[24] Dalang, R., Khoshnevisan, D., Mueller, C., Nualart, D., and Xiao, Y. *A minicourse on stochastic partial differential equations*, vol. 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[25] de Bouard, A., and Debussche, A. Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation. *Appl. Math. Optim. 54*, 3 (2006), 369–399.

[26] Debussche, A. Weak approximation of stochastic partial differential equations: the nonlinear case. *Math. Comp. 80*, 273 (2011), 89–117.

[27] Debussche, A., and Printems, J. Weak order for the discretization of the stochastic heat equation. *Math. Comp. 78*, 266 (2009), 845–863.

[28] Deya, A. Numerical schemes for the rough heat equation. *arXiv:1003.0587v2* (2010), 30 pages.

[29] Doersek, P., and Teichmann, J. A semigroup point of view on splitting schemes for stochastic (partial) differential equations. *arXiv:1011.2651v1* (2010), 32 pages.

[30] Du, Q., and Zhang, T. Numerical approximation of some linear stochastic partial differential equations driven by special additive noises. *SIAM J. Numer. Anal. 40*, 4 (2002), 1421–1445 (electronic).

[31] Filipović, D., Tappe, S., and Teichmann, J. Jump-diffusions in Hilbert spaces: existence, stability and numerics. *Stochastics 82*, 5 (2010), 475–520.

[32] Geissert, M., Kovács, M., and Larsson, S. Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise. *BIT 49*, 2 (2009), 343–356.

[33] Gradinaru, M., Nourdin, I., and Tindel, S. Ito's- and Tanaka's-type formulae for the stochastic heat equation: the linear case. *J. Funct. Anal. 228*, 1 (2005), 114–143.

[34] Grecksch, W., and Kloeden, P. E. Time-discretised Galerkin approximations of parabolic stochastic PDEs. *Bull. Austral. Math. Soc. 54*, 1 (1996), 79–85.

[35] Grecksch, W., and Tudor, C. *Stochastic Evolution Equations: A Hilbert Space Approach.* Wiley, New York, 1996.

[36] Gyöngy, I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal. 11*, 1 (1999), 1–37.

[37] Gyöngy, I., and Krylov, N. On the splitting-up method and stochastic partial differential equations. *Ann. Probab. 31*, 2 (2003), 564–591.

[38] Gyöngy, I., and Krylov, N. V. Itô formula in Banach spaces. In *Stochastic differential systems (Visegrád, 1980)*, vol. 36 of *Lecture Notes in Control and Information Sci.* Springer, Berlin, 1981, pp. 69–73.

[39] Gyöngy, I., and Krylov, N. V. On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces. *Stochastics 6*, 3-4 (1981/82), 153–173.

[40] Hairer, M. An Introduction to Stochastic PDEs. *arXiv:0907.4178v1* (2009), 78 pages.

[41] Hausenblas, E. Numerical analysis of semilinear stochastic evolution equations in Banach spaces. *J. Comput. Appl. Math. 147*, 2 (2002), 485–516.

[42] Hausenblas, E. Approximation for semilinear stochastic evolution equations. *Potential Anal. 18*, 2 (2003), 141–186.

[43] Hausenblas, E. Weak approximation for semilinear stochastic evolution equations. In *Stochastic analysis and related topics VIII*, vol. 53 of *Progr. Probab.* Birkhäuser, Basel, 2003, pp. 111–128.

[44] Hausenblas, E. Weak approximation of the stochastic wave equation. *J. Comput. Appl. Math. 235*, 2 (2010), 33–58.

[45] Hausenblas, E., and Seidler, J. A note on maximal inequality for stochastic convolutions. *Czechoslovak Math. J. 51(126)*, 4 (2001), 785–790.

[46] Hausenblas, E., and Seidler, J. Stochastic convolutions driven by martingales: maximal inequalities and exponential integrability. *Stoch. Anal. Appl. 26*, 1 (2008), 98–119.

[47] Holden, H., Oeksendal, B., Uboe, J., and Zhang, T. *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach.* Universitext. Springer-Verlag, Berlin, 2009.

[48] Itô, K. On stochastic differential equations. *Mem. Amer. Math. Soc. 4* (1951), 1–51.

[49] Jentzen, A. Taylor expansions of solutions of stochastic partial differential equations. *Discrete Contin. Dyn. Syst. Ser. B 14*, 2 (2010), 515–557.

[50] Jentzen, A. Higher order pathwise numerical approximations of SPDEs with additive noise. *SIAM J. Numer. Anal. 49*, 2 (2011), 642–667 (electronic).

[51] Jentzen, A., and Kloeden, P. *Taylor Approximations for Stochastic Partial Differential Equations*, vol. 83 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2012.

[52] Jentzen, A., and Kloeden, P. E. The numerical approximation of stochastic partial differential equations. *Milan J. Math. 77*, 1 (2009), 205–244.

[53] Jentzen, A., and Kloeden, P. E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465*, 2102 (2009), 649–667.

[54] Jentzen, A., and Kloeden, P. E. Taylor expansions of solutions of stochastic partial differential equations with additive noise. *Ann. Probab. 38*, 2 (2010), 532–569.

[55] Jentzen, A., Kloeden, P. E., and Winkel, G. Efficient simulation of nonlinear parabolic SPDEs with additive noise. *Ann. Appl. Probab. 21*, 3 (2011), 908–950.

[56] Jentzen, A., and Röckner, M. Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. *J. Differential Equations 252*, 1 (2012), 114–136.

[57] Kallianpur, G., and Xiong, J. *Stochastic Differential Equations in Infinite Dimensional Spaces*, vol. 26 of *Lecture Notes-Monographs.* Inst of Mathematical Statistic, 1995.

[58] Klenke, A. *Probability theory.* Universitext. Springer-Verlag London Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original.

[59] Kloeden, P. E., and Platen, E. *Numerical solution of stochastic differential equations*, vol. 23 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin, 1992.

[60] Kloeden, P. E., and Shott, S. Linear-implicit strong schemes for Itô-Galerkin approximations of stochastic PDEs. *J. Appl. Math. Stochastic Anal. 14*, 1 (2001), 47–53. Special issue: Advances in applied stochastics.

[61] Kotelenez, P. *Stochastic Ordinary and Stochastic Partial Differential Equations*, vol. 58 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2008.

[62] Kovács, M., Larsson, S., and Lindgren, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise. *BIT Numerical Mathematics* (2011), 24 pages.

[63] Kovács, M., Larsson, S., and Lindgren, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II. Fully discrete schemes. *arXiv:1203.2029v1* (2012), 24 pages.

[64] Kruse, R. Optimal error estimates of Galerkin finite element methods for stochastic partical differential equations with multiplicative noise. *arXiv:1103.4504v1* (2011), 30 pages.

[65] Kruse, R. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations.* Bielefeld University, Bielefeld, Germany, 2012. Dissertation.

[66] Kruse, R., and Larsson, S. Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise. *arXiv:1109.6487v1* (2011), 19 pages.

[67] Krylov, N. V., Röckner, M., and Zabczyk, J. *Stochastic PDE's and Kolmogorov equations in infinite dimensions*, vol. 1715 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999. Lectures given at the 2nd C.I.M.E. Session held in Cetraro, August 24–September 1, 1998, Edited by G. Da Prato, Fondazione C.I.M.E.. [C.I.M.E. Foundation].

[68] Krylov, N. V., and Rozovskiĭ, B. L. Stochastic evolution equations. In *Current problems in mathematics, Vol. 14 (Russian)*. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979, pp. 71–147, 256.

[69] Lanconelli, A. White noise approach to the Itô formula for the stochastic heat equation. *Commun. Stoch. Anal. 1*, 2 (2007), 311–320.

[70] Lanconelli, A. A remark on the renormalized square of the solution of the stochastic heat equation and on its associated evolution. *Infin. Dimens. Anal. Quantum Probab. Relat. Top. 12*, 3 (2009), 497–502.

[71] Lang, A., Chow, P.-L., and Potthoff, J. Almost sure convergence for a semidiscrete Milstein scheme for SPDEs of Zakai type. *Stochastics 82*, 1-3 (2010), 315–326.

[72] León, J. A., and Tindel, S. Itô's formula for linear fractional PDEs. *Stochastics 80*, 5 (2008), 427–450.

[73] Lindner, F., and Schilling, R. L. Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *arXiv:0911.4681v2* (2010), 29 pages.

[74] Lord, G. J., and Rougemont, J. A numerical scheme for stochastic PDEs with Gevrey regularity. *IMA J. Numer. Anal. 24*, 4 (2004), 587–604.

[75] Lord, G. J., and Tambue, A. A modified semi-implict Euler-Maruyama Scheme for finite element discretization of SPDEs. *arXiv:1004.1998v1* (2010), 23 pages.

[76] Lord, G. J., and Tambue, A. Stochastic Exponential Integrators for a Finite Element Discretization of SPDEs. *arXiv:1005.5315* (2010), 31 pages.

[77] Lord, G. J., and Tambue, A. Stochastic exponential integrators for finite element discretization of SPDEs for multiplicative and additive noise. *arXiv:1103.1986v2* (2011), 23 pages.

[78] Ma, Z.-M., and Röckner, M. *Introduction to the Theory of (Non Symmetric) Dirichlet Forms.* Springer-Verlag, Berlin, 1992.

[79] Matthias Morzfeld, Xuemin Tu, E. A., and Chorin, A. J. A random map implementation of implicit filters. *arXiv:1102.4375v3* (2011), 34 pages.

[80] Mishura, Y. S., and Shevchenko, G. M. Approximation schemes for stochastic differential equations in a Hilbert space. *Teor. Veroyatn. Primen. 51*, 3 (2006), 476–495.

[81] Müller-Gronbach, T., and Ritter, K. An implicit Euler scheme with non-uniform time discretization for heat equations with multiplicative noise. *BIT 47*, 2 (2007), 393–418.

[82] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. In *Monte Carlo and quasi-Monte Carlo methods 2006*. Springer, Berlin, 2007, pp. 577–589.

[83] Ocone, D., and Pardoux, É. A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations. *Ann. Inst. H. Poincaré Probab. Statist. 25*, 1 (1989), 39–71.

[84] Oeksendal, B. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag, Berlin, 2000.

[85] Pardoux, É. Sur des équations aux dérivées partielles stochastiques monotones. *C. R. Acad. Sci. Paris Sér. A-B 275* (1972), A101–A103.

[86] Pardoux, É. Équations aux dérivées partielles stochastiques de type monotone. *Séminaire sur les Équations aux Dérivées Partielles 3*, 2 (1975), 1–10.

[87] Pardoux, É. *Équations aux dérivées partielles stochastiques non lineaires monotones. Etude de solutions fortes de type Itô.* Sci. Math. Univ. Paris Sud., Paris, France, 1975.

[88] Pardoux, É. Two-sided stochastic calculus for SPDEs. In *Stochastic partial differential equations and applications (Trento, 1985)*, vol. 1236 of *Lecture Notes in Math.* Springer, Berlin, 1987, pp. 200–207.

[89] Peszat, S., and Zabczyk, J. *Stochastic partial differential equations with Lévy noise*, vol. 113 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.

[90] Prévôt, C., and Röckner, M. *A concise course on stochastic partial differential equations*, vol. 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[91] Ren, J., Röckner, M., and Wang, F.-Y. Stochastic generalized porous media and fast diffusion equations. *J. Differential Equations 238*, 1 (2007), 118–152.

[92] Renardy, M., and Rogers, R. C. *An introduction to partial differential equations*, vol. 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1993.

[93] Röckner, M. Lp-analysis of finite and infinite dimensional diffusions. In *Stochastic PDEs and Kolmogorov Equations in Infinite Dimensions*, vol. 1715 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 65–116.

[94] Röckner, M., and Sobol, Z. Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. *Ann. Probab. 34*, 2 (2006), 663–727.

[95] Rozovskiĭ, B. L. *Stochastic evolution systems*, vol. 35 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. Linear theory and applications to nonlinear filtering, Translated from the Russian by A. Yarkho.

[96] Sanz-Solé, M. *Malliavin Calculus with Applications to Stochastic Partial Differential Equations.* EFPL Press, Lausanne, 1970.

[97] Shardlow, T. Numerical methods for stochastic parabolic PDEs. *Numer. Funct. Anal. Optim. 20*, 1-2 (1999), 121–145.

[98] Sz.-Nagy, B. Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. Szeged 15* (1953), 87–92.

[99] Sz.-Nagy, B. Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe. *Acta Sci. Math. Szeged 15* (1954), 104–114.

[100] Sz.-Nagy, B., and Foiaş, C. *Harmonic analysis of operators on Hilbert space.* Translated from the French and revised. North-Holland Publishing Co., Amsterdam, 1970.

[101] Teichmann, J. Another approach to some rough and stochastic partial differential equations. *arXiv:0908.2814v1* (2009), 10 pages.

[102] van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Stochastic integration in UMD Banach spaces. *Ann. Probab. 35*, 4 (2007), 1438–1478.

[103] van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal. 255*, 4 (2008), 940–993.

[104] Van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Maximal Lp-regularity for stochastic evolution equations. *arXiv:1101.3504v3* (2011), 43 pages.

[105] Walsh, J. B. *An introduction to stochastic partial differential equations*, vol. 1180 of *Lecture Notes in Mathematics*. Springer, Berlin, 1986.

[106] Walsh, J. B. Finite element methods for parabolic stochastic PDE's. *Potential Anal. 23*, 1 (2005), 1–43.

[107] Wang, X., and Gan, S. A Runge-Kutta type scheme for nonlinear stochastic partial differential equations with multiplicative trace class noise. *arXiv:1106.1889v2* (2012), 24 pages.

[108] Xu, S., and Duan, J. A Taylor expansion approach for solving partial differential equations with random Neumann boundary conditions. *Appl. Math. Comput. 217*, 23 (2011), 9532–9542.

[109] Zabczyk, J. Parabolic equations on Hilbert spaces. In *Stochastic PDEs and Kolmogorov Equations in Infinite Dimensions*, vol. 1715 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 117–239.

[110] Zambotti, L. Itô-Tanaka's formula for stochastic partial differential equations driven by additive space-time white noise. In *Stochastic partial differential equations and applications—VII*, vol. 245 of *Lect. Notes Pure Appl. Math.* Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 337–347.

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