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Abstract

In this paper, a new Petrov-Galerkin formulation is presented for solving convection-dominated problems. The method developed achieves the quasi-optimal convergence rates when the solution is regular and provides the necessary stability to avoid spurious oscillations when strong gradients are presented. Such important properties allow the use of $\mathbf{p}$ refinement to improve the solution in regions with discontinuities because of the stability engendered by the new Petrov-Galerkin method. In this matter, a proper evaluation of the intrinsic time scale function, appearing in the design of this method, is crucial to guarantee the required accuracy.

1. Introduction

It is well known that the convection-dominated flows present numerical difficulties associated to the representation of steep gradients occurring when boundary layers are presented. Considerable success has been achieved by using Petrov-Galerkin finite element models mainly due to the possibility of using discontinuous weighting functions as first proposed by Brooks and Hughes [1]. They designed the *Streamline Upwind Petrov-Galerkin method* (SUPG) which introduces consistently an additional stability term in the upwind direction. The result is a method with good stability properties and accuracy if the exact solution is regular: optimal error estimates are obtained for the derivatives in the streamline direction and quasi-optimal error estimates are obtained for the function [2] showing convergence improvement over the Galerkin method. For non-regular solutions these estimates still are confirmed out of neighborhoods containing sharp layers, where spurious oscillations remain. In order to prevent those "wiggles" the CAU method (*Consistent Approximate Upwind method)* was designed in [3] by introducing the concept of the approximate upwind direction, meaning that the real (physical) streamline is not always the appropriate upwind direction [3]. Practically such an idea leads to a method that keeps the perturbation term

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provided by the SUPG and adds a discontinuity-capturing perturbation, which is henceforth referred as the CAU perturbation. It provides an extra control over the derivatives in the direction of the approximate gradient and, as it is by design proportional to the residual of the approximate solution, it is supposed to vanish in smooth regions. This means that, as the CAU perturbation term is introduced due to an inability of the SUPG to prevent oscillations near boundary layers, it should act only in those regions in order to keep the SUPG accuracy in smooth regions. However, it was observed in [4] a loss of accuracy of the CAU's approximation when compared with the SUPG's if the exact solution is regular. This fact was the motivation for the development of a new method that recovers the accuracy of the SUPG for regular solutions and keeps the stability of the CAU in sharp layers. This aim is achieved by weighting the CAU's approximate upwind direction such that the regularity of the approximate solution is implicitly taken into account. We also derive here appropriate intrinsic time scale functions, dependent on the hierarchical interpolation order, such that quasi-optimal error estimates are attained for regular solutions and higher interpolation orders.

An outline of this paper is as follows. In section 2, we present the convection-dominated problem and its variational formulation. In section 3, we recall some Petrov-Galerkin methods, including the SUPG and the CAU methods. We also present in this section the new Petrov-Galerkin method, a variant of the CAU. In section 4, we discuss the definition of the intrinsic time scale functions appearing in the design of those methods. Numerical experiments are conducted in section 5 in order to demonstrate the efficiency of the proposed scheme as compared to the SUPG and the CAU. Conclusions are drawn in section 6.

2. Problem Statement

In this work, we are interested in solving the stationary linear convection-dominated convection-diffusion problem of the form

$$\mathbf{u} \cdot \nabla \varphi + \nabla \cdot (-\mathbf{K} \nabla \varphi) + \sigma \varphi = f(\mathbf{x}) \qquad \text{in} \quad \Omega \tag{1}$$

with the boundary conditions

$$\begin{aligned} \varphi(\mathbf{x}) &= g(\mathbf{x}) \; ; \qquad && \mathbf{x} \in \Gamma_g \\ -\mathbf{K} \nabla \varphi \cdot \mathbf{n} &= q(\mathbf{x}) \; ; \qquad && \mathbf{x} \in \Gamma_q \end{aligned} \tag{2}$$

where the bounded domain $\Omega \in \Re^n$ has a smooth boundary $\Gamma = \Gamma_g \cup \Gamma_q$, $\Gamma_g \cap \Gamma_q = \emptyset$, and an outward unit normal $\mathbf{n}$. The unknown field $\varphi = \varphi(\mathbf{x})$ is the physical quantity to be transported by a flow characterized by the velocity field $\mathbf{u} = (u_1, \ldots, u_n)$, the (small) diffusion tensor $\mathbf{K} = \mathbf{K}(\mathbf{x})$ and by the reaction term $\sigma = \sigma(\mathbf{x})$, subject to the source term $f(\mathbf{x})$. The functions $g(\mathbf{x})$ and $q(\mathbf{x})$ are given data.

It is well known that if the boundary data $g(\mathbf{x})$ is discontinuous there may be regions in Ω, called layers, where the solution φ varies dramatically. The width of such layers depends on the amount of diffusion present in the fluid flow, being very small for convection-dominated problems. In this case, the use of the standard Galerkin method produces globally oscillating approximate solution unless an extremely fine mesh is used, which most of the time leads to an unbearable computational cost for practical purposes. To resolve accurately those sharp layers is the major concern in this work. To this end, let us first introduce some notations.

Let

$$\mathcal{S} = \left\{ \phi(\mathbf{x}) : \phi \in H^1(\Omega), \, \phi|_{\Gamma_g} = g \right\} \tag{3}$$

and

$$\mathcal{V} = \left\{\theta(\mathbf{x}) : \theta \in H^1(\Omega), \theta|_{\Gamma_g} = 0\right\} \tag{4}$$

be the set of kinematically admissible functions and the space of admissible variations, respectively, where $H^s(\Omega)$ is the usual Sobolev space. Let $(\phi, \theta) = (\phi, \theta)_\Omega = \int_\Omega \phi\theta \, d\mathbf{x}$ and $\|\phi\|_\Omega^2 = (\phi, \phi)$.

The variational formulation for (1) consists of:

$$\boxed{\begin{array}{l}\text{Find } \varphi \in \mathcal{S} \text{ such that} \\ (\mathbf{u}\cdot\nabla\varphi, \theta) + (\mathbf{K}\nabla\varphi, \nabla\theta) + (\sigma\varphi, \theta) - (f, \theta) - (q, \theta)_{\Gamma_q} = 0 \ , \quad \forall\theta \in \mathcal{V},\end{array}} \tag{5}$$

where $(q, \theta)_{\Gamma_q} = \int_{\Gamma_q} q\theta \, ds$.

3. Approximate Solution

Let us consider a finite element partition π^h of triangular elements Ω_e such that $\overline{\Omega} = \bigcup_{e=1}^{N_e} \overline{\Omega}_e$ and $\bigcap_{e=1}^{N_e} \Omega_e = \emptyset$, where N_e is the total number of elements in π^h. The finite dimensional counterparts of (3) and (4) are

$$\mathcal{S}^h = \left\{\phi^h \in C^0(\Omega), \ \phi^h\Big|_{\Omega_e} \in P_e^k \ , \ \forall\Omega_e \in \pi^h, \ \phi^h\Big|_{\Gamma_g} = g\right\} \tag{6}$$

and

$$\mathcal{V}^h = \left\{\theta^h \in C^0(\Omega), \ \theta^h\Big|_{\Omega_e} \in P_e^k \ , \ \forall\Omega_e \in \pi^h, \ \theta^h\Big|_{\Gamma_g} = 0\right\} \tag{7}$$

where P_e^k is the space of polynomials of degree less or equal to k and the subscript e denotes the restriction of a given function to the element Ω_e.

With these definitions, the Petrov-Galerkin approximation of problem (5) is the following;

$$\boxed{\begin{array}{l}\text{Find } \varphi^h \in \mathcal{S}^h \text{ such that} \\ G\left(\varphi^h, \theta^h\right) + \sum_{e=1}^{N_e} \left(R_e\left(\varphi^h\right), p_e^h\right)_{\Omega_e} = F\left(\theta^h\right) \ , \qquad \forall\theta^h \in \mathcal{V}^h \ ,\end{array}} \tag{8}$$

where

$$R_e\left(\varphi^h\right) = \mathbf{u}\cdot\nabla\varphi^h + \nabla\cdot\left(-\mathbf{K}\nabla\varphi^h\right) + \sigma\varphi^h - f \qquad \text{in} \quad \Omega_e \tag{9}$$

is the residual of the approximate solution and

$$G\left(\varphi^h, \theta^h\right) = \left(\mathbf{u}\cdot\nabla\varphi^h, \theta^h\right) + \left(\mathbf{K}\nabla\varphi^h, \nabla\theta^h\right) + \left(\sigma\varphi^h, \theta^h\right) \ ,$$

$$F\left(\theta^h\right) = \left(f, \theta^h\right) + \left(q, \theta^h\right)_{\Gamma_q} \ .$$

Remarks:

1. in the formulation (8), the space of weighting functions is constructed by adding to the standard Galerkin weighting function $\theta^h \in \mathcal{V}^h$ a perturbation p_e^h in each element Ω_e. Different choices of this perturbation generate different Petrov-Galerkin methods but all of them sharing the consistency property, in the sense that the exact solution satisfies the approximate problem;

2. the operator $G(\cdot,\cdot)$ defined previously stands for the Galerkin contribution to (8).

3.1. The SUPG Method

As described in [1], the SUPG method introduces a discontinuous perturbation in the streamline direction defined as

$$p_e^h = \tau_e^s \, \mathbf{u}_e \cdot \nabla\theta^h \ , \qquad e = 1, \ldots, N_e \ , \tag{10}$$

where τ_e^s is the intrinsic time scale function (or upwind function), which is given by

$$\tau_e^s = \frac{\xi_e h_e}{2\,|\mathbf{u}_e|} \ , \tag{11}$$

with h_e being the characteristic element length in the streamline direction and ξ_e the non-dimensional numerical diffusivity, whose expression will be discussed in the next section. Rewriting the variational formulation (8) introducing the SUPG operator - $S(\cdot,\cdot)$ - we obtain:

$$\boxed{\begin{array}{l} \text{Find } \varphi^h \in \mathcal{S}^h \text{ such that} \\ G\left(\varphi^h, \theta^h\right) + S\left(\varphi^h, \theta^h\right) = \overline{F}\left(\theta^h\right) \ , \qquad \forall \theta^h \in \mathcal{V}^h \ , \end{array}} \tag{12}$$

where

$$S\left(\varphi^h, \theta^h\right) = \sum_{e=1}^{N_e} \left\{ \left(\tau_e^s \, \mathbf{D}\nabla\varphi^h, \nabla\theta^h\right)_{\Omega_e} - \left(\nabla\cdot\mathbf{K}\nabla\varphi^h, \tau_e^s \, \mathbf{u}_e\cdot\nabla\theta^h\right)_{\Omega_e} \right.$$
$$\left. + \quad \left(\sigma\varphi^h, \tau_e^s \, \mathbf{u}_e\cdot\nabla\theta^h\right)_{\Omega_e} \right\} \ , \tag{13}$$

$$\overline{F}\left(\theta^h\right) = F\left(\theta^h\right) + \sum_{e=1}^{N_e} \left(f . \tau_e^s \, \mathbf{u}_e \cdot \nabla\theta^h\right)_{\Omega_e} \ . \tag{14}$$

In the above expression $\mathbf{D} = \mathbf{u}_e \otimes \mathbf{u}_e$, $\otimes$ denoting the tensor product. It is exactly the quadratic term containing the tensor $\mathbf{D}$ which is responsible for the additional stability engendered by the SUPG, as shown in [2]. On the other hand, the function τ_e^s must satisfy some requirements such that the second term under summation in (13) does not decrease the stability (see [2] for more details).

Assuming only essential boundary conditions ($\Gamma_q = \Gamma$) for simplicity and defining $e = \varphi - \varphi^h$, where φ denotes the exact solution of (1), the main results in [2] concerning the stability of the SUPG using a mesh with mesh length h are:

- the bilinear form $G(\cdot,\cdot)+S(\cdot,\cdot)$ satisfies

$$G\left(\varphi^h,\varphi^h\right)+S\left(\varphi^h,\varphi^h\right)\geq c\left|\left|\left|\varphi^h\right|\right|\right|^2 \tag{15}$$

where c is a constant independent of φ and the mesh size h and $|||\cdot|||$ is the mesh-dependent norm given by

$$|||\psi|||^2=(\mathbf{K}\nabla\psi,\nabla\psi)+(\sigma\psi,\psi)+\sum_{e=1}^{N_e}\left(\tau_e^s\ \mathbf{D}\nabla\varphi^h,\nabla\theta^h\right)_{\Omega_e}\ ; \tag{16}$$

- if $\varphi\in H^r(\Omega)$ the following error estimate holds:

$$|||e|||\leq C\ h^{\nu-\frac{1}{2}}\ |\varphi|_r\ , \tag{17}$$

where $|\cdot|_r$ denotes the semi-norm in the Sobolev space $H^r(\Omega)$ and $\nu=\min(k+1,r)$.

3.2. The CAU Method

As mentioned before, the approximation solution provided by the formulation (12) is not free of spurious oscillations when boundary layers are presented. They are precluded by adding to the left hand side of (12) the operator

$$C\left(\varphi^h,\theta^h\right)=\sum_{e=1}^{N_e}\left(\tau_e^c\ \mathbf{C}\nabla\varphi^h,\nabla\theta^h\right)_{\Omega_e} \tag{18}$$

where $\mathbf{C}=\left(\mathbf{u}_e-\mathbf{v}_e^h\right)\otimes\left(\mathbf{u}_e-\mathbf{v}_e^h\right)$. This operator plays the role of a discontinuity-capturing operator, controlling the derivatives in the direction of $\nabla\varphi^h$. In its definition, an auxiliary vector field $\mathbf{v}^h$ is introduced, which was designed in [3] such that, in each element Ω_e, $e=1,\ldots,N_e$,

$$\left|\mathbf{u}_e-\mathbf{v}_e^h\right|^2\ \leq\ \left|\mathbf{u}_e-\vartheta_e^h\right|^2\ ,\qquad\forall\vartheta_e^h\in Q^h\ , \tag{19}$$

where

$$Q^h=\left\{\vartheta_e^h\ ;\ \vartheta_e^h\cdot\nabla\varphi^h-\nabla\cdot\mathbf{K}\nabla\varphi^h+\sigma\varphi^h-f=0\text{ in each }\Omega_e\right\}\ . \tag{20}$$

This condition ensures that $\varphi^h\underset{h\to0}{\longrightarrow}\varphi\implies\mathbf{v}^h\underset{h\to0}{\longrightarrow}\mathbf{u}$ and leads to

$$\begin{aligned}&\mathbf{v}_e^h=\mathbf{u}_e &&\text{if}\quad\left|\nabla\varphi^h\right|=0\ ;\\ &\mathbf{v}_e^h=\mathbf{u}_e-\frac{R_e\left(\varphi^h\right)}{\left|\nabla\varphi^h\right|^2}\nabla\varphi^h &&\text{if}\quad\left|\nabla\varphi^h\right|\neq0\ .\end{aligned} \tag{21}$$

Moreover, from (1) and (20), the CAU method imposes implicitly the following restriction

$$\left(\mathbf{u}_e-\mathbf{v}_e^h\right)\cdot\nabla\varphi^h=R_e\left(\varphi^h\right) \tag{22}$$

which shows that the operator (18) can also be written either

$$C\left(\varphi^h,\theta^h\right)=\sum_{e=1}^{N_e}\left(R_e\left(\varphi^h\right),\tau_e^c\ \left(\mathbf{u}_e-\mathbf{v}_e^h\right)\cdot\nabla\theta^h\right)_{\Omega_e} \tag{23}$$

or

$$C\left(\varphi^h,\theta^h\right)=\sum_{e=1}^{N_e}\left(\left(\mathbf{u}_e-\mathbf{v}_e^h\right)\cdot\nabla\varphi^h,\tau_e^c\ \left(\mathbf{u}_e-\mathbf{v}_e^h\right)\cdot\nabla\theta^h\right)_{\Omega_e}\ . \tag{24}$$

Remarks:

1. this method is non-linear even if the original problem is linear since the quantity $\left(\mathbf{u}_e - \mathbf{v}_e^h\right)$ depends on the approximate solution φ^h;

2. the CAU method can be seen as introducing the perturbation $p_e^h = \mathbf{u}_e^h \cdot \nabla\theta^h$, $e = 1, \dots N_e$, in the approximate upwind direction $\mathbf{u}_e^h$ given by

$$\mathbf{u}_e^h = \tau_e^s \ \mathbf{u}_e + \tau_e^c \ \left(\mathbf{u}_e - \mathbf{v}_e^h\right) \ , \tag{25}$$

with $\mathbf{v}_e^h$ as defined in (21). Thus, ideally when $R_e\left(\varphi^h\right) \underset{h\to 0}{\longrightarrow} 0$, $\left|\mathbf{u}_e - \mathbf{v}_e^h\right| \underset{h\to 0}{\longrightarrow} 0$ and $\mathbf{u}_e^h \underset{h\to 0}{\longrightarrow} \tau_e^s \ \mathbf{u}_e$, recovering the SUPG perturbation;

3. eventually, the direction of $\left(\mathbf{u}_e - \mathbf{v}_e^h\right)$ - the direction of $\nabla\varphi^h$ - can coincide with the streamline direction, leading to an over-diffusive approximate solution because of the doubling effect caused by the simultaneous action of $S(\cdot,\cdot)$ and $C(\cdot,\cdot)$ over the same direction. To avoid such effect, it was suggested in [5] to take:

$$\tau_e^c = \max\{0, \overline{\tau}_e^c - \tau_e^s\} \ , \tag{26}$$

where $\overline{\tau}_e^c$ is the intrinsic time scale function associated with the CAU operator, determined as

$$\overline{\tau}_e^c = \frac{\xi_e^c h_e^c}{2\left|\mathbf{u}_e - \mathbf{v}_e^h\right|} \tag{27}$$

where h_e^c is the characteristic element length in the $\left(\mathbf{u}_e - \mathbf{v}_e^h\right)$ direction and ξ_e^c is the non-dimensional numerical diffusivity associated with the CAU operator. Rewriting (26), it follows that

$$\begin{aligned} &\tau_e^c = 0 && \text{if } \frac{\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}{|\mathbf{u}_e|} \geq \frac{\xi_e^c h_e^c}{\xi_e h_e} \ , \\ &\tau_e^c = \frac{\xi_e h_e}{2\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}\left\{\frac{\xi_e^c h_e^c}{\xi_e h_e} - \frac{\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}{|\mathbf{u}_e|}\right\} && \text{otherwise} \ ; \end{aligned} \tag{28}$$

4. no requirement was compelled about the relative orientation between the approximate streamline direction $\mathbf{v}_e^h$ and the real streamline $\mathbf{u}_e$. However, it is quite obvious that the streamline $\mathbf{v}_e^h$ should have no component in the direction opposite to the real streamline. Indeed, some numerical experiments have shown that a loss of stability can occur for distorted element geometries when $\mathbf{u}_e \cdot \mathbf{v}_e^h \leq 0$. Thus, in order to assure that

$$\mathbf{u}_e \cdot \mathbf{v}_e^h > 0 \quad \text{in each element } \Omega_e \ , \tag{29}$$

a variant of the CAU method was introduced in [4], called the VCAU method. In the VCAU the latter condition is automatically satisfied by changing τ_e^c for a new function τ_e^{vc} such that in each element we have

$$\begin{aligned} &\tau_e^{vc} = 0 && \text{if } \frac{\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}{|\mathbf{u}_e|} \geq 1 \ , \\ &\tau_e^{vc} = \frac{\xi_e h_e}{2\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}\left\{1 - \frac{\left|\mathbf{u}_e - \mathbf{v}_e^h\right|}{|\mathbf{u}_e|}\right\} && \text{otherwise} \ ; \end{aligned} \tag{30}$$

5. it was shown in [4] that in the presence of either source and transient terms or internal and/or external boundary layers, the VCAU's approximate solutions are always stable, much better than those obtained with the SUPG [1] or the combination of the SUPG and discontinuity-capturing proposed in [5]. However, a loss of accuracy was observed when compared with the quasi-optimal SUPG's approximations for regular problems with smooth exact solutions [4]. The idea of the new method which will be derived in the following is to use the same CAU's approximate upwind direction given in (25), incorporating in the VCAU's upwind function, τ_e^{vc}, an additional control that takes into account the regularity of the solution.

3.3. Recovering the Accuracy of the CAU - The RVCAU Method

In the design of the approximate upwind direction (25), the term $\left(\mathbf{u}_e - \mathbf{v}_e^h\right)$ is the one responsible for controlling over the approximate gradient direction, which is crucial in avoiding localized spurious oscillations near sharp layers. This term is weighted by the upwind function τ_e^{vc} as defined in (30) which, in spite of being designed to fulfill some stability requirements, is primarily responsible for the accuracy of the method. Ideally, this function should vanish when (and where) the solution is regular in order to recover the quasi-optimal rates provided by the SUPG. In that case, this means that the difference $(\overline{\tau}_e^c - \tau_e^s)$ in (26) should be less or equal to zero. This approach was used in [4] to modify τ_e^{vc} by using a feedback function which takes into account the regularity of the solution. The method generated preserves the accuracy of the SUPG's regular approximations but its extension for systems of equations, like the Navier-Stokes equations, is not straightforward. On the other hand, the CAU method has already been successfully generalized for systems in [6]. Besides, we claim that we can get the same good behavior by subtracting from the VCAU's upwind function the effect produced by the SUPG operator over the approximate gradient direction. This means that the loss of accuracy is mainly associated with the inappropriate procedure currently used to avoid the doubling effect.

Let us call η_e the function to be designed to compensate the mentioned effect. Using (13), (14) and (18) we have

$$\eta_e \left(\mathbf{u}_e - \mathbf{v}_e^h\right) \otimes \left(\mathbf{u}_e - \mathbf{v}_e^h\right) \nabla\varphi^h \cdot \nabla\varphi^h = \tau_e^s \; R_e\left(\varphi^h\right) \mathbf{u}_e \cdot \nabla\varphi^h \; .$$

Using now the restriction (22) we obtain

$$\eta_e = \frac{\mathbf{u}_e \cdot \nabla\varphi^h}{R_e\left(\varphi^h\right)} \; \tau_e^s \; . \tag{31}$$

Notice that η_e should not be less than τ_e^s in order to recover (30) when the direction of the real streamline $\mathbf{u}_e$ and the approximate gradient direction, the direction of $\left(\mathbf{u}_e - \mathbf{v}_e^h\right)$, coincide. Thus, we define

$$\overline{\eta}_e = \tau_e^s \; \max\left(1, \eta_e\right) \qquad , \; e = 1, \ldots, N_e \; . \tag{32}$$

Hereby, the new upwind function in each element is

$$\tau_e^{new} = \max\left\{ 0 \; , \; \frac{\xi_e h_e}{2\left|\mathbf{u}_e - \mathbf{v}_e^h\right|} \; - \; \overline{\eta}_e \right\} \tag{33}$$

providing a reduction of the amount of artificial diffusivity over the approximate gradient direction since $\overline{\eta}_e$ is always positive. Thus, the new variant of the CAU method is given by:

$$
\boxed{\begin{array}{l}\text{Find } \varphi^h \in \mathcal{S}^h \text{ such that} \\ G\left(\varphi^h, \theta^h\right) + S\left(\varphi^h, \theta^h\right) + C^{new}\left(\varphi^h, \theta^h\right) = \overline{F}\left(\theta^h\right) , \qquad \forall \theta^h \in \mathcal{V}^h ,\end{array}} \tag{34}
$$

where

$$
C^{new}\left(\varphi^h, \theta^h\right) = \sum_{e=1}^{N_e} \left(\tau_e^{new}\ \mathbf{C}\nabla\varphi^h, \nabla\theta^h\right)_{\Omega_e} . \tag{35}
$$

<u>Remark:</u> the stability of this new method follows directly from that of the SUPG, i.e.,

$$
G\left(\varphi^h, \varphi^h\right) + S\left(\varphi^h, \varphi^h\right) + C^{new}\left(\varphi^h, \varphi^h\right) \geq c\left|\left|\left|\varphi^h\right|\right|\right|^2 + \sum_{e=1}^{N_e} \left(\tau_e^{new}\ \mathbf{C}\nabla\varphi^h, \nabla\varphi^h\right)_{\Omega_e} . \tag{36}
$$

Clearly, the method (34) is more coercive than (12): the perturbation terms that emanate by the concept of the approximate upwind direction are in fact adding the term (35) to the SUPG's perturbation, providing the improvement of the solution stability in the presence of sharp layers. Otherwise, if the solution is regular, this term vanishes and (15) is recovered.

In section 5 some numerical results are presented to demonstrate the very good numerical performance of this new variant of the VCAU, which is referred here as **RVCAU** because of its property of recovering the convergence rates for smooth solutions, to approximate regular as well as solutions with sharp boundary layers.

4. The Upwind Parameter for Higher Order Approximations

The development of accurate Petrov-Galerkin procedures for convection-dominated problems is strongly dependent on the design of the intrinsic time scale function. In particular, for the RVCAU, the parameters to be estimated are those associated with the choice of the intrinsic time scale function of the SUPG - τ_e^s (11), i.e., h_e and ξ_e. Besides, it is well known that choosing

$$
\xi = \xi_e = \xi_{opt} = \coth\, Pe \;-\; \frac{1}{Pe} , \qquad Pe = \frac{h_e\left|\mathbf{u}_e\right|}{2\left|\mathbf{K}_e\right|} , \tag{37}
$$

the SUPG approximation of the one-dimensional steady-state case with no source term in which $\mathbf{u}$ and $\mathbf{K}$ are constant, using a regular mesh with linear elements of length $h_e = h$, is a nodally exact solution [7]. This also occurs for the RVCAU approximation since, in this particular case, the streamline direction coincides with the direction of the approximate gradient, implying that τ_e^{new} vanishes.

In the definition (37), the element Peclet number Pe, which measures the relative importance between the convective and diffusive phenomena, is used to build the doubly asymptotic behavior of ξ: it goes to one when Pe is large (the convection-dominated case) and vanishes when Pe is small, meaning that the necessary stability is already provided by the physical diffusion $\mathbf{K}$. Although the nodally exact property is not extended to the case of non-constant coefficients and irregular meshes, the function ξ_{opt} still works for each element individually [8]. In addition to that, the definition (11) can also be used for multidimensional case, provided a reasonable definition of h_e is available [1]. The main question here is: *how should ξ_e (or τ_e^s) be defined if interpolation orders other than one are used?.*

Based on numerical experiments using quadratic elements, it was suggested in [9] that approximately the half value of ξ_{opt} should be used to provide the stability using the Galerkin-Least Square method, i.e.,

$$\begin{aligned} k = 1 \quad &\longrightarrow \quad \xi_1 = \xi_{opt} \; ; \\ k = 2 \quad &\longrightarrow \quad \xi_2 = \tfrac{1}{2}\, \xi_1 \; . \end{aligned} \tag{38}$$

In general, and leading to the same result, it was proposed in [8] to extend the Petrov-Galerkin perturbation function using for quadratic elements half of the one used for linear elements. In [10] that choice was justified by studying exact discrete solutions for one-dimensional quadratic and hierarchical elements, using the SUPG formulation. Although the procedure used there does not seem feasible for interpolation orders greater than two, as pointed out by the authors, they showed that it is necessary to have two different non-dimensional diffusivity functions for each element, one for the extreme nodes of the element and other for the central node, so as to achieve nodally exact results. They also mentioned the possibility of using a unique function and, in this case, the best choice happens to be the same as that one heuristically obtained in [8, 9].

An interesting conclusion in those studies is that the non-dimensional numerical diffusivity function is very sensitive to the order of the element. Thus, its definition must depend in some sense on the interpolation order in each element. We have performed some experiments using a unique function for every element degree of freedom and the results suggest that we can generalize the upper bound of ξ for the convection-dominated case, when $Pe \rightarrow \infty$, as a function of the hierarchical order of interpolation k in the following way: denoting p_e as the order of the element e, $e = 1, \ldots, N_e$, when $Pe \rightarrow \infty$ we should select

$$\begin{aligned} p_e = 1 \quad &\longrightarrow \quad \xi_1 = \xi_{opt} = 1 \; ; \\ p_e = 2 \quad &\longrightarrow \quad \xi_2 = \tfrac{1}{2} \; ; \\ \vdots \quad & \quad\quad\quad \vdots \\ p_e = k \quad &\longrightarrow \quad \xi_k = \tfrac{1}{k} \; . \end{aligned} \tag{39}$$

<u>**Remark:**</u> some very interesting results yield of this choice:

- the control in the streamline direction engendered by the SUPG method depends, in each element, on the factor h_e/p_e. The analysis of this method for non-constant polynomial interpolation order, following the same procedure used in [2] and applying the Babuska and Suri's Lemma [11] leads to

$$|||\psi|||_{\Omega_e} \leq C \, \frac{h_e^{\nu - \frac{1}{2}}}{p_e^{r - \frac{1}{2}}} \, |\varphi|_r \; , \tag{40}$$

assuming that φ has enough regularity. This relation means that the error estimate for the function is quasi-optimal with a gap of $(h_e/p_e)^{\frac{1}{2}}$, keeping the optimal estimates for derivatives in the streamline direction;

- the method (34) attains the same quasi-optimal convergence rates of the SUPG for globally smooth solutions, as it will be shown by the numerical examples in the next section;

- since the additional control introduced by (35) depends on the element size as well as the element order, high-order interpolating can also be used successfully to improve the approximation of boundary-layers; this opens a challenging possibility of combining the RVCAU method with *hp* adaptive refinement schemes in order to solve problems presenting regions with regular as well as non-regular solutions;

- obviously it is possible to infer other definitions than (39) for the asymptotic limit of ξ. Indirectly this was done in [12] where *a priori* estimate error was performed for *hp*-versions of the discontinuous Galerkin approximation of purely convective problems. In that case, the inverse of the square of the spectral order was chosen. Although this choice when applied to (34) leads to almost the same convergence rates for smooth solutions, the stability decreases too fast in the presence of sharp gradients when the interpolation order grows.

5. Numerical Examples

In this section we present numerical results obtained by applying the proposed methodology to the solution of a variety of convection-dominated problems in which the medium is assumed homogeneous and isotropic with a constant physical diffusivity $\mathbf{K} = 10^{-8}$ and no reaction term, solved in a unit square domain $\Omega = (0,1)\text{x}(0,1)$.

The non-linear CAU and RVCAU formulations are solved using the SUPG's approximate solution as the initial guess and delaying the $\mathbf{C}$ operator one iteration until the following convergence criteria is satisfied:

$$\max_{i=1}^{ntdof} \left|\varphi_n^{h^i} - \varphi_{n-1}^{h^i}\right| < 10^{-3} \ ,$$

where $ntdof$ is the total number of degree of freedom and n is the iteration number.

5.1. Example 1: "Sine function"

To verify the convergence rates, in this first example the source term $f(\mathbf{x})$ was chosen so that the exact solution is the continuous function

$$\varphi(\mathbf{x}) = \sin(2\pi x)$$

with $\mathbf{u}^t = (1,0)$ and an inflow $g = 0$. Figure 1 shows the error measure in the norm (16) as a function of the characteristic length h. Three regular triangular meshes are used, corresponding to a uniformly divided unit square with 4, 8 and 16 divisions. The slope of the lines corresponding to a fixed value of the interpolation order k confirms the expected $k + \frac{1}{2}$ rate. In Figure 2 the error is shown as a function of the number of unknowns. The h-refinements are shown by the solid lines for fixed k and the p-refinements are shown by the dashed lines. It is clear that an improvement in accuracy is obtained by using higher-order elements for the same number of unknowns.

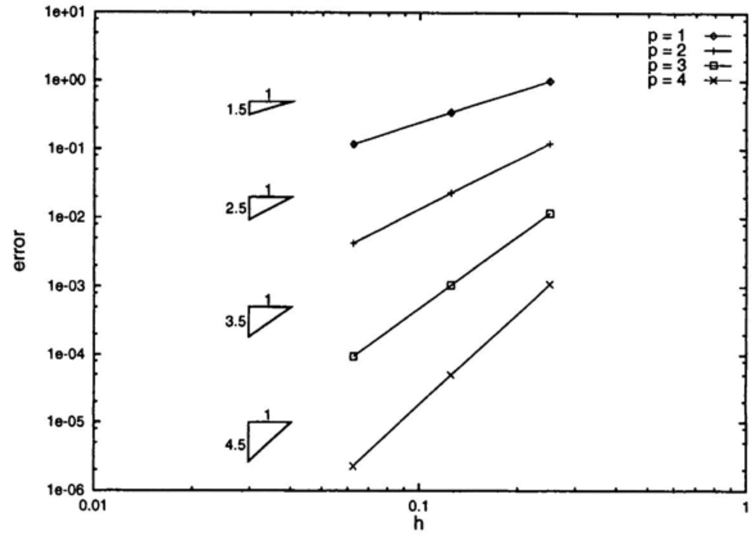

Fig. 1. "Sine function": convergence rates for RVCAU

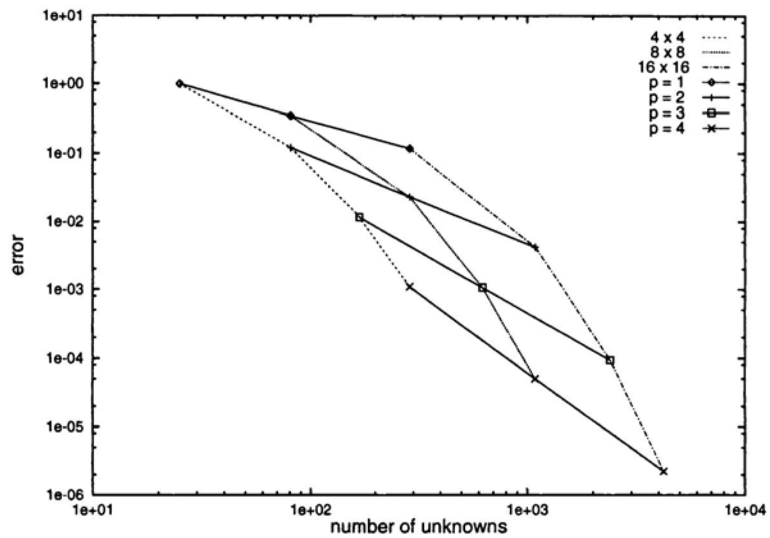

Fig. 2. "Sine function": convergence rates for RVCAU

5.2. Example 2: "Roof problem"

This example deals with a source term defined as

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad 0 < x \leq 0.5 \\ \\ -1 & \text{if} \quad 0 < x \leq 0.5 \end{cases},$$

with homogeneous Dirichlet boundary conditions on all the sides of the domain and a velocity field $\mathbf{u}^t = (1, 0)$. The exact solution consists of by two inclined planes, symmetric about

$x = 0.5$. Because of the boundary conditions, external boundary layers are formed along $y = 0$ and $y = 1$. The approximate solutions along $x = 0.5$ obtained by using the SUPG, the CAU and the RVCAU methods are depicted in Figure 3 using a mesh with 10 divisions in each side of the domain and linear elements. We see that both CAU and RVCAU do not present localized oscillations near the boundary layers, typical of the SUPG solution. Moreover, the results in Figure 4 indicate a remarkable improvement in the accuracy of the CAU and RVCAU's approximate solution when the interpolation order is increased. However, a different behavior is observed for the SUPG's solution since the oscillations become greater.

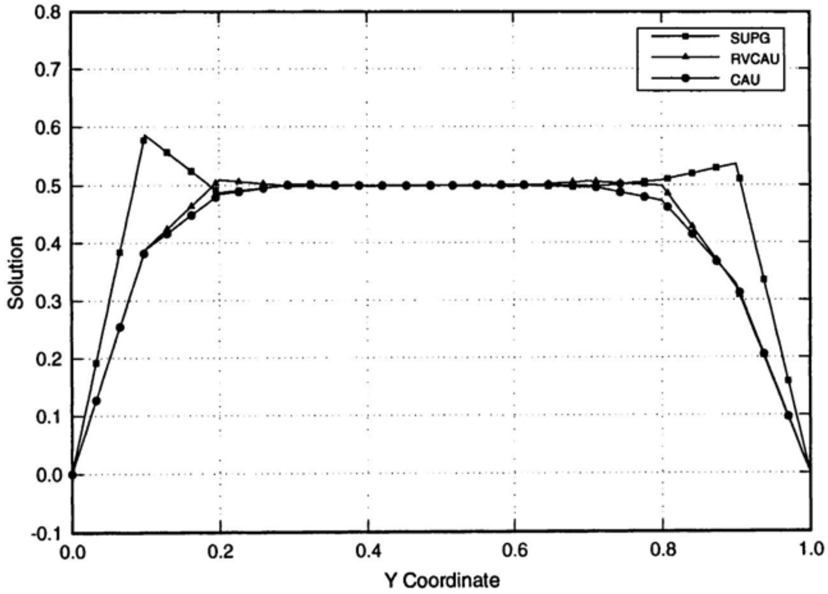

Fig. 3. "Roof problem": solution at $x = 0.5$, $p = 1$.

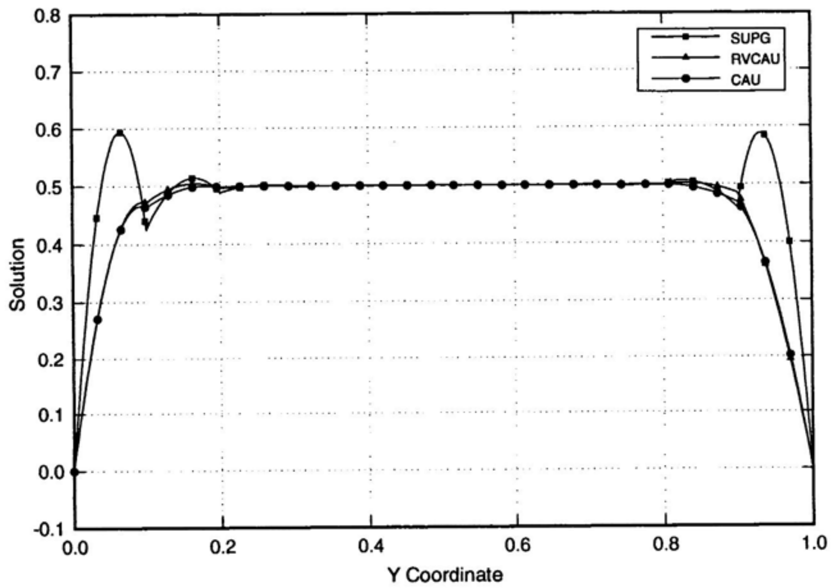

Fig. 4. "Roof problem": solution at $x = 0.5$, $p = 2$.

In smooth regions as, for example, along $y = 0.5$, both the RVCAU and SUPG methods lead to the same solution (the exact one). A more refined mesh (20 divisions in each side) improves the solution near the boundary layer using the RVCAU but is not fine enough to prevent the oscillations appearing in the SUPG solution as shown in Figure 5.

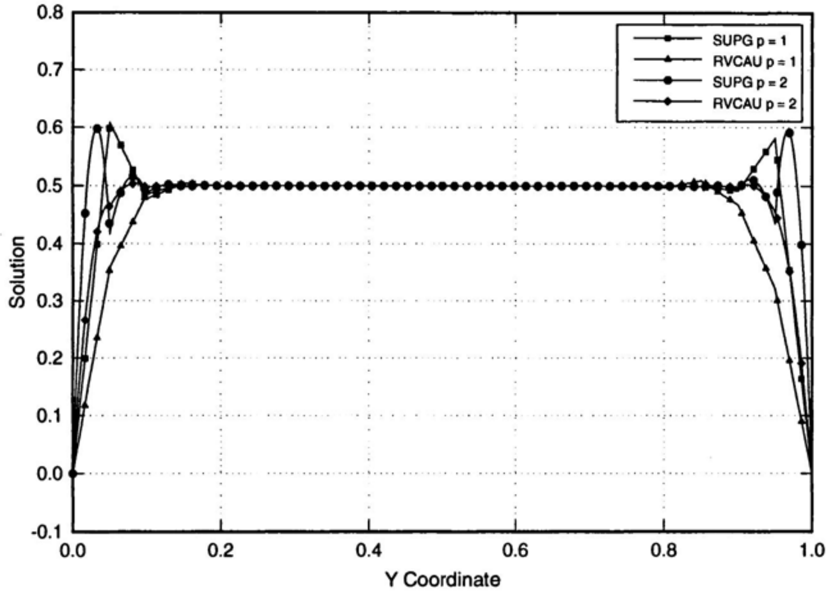

Fig. 5. "Roof problem": solution at $x = 0.5$.

5.3. Example 3: "Advection skew to the mesh"

We consider now the well known problem of the advection skewed with respect to the mesh. In this problem, $f(\mathbf{x}) = 0$, $\mathbf{u}^t = (1,1)$ and the following Dirichlet boundary conditions are assumed:

$$\varphi(x,1) = 0 \; ;$$

$$\varphi(x,0) = 1 \; ;$$

$$\varphi(0,y) = 1 \quad \text{if} \quad y \leq 0.2 \; ;$$

$$\varphi(0,y) = 0 \quad \text{if} \quad y > 0.2 \; ,$$

implying that the exact solution presents an internal boundary layer.

The numerical results for the mid-section $y = 0.5$ are shown in Figures 6-8, obtained for a mesh with 10 divisions in each side of the domain. These results point out the improvement in representing boundary layers when using p-refinement provided we have a stable method, free of spurious oscillations (Figure 6). Clearly, this behavior does nor occur for the SUPG method (Figure 7) and in Figure 8 these methods are compared for $p = 4$. The apparent improvement in the solution produced when increasing the interpolation order near sharp gradients suggests that the algorithm can be a very useful tool in delivering accurate solutions in regions containing discontinuities or high gradients.

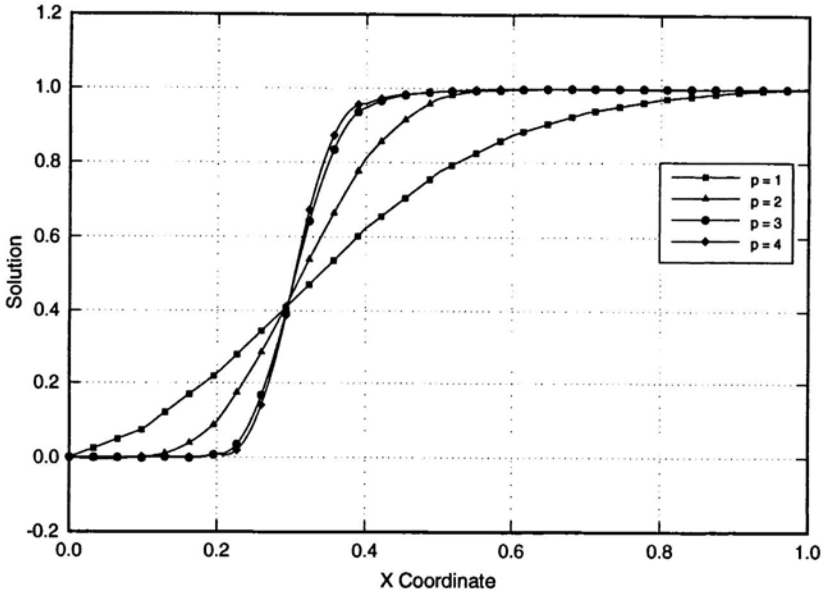

Fig. 6. "Advection skew to the mesh": RVCAU's approximation at $y = 0.5$.

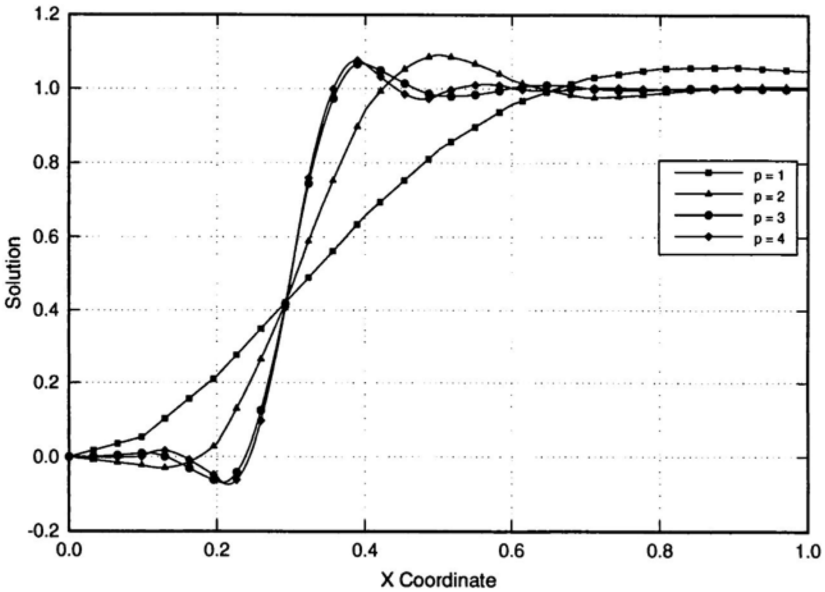

Fig. 7. "Advection skew to the mesh": SUPG's approximation at $y = 0.5$.

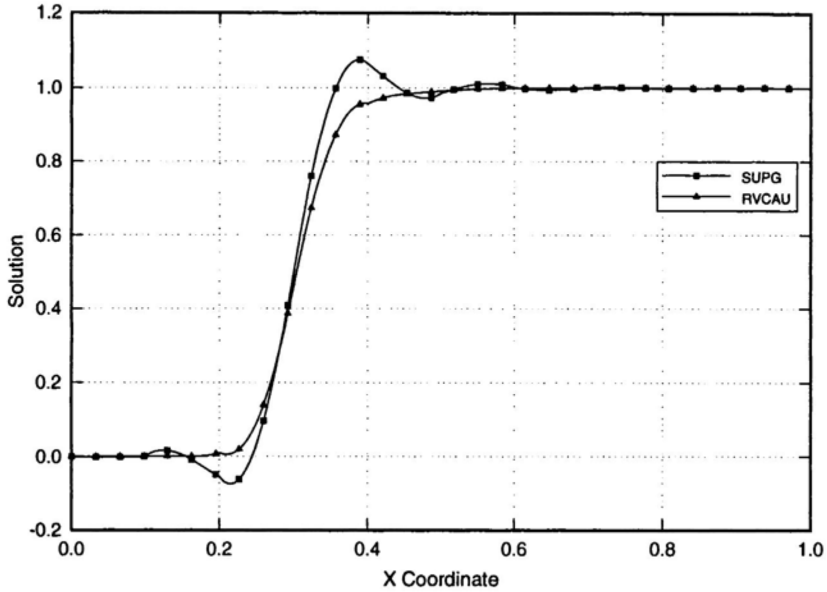

Fig. 8. "Advection skew to the mesh": comparison between RVCAU's and SUPG's approximations at $y = 0.5$ for $p = 4$.

6. Conclusions

In this paper, a new Petrov-Galerkin formulation is designed, which is derived from the CAU method [3] by changing the approximate upwind direction. Applying the upwind functions heuristically designed for higher order elements, this method achieves the same good stability properties of the CAU method in representing steep gradients as well as the accuracy of the SUPG method for smooth solutions. Moreover, remarkable improvement is observed in the approximation of boundary layers when this method is combined with a p-refinement scheme. Indeed, this fact opens a fascinating possibility of combining this stable method with a *hp* adaptive refinement technique so as to solve more accurately and faster a great variety of problems in computational fluid mechanics when convection phenomena are dominated.

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