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A New One-Dimensional Finite Volume Method for Hyperbolic Conservation Laws

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Abstract

In this paper, a new one-dimensional Finite Volume Method for Hyperbolic Conservation Laws is presented. The method consists in an improved numerical inter-cell flux function at the element interface. To back theoretically the method, necessary components for convergence are presented. Therefore, it is proved that the method is consistent with the P.D.E and that it is monotone with respect its variables. Moreover, to validate the approach and show its efficiency, we compute several one-dimensional test problems with discontinuous solutions and we make comparisons with traditional methods. The results show an improvement on the non-oscillatory shock-capturing properties based on the new approach.

Keywords: Finite volume method • Numerical flux • Conservation laws • Non-oscillatory approximation • Method exactly conservative • Higher order schemes

Introduction

The general finite volume method for hyperbolic systems has the form $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n\right)$ where $f_{i-\frac{1}{2}}^n$ is some approximation to the average flux along $x = x_{i-\frac{1}{2}}$. Therefore, the main ingredient for finite volume methods is to define the numerical flux, $f_{i+\frac{1}{2}}$, at the cell interfaces as functions of the cell-average Q^n_i, since for finite volume framework only the cell-averages are known, [1,2]. So that the specific method depends on how we choose the numerical flux function.

There are several ways in which the numerical flux might be defined. For instance, wishing to define the average flux at $x = x_{i-\frac{1}{2}}$, based on the data Q_{i-1}^n and Q^n_i to the left and right of this point, considering as first attempt the simple arithmetic average, we get the so called central flux which gets the form $f_{i-\frac{1}{2}}^n = f(Q_{i-1}^n, Q_i^n) = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right]$.

The numerical method that is obtained from the referred numerical flux is called central finite volume method which can be written as $Q_i^{n+1} = Q_i^n - \frac{1}{2}(f(Q_{i+1}^n) - f(Q_i^n))$. This method, in general, is unstable for hyperbolic problems, even if the time step is small enough such that the CFL condition is satisfied [3].

An improvement on the central numerical flux is the classical Lax-Friedrichs flux of the form $f(Q_{i-1}^n, Q_i^n) = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right] - \frac{1}{2}\frac{\Delta x}{\Delta t}(Q_i^n - Q_{i-1}^n)$, which, as it can be seen, is the central flux modified by adding another therm. The Lax-Friedrichs method is stable for a linear hyperbolic equation provided $v \leq 1$, where v is the Courant number [3]. Other improvements took place.

An improvement on the Lax-Friedrichs method, in turn, was reached by replacing the value $\frac{\Delta x}{\Delta t}$ by a locally determined value $a = max(|\ f'(Q_{i-1}^n)\ |, |\ f'(Q_i^n)\ |)$ to get the so-called Local Lax-Friedrichs numerical flux, $f(Q_{i-1}^n, Q_i^n) = \frac{1}{2}\left[f(Q_{i-1}^n) + f(Q_i^n)\right] - \frac{1}{2}a(Q_i^n - Q_{i-1}^n)$.

Over the last few years it was a concern for fluid dynamics researchers to introduce new and innovative approaches. Regarding the finite volume methods several improvements where curried out by researchers. For instance, in 2000, [4] presented the improvements on centred methods for conservation laws, by constructing two families of centred total variation dimishing (TVD) schemes and extended them to nonlinear systems. More recently, in 2020, [5] presented a new modified Local Lax-Friedrichs scheme for scalar conservation laws with discontinuous flux. The modifying is based on the (A – B) type entropy solutions, defined by [6]. In this paper our aim is to construct a new numerical flux based on the traditional numerical fluxes.

Finite Volume Methods

To give an overview of the finite volume method, it is important to emphasize that for the conservation laws it is important to write the method in the conservation form, which ensures the correct approximation for the weak solutions, [3]. As for discontinuous solutions, the integral form is the correct way to model the conservation laws, our aim here is to derive the conservation laws directly from its integral form. Consider the following initial value problem:

PDE: $\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}f(Q) = 0$, $x \in [x_L, xR]$, $U \in \Omega_U \subset R^m$

IC: $Q(x,0) = h(x)$, (2.1)

BC: $Q(x_L,t) = B_L(t)$, $Q(x_R,t) = B_R(t)$,

where Q is the vector of the conservative variables, $f(Q)$ is the flux vector, h is the initial condition and the B_L and B_R are the boundary conditions on the left and right.

In this paper, we restrict our consideration to simple model problems, assuming that the physical principles of mass, momentum and energy are satisfied.

For one-dimensional problems, the finite volume method consists of sub-diving the spatial domain into intervals called finite volumes or grid cells.

Denoting the i_{th} grid cell by

$$\mathcal{C}_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]. \quad (2.2)$$

The value Q^n_i is the approximation of the average value over the i_{th} grid cell at the current time $t = t^n$, and reads

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$$Q_i^n \approx \frac{1}{\Delta x}\int_{\mathcal{C}_i} Q(x,t^n)dx, \tag{2.3}$$

where $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$.

Up to now and on, our concern is to derive the integral form of the conservation laws.

Integrating the PDE (2.1) over the spatial grid cell, we obtain

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial t}Q(x,t)dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial t}f(Q(x,t))dx = 0 \tag{2.4}$$

Assuming that *Q* and *f*(*Q*) are smooth, we have

$$\frac{\partial}{\partial t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x,t)dx + f(Q(x,t))\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \tag{2.5}$$

Hence, we get

$$\frac{\partial}{\partial t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x,t)dx = f(Q(x_{i-\frac{1}{2}},t)) - f(Q(x_{i+\frac{1}{2}},t)) \tag{2.6}$$

Defining $[t^n,t^{n+1}]$ as the grid cell in time, and using the expression 2.6 to develop a time marching algorithm, we have

$$\int_{t^n}^{t^{n+1}}\left[\frac{\partial}{\partial t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x,t)dx\right]dt = \int_{t^n}^{t^{n+1}}[f(Q(x_{i-\frac{1}{2}},t)) - f(Q(x_{i+\frac{1}{2}},t))]dt \tag{2.7}$$

which gives

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x,t^{n+1})dx = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} Q(x,t^n)dx - \int_{t^n}^{t^{n+1}}[f(Q(x_{i+\frac{1}{2}},t))dt - f(Q(x_{i-\frac{1}{2}},t))dt \tag{2.8}$$

Dividing 2.8 by Δ*x*, yelds

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x,t^{n+1})dx = \frac{1}{\Delta x}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} Q(x,t^n)dx - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(Q(x_{i+\frac{1}{2}},t))dt - \int_{t^n}^{t^{n+1}} f(Q(x_{i-\frac{1}{2}},t))dt\right] \tag{2.9}$$

As Q^n_i is the approximation of the average value over the i_{th} grid cell at current time $t = t^n$, reads

$$Q_i^n \approx \frac{1}{\Delta x}\int_{\mathcal{C}_i} Q(x,t^n)dx \tag{2.10}$$

Similarly, defining the time-average flux computed at the cell interface $x_{i+\frac{1}{2}}$ as

$$f_{i+\frac{1}{2}} \approx \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} f\Big(Q(x_{i+\frac{1}{2}},t)\Big)dt \tag{2.11}$$

Substituting (2.10) and (2.11) in (2.9) it is derived the exact relation, which is the reformulation of the principle of the integration conservation:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right) \tag{2.12}$$

The time step Δ*t* satisfies the condition $\Delta t \leq \frac{\Delta x}{Smax}$, where *Smax* is the largest wave speed present throughout the domain at time $t = t^n$, [3,7]. The size of the spatial discretization Δ*x* is chosen according to the desired accuracy. However, the size of the time step Δ*t* has to be chosen on the condition

$$\Delta t = CFL\frac{\Delta x}{Smax} \tag{2.13}$$

where *CFL* is the Courant-Friedrichs-Lewy number, which satisfies the condition $0 < CFL \leq 1$.

More details about *CFL* condition will be discussed in this paper in the next sections. Equation (2.12) tells us how the cell-average of *Q* from (2.10) should be updated exactly in one time step.

In general, it is not possible to evaluate the time integrals on the right-hand side of (2.11), once $Q(x_{i-\frac{1}{2}})$ and $Q(x_{i+\frac{1}{2}})$ vary with time along each edge of the cell, and, on the other hand, there is not the exact solution to work with. Therefore, there is a need of studying numerical methods in the conservation form (2.12).

Construction of the finite volume methods

The discrete method that we are going to construct is based on the integral conservation over finite volume control volumes. For this reason, it is called finite volume method. The main feature is to define the numerical flux, $f_{i+\frac{1}{2}}$, at the cell interfaces as a function of the cell-averages Q^n_i and $Q^n_{i\pm1}$ once only the cell-average is known. In the finite volume framework instead of to get a discrete value of *Q* at a spatial point *x*, we get its integral average over the spatial control volume. However, in order to compute the flux at the element interfaces, it is needed the so-called reconstruction step, which consists in defining the values of *Q* at the interfaces $x_{i\pm\frac{1}{2}}$. For the most simple case, the solution is supposed to be piece-wise constant with $Q^n_i \in C_i$, which leads to discontinuities at the cell interfaces, since the numerical solution has two values at the interface, namely $Q^-_{i+\frac{1}{2}} = Q^n_i$ and $Q^+_{i+\frac{1}{2}} = Q^n_{i+1}$, where $Q^-_{i+\frac{1}{2}}$ and $Q^+_{i+\frac{1}{2}}$ denote the solution at the left and at the right of the interface $x_{i+\frac{1}{2}}$, respectively.

Then, it is reasonable to suppose that to solve the referred discontinuities it is needed a function $f_{i+\frac{1}{2}}$ of both states $Q^-_{i+\frac{1}{2}}$ and $Q^+_{i+\frac{1}{2}}$, called flux. So, we might use the formula of the form

$$f^n_{i+\frac{1}{2}} = f\left(Q^-_{i+\frac{1}{2}}, Q^+_{i+\frac{1}{2}}\right) = f\left(Q^n_{i-1}, Q^n_i\right) \tag{2.15}$$

Therefore the method (2.12)becomes

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\Big[f(Q^n_i, Q^n_{i+1}) - f(Q^n_{i-1}, Q^n_i)\Big] \tag{2.16}$$

Properties of the finite volume method

Exact conservation: The exact conservative property is one of the very important features of the finite volume methods. This is justified by the fact that the variation in time of a quantity, whose evolution obeys a conservation law (2.1), is only given by the exchange thought the element interfaces. A method is conservative when it can be written as follows:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) \tag{2.17}$$

with

$$f_{i+\frac{1}{2}} = f_{i+\frac{1}{2}}\left(Q^n_{i-l}, Q^n_{i-l+1}, \cdots, Q^n_i, Q^n_{i+1}, \cdots, Q^n_{i+r}\right), l, r \geq 0.$$

Consistency with the P.D.E.: A conservative method (2.17) is consistent if

$$f_{i+\frac{1}{2}}\left(Q, Q, \cdots, Q\right) = f(Q) \tag{2.18}$$

Convergence to weak solutions: According to Lax-Wendroff Theorem (), if a numerical solution of a conservative method (2.17) converges to a weak solution of the conservation law. It is necessary to the method to be stable and consistent to converge.

Monotonicity: A necessary condition for the finite volume method to be monotone for a scalar P.D.E. $Q_t + f_x = 0$ is given by $\frac{\partial}{\partial Q^n_i} f_{i+\frac{1}{2}}(Q^n_i, Q^n_{i+n}) \geq 0$ and $\frac{\partial}{\partial Q^n_{i+1}} f_{i+\frac{1}{2}}(Q^n_i, Q^n_{i+n}) \leq 0$.

WENO Methods

High-order accurate weighted essentially non oscillatory (WENO)schemes have gained popularity in numerical solutions of hyperbolic partial differential equations (PDEs) and other convection-dominated problems. WENO methods refer a class of nonlinear finite volume or finite difference methods which can

approximate numerically solutions of hyperbolic conservation laws and other convection-dominated problems with high-order accuracy in smooth regions and essentially non-oscillatory transition for solution discontinuities. In this paper a one-dimensional case for an arbitrary high-order non-oscillatory finite volume scheme is applied. For more details we refer the reader to [8-10].

A new Finite Volume Method for Hyperbolic Conservation Laws

Basic numerical fluxes: The choice of the numerical flux can be done as follows:

(a) The Central Flux

Let us define the average flux at the point $x_{i+\frac{1}{2}}$ on the basis of Q^n_i and Q^n_{i+1} to the left and the right, respectively, of the referred point. Attempting the simple arithmetic average can lead to the so-called Central Flux, which reads:

$$f^c_{i+\frac{1}{2}} = \frac{1}{2}\Big(f(Q^n_{i+1}) + f(Q^n_i)\Big) \quad (4.1)$$

Using it in 2.17, we obtain the Central Method

$$Q^{n+1}_i = Q^n_i - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f(Q^n_{i+1}) - f(Q^n_{i-1})\right! \quad (4.2)$$

This method, in general, is unstable for hyperbolic problems, even if the time step is small enough such that the CFL condition, more frequently called Courant Number, is satisfied, [3].

(b) The Lax-Friedrichs Flux

It is defined as!

$$f^{LF}_{i+\frac{1}{2}} = \frac{1}{2}\left(f(Q^n_{i+1}) + f(Q^n_i)\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}\left(Q^n_{i+1} - Q^n_i\right. \quad (4.3)$$

taking $f^{LF}_{i-\frac{1}{2}} = \frac{1}{2}\Big(f(Q^n_i) + f(Q^n_{i-1})\Big) - \frac{1}{2}\frac{\Delta x}{\Delta t}\Big(Q^n_i - Q^n_{i-1}\Big)$ and recalling the method (2.17), it leads to the classical Lax-Friedrichs Method given by

$$Q^{n+1}_i = \frac{1}{2}\Big(Q^n_{i-1} + Q^n_{i+1}\Big) - \frac{1}{2}\frac{\Delta t}{\Delta x}\Big(f(Q^n_{i+1}) - f(Q^n_{i-1})\Big) \quad (4.4)$$

It can be seen that this method is similar to the central method (4.4) with the only difference that the value Q^n_i is replaced by the average $\frac{1}{2}\Big(Q^n_{i-1} + Q^n_{i+1}\Big)$.

(c) The Local Lax-Friedrichs Flux

It is an improvement for the Lax-Friedrichs Method which is the result of the replacing $\frac{\Delta x}{\Delta t}$ by a $\alpha_{i+\frac{1}{2}} = max\left(\Big|f'(Q_i)\Big|, \Big|f'(Q_{i+1})\Big|\right.$ locally determined value,

$$f^{LLF}_{i+\frac{1}{2}} = f(Q^n_i, Q^n_{i+1}) = \frac{1}{2}\Big(f(Q^n_{i+1}) + f(Q^n_i)\Big) - \frac{1}{2}\alpha_{i+\frac{1}{2}}\Big(Q^n_{i+1} - Q^n_i\Big) \quad (4.5)$$

The resulting method is known as Rusanov's or Local Lax-Friedrichs method, [12], [4].

(d) The Lax-Wendroff Flux

There are two possible versions of the Lax-Wendroff Flux, namely:

$$f^{LW1}_{i+\frac{1}{2}} = \frac{1}{2}\Big(f(Q^n_{i+1}) + f(Q^n_i)\Big) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2_{i+\frac{1}{2}}\Big(Q^n_{i+1} - Q^n_i\Big) \quad (4.6)$$

where $A^n_{i+\frac{1}{2}}$ is an average Jacobian matrix defined at $x_{i+\frac{1}{2}}$.

The second version is given by two stage procedure

$$f^{LW2}_{i+\frac{1}{2}} = f\Big(Q^{LW}_{i+\frac{1}{2}}\Big) \quad (4.7)$$

where the state $Q^{LW}_{i+\frac{1}{2}}$ is computed from

$$Q^{LW}_{i+\frac{1}{2}} = \frac{1}{2}\Big(Q^n_{i+1} + Q^n_i\Big) - \frac{1}{2}\frac{\Delta t}{\Delta x}\Big(f(Q^n_{i+1}) - f(Q^n_i)\Big) \quad (4.8)$$

The proposed new numerical flux: The proposed improved numerical flux, denoted by f^{NF}_{i+1}, is the average of the Local Lax-Friedrichs Flux, 2 (4.5), and the first version of the Lax-Wendroff Flux, (4.6), and takes the form:

$$f^{NF}_{i+\frac{1}{2}} = \frac{1}{2}\left[f^{LLF}_{i+\frac{1}{2}}(Q^n_{i+1}, Q^n_i) + f^{LW1}_{i+\frac{1}{2}}(Q^n_{i+1}, Q^n_i)\right] \quad (4.9)$$

It can be written as:

$$D_{i+\frac{1}{2}} = \left(\frac{\Delta x\alpha_{i+\frac{1}{2}} + \Delta t A^2_{i+\frac{1}{2}}}{2\Delta x}\right.$$

$$f^{NF}_{i+\frac{1}{2}} = \frac{1}{2}\Big[f(Q^n_{i+1}) + f(Q^n_i)\Big] - \frac{1}{2}D_{i+\frac{1}{2}}\Big[Q^n_{i+1} - Q^n_i\Big] \quad (4.10)$$

!

where. The proposed numerical flux (4.10) leads to the following numerical method:

$$Q^{n+1}_i = Q^n_i - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\Big(f(Q^n_{i+1}) - f(Q^n_{i-1})\Big) - D_{i+\frac{1}{2}}\Big(Q^n_{i+1} - 2Q^n_i + Q^n_{i-1}\Big)\right] \quad (4.11)$$

Proofs: In this section we prove that the proposed numerical method satisfies the main properties of finite volume method, enounced in section 2.2., as follows:

1. Exact Conservation

By construction, the method is exactly conservative;

2. Consistency with the P.D.E.

A conservative method (2.17) is consistency if

$$f_{i+\frac{1}{2}}(Q, Q, \cdots, Q) = f(Q) \quad (4.12)$$

Proof: *Given the proposed numerical flux*

$$f^{NF}_{i+\frac{1}{2}} = \frac{1}{2}\Big[f(Q^n_{i+1}) + f(Q^n_i)\Big] - \frac{1}{2}D_{i+\frac{1}{2}}\Big[Q^n_{i+1} - Q^n_i\Big]; \quad (4.13)$$

we have to prove that (4.13) satisfies (4.12).

We have that

$$f^{NF}_{i+\frac{1}{2}}(Q, Q) = \frac{1}{2}\Big[f(Q) + f(Q)\Big] - \frac{1}{2}D_{i+\frac{1}{2}}\Big[Q - Q\Big]$$

; (4.14)

which gives

$$f^{NF}_{i+\frac{1}{2}}(Q, Q) = \frac{1}{2}\Big[2f(Q)\Big] - \frac{1}{2}D_{i+\frac{1}{2}} \times \Big[0\Big]. \quad (4.15)$$

Then, from 4.15 we have that

$$f^{NF}_{i+\frac{1}{2}}(Q, Q) = f(Q); \quad (4.16)$$

which proves that the proposed method is consistent.

Monotonicity

Consider the method (4.11) written in a compact form, [1]:

$$Q^{n+1}_i = H(Q^n_{i-l}, Q^n_{i-l+1}, \cdots, Q^n_{i-1}, Q^n_i, Q^n_{i+1}, \cdots, Q^n_{i+r-1}, Q^n_{i+r}), \quad (4.17)$$

where *H* is a linear or nonlinear operator.

The method is monotone if *H* is a nondecreasing function with respect to all its arguments. That is

$$\frac{\partial H}{\partial Q^n_k} \geq 0, \qquad i - i \leq k \leq i + r \quad (4.18)$$

To prove the monotonicity, let us recall the proposed numerical method (4.11)

and, for simplicity, let us consider the advection equation

$Q_t + \bar{u}Q_x = 0$

as application.

Applying the proposed method (4.11) to the advection equation (4.19), we obtain:

$$Q_i^{n+1} = Q_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(\bar{u}Q_{i+1}^n - \bar{u}Q_{i-1}^n\right)\right) - D_{i+\frac{1}{2}}\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right) \quad (4.20)$$

Factoring with respect to Q_{i-1}^n, Q_i^n and Q_{i+1}^n, we get:

$$Q_i^{n+1} = \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\bar{u} + \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right)Q_{i-1}^n + \left(1 - \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right)Q_i^n + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\bar{u}\right)Q_{i+1}^n \quad (4.21)$$

Writing (4.21) in a compact form, using the operator H, we have:

$$H(Q_{i-1}^n, Q_i^n, Q_{i+1}^n) = \frac{1}{2}\left(\nu + \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right)Q_{i-1}^n + \left(1 - \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right)Q_i^n + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} - \nu\right)Q_{i+1}^n \quad (4.22)$$

where $\nu = \frac{\Delta t}{\Delta x}\bar{u}$ is the Courant Number, [1].

Here, our aim is to prove that

$$\frac{\partial H}{\partial Q_k^n} \geq 0, \text{ for } i-1 \leq k \leq i+1, \quad (4.23)$$

under the Courant Number condition (2.14).

Proof: From (4.22) and (4.23) we have that

$$\frac{\partial H}{\partial Q_k^n} \geq 0 \text{ then } \frac{\partial H}{\partial Q_{i+1}^n} \geq 0, \quad \frac{\partial H}{\partial Q_i^n} \geq 0 \quad \text{and}$$

$$\frac{\partial H}{\partial Q_{i+1}^n} \geq 0, \quad \forall c: 0 < \nu \leq 1. \quad (4.24)$$

We can rewrite (4.24) as

$$\frac{\partial H}{\partial Q_k^n} \geq 0 \text{ then}$$

$$\frac{1}{2}\left(\nu + \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right) \geq 0, \quad \left(1 - \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right) \geq 0 \text{ and } \frac{1}{2}\left(\frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} - \nu\right) \geq 0, \quad (4.25)$$

under the condition $0 < v \leq 1$.

(a) Relation 1:

$$\frac{1}{2}\left(c + \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right) \geq 0, \quad \forall c: 0 < \nu \leq 1; \quad (4.26)$$

(b) Relation 2:

$$\left(1 - \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}}\right) \geq 0 \quad (4.27)$$

Then $\frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} \leq 1 \quad (4.28)$

(c) Relation 3:

$$\frac{1}{2}\left(\frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} - \nu\right) \geq 0 \quad (4.29)$$

Then $\nu \leq \frac{\Delta}{\Delta x}^{t}D_{i+\frac{1}{2}}$ (4.30)

From relations (4.28) and (4.30) we have that

$$\nu \leq \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} \text{ and } \frac{\Delta t}{\Delta x}D_{i+\frac{1}{2}} \leq 1 \quad (4.31)$$

And then by the transitive relation we have that

$v \leq 1$ (4.32)

Once v is a positive real number, from (4.32), we deduce that

$0 < v \leq 1$ (4.33)

Therefore, we conclude that the proposed finite volume method is monotone under the C.F.L. condition, (2.14).

Numerical Experiments

In this section the proposed numerical method is tested by comparing it numerically with the Rusanov's flux, popularly known as the Local Lax-Friendrichs flux (LLF).

The one-dimensional system of the Euler equations

$$\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}f(Q) = 0 \quad (5.1)$$

where $Q = \begin{pmatrix}\rho \\ \rho u \\ \rho E\end{pmatrix}$ and $f(Q) = \begin{pmatrix}\rho u \\ \rho u^2 + P \\ u(\rho E + P)\end{pmatrix}$, will be solved.

Here, ρ is the density, u the velocity, P is the pressure, $E = \frac{1}{2}\rho u^2 + \frac{P}{(\gamma - 1)}$ is the total energy and γ is the ratio of specific heat, which is taken as $\gamma = 1.4$. The following Riemann type initial conditions are imposed:

$$q(x,0) = \begin{cases} q_L, & \text{if } x < 0 \\ q_R, & \text{if } x > 0 \end{cases} \quad (5.2).$$

The four test problems used in this paper were taken from [9]. These test problems were computed in [9] for a fully three-dimensional setting, to test the quadrature-free non-oscillatory finite volume schemes on unstructured meshes for nonlinear hyperbolic conservation laws.

Given the computational domain $[a,b]$ discretized into L computing cells with length Δx, there are required conditions at the boundaries $x = a$ and $x = b$. In this paper we consider the transitive boundary conditions, [3,7], for the test problems. All results are obtained for the CFL numer=0.5. Once our aim in this section is to show whether the proposed new scheme can reduce significantly the oscillations verified in most of the finite volume methods when it comes to compute these kind of problems, for each problem we chose the most challenging cases, as it follows:

Example 1: The Sod shock tube problem

This is a modification of the standard Sod problem.

- Initial conditions: $\begin{pmatrix}\rho_L \\ u_L \\ P_L\end{pmatrix} = \begin{pmatrix}1 \\ 0.75 \\ 1\end{pmatrix}$ and $\begin{pmatrix}\rho_R \\ u_R \\ P_R\end{pmatrix} = \begin{pmatrix}0.125 \\ 0.0 \\ 0.1\end{pmatrix}$ separated by a discontinuity at $x = 0.5$
- Computational domain: From $x = 0.0$ and $x = 1.0$ divided in 200 cells.
- Results: We compute the test problem 1 at time $t = 0.20$ and the results are shown in (Figures 1 and 2). To show the performance of the proposed new scheme, in this example, we compute the energy case, once we found it as the most challenging among all the cases. The result show clearly that the new scheme performs better than the traditional LLF.

Example 2: The Lax shock tube problem

This problem was proposed by Lax and can be found in Shu and Jiang [9].

- Initial conditions: $\begin{pmatrix}\rho_L \\ u_L \\ P_L\end{pmatrix} = \begin{pmatrix}0.445 \\ 0.698 \\ 3.528\end{pmatrix}$ and $\begin{pmatrix}\rho_R \\ u_R \\ P_R\end{pmatrix} = \begin{pmatrix}0.5 \\ 0.0 \\ 0.571\end{pmatrix}$ separated by a discontinuity at $x = 0.5$
- Computational domain: From $x = 0.0$ and $x = 1.0$ divided in 200 cells.
- Results: We compute the test problem 2 at time $t = 0.14$ and the results are shown in (Figures 3 and 4). In this test problem, we have chosen the density and the velocity cases. From the density and the

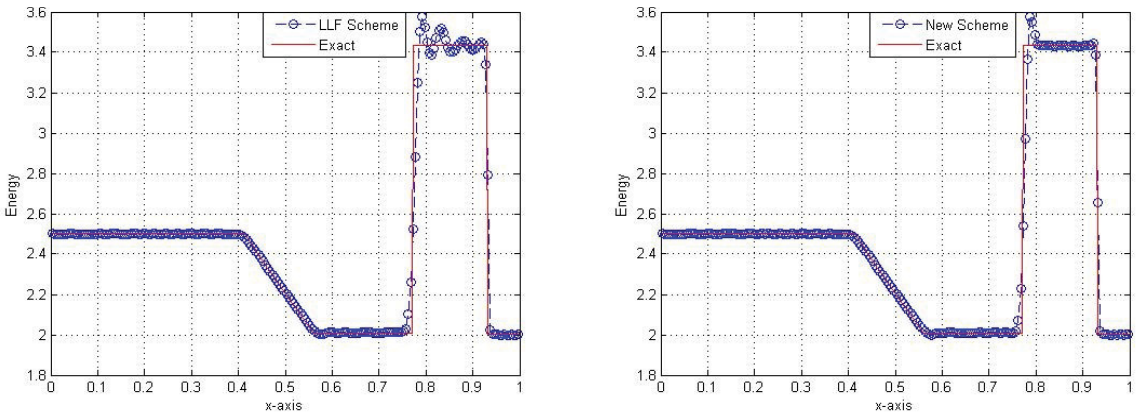

Figure 1. Results for the test problem 1: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Energy profile.

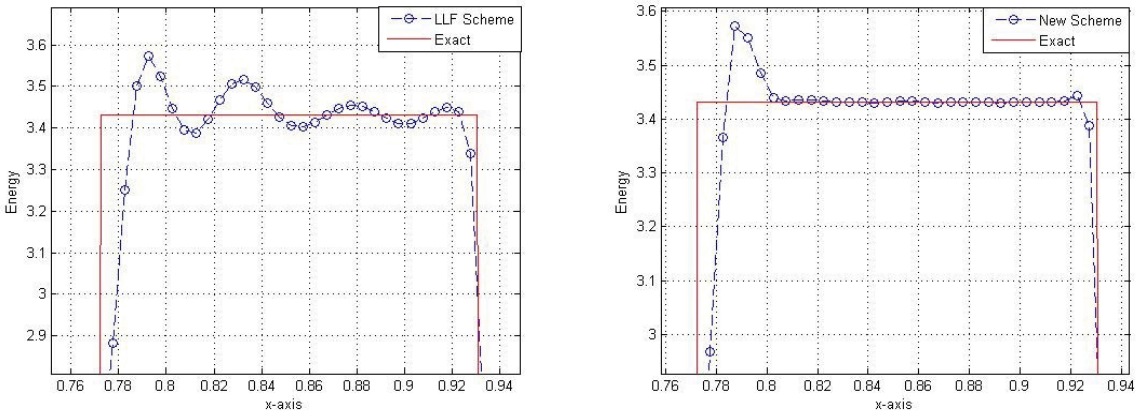

Figure 2. Results for the test problem 1: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Partial energy profile, magnified.

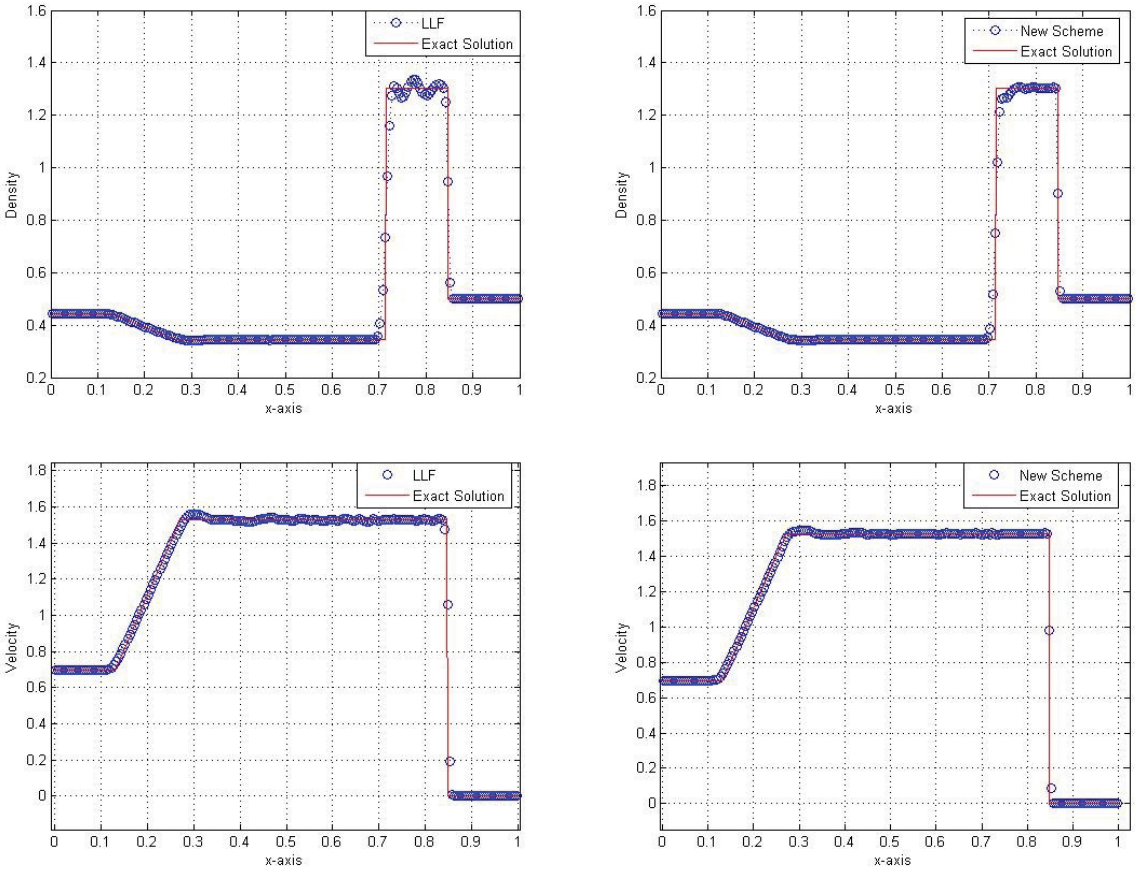

Figure 3. Results for the test problem 2: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Density and velocity profiles.

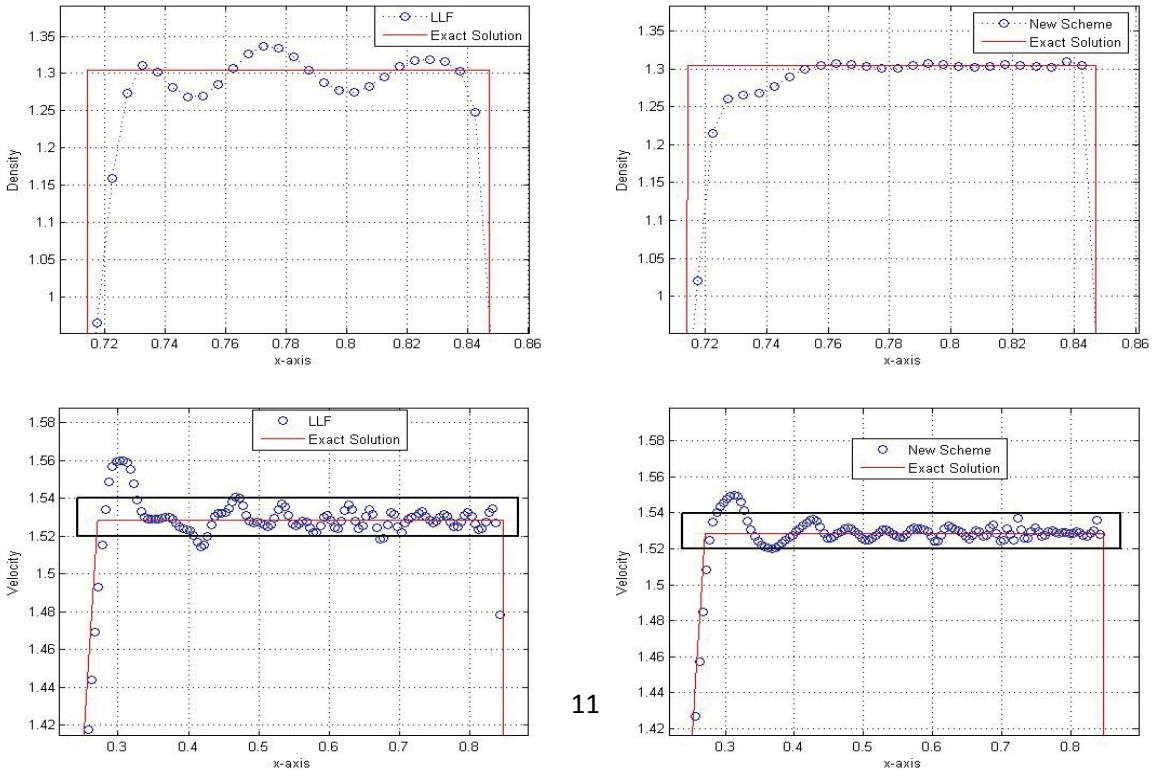

Figure 4. Results for the test problem 2: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Partial Density and velocity profiles, magnified.

velocity profiles we can see that the proposed new scheme performs better than the LLF scheme.

Example 3: With high pressure jump

This is a test problem with high pressure jump. This example can be considered as a more challenging Riemann test problem. It was introduced to prove that the proposed scheme is capable to capture discontinuities very closer to each other and also to deal with very severe pressure jumps without producing negative pressures. The initial condition jumps over five order of magnitude, from 10^3 to 10^{-2}.

- Initial conditions: $\begin{pmatrix}\rho_L\\ u_L\\ P_L\end{pmatrix} = \begin{pmatrix}1.0\\ 0.0\\ 1000\end{pmatrix}$ and $\begin{pmatrix}\rho_R\\ u_R\\ P_R\end{pmatrix} = \begin{pmatrix}1.0\\ 0.0\\ 0.01\end{pmatrix}$

  separated by a discontinuity at $x = 0.5$.
- Computational domain: From $x = 0.0$ and $x = 1.0$ divided in 200 cells.
- Results: We compute the test problem 3 at time $t = 0.012$ and the results are shown in (Figures 5 and 6.) For this problem we haven chosen the velocity and the pressure cases. The results show that on one hand the new scheme reduces significantly the oscillations and on the other hand it does not produce negative pressures.

Example 4: Slowly moving shock wave

The particular feature of this test problem is that it is a very slowly moving shock wave which can lead to spurious oscillations.

- Initial conditions: $\begin{pmatrix}\rho_L\\ u_L\\ P_L\end{pmatrix} = \begin{pmatrix}5.99924\\ 19.5975\\ 1.460895\end{pmatrix}$ and $\begin{pmatrix}\rho_R\\ u_R\\ P_R\end{pmatrix} = \begin{pmatrix}5.99242\\ -6.19633\\ 46.0950\end{pmatrix}$

  separated by a discontinuity at $x = 0.5$.
- Computational domain: From $x = 0.0$ and $x = 1.20$ divided in 200 cells.
- Results: We compute the test problem 3 at time $t = 0.035$ and the results are shown in (Figure 7 and 8). For this test problem we computed the density and the energy cases. From the results it is easy to see that the new scheme performs better when it comes to deal with the spurious oscillations.

Results and Discussions

Four test problems are carried out for numerical experiments. For each one the test problems the most challenging cases, concerning to the oscillations, are computed and the results are described as follows:

Test problem 1: The density case is computed. As it can be seen, almost all the oscillations verified by using LLF scheme are sorted out when the proposed new scheme is applied.

Test problem 2: Density and velocity cases are computed. According to the results, the new scheme performs very well for the density case and for the velocity case it reduces significantly the oscillations compared to the result given by the LLF scheme.

Test problem 3: Velocity and pressure cases are computed. The results show that the new scheme reduces significantly the oscillations, is more accurate and does not produce negative pressures.

Test problem 4: Density and energy cases are computed. For both cases it is possible to see that the new scheme practically eliminates the all the oscillations that appear when the traditional LLF scheme is used.

Concluding Remarks

In this paper, a review of finite volume methods has been made. Based on the traditional numerical fluxes, the Local Lax-friedrichs and the Lax-Wendroff fluxes, a new numerical flux and consequently a new finite volume method were constructed. It was proved that the proposed new numerical method satisfies the properties of a finite volume method. Several challenging test problems, especially those with discontinuities solutions for Euler Equations,

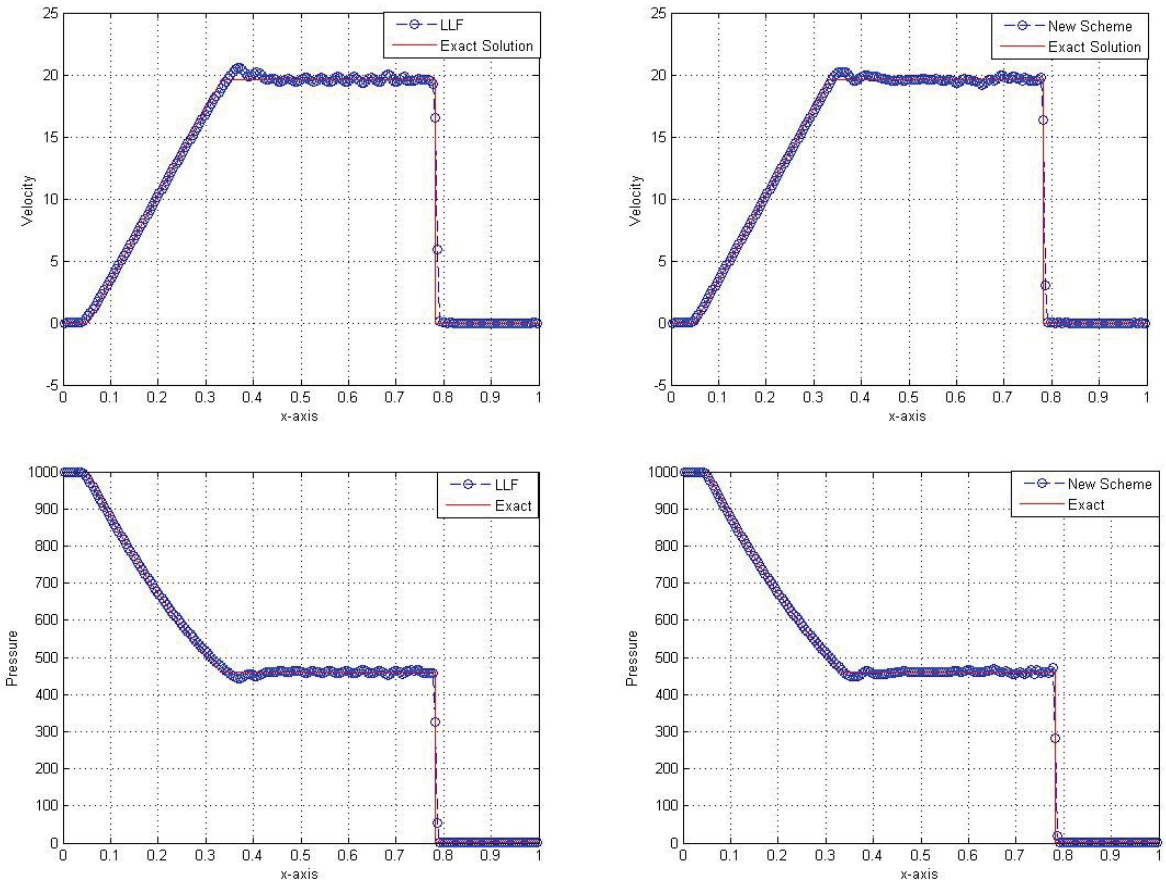

Figure 5. Results for the test problem 3: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Velocity and pressure profiles.

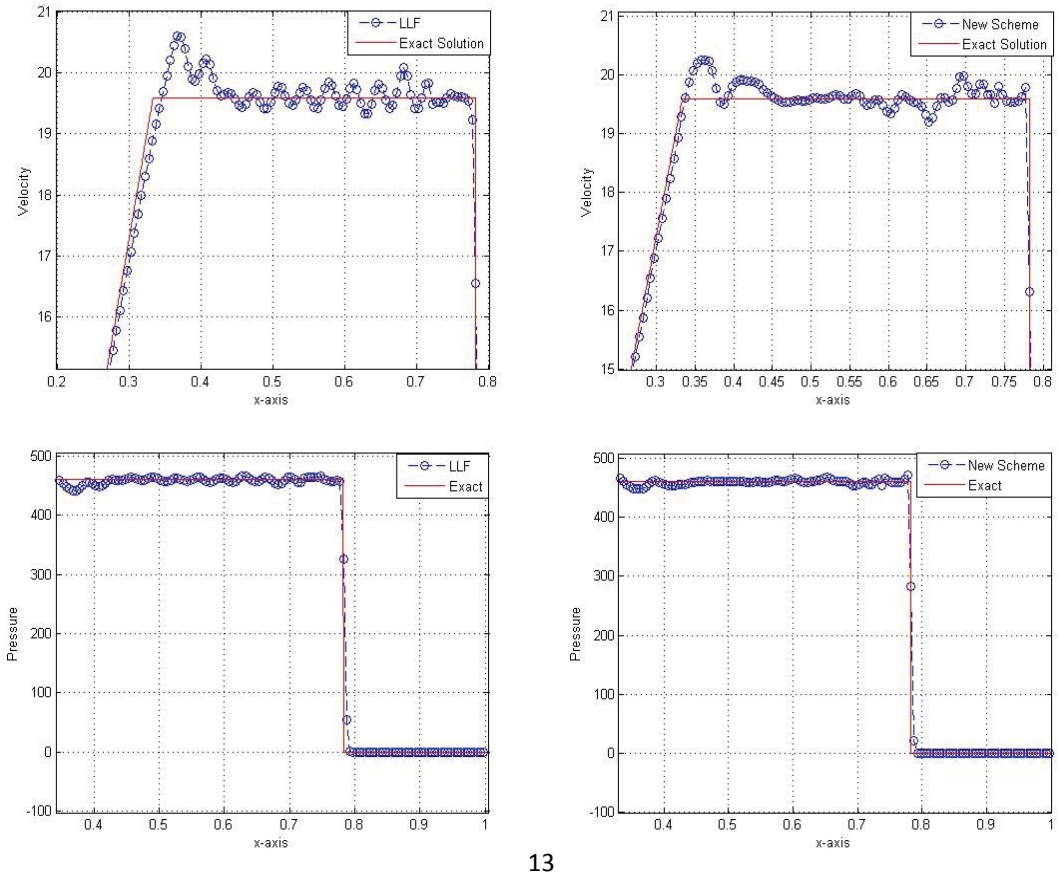

Figure 6: Results for the test problem 3: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Partial Velocity and pressure profiles, magnified.

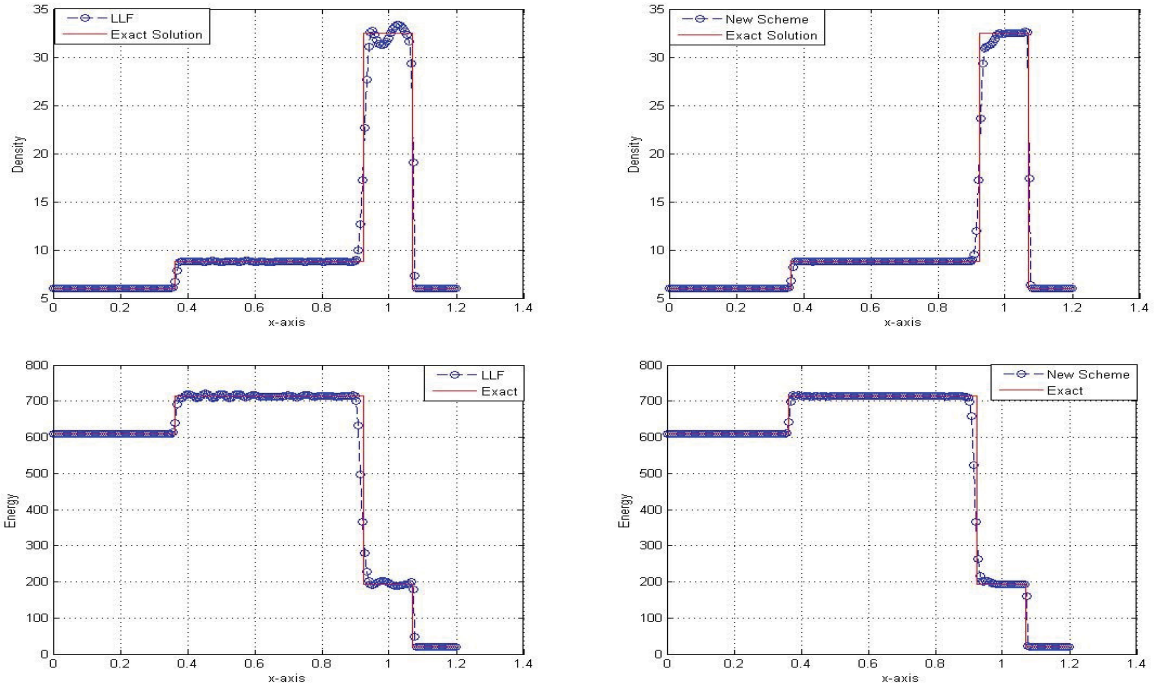

Figure 7. Results for the test problem 4: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Density and energy profiles.

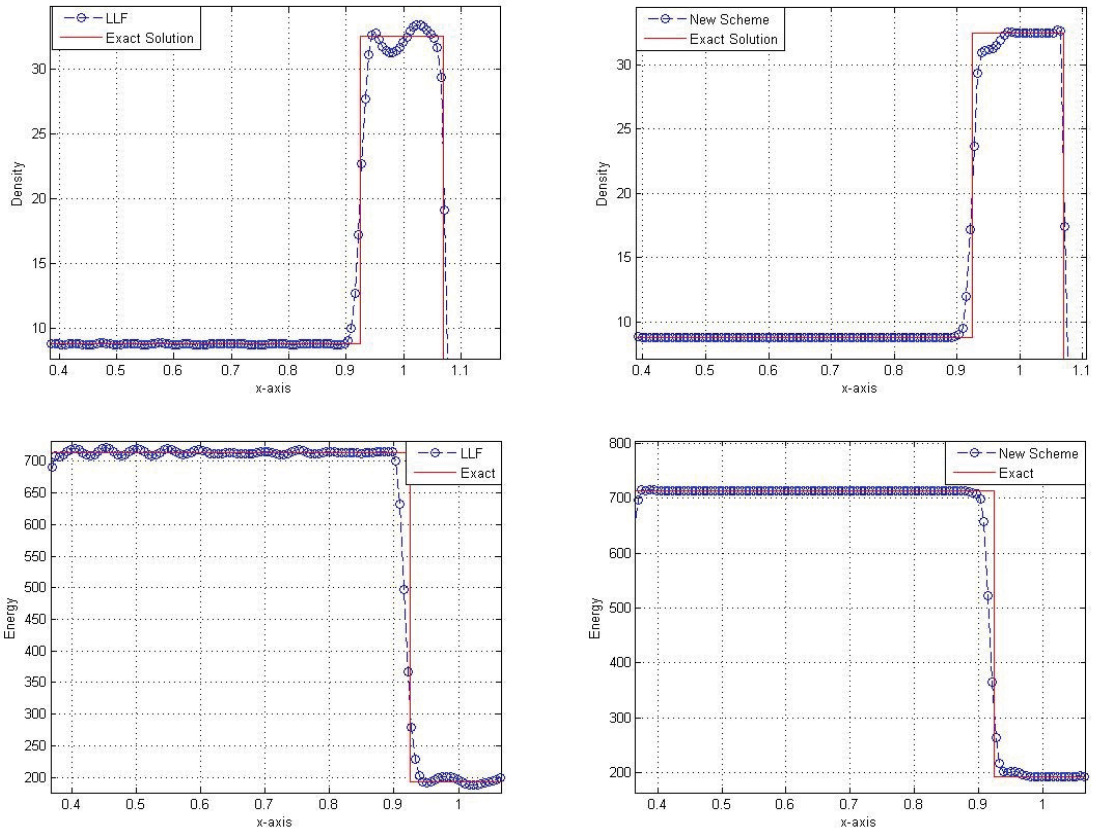

Figure 8. Results for the test problem 4: Performance of the LLF inter-cell flux (left) and the proposed new inter-cell flux(right). Partial Density and energy profiles, magnified.

were curried out to test the performance of the proposed new scheme. In general, the numerical results show that the proposed new scheme performs better than the traditional LLF scheme mainly when it comes to deal with the spurious oscillations.

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