

A NON-CONFORMING GENERALIZED FINITE ELEMENT METHOD FOR TRANSMISSION PROBLEMS

ANNA L. MAZZUCATO*, VICTOR NISTOR†, AND QINGQIN QU‡

Abstract. We obtain “quasi-optimal rates of convergence” for transmission (interface) problems on domains with smooth, curved boundaries using a non-conforming Generalized Finite Element Method (GFEM). More precisely, we study the strongly elliptic problem $Pu := -\sum \partial_j(A^{ij}\partial_i u) = f$ in a smooth bounded domain Ω with Dirichlet boundary conditions. The coefficients A^{ij} are piecewise smooth, possibly with jump discontinuities along a smooth, closed surface Γ , called the *interface*, which does not intersect the boundary of the domain. We consider a sequence of approximation spaces S_μ satisfying two conditions: (1) *nearly zero boundary and interface matching*, (2) *approximability*, which are similar to those in Babuška, Nistor, and Tarfulea, *J. Comput. Appl. Math.* (2008). Then, if $u_\mu \in S_\mu$, $\mu \geq 1$, is a sequence of Galerkin approximations of the solution u to the interface problem, the approximation error $\|u - u_\mu\|_{\hat{H}^1(\Omega)}$ is of order $O(h_\mu^m)$, where h_μ^m is the typical size of the elements in S_μ and \hat{H}^1 is the Sobolev space of functions in H^1 on each side of the interface. We give an explicit construction of GFEM spaces S_μ for which our two assumptions are satisfied, and hence for which the quasi-optimal rates of convergence hold and present a numerical test.

Key words. strongly elliptic equations, interface problems, transmission conditions, partition of unity, generalized finite element, optimal rate of convergence

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Introduction. Let us consider the second order elliptic equation

$$Pu := -\operatorname{div}(A\nabla u) := -\sum_{i,j=1}^n \partial_j(A^{ij}\partial_i u) = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad (0.1)$$

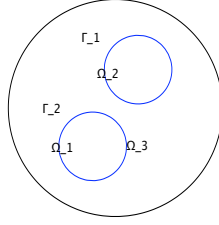
where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$ and P is a second-order, scalar differential operator with real coefficients. We assume that the coefficient matrix $A = (A^{ij})_{n \times n}$ is piecewise smooth with only jump discontinuities across a finite number of smooth, closed surfaces, the union of which is called the *interface*, and is denoted by Γ . In this work, we assume that the interface does not intersect the boundary of Ω , $\partial\Omega \cap \Gamma = \emptyset$. The interface partitions Ω into a finite number of smooth subdomains Ω_j . We say that u is locally strong solution of (0.1) if $u \in C^2(\Omega_j) \cap C(\bar{\Omega}_j)$ solves (0.1) pointwise away from the interface. When the coefficients are piecewise smooth, any locally strong solution is a weak solution if it satisfies certain *jump* or *transmission conditions* at the interface Γ . (See Equation (0.8) below.)

The problem (0.1) is then called a *transmission* (or *interface*) *problem*, and it appears in many practical applications. In particular, transmission problems arise when the underlying physical problem is formulated for a body that consists of more than one material. Because of this connection, extensive research has been done on the transmission problem. For instance, the classical approach using isoparametric elements was extended to interfaces by Barrett and Elliott in [10]. This work extends the earlier work on the Dirichlet problem [9] by the same authors and underscores the similarities between the treatment of boundary and interface problems

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FIG. 0.1. *Domain with smooth boundary and smooth interface*

using isoparametric elements. This similarity is also apparent in the present work for GFEM. Optimal a priori estimates for higher-order FEMs were obtained by Li, Melenk, Wohlmuth, and Zhu in [27]. In that paper, the authors consider the broken Sobolev spaces $\hat{H}(\Omega)$ in the treatment of their error estimates. The broken Sobolev spaces and their weighted analogues also play an important role in the regularity estimates in polygonal and polyhedral domains for interface problems obtained in [26, 40], where also applications to h^m -quasi-optimal rates of convergence for FEM were obtained. Multigrid methods for solving interface problems were obtained in [13, 41, 42, 43]. A systematic treatment of transmission problems is provided in [29]. See also [16, 24, 23, 28, 31, 22] and the references cited therein.

Let us now explain our framework in more detail. We assume that the coefficients A^{ij} are bounded and that the resulting operator is uniformly strongly elliptic, that is:

$$\alpha \|\xi\|^2 \leq \sum_{i,j=1}^n A^{ij}(x) \xi_i \xi_j \leq C \|\xi\|^2, \quad (0.2)$$

for some constants $\alpha > 0$ and $C > 0$ independent of $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$. We also assume that we are given a decomposition

$$\bar{\Omega} = \bigcup_{j=1}^K \bar{\Omega}_j \subset \mathbb{R}^n, \quad (0.3)$$

where Ω_j are disjoint subdomains *with smooth boundaries* that define the interface $\Gamma = (\bigcup_{j=1}^K \partial\Omega_j) \setminus \partial\Omega$. We further assume Γ to be the union of a finite number of closed, smooth simple surfaces Γ_j that do not intersect each other nor the boundary $\partial\Omega$. See Figure 0.1 for a typical example, in which $K = 3$ and Γ is the disjoint union of its two disjoint components $\Gamma_1 \equiv \partial\Omega_1$ and $\Gamma_2 \equiv \partial\Omega_2$.

Our paper provides a treatment of the transmission problem (0.1) that yields the theoretical *h^m -quasi-optimal rate of convergence* for an associated discretization. In general, the solutions of transmission problems have lower regularity in the usual Sobolev spaces. We shall therefore introduce the *broken Sobolev spaces* $\hat{H}^p(\Omega)$ that are better adapted to transmission problems. They are defined by:

$$\hat{H}^p(\Omega) := \{u \in L^2(\Omega), u \in H^p(\Omega_j), \text{ for } j = 1, \dots, K\}, \quad (0.4)$$

with norm and seminorm respectively

$$\|u\|_{\hat{H}^p(\Omega)}^2 = \sum_{j=1}^K \|u\|_{H^p(\Omega_j)}^2, \quad \text{and} \quad |u|_{\hat{H}^p(\Omega)}^2 = \sum_{j=1}^K |u|_{H^p(\Omega_j)}^2. \quad (0.5)$$

Also, we shall denote by

$$\|u\|^2 := \|u\|_{L^2(\partial\Omega)}^2 + \|u_+ - u_-\|_{L^2(\Gamma)}^2, \quad (0.6)$$

where u_+, u_- are the traces of $u \in \hat{H}^1(\Omega)$ on each side of the interface. The trace operator is well-defined on $\hat{H}^s(\Omega)$, $s > 1/2$, as it is well defined on $H^s(\Omega_j)$ for each $j = 1, \dots, K$. We observe that, by density of $C^\infty(\bar{\Omega}_j)$ in $H^s(\Omega_j)$, the trace agrees with the non-tangential limit of u at $\partial\Omega_j$, i.e., on one side of the interface. (For a definition of non-tangential limit, we refer e.g. to [36, page 197].)

We shall consider a sequence $S_\mu \subset \hat{H}^1(\Omega)$ of generalized finite element (GFE) spaces satisfying

$$\dim(S_\mu) \sim h_\mu^{-n}, \quad (0.7)$$

where n is the dimension of the space, and $h_\mu \rightarrow 0$ is the typical size of the elements in S_μ , as in [7], for example. We will denote by u_μ the projection of the solution u to (0.1) onto S_μ in the \hat{H}^1 -seminorm (see Equation (1.4)). We shall further assume that the spaces S_μ are given in terms of a *fixed integer* m , which plays the role of the approximation order in this work and satisfy Assumptions 1.3 (nearly zero boundary conditions and nearly interface matching) and 1.4 (approximability). Under these conditions, we obtain the theoretical h^m -quasi-optimal rate of convergence, which is the main result of this paper.

THEOREM 1. *Let $S_\mu \subset \hat{H}^1(\Omega)$ be a sequence of finite-dimensional spaces, satisfying the nearly zero boundary conditions and interface matching, and the approximability condition, for a sequence $h_\mu \rightarrow 0$. If $f \in \hat{H}^{m-1}(\Omega)$ and $m \geq 1$, then there exists a constant $C > 0$, independent of f and μ , such that*

$$\|u - u_\mu\|_{\hat{H}^1(\Omega)} \leq Ch_\mu^m \|f\|_{\hat{H}^{m-1}(\Omega)},$$

where u is the exact solution to (0.1), and u_μ is its approximation in the space S_μ .

It is known that equation (0.1) has a unique (weak) solution in $H_0^1(\Omega) := H^1(\Omega) \cap \{u|_{\partial\Omega} = 0\}$. As already recalled, the discontinuities of the coefficient matrix A lead to the so called ‘‘transmission’’ or ‘‘jump conditions’’ at the interface Γ :

$$u_+ = u_-, \quad D_{\nu^+}^P u = D_{\nu^-}^P u, \quad (0.8)$$

which are necessary conditions for any locally strong solution away of the interface to be a weak solution. Above, we label the traces (or non-tangential limits) u_+, u_- of u at each side of the interface, and denote the respective conormal derivatives by $D_{\nu^+}^P$ and $D_{\nu^-}^P$, given by:

$$D_{\nu^\pm}^P u := \sum_{i,j} \nu_i A^{i,j} \partial_j u_\pm = \nu \cdot A \cdot \nabla u_\pm, \quad (0.9)$$

where ν is the outer unit normal vector to the interface Γ . By outer normal here, we mean the outer normal to each subdomain Ω_j . The labeling \pm is only for notational convenience and plays no role.

In this article, we study the problem (0.1) using the Generalized Finite Element method (GFEM), extending the work in [6] and [7] to transmission problems. The GFEM is a generalization of the meshless methods based on partitions of unity. The support of the partition of unity need not depend on any mesh or on the geometry of the domain Ω . As an extension of FEM, the GFEM allows great flexibility

in constructing trial spaces. In particular, the GFEM allows one to include a priori knowledge about the local behavior of the solution, and gives the option of constructing trial spaces of any desired regularity. For more information on the GFEM and associated partition of unity, see [1, 3, 4, 5, 8, 37] and references cited therein. Recent work showcases some of the advantages of the GFEM, as it applies to various fields [18, 19, 34, 38].

We next briefly review the main ideas of the GFEM. Given overlapping patches $\{\omega_j\}$ such that $\bar{\Omega} = \cup_{j=1}^N \bar{\omega}_j$, let $\{\psi_j\}$ be a partition of unity subordinate to these patches. On each patch, let function spaces Ψ_j reflect the local approximability. Then the global GFE space S is given by

$$S = \left\{ \sum_j \psi_j v_j, v_j \in \Psi_j \right\}. \quad (0.10)$$

Our local approximation spaces are polynomials or piecewise polynomials, depending on whether the patch intersects the boundary or the interface, which makes their implementation easier to carry out than for non-polynomial spaces.

Similarly to the case of the usual FEM, one of the major problems in the implementation of meshless methods is the enforcement of Dirichlet boundary conditions. As in the paper [7], we address the problem of enforcing Dirichlet boundary conditions in the GFEM framework. The classical FEM assumes that the trial subspace functions fulfill the boundary conditions. Nevertheless, in practice the construction implies difficulties when the boundary is curved. Nitsche's approach is to use subspace with nearly zero boundary conditions. This idea was first outlined by Nitsche [30], and further studied by Berger, Scott and Strang [11] and Scott [33], who called it "interpolated boundary conditions". In this paper, we also use nearly zero boundary conditions. As discussed in Section 4, this method of enforcing boundary conditions is straightforward to realize in practice. In fact, our code for the GFEM is very simple (available on the authors' webpage).

For transmission problems an additional difficulty arises from the interface. Because of the loss of regularity in the problem (0.1), in order to get a good local approximation space, one needs to include transmission conditions in the construction of the space. Again, it is difficult to impose the transmission conditions exactly on the approximate solution, when the interface is curved. We employ polynomials to approximate the interface, and then construct the local approximation spaces of piecewise polynomials that are *nearly continuous* across the approximate interface and allow jumps in their derivatives. While superficially, the difficulties encountered in enforcing the Dirichlet boundary conditions are mirrored in similar difficulties in enforcing the transmission conditions at the interface, in fact, both theoretically and from an implementation point of view, the treatment of interfaces requires a new approach to deal with the discontinuities of the trial functions across the interface. The approach we used has certain points in common with the Isoparametric Finite Element Method (IFEM) since both of them use polynomials to approximate the boundary. But our construction is also different from the IFEM. See [15, 17], or [35] for the recent work and applications on IFEM. In the IFEM the domain is approximated by Ω_h and the approximate solution u_h is sought in a subspace $V_h \subset H_0^1(\Omega_h)$, so the boundary condition $u = 0$ on $\partial\Omega$ is approximated by $u_h = 0$ on $\partial\Omega_h$. Differently than in the IFEM however, we do not need to map to a reference domain, which may entail a non-polynomial change of coordinates.

Although we restrict our attention in this paper to scalar equations with Dirichlet boundary conditions, our method can be generalized to other type of boundary

conditions, such as Neumann boundary conditions, and to strongly elliptic systems. Non-homogeneous boundary conditions can be treated exactly like in [7], since the interface does not touch the boundary.

The paper is organized as follows. In the first section, we show that Assumptions 1.3 and 1.4 lead to h^m -quasi-optimal rates of convergence. In Section 2 we introduce the GFE spaces used in this paper, in particular, we introduce the conditions A(h_μ), B, C, and D satisfied by the data that define our GFE spaces. In Section 3, we prove that conditions A(h_μ), B, C, and D imply Assumptions 1.3 and 1.4. In Section 4, numerical tests are provided to confirm our theoretical result.

1. Approximate Dirichlet boundary conditions. In this section we prove our main approximation result (Theorem 1.11) under the two Assumptions formulated below.

We recall that $\bar{\Omega} = \cup_{j=1}^K \bar{\Omega}_j$ is a bounded domain that is decomposed as the union of the closures of the disjoint domains Ω_j . We also recall that the set $\Gamma := \cup_{j=1}^K \partial\Omega_j \setminus \partial\Omega$ is called the *interface*. We shall fix in what follows $m \in \mathbb{N} = \{1, 2, \dots\}$, which will play the role of the order of approximation.

In this paper, we shall assume that both the boundary $\partial\Omega$ and Γ are smooth and do not intersect each other: $\partial\Omega \cap \Gamma = \emptyset$.

As already mentioned, any locally strong solution to (0.1) (i.e., any solution in $C^2(\Omega_j) \cap C(\bar{\Omega}_j)$) satisfies the Dirichlet problem for equation (0.1) in weak form only if it satisfies the following interface problem

$$\left\{ \begin{array}{ll} Pu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_+ = u_- & \text{on } \Gamma \\ D_{\nu^+}^P u = D_{\nu^-}^P u & \text{on } \Gamma, \end{array} \right. \quad (1.1)$$

where u_+ , u_- are the traces of u on each side of the interface and $D_{\nu^+}^P u$, $D_{\nu^-}^P u$ their conormal derivatives (given in (0.9)). The transmission conditions are hence a consequence of the weak formulation, and will always be considered as part of Equation (1.1).

We want to approximate u with functions $u_\mu \in S_\mu$, $\mu \in \mathbb{N}$, where $S_\mu \subset \hat{H}^1(\Omega)$ is a sequence of finite dimensional subspaces that satisfy Assumptions 1.3 and 1.4 below.

First, let us recall the *broken Sobolev spaces* given in (0.4) and (0.5). Let

$$B(u, v) = \sum_{i,j}^n \sum_k^K \int_{\Omega_k} A^{ij} \partial_i u \partial_j v dx \quad (1.2)$$

be the bilinear form associated with problem (0.1). From the elliptic property (0.2), we have the following coercivity estimate

$$\alpha |v|_{\hat{H}^1(\Omega)}^2 \leq B(v, v), \text{ for } v \in \hat{H}^1(\Omega). \quad (1.3)$$

We also have the following continuity property

$$B(u, v) \leq |u|_{\hat{H}^1(\Omega)} |v|_{\hat{H}^1(\Omega)}.$$

Assume now that we are given a sequence of finite-dimensional trial spaces $S_\mu \subset \hat{H}^1(\Omega)$, and a numeric sequence $h_\mu > 0$, $\mu \in \mathbb{N}$, $h_\mu \rightarrow 0$ as $\mu \rightarrow \infty$.

REMARK 1.1. *In the explicit construction of the spaces S_μ given later in the paper, h_μ will be the maximum size of the elements in the approximation space. We do not impose directly any condition on the minimum size of the patches, so no quasi-uniform condition is required. However, the conditions on the partition of unity may imply a minimum condition on the diameters of the patches.*

Then we define the discrete solution u_μ for μ large, as the solution of

$$B(u_\mu, v_\mu) = \int_{\Omega} f(x)v_\mu(x)dx, \text{ for all } v_\mu \in S_\mu. \quad (1.4)$$

This is *essentially* the standard way to define the approximate solution for an elliptic problem. But note that, since the elements in S_μ do not exactly satisfy the Dirichlet boundary conditions on $\partial\Omega$ or the transmission conditions on Γ , the approximate solution u_μ is not a variational solution of the Dirichlet problem for (0.1) in the space S_μ , contrary to what happens for conforming methods.

REMARK 1.2. *In principle, one needs to know the geometry of Ω exactly in order to define the bilinear form B . In practice, even when the geometry is not known, it is usually given with some degree of accuracy. The relation between the influence of the approximation of the geometry, in particular the quadrature method used to numerically compute B , on the accuracy of the method is an interesting and important point. Results in this direction were obtained in [27, Corollary 4.9], where optimal error estimates were obtained taking into account also the error made by the approximation of the bilinear B due to the limited knowledge of the interface. In that paper, also more freedom was allowed in approximating the interface. See also [10] for a related analysis, where isoparametric elements were used. These works deal with the Finite Element Method. Clearly this problem deserves further study in the case of the Generalized Finite Element Method. Also, it may be possible to use “boundary fitting curves” (transfinite blending functions) to match the boundary exactly (we owe this remark to J. Melenk). That approach would require a different treatment of the approximation issues than the one in this paper. We hope to address these points in future work.*

Recall that we have denoted by $\| |w| \|^2 = \|w\|_{L^2(\partial\Omega)}^2 + \|w_+ - w_-\|_{L^2(\Gamma)}^2$.

Given a function $w \in \hat{H}^1(\Omega)$, we also define $\llbracket w \rrbracket : \partial\Omega \cup \Gamma \rightarrow \mathbb{C}$ as the function that is equal to w on the boundary and to the jump $w_+ - w_-$ on the interface Γ in trace sense. Thus $\| |w| \| = \|\llbracket w \rrbracket\|_{L^2(\partial\Omega \cup \Gamma)}$.

We are now ready to state the conditions that the trial spaces S_μ must satisfy in order for the quasi-optimal rates of convergence for the approximate solution sequence to hold. We recall that m is a fixed integer, which plays the role of the *approximation order*.

ASSUMPTION 1.3. *Nearly zero boundary and nearly interface matching condition: For each fixed $m \geq 1$, there exists $C > 0$, independent of μ and v_μ , such that*

$$\| |v_\mu| \| \leq Ch_\mu^m \|v_\mu\|_{\hat{H}^1(\Omega)}, \quad \forall \mu.$$

ASSUMPTION 1.4. *Approximability: For each fixed $m \geq 1$, there exists $C > 0$ such that for any $0 \leq r \leq m$, any $u \in \hat{H}^{r+1}(\Omega) \cap H_0^1(\Omega)$, and any μ , there exists $u_I \in S_\mu$ satisfying*

$$\|u - u_I\|_{\hat{H}^1(\Omega)} \leq Ch_\mu^r \|u\|_{\hat{H}^{r+1}(\Omega)}.$$

In Section 2, we will give an example of a sequence of GFE spaces S_μ that satisfies these two assumptions (see Condition **A**(h_μ), Condition **B**, Condition **C** and Condition **D**).

REMARK 1.5. *In this paper, we shall use the convention that C indicates a generic positive constant, independent of μ , which may be different each time when it is used and may depend on the approximation integer m .*

We follow [7, 30, 33] in proving our main result from these two assumptions. The proof is a consequence of several lemmas.

We start by establishing a needed norm equivalence in the finite dimensional trial spaces S_μ . The following variant of the classical Poincaré's inequality that takes into account the jumps at the interface Γ can be proved as in the classical case [15]. More details can be found [10, 12], where some very similar results have been proved. See also [14].

LEMMA 1.6. *There exists a constant C depending only on Ω such that*

$$\|u\|_{\hat{H}^1(\Omega)}^2 \leq C[|u|_{\hat{H}^1(\Omega)}^2 + \|u\|^2]. \quad (1.5)$$

for all $u \in \hat{H}^1(\Omega)$.

We are ready not to prove the following equivalence of norms.

LEMMA 1.7. *Given a sequence of trial spaces S_μ satisfying Assumptions 1.3 and 1.4, there exists $C > 0$ independent of μ , such that for any μ large enough and all $v_\mu \in S_\mu$,*

$$\|v_\mu\|_{\hat{H}^1(\Omega)} \leq C|v_\mu|_{\hat{H}^1(\Omega)}.$$

In particular, $|v_\mu|_{\hat{H}^1(\Omega)}$ and $\|v_\mu\|_{\hat{H}^1(\Omega)}$ are equivalent norms on S_μ .

Proof. By definition, $|v_\mu|_{\hat{H}^1(\Omega)} \leq \|v_\mu\|_{\hat{H}^1(\Omega)}$. To establish the opposite inequality, we observe that from Lemma 1.6,

$$\begin{aligned} \|v_\mu\|_{\hat{H}^1(\Omega)}^2 &\leq C[|v_\mu|_{\hat{H}^1(\Omega)}^2 + \|v_\mu\|^2] \\ &\leq C|v_\mu|_{\hat{H}^1(\Omega)}^2 + Ch_\mu^{2m}\|v_\mu\|_{\hat{H}^1(\Omega)}^2, \end{aligned}$$

where the last inequality follows from Assumption 1.3. Therefore,

$$\|v_\mu\|_{\hat{H}^1(\Omega)} \leq C(1 - Ch_\mu^{2m})^{-1/2}|v_\mu|_{\hat{H}^1(\Omega)}.$$

Since $h_\mu \rightarrow 0$ as $\mu \rightarrow \infty$ by hypothesis, for μ large enough $(1 - Ch_\mu^{2m}) \geq 1/2$, which gives the desired estimate. \square

We now turn to study the problem (1.1) in the broken Sobolev spaces. Recall that D_ν^P is the conormal derivative in the sense of trace at the boundary $\partial\Omega$, while $D_{\nu^+}^P$ and $D_{\nu^-}^P$ are the conormal derivatives on each side of the interface Γ , defined in equation (0.9).

Let $w_\mu \in S_\mu$, for μ large, be the solution of the following variational problem: for all $v_\mu \in S_\mu$,

$$B(w_\mu, v_\mu) = \int_{\partial\Omega \cup \Gamma} D_\nu^P u[[v_\mu]] dS(x), \quad (1.6)$$

where u is the solution of (1.1) and $D_\nu^P u$ on Γ is given by $D_{\nu^+}^P u = D_{\nu^-}^P u$ in view of the transmission conditions that u must satisfy.

By Lemma 1.7, the Lax-Milgram Lemma (see e.g. [17]), and inequality (1.3), there exists a unique solution to problem (1.4) and (1.6).

LEMMA 1.8. *Let u be the solution of the transmission problem (1.1) and let u_μ , and w_μ be as in Equation (1.4) and (1.6). Then $B(u - u_\mu - w_\mu, v_\mu) = 0$ for all $v_\mu \in S_\mu$, and hence*

$$|u - u_\mu - w_\mu|_{\hat{H}^1(\Omega)} \leq C|u - v_\mu|_{\hat{H}^1(\Omega)} \text{ for all } v_\mu \in S_\mu.$$

Proof. Integrating by parts and using the transmission conditions for u on the interface Γ , we have for all $v_\mu \in S_\mu$,

$$\begin{aligned} B(u, v_\mu) &:= \sum_{i,j} \sum_k \int_{\Omega_k} A^{ij} \partial_i u \partial_j v_\mu dx \\ &= - \sum_{i,j} \sum_k \int_{\Omega_k} \partial_j (A^{ij} \partial_i u) v_\mu dx + \int_{\partial\Omega \cup \Gamma} D_\nu^P u \llbracket v_\mu \rrbracket dS(x) \\ &= \int_{\Omega} f v_\mu dx + \int_{\partial\Omega \cup \Gamma} D_\nu^P u \llbracket v_\mu \rrbracket dS(x) = B(u_\mu + w_\mu, v_\mu). \end{aligned}$$

Using the continuity and coercivity of the bilinear form B , we obtain that, for all $v_\mu \in S_\mu$

$$\begin{aligned} \alpha |u - u_\mu - w_\mu|_{\hat{H}^1(\Omega)}^2 &\leq B(u - u_\mu - w_\mu, u - u_\mu - w_\mu) = B(u - u_\mu - w_\mu, u - v_\mu) \\ &\leq C|u - u_\mu - w_\mu|_{\hat{H}^1(\Omega)} |u - v_\mu|_{\hat{H}^1(\Omega)}, \end{aligned}$$

which gives the desired estimate. \square

We assume now that the solution u of (1.1) is in $\hat{H}^2(\Omega)$. We will show later that this is the case for the problem at hand.

LEMMA 1.9. *For μ sufficiently large, the solution w_μ of problem (1.6) satisfies*

$$\|w_\mu\|_{\hat{H}^1(\Omega)} \leq Ch_\mu^m \|u\|_{\hat{H}^2(\Omega)},$$

where C is a constant independent of μ and u .

Proof. From Lemma 1.7 and inequality (1.3) we have:

$$\begin{aligned} \|w_\mu\|_{\hat{H}^1(\Omega)}^2 &\leq C|w_\mu|_{\hat{H}^1(\Omega)}^2 \leq C\alpha^{-1} B(w_\mu, w_\mu) \\ &= C\alpha^{-1} \int_{\partial\Omega \cup \Gamma} D_\nu^P u \llbracket w_\mu \rrbracket dS(x) \\ &\leq C \|D_\nu^P u\|_{L^2(\partial\Omega \cup \Gamma)} \|\llbracket w_\mu \rrbracket\|_{L^2(\partial\Omega \cup \Gamma)} \\ &\leq C \sum_j \|D_\nu^P u\|_{L^2(\partial\Omega_j)} \|w_\mu\| \\ &\leq Ch_\mu^m \sum_j \|u\|_{\hat{H}^2(\Omega_j)} \|w_\mu\|_{\hat{H}^1(\Omega)} \\ &\leq Ch_\mu^m \|u\|_{\hat{H}^2(\Omega)} \|w_\mu\|_{\hat{H}^1(\Omega)}, \end{aligned}$$

where we have used Assumption 1.3, the coercivity of B (Equation (1.3)), and the trace estimate $\|D_\nu^P u\|_{L^2(\partial\Omega_j)} \leq C\|u\|_{\hat{H}^2(\Omega_j)}$. \square

LEMMA 1.10. For μ sufficiently large, the solution u_μ of problem (1.4) satisfies

$$\|u_\mu\|_{\hat{H}^1(\Omega)} \leq C\|u\|_{\hat{H}^2(\Omega)}$$

where C is a constant independent of μ and u .

Proof. Previous lemmas give

$$\begin{aligned} \|u_\mu\|_{\hat{H}^1(\Omega)}^2 &\leq C|u_\mu|_{\hat{H}^1(\Omega)}^2 \leq C\alpha^{-1}B(u_\mu, u_\mu) \\ &= C\alpha^{-1}[B(u, u_\mu) - B(w_\mu, u_\mu)] \\ &\leq C\|u\|_{\hat{H}^1(\Omega)}\|u_\mu\|_{\hat{H}^1(\Omega)} + C\|D_\nu u(x)\|_{L^2(\partial\Omega \cup \Gamma)}\|u_\mu\| \\ &\leq C\|u\|_{\hat{H}^2(\Omega)}\|u_\mu\|_{\hat{H}^1(\Omega)} + Ch_\mu^m\|u\|_{\hat{H}^2(\Omega)}\|u_\mu\|_{\hat{H}^1(\Omega)} \\ &\leq C\|u\|_{\hat{H}^2(\Omega)}\|u_\mu\|_{\hat{H}^1(\Omega)} \leq C\|u\|_{\hat{H}^2(\Omega)}^2. \end{aligned}$$

In the second inequality, we used Assumption 1.3 and, in each subdomain, the trace estimate $\|D_\nu^p u(x)\|_{L^2(\partial\Omega_j)} \leq C\|u\|_{\hat{H}^2(\Omega_j)}$. \square

We are now ready to prove our main result.

THEOREM 1.11. Given $m \geq 1$, let $S_\mu \subset \hat{H}^1(\Omega)$ be a sequence of finite dimensional subspaces satisfying Assumptions 1.3 and 1.4 for a sequence $h_\mu \rightarrow 0$. Let $f \in \hat{H}^{p-1}(\Omega)$, $1 \leq p \leq m$. Then the unique solutions u and u_μ of problems (1.1) and (1.4), respectively, satisfy

$$\|u - u_\mu\|_{\hat{H}^1(\Omega)} \leq Ch_\mu^p\|u\|_{\hat{H}^{p+1}(\Omega)} \leq C_1h_\mu^p\|f\|_{\hat{H}^{p-1}(\Omega)},$$

for a constant C independent of μ and f .

Proof. Since $f \in \hat{H}^{p-1}(\Omega)$ and each Ω_j is a smooth domain with smooth boundary, by regularity results in [32], $u \in \hat{H}^{p+1}(\Omega)$, and $\|u\|_{\hat{H}^{p+1}(\Omega)} \leq C\|f\|_{\hat{H}^{p-1}(\Omega)}$. We assume $p = m$, for simplicity. Lemma 1.8, Lemma 1.9, and Assumption 1.4 give

$$\begin{aligned} |u - u_\mu|_{\hat{H}^1(\Omega)} &\leq |u - u_\mu - w_\mu|_{\hat{H}^1(\Omega)} + |w_\mu|_{\hat{H}^1(\Omega)} \\ &\leq |u - u_I|_{\hat{H}^1(\Omega)} + Ch_\mu^m\|u\|_{\hat{H}^2(\Omega)} \\ &\leq Ch_\mu^m\|u\|_{\hat{H}^{m+1}(\Omega)} + Ch_\mu^m\|u\|_{\hat{H}^2(\Omega)} \leq Ch_\mu^m\|u\|_{\hat{H}^{m+1}(\Omega)}. \end{aligned}$$

From Lemma 1.6 and Assumption 1.3, then

$$\begin{aligned} \|u - u_\mu\|_{\hat{H}^1(\Omega)}^2 &\leq C(|u - u_\mu|_{\hat{H}^1(\Omega)}^2 + \|u_\mu\|) \\ &\leq Ch_\mu^m\|u\|_{\hat{H}^{m+1}(\Omega)} + Ch_\mu^m\|u_\mu\|_{\hat{H}^1(\Omega)} \\ &\leq Ch_\mu^m\|u\|_{\hat{H}^{m+1}(\Omega)} \leq C_1h_\mu^m\|f\|_{\hat{H}^{m-1}(\Omega)}. \end{aligned}$$

The proof is now complete. \square

We therefore obtain a quasi-optimal rate of convergence for the approximate solution $u_\mu \in S_\mu$.

2. GFE spaces. In this section, we discuss the construction of a sequence of GFE spaces S_μ that satisfies Assumptions 1.3 and 1.4. Our choice of the sequence S_μ is determined by a sequence of data $\Sigma = \{\Omega_j, \phi_j, \Psi_j, \omega_j^*\}$, described next. Later in this section (see Subsection 2.2), we recast the assumptions on the spaces S_μ as conditions, specifically Condition **A**(h_μ), **B**, **C**, and **D** below, on the data.

We follow the notations and basic techniques from [6] and [7].

2.1. Definition of GFE spaces. We begin by recalling the properties of the partition of unity on which the GFE spaces are based.

DEFINITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open set and $\{\omega_j\}_{j=1}^N$ be an open cover of Ω such that any $x \in \Omega$ belongs to at most κ of the sets ω_j . The covering sets $\{\omega_j\}$ are called patches.*

Let $\{\phi_j\}$ be a partition of unity consisting of $W^{m,\infty}(\Omega)$ functions subordinated to the covering $\{\omega_j\}$. That is,

- (i) *supp $(\phi_j) \subset \bar{\omega}_j$, and*
- (ii) *$\sum_{j=1}^N \phi_j = 1$, in Ω .*

The collection $\{\phi_j\}$ is then called a $(\kappa, C_0, C_1, \dots, C_m)$ partition of unity, if

$$\|\partial^\alpha \phi_j\|_{L^\infty(\Omega)} \leq C_k / (\text{diam}(\omega_j))^k, \quad k = |\alpha| \leq m,$$

for any $j = 1, \dots, N$.

We will also need the following basic concept (see e.g. [15]).

DEFINITION 2.2. *A set ω is star-shaped with respect to $\omega^* \subset \omega$ if, for every $x \in \omega$ and every $y \in \omega^*$, the segment with end points x and y is completely contained in ω .*

We next assume that we are given linear subspaces $\Psi_j \subset H^{m+1}(\omega_j)$ if ω_j does not touch the interface, and $\Psi_j \subset \hat{H}^{m+1}(\omega_j)$ if ω_j does touch the interface, for $j = 1, \dots, N$. The space $\{\Psi_j\}$ are called *local approximation spaces* and they will be used to define the GFE space

$$S = S_{GFE} := \left\{ w = \sum_{j=1}^N \phi_j v_j, \quad v_j \in \Psi_j \right\} \subset \hat{H}^1(\Omega).$$

We will take the patches ω_j to be star-shaped with respect to given ω_j^* , to be chosen later. Then, the set $\{\Omega_j, \phi_j, \Psi_j, \omega_j^*\}$ will be called *the set of data defining the GFE space S* .

REMARK 2.3. *The usual piecewise-linear hat functions satisfy the conditions of a (κ, C_0, C_1) partition of unity (Definition 2.1). A more general choice of a partition of unity is given by the following well-known procedure. Let $\{\psi_j\}$ be a collection of functions which are supported on the patches $\{\omega_j\}$. Then setting $\phi_j := \frac{\psi_j}{\sum_j \psi_j}$ yields a partition of unity, the elements of which are called Shepard functions. In practice, in order to solve Equation (1.4), the basis for the test and trial spaces needs to be found. One difficulty in finding a basis for the trial spaces is that, if $\{v_j\}$ is a basis of functions of the local spaces Ψ_j , the functions $\{\phi_j v_i\}$ may be linearly dependent or nearly linearly dependent. One can avoid the linear dependence if we use a ‘‘flat-top’’ partition of unity (we refer the reader to [3, 5].) Our Condition **C** in the next subsection guarantees this linear independence.*

We shall need the following standard Lemma (see for example [5]).

LEMMA 2.4. *Let $\{\psi_j\}$ be a countable collection of measurable functions defined on an open set W and fix $s \geq 0$. Assume that there exists an integer κ such that any point $x \in W$ can belong to no more than κ of the sets $\text{supp}(\psi_j)$. Let $f = \sum_j \psi_j$. Then there exists a constant $C > 0$, depending only on κ , such that $\|f\|_{H^s(W)}^2 \leq C \sum_j \|\psi_j\|_{H^s(W)}^2$.*

2.2. Conditions on GFE data defining S_μ . In this subsection we introduce some conditions on the data defining a GFE space. Let \tilde{h} be a constant that is small enough.

For the definition of the local approximation spaces near the interface Γ or the boundary $\partial\Omega$, we shall need the following construction. For each j such that $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$, let us choose $x_j \in \bar{\omega}_j \cap \partial\Omega$ and define local coordinates near x_j such that x_j is mapped to $0 \in \mathbb{R}^n$ and the tangent space to $\partial\Omega$ at x_j is mapped to $\{x_n = 0\} = \mathbb{R}^{n-1}$. For \tilde{h} small, if the diameter of ω_j is less than \tilde{h} , we can assume that $\bar{\omega}_j \cap \partial\Omega$ is contained in the graph of a smooth function $g_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. In the special coordinates chosen above, we denote $\bar{x} = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, so that $x = (\bar{x}, x_n) \in \mathbb{R}^d$. Denote by d_j the diameter of ω_j and let $q_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be a polynomial of order m such that

$$|g_j(\bar{x}) - q_j(\bar{x})| \leq Cd_j^{m+1}, \quad |\nabla g_j - \nabla q_j| \leq Cd_j^m, \quad (2.1)$$

for all $(\bar{x}, x_n) \in \omega_j$. We can choose the function q_j such that $\partial^\alpha q_j(0) = \partial^\alpha g_j(0)$, for all $|\alpha| \leq m$. We then denote by $\tilde{q}_j : \mathbb{R}^n \rightarrow \mathbb{R}^d$ the bijective map

$$\tilde{q}_j(x) = \tilde{q}_j(\bar{x}, x_n) = (\bar{x}, x_n + q_j(\bar{x})). \quad (2.2)$$

The existence of such function q_j follows by Taylor expanding the local boundary distance function. For example, let Ω be the unit disk in \mathbb{R}^2 . We choose $m = 1$ and local coordinates (x, y) at the point $x_0 = (1, 0)$. We clearly have $g_j(x) = 1 - \sqrt{1 - x^2}$ for $0 \leq x \leq d_j$. Then, we can choose $q_j(x) = x$, where $0 \leq x \leq d_j$. By Taylor expanding g_j in local coordinates $g_j(x) = q_j(x) + \frac{g_j''(0)}{2}x^2 + \dots$, so that $|g_j(x) - q_j(x)| \leq Cd_j^2$. Similarly, $|\nabla g_j - \nabla q_j| \leq Cd_j$ also holds.

We also choose a point $x_i \in \omega_i \cap \Gamma$, if this intersection is not empty. We then again choose local coordinates such that x_i is mapped to 0 and the tangent space to Γ at x_i is mapped to $\mathbb{R}^{n-1} = \{x_n = 0\}$. For \tilde{h} small, we can assume that $\Gamma \cap \omega_i$ is contained in the graph of a smooth function $f_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. Let $r_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be a polynomial of order m satisfying:

$$|f_i(\bar{x}) - r_i(\bar{x})| \leq Cd_i^{m+1}, \quad |\nabla f_i - \nabla r_i| \leq Cd_i^m, \quad (2.3)$$

for all $(\bar{x}, x_n) \in \omega_j$. We also denote by $\tilde{r}_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the map

$$\tilde{r}_i(x) := \tilde{r}_i(\bar{x}, x_n) = (\bar{x}, x_n + r_i(\bar{x})), \quad (2.4)$$

and set $\omega_j \setminus \Gamma = \omega_j^+ \cup \omega_j^-$ to be the decomposition of ω_j defined by the interface Γ .

An example of functions r_j satisfying (2.3) can be constructed in the same way as for the case of the disk described above (take. e.g. Ω to be the unit disk and Γ to be a smaller concentric circle to the unit circle). We also remark that, since $\tilde{q}_j^{-1}(x) = (\bar{x}, x_n - q_j(\bar{x}))$ and similarly for \tilde{r}_j^{-1} , both inverse maps are still polynomial in local coordinates at x_j .

Condition A(h): We have that $\bar{\Omega} = \cup_{j=1}^N \bar{\omega}_j$. The set ω_j is open cover (patch) of diameter $d_j \leq h < 1$ and $\omega_j^* \subset \omega_j$ is an open ball of diameter $\geq \sigma d_j$ such that ω_j is star-shaped with respect to ω_j^* . If $\Gamma \cap \omega_j \neq \emptyset$, we require that Γ divides ω_j into two subsets ω_j^+ and ω_j^- , each of which is star-shaped with respect to a (possibly different) ball of diameter $\geq \sigma d_j$ contained in ω_j^* .

Condition B: The family of functions $\{\phi_j\}_{j=1}^N$ is a $(\kappa, C_0, C_1, \dots, C_m)$ partition of unity (see Definition 2.1).

Condition C: We have $\phi_j = 1$ on ω_j^* for all $j = 1, \dots, N$ for which $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$.

The following condition defines the local approximation spaces Ψ_j . Let us denote by \mathcal{P}_m the space of polynomials of order at most m in n variables.

Condition D:

- (i) We have $\Psi_j = \mathcal{P}_m$ if $\bar{\omega}_j \cap \partial\Omega = \emptyset$ and $\bar{\omega}_j \cap \Gamma = \emptyset$.
- (ii) If $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$ and $\bar{\omega}_j \cap \Gamma = \emptyset$, then Ψ_j consists of function w of the form

$$w(x) = (p \circ \tilde{q}_j^{-1})(x), \quad p \in \mathcal{P}_m, \text{ such that } p(\bar{x}, 0) = 0. \quad (2.5)$$

- (iii) If $\bar{\omega}_j \cap \Gamma \neq \emptyset$, then Ψ_j consists of functions w of the form

$$w(x) = \begin{cases} p(x) + (p^+ \circ \tilde{r}_j^{-1})(x) & \text{in } \omega_j^+, \\ p(x) & \text{in } \omega_j^-, \end{cases} \quad (2.6)$$

where $p, p^+ \in \mathcal{P}_m$ and $p^+(\bar{x}, 0) = 0$.

REMARK 2.5. *An equivalent form for the condition $p \in \mathcal{P}_m, p(\bar{x}, 0) = 0$ in part (ii) above is that $p = x_n p_1, p_1 \in \mathcal{P}_{m-1}$. Since $\tilde{q}_j^{-1}(x) = (\bar{x}, x_n - q_j(\bar{x}))$, we obtain $w(\bar{x}, x_n) = (x_n - q_j(\bar{x}))(p_1 \circ \tilde{q}_j^{-1})(x)$. It is not difficult to see that Ψ_j is a subset of the space of polynomials in (ii). We observe that $p \circ \tilde{q}_j^{-1}$ is zero only across the approximate boundary defined by r_j , so the elements of Ψ_j are “nearly zero” across the boundary $\partial\Omega$.*

REMARK 2.6. *The definition of w in (2.6) is equivalent to*

$$w(x) = \begin{cases} p(x) + (x_n - r_j(\bar{x}))(p_1 \circ \tilde{r}_j^{-1})(x), & \text{in } \omega_j^+, \\ p(x) & \text{in } \omega_j^-, \end{cases} \quad (2.7)$$

where $p_1 \in \mathcal{P}^{m-1}$. It is not difficult to see that Ψ_j is a subset of the space of piecewise polynomials in (iii). We observe that $p^+ \circ \tilde{r}_j^{-1}$ is zero only across the approximate interface defined by r_j , and not Γ . So w jumps across Γ and the elements of Ψ_j are “nearly continuous” across the interface Γ , may have jump discontinuities in the derivatives across Γ , such that globally $\Psi_j \in \dot{H}^{m+1}(\Omega)$.

Condition **D** implies the following property:

LEMMA 2.7. *Assume **D** is satisfied. Let $0 \leq r \leq m+1$ be fixed. There exists a constant $C > 0$, depending only on σ, m, n and r , such that, for any patch $\bar{\omega}_j \cap \Gamma = \emptyset$, for any ball $\omega^* \subset \omega_j$ of diameter $\geq \sigma d_j$, and for any $w \in \Psi_j$, we have*

$$\|w\|_{H^r(\omega_j)} \leq C \|w\|_{H^r(\omega^*)}.$$

Similarly, let $\bar{\omega}_j = \bar{\omega}^+ \cup \bar{\omega}^-$ be the decomposition of ω_j defined by Γ . Then there exists a constant $C > 0$ such that for any $\bar{\omega}_j \cap \Gamma \neq \emptyset$, any $w \in \Psi_j$, any ball $\omega^{+*} \subset \omega_j^+$ of diameter $\geq \sigma d_j$, and any ball $\omega^{-*} \subset \omega_j^-$ of diameter $\geq \sigma d_j$, we have

$$\|w^+\|_{H^r(\omega_j^+)} + \|w^-\|_{H^r(\omega_j^-)} \leq C (\|w^+\|_{H^r(\omega^{+*})} + \|w^-\|_{H^r(\omega^{-*})}).$$

Proof. Let x_0, a be the center and radius of ω^* , so $\sigma d_j \leq a \leq d_j/2$. We can assume $x_0 = 0$. Then $B(0, d_j)$ is a ball which contains ω_j . We notice that, since w is a polynomial, it extends to a smooth function on $B(0, d_j)$. If $w \in \mathcal{P}_m$, then

$$\begin{aligned} \|w\|_{H^r(\omega_j)}^2 &\leq C \int_{B(0, d_j)} \sum_{\alpha \leq r} |\partial^\alpha w|^2(t) dt \\ &\leq \left(\frac{d_j}{a}\right) C \int_{\omega^*} \sum_{\alpha \leq r} |\partial^\alpha w|^2\left(\frac{d_j}{a}x\right) dx \leq C \|w\|_{H^r(\omega^*)}^2, \end{aligned}$$

where C only depends on σ, m, n , and r . Here, we recall, m is a fixed integer and n is the dimension of space. Above, we have used a change of variables and the fact that w is a polynomial.

When ω_j intersects the interface Γ , we can derive similar inequalities in each subdomain ω_j^+ , and ω_j^- owing to the Condition **A**(h) and the fact that w is a piecewise polynomial across the interface Γ in ω_j . The proof is now complete. \square

2.3. Inverse inequalities. We will need the following inverse inequality (see for example [6]).

LEMMA 2.8 (Inverse Inequality). *There exists $C > 0$, depending only on r, s , and m , such that*

$$\|p\|_{H^s(\omega_j)} \leq C(d_j)^{r-s} \|p\|_{H^r(\omega_j)},$$

for all $0 \leq r \leq s \leq m$, all $j \in \mathbb{Z}_+$, and all polynomials p of degree $\leq m$, where $d_j = \text{diam}(\omega_j)$.

From this lemma and Condition **A**(h), we immediately obtain the following generalization.

LEMMA 2.9. *There exists $C > 0$, depending only on r, s , and m , such that*

$$\|p\|_{H^s(\omega_j^\pm)} \leq C(d_j)^{r-s} \|p\|_{H^r(\omega_j^\pm)},$$

for any function $p : \omega_j \rightarrow \mathbb{R}$ that is given by a polynomial of degree $\leq m$ on each side ω_j^\pm of the interface Γ and for any $j \in \mathbb{Z}_+$, and $0 \leq r \leq s \leq m$.

2.4. Conditions on the sequence of GFE data defining S_μ . Our goal is to define a sequence of data sets $\Sigma_\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}$, yielding GFE-spaces S_μ , as in the previous Subsection 2.1, by

$$S_\mu = \left\{ \sum_j^{N_\mu} \phi_j^\mu v_j, v_j \in \Psi_j^\mu \right\} \subset \hat{H}^1(\Omega). \quad (2.8)$$

Let us denote by h_μ the maximum of the diameters of the patches ω_j^μ defining S_μ . Also, let $\tilde{h} = \max h_\mu$. Note that by assuming $h_\mu \rightarrow 0$ and by possibly excluding the first few spaces S_μ in the sequence we can assume that \tilde{h} is sufficiently small. For notational convenience, whenever μ is fixed, we drop it from the notation in what follows.

We are ready now to introduce the conditions on the sequence of data Σ_μ used to define the flat top GFE spaces S_μ . Namely, we assume that there exist constants C_j, σ, κ which will be called *structural constants* and a sequence $h_\mu \rightarrow 0$, as $\mu \rightarrow \infty$, such that, for any μ , Σ_μ satisfies Conditions **A**(h_μ), **B**, **C**, and **D**. (In particular, $d_j^\mu \leq h_\mu$, with $d_j^\mu = \text{diam}(\omega_j^\mu)$.) Note that if we denote, as before, by N_μ the number of elements of the μ th partition of unity, then $N_\mu \rightarrow \infty$ as $\mu \rightarrow \infty$.

Flat top GFE spaces were considered before by [3, 21] and in other papers. The conditioning of the resulting linear system was studied in [25]. Let us choose a basis v_j^μ of Ψ_j^μ and consider $\{v_j^\mu \phi_j^\mu\}$, which is a system of generators of S_μ . In general, this system will not form a basis of S_μ , and if we define the stiffness matrix using this system, the resulting system will be singular. The matrix may not be singular if exact integration is used, but become singular due to errors arising from numerical integration. More details can be founded in [1, 21, 20].

3. Properties of the GFE spaces S_μ . The main result of this section is that a sequence S_μ of GFE spaces obtained from data satisfying Conditions **A**(h_μ), **B**, **C**, and **D** given in the previous section also satisfies Assumptions 1.3 and 1.4 of Section 2. Whenever no confusion can arise, we will omit the explicit dependence on μ . We also assume throughout that h_μ is chosen sufficiently small.

We recall the functions g_j , q_j , f_i , and $r_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ defined in the previous section. Let \tilde{q}_j , $\tilde{r}_i(x)$ be defined by equation (2.2), (2.4), and similarly define

$$\begin{aligned}\tilde{g}_j(x) &= \tilde{g}_j(\bar{x}, x_n) = (\bar{x}, x_n + g_j(\bar{x})), \\ \tilde{f}_i(x) &= \tilde{f}_i(\bar{x}, x_n) = (\bar{x}, x_n + f_i(\bar{x})).\end{aligned}$$

We then have

$$\begin{aligned}\tilde{g}_j^{-1}(x) &= (\bar{x}, x_n - g_j(\bar{x})), & \tilde{f}_i^{-1}(x) &= (\bar{x}, x_n - f_i(\bar{x})), \\ \tilde{q}_j^{-1}(x) &= (\bar{x}, x_n - q_j(\bar{x})), & \tilde{r}_i^{-1}(x) &= (\bar{x}, x_n - r_i(\bar{x})).\end{aligned}$$

We will need the following result from [6].

LEMMA 3.1. *There exists a constant $C > 0$ depending only on the structural constants such that for any j satisfying $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$ and any polynomial p of degree m , we have*

$$\begin{aligned}\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} &\leq Cd_j^{m+1} \|p\|_{H^1(\omega_j)}, \quad \text{and} \\ \|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} &\leq Cd_j^m \|p\|_{H^1(\omega_j)}.\end{aligned}$$

Similarly, we have the following result.

LEMMA 3.2. *There exists a constant $C > 0$ depending only on the structural constants such that for each i satisfying $\bar{\omega}_i \cap \Gamma \neq \emptyset$ and any p which is a polynomial of degree m in ω_i^+ , we have*

$$\begin{aligned}\|p \circ \tilde{f}_i^{-1} - p \circ \tilde{r}_i^{-1}\|_{L^2(\omega_i^+)} &\leq Cd_i^{m+1} \|p\|_{H^1(\omega_i^+)}, \quad \text{and} \\ \|p \circ \tilde{f}_i^{-1} - p \circ \tilde{r}_i^{-1}\|_{H^1(\omega_i^+)} &\leq Cd_i^m \|p\|_{H^1(\omega_i^+)}.\end{aligned}$$

These two inequalities also hold in ω_i^- .

Proof. We apply Lemma 3.1 to p on ω_i^+ , which is possible since $\omega_i \subset \Omega$, in other words, in the above lemma ω_i is on one side of the boundary, in the same way as ω_i^\pm is on one side of Γ . We then proceed similarly in ω_i^- . \square

These lemmas give the following corollaries, the first near the boundary $\partial\Omega$, the second across the interface Γ . (For a definition of the Sobolev spaces $H^s(\partial\Omega)$, $s \in \mathbb{R}$, and their associated norm, we refer for example to [39].)

COROLLARY 3.3. *Let $p \in \mathcal{P}_m$, then for any j such that $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$, it holds*

$$\|\phi_j(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1})\|_{H^1(\omega_j)} \leq Cd_j^m \|p\|_{H^1(\omega_j)}.$$

If in addition $p \in \mathcal{P}_m$ vanishes on $\{x_n = 0\}$, we then have

$$\|\phi_j(p \circ \tilde{q}_j^{-1})\|_{H^{1/2}(\partial\Omega)} \leq Cd_j^m \|p\|_{H^1(\omega_j^*)}.$$

Above C is a constant independent of p, μ , and j , but dependent on m .

Proof. The first inequality follows directly from Lemma 3.1. The second inequality is proven in the same way as in [6], given that it is a local property near the boundary. \square

COROLLARY 3.4. *Let $\omega_j \cap \Gamma \neq \emptyset$ and let $p : \omega_j \rightarrow \mathbb{R}$ be given by a piecewise polynomial function that equals a polynomial on each side of the interface Γ . Then, there exists a constant C independent of p, μ , and j , but dependent on the structural constants, such that*

$$\|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{\hat{H}^1(\omega_j)} \leq Cd_j^m \|p\|_{\hat{H}^1(\omega_j)}.$$

If in addition p vanishes on $\{x_n = 0\}$, we then have

$$\|\phi_j(p \circ \tilde{r}_j^{-1})\|_{H^{1/2}(\Gamma \cap \omega_j)} \leq Cd_j^m \|p\|_{\hat{H}^1(\omega_j^*)}.$$

We remark that in the above corollary p may change patch by patch along Γ .

Proof. Using Lemma 3.2, Condition **B** or Definition 2.1, gives

$$\begin{aligned} \|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{H^1(\omega_j^+)} &\leq \|\phi_j\|_{L^\infty(\omega_j^+)} \|p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1}\|_{H^1(\omega_j^+)} \\ &\quad + \sum_{i=1}^n \|\partial_i \phi_j\|_{L^\infty(\omega_j^+)} \|p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1}\|_{L^2(\omega_j^+)} \\ &\leq Cd_j^m \|p\|_{H^1(\omega_j^+)} + Cd_j^{-1} d_j^{m+1} \|p\|_{H^1(\omega_j^+)} \leq Cd_j^m \|p\|_{H^1(\omega_j^+)}. \end{aligned}$$

Since a similar estimate holds in ω_j^- , we obtain the first inequality in the broken Sobolev space $\hat{H}^1(\omega_j)$.

We next observe that the hypotheses on Γ and $\partial\Omega$ imply that we can extend any function in $H^m(\Omega_k)$, $k = 1, \dots, K$, where Ω_k are as in (0.3), to a function in $H^m(\Omega)$ with control on the norm in terms only of the geometry of Γ and Ω .

Then, if p vanishes on $\{x_n = 0\}$, we have $\phi_j(p \circ \tilde{f}_j^{-1}) = 0$, and consequently

$$\begin{aligned} \|\phi_j(p \circ \tilde{r}_j^{-1})\|_{H^{1/2}(\Gamma \cap \omega_j)} &= \|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{H^{1/2}(\Gamma \cap \omega_j)} \\ &\leq \|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{H^{1/2}(\Gamma)} \leq C \|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{\hat{H}^1(\Omega)} \\ &\leq C \|\phi_j(p \circ \tilde{r}_j^{-1} - p \circ \tilde{f}_j^{-1})\|_{\hat{H}^1(\omega_j)} \leq Cd_j^m \|p\|_{\hat{H}^1(\omega_j)} \leq Cd_j^m \|p\|_{\hat{H}^1(\omega_j^*)}, \end{aligned}$$

where we used the trace theorem on each side of the interface and Lemma 2.7. By construction C above depends only on the trace constant on the Ω_k , and the structural constants by Condition **B**. \square

Now we are ready to prove that our first assumption, that is nearly zero boundary conditions and interface matching, is satisfied by the sequence of GFE-space S_μ introduced in Section 2.

PROPOSITION 3.5. *Let $S_\mu \subset \hat{H}^1(\Omega)$ be the sequence of GFE spaces defined by the data Σ_μ satisfying conditions $A(h_\mu)$, B , C and D . Then the sequence S_μ satisfies Assumption 1.3.*

Proof. We first show that elements in S_μ have nearly zero boundary values. Let $v_j \in \Psi_j^\mu$ and $w = \sum \phi_j v_j \in S_\mu$. We may assume $v_j = 0$ if $\bar{\omega}_j \cap \partial\Omega = \emptyset$. Then

$v_j = p_j \circ \tilde{q}_j^{-1}$, for some $p_j \in \mathcal{P}_m$ vanishing on \mathbb{R}^{n-1} (hence $p_j \circ \tilde{g}_j^{-1} = 0$.) Therefore,

$$\begin{aligned} \|w\|_{L^2(\partial\Omega)}^2 &\leq C \sum_j \|\phi_j v_j\|_{L^2(\partial\Omega)}^2 = C \sum_j \|\phi_j(p_j \circ \tilde{q}_j^{-1})\|_{L^2(\partial\Omega)}^2 \\ &\leq C \sum_j d_j^{2m} \|p_j\|_{H^1(\omega_j)}^2 \leq Ch_\mu^{2m} \sum_j \|p_j\|_{H^1(\omega_j^*)}^2 \leq Ch_\mu^{2m} \sum_j \|v_j\|_{H^1(\omega_j^*)}^2. \end{aligned}$$

Here we have used Lemma 2.4 and Corollary 3.3. By Condition **C**, $\phi_j^\mu = 1$ on ω_j^* and hence the sets ω_j^* do not intersect. Therefore $\sum_j \|v_j\|_{H^1(\omega_j^*)}^2 = \|w\|_{H^1(\cup\omega_j^*)}^2$ and

$$\|w\|_{L^2(\partial\Omega)}^2 \leq Ch_\mu^{2m} \|w\|_{H^1(\cup\omega_j^*)}^2 \leq Ch_\mu^{2m} \|w\|_{\hat{H}^1(\Omega)}^2.$$

We now proceed to estimate the jump on the interface Γ using a similar reasoning. Indeed, as before, let $v_j \in \Psi_j^\mu$ and let $w = \sum \phi_j v_j \in S_\mu$. We may assume again that $v_j = 0$ if $\bar{\omega}_j \cap \Gamma = \emptyset$. Then,

$$v_j = p_j + (p_j^+ \circ \tilde{q}_j^{-1})1_{\omega_j^+},$$

where $1_{\omega_j^+}$ is the characteristic function of ω_j^+ , $p_j, p_j^+ \in \mathcal{P}_m$, and p_j^+ vanishes on \mathbb{R}^{n-1} by Condition **D**. Therefore,

$$p_j^+ \circ \tilde{f}_j^{-1}(x) = p_j^+(\bar{x}, x_n - f_j(\bar{x})) = 0$$

on Γ and we obtain

$$\begin{aligned} \|w_+ - w_-\|_{L^2(\Gamma)}^2 &\leq \sum_j \|\phi_j(p_j^+ \circ \tilde{r}_j^{-1})\|_{L^2(\Gamma \cap \omega_j)}^2 \\ &= \sum_j \|\phi_j(p_j^+ \circ (\tilde{r}_j^{-1} - \tilde{f}_j^{-1}))\|_{L^2(\Gamma \cap \omega_j)}^2 \\ &\leq C \sum_j d_j^{2m} \|p_j^+\|_{H^1(\tilde{f}_j^{-1}(\omega_j^*))}^2 \quad \text{by Corollary 3.4} \\ &\leq Ch_\mu^{2m} \|w\|_{\hat{H}^1(\Omega)}^2, \end{aligned}$$

where the last step is proved as in the first part of the proof. The proof is now complete. \square

Next we are going to prove that the sequence S_μ also satisfies our second assumption, that is, the assumption of approximability, using Lemmas 3.1 and 3.2 and the following result.

LEMMA 3.6. (i) *Let $\bar{\omega}_j \cap \Gamma = \emptyset$, and let $u \in H^{m+1}(\omega_j)$ satisfy $u = 0$ on $\bar{\omega}_j \cap \partial\Omega$. Then there exists a polynomial $w \in \Psi_j^\mu$ such that*

$$\begin{aligned} \|u - w\|_{H^1(\omega_j)} &\leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}, \quad \text{and} \\ \|u - w\|_{L^2(\omega_j)} &\leq Cd_j^{m+1} \|u\|_{H^{m+1}(\omega_j)} \end{aligned}$$

for a constant C independent of u, μ , and j .

(ii) *Let $\bar{\omega}_j \cap \Gamma \neq \emptyset$, and let $u \in \hat{H}^{m+1}(\omega_i) \cap H^1(\Omega)$. Then there exists $w \in \Psi_i^\mu$ such that*

$$\begin{aligned} \|u - w\|_{\hat{H}^1(\omega_i)} &\leq Cd_i^m \|u\|_{\hat{H}^{m+1}(\omega_i)}, \quad \text{and} \\ \|u - w\|_{L^2(\omega_i)} &\leq Cd_i^{m+1} \|u\|_{\hat{H}^{m+1}(\omega_i)}. \end{aligned}$$

Proof. (i) Let $v = u \circ \tilde{g}_j = u(\bar{x}, x_n + g_j(\bar{x}))$. Since $u|_{\partial\Omega} = u(\bar{x}, g_j(\bar{x})) = 0$, we have that $v = 0$ on $\{x_n = 0\} \cap \tilde{g}_j^{-1}(\omega_j)$, where $\tilde{g}_j^{-1}(x) = (\bar{x}, x_n - g_j(\bar{x}))$. Without loss of generality, we may assume for h_μ small enough that $\tilde{g}_j^{-1}(\omega_j)$ lies on one side of \mathbb{R}^{n-1} , for example $U_+ := \tilde{g}_j^{-1}(\omega_j) \subset \{x, x_n \geq 0\}$. Let U be the union of the closure of $\tilde{g}_j^{-1}(\omega_j)$ and of its symmetric subset with respect to \mathbb{R}^{n-1} . Define $v_1 \in H^1(U)$ to be odd extension of v , and let p_1 be the $H^1(U)$ - projection of v_1 onto the subspace \mathcal{P}_m of polynomials of degree $\leq m$. It is easy to see that p_1 is odd and $p_1 = 0$ on \mathbb{R}^{n-1} . From standard approximation results [15], it follows that

$$\|v - p_1\|_{H^1(U_+)} \leq Cd_j^m \|v\|_{H^{m+1}(U_+)}.$$

Then,

$$\begin{aligned} \|u - p_1 \circ \tilde{g}_j^{-1}\|_{H^1(\omega_j)} &\leq \|v - p_1\|_{H^1(U_+)} \\ &\leq Cd_j^m \|v_1\|_{H^{m+1}(U_+)} \leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}. \end{aligned}$$

Let $w = p_1 \circ \tilde{q}_j^{-1}$, then $w \in \Psi_j^\mu$ and

$$\begin{aligned} \|u - w\|_{H^1(\omega_j)} &\leq \|u - p_1 \circ \tilde{g}_j^{-1}\|_{H^1(\omega_j)} + \|p_1 \circ \tilde{g}_j^{-1} - p_1 \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \\ &\leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)} + Cd_j^m \|p_1\|_{H^1(\omega_j)} \leq Cd_j^m \|u\|_{H^{m+1}(\omega_j)}, \end{aligned}$$

where we have used Lemma 3.1.

Since both v_1 and p_1 vanish for $x_n = 0$, Poincaré's inequality gives

$$\|v_1 - p_1\|_{L^2(U_+)} \leq Cd_j \|v_1 - p_1\|_{H^1(U_+)} \leq Cd_j^{m+1} \|v_1\|_{H^{m+1}(\omega_j)}.$$

Hence,

$$\begin{aligned} \|u - w\|_{L^2(\omega_j)} &\leq \|u - p_1 \circ \tilde{g}_j^{-1}\|_{L^2(\omega_j)} + \|p_1 \circ \tilde{g}_j^{-1} - p_1 \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \\ &\leq \|v_1 - p_1\|_{L^2(U)} + Cd_j^{m+1} \|p_1\|_{H^1(\omega_j)} \leq Cd_j^{m+1} \|u\|_{H^{m+1}(\omega_j)}. \end{aligned}$$

This completes the proof of the first part of this lemma.

(ii) We now proceed to the proof of the second part. We assume that ω_i intersects the interface Γ . We know that $u \in \hat{H}^{m+1}(\omega_i)$. That is, $u \in H^{m+1}(\omega_i^+)$ and $u \in H^{m+1}(\omega_i^-)$, by the definition of the broken Sobolev spaces \hat{H} . Standard extension theorems (see e.g. [39, pages 284-286] for more details). To see that the constants do not depend on i , we notice that we can embed each ω_i in a fixed ball with chosen point of the interface mapping to the origin. We can then apply a diffeomorphism that straightens the interface. As μ increases, this diffeomorphism approaches a linear transformation, which can be assumed to be an orthogonal transformation, thus preserving the Sobolev norm. We can then apply the extension theorem for a half space with uniform control on the Sobolev norm. The constant in our extension theorem for the sets $\omega_i^\pm \subset \omega_i$ can then be chosen to be a multiple of the constant appearing in the extension for half-space. Alternatively, we can employ a global extension from one side of Γ as in the proof of Lemma 3.4.

We now show that there exists $v \in H^{m+1}(\omega_i)$ such that $u = v$ on ω_i^- and

$$\|v\|_{H^{m+1}(\omega_i)} \leq C \|u\|_{H^{m+1}(\omega_i^-)}. \quad (3.1)$$

By standard approximation results (see [15, Section 4.1], for example), there exists $p \in \mathcal{P}_m$ such that

$$\begin{aligned} \|v - p\|_{H^k(\omega_i)} &\leq Cd_i^{m+1-k} \|v\|_{H^{m+1}(\omega_i)} \\ &\leq Cd_i^{m+1-k} \|u\|_{H^{m+1}(\omega_i^-)} \leq Cd_i^{m+1-k} \|u\|_{\hat{H}^{m+1}(\omega_i)} \end{aligned} \quad (3.2)$$

for $k = 0, 1, \dots, m+1$.

We next define $v^+ := (u - v) \circ \tilde{f}_i$ on $\tilde{f}_i^{-1}(\omega_i)$. Since $u = v$ on ω_i^- , $v^+ \equiv 0$ on $\tilde{f}_i^{-1}(\omega_i^-)$, and in particular on \mathbb{R}^{n-1} , we obtain

$$\|v^+\|_{H^{m+1}(\tilde{f}_i^{-1}(\omega_i^+))} \leq C \|u\|_{\hat{H}^{m+1}(\omega_i)}, \quad (3.3)$$

owing to (3.1).

We now proceed as in the proof of (i). Namely, let p^+ be defined in the same way as p_1 was defined in the proof of (i), but using v^+ instead of v . In particular, p^+ is odd and hence $p^+ = 0$ on \mathbb{R}^{n-1} . As before,

$$\|v^+ - p^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} \leq Cd_i^m \|v^+\|_{H^{m+1}(\tilde{f}_i^{-1}(\omega_i^+))}. \quad (3.4)$$

Then inequalities 3.1, 3.2, 3.3, and 3.4 give

$$\begin{aligned} \|(u - p) - p^+ \circ \tilde{f}_i^{-1}\|_{H^1(\omega_i^+)} &\leq C(\|u - v - p^+ \circ \tilde{f}_i^{-1}\|_{H^1(\omega_i^+)} + \|v - p\|_{H^1(\omega_i^+)}) \\ &\leq C(\|v^+ - p^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} + \|v - p\|_{H^1(\omega_i^+)}) \\ &\leq Cd_i^m \|v^+\|_{H^{m+1}(\tilde{f}_i^{-1}(\omega_i^+))} + Cd_i^m \|u\|_{\hat{H}^{m+1}(\omega_i)} \leq Cd_i^m \|u\|_{\hat{H}^{m+1}(\omega_i)}. \end{aligned}$$

Next, let $w = p + p^+ \circ \tilde{r}_i^{-1} 1_{\omega_i^+}$, where $1_{\omega_i^+}$ is the characteristic function of the set ω_i^+ . Then

$$\begin{aligned} \|u - w\|_{H^1(\omega_i^+)} &\leq \|(u - p) - p^+ \circ \tilde{f}_i^{-1}\|_{H^1(\omega_i^+)} + \|p^+ \circ \tilde{f}_i^{-1} - p^+ \circ \tilde{r}_i^{-1}\|_{H^1(\omega_i^+)} \\ &\leq Cd_i^m \|u\|_{\hat{H}^{m+1}(\omega_i)} \end{aligned}$$

where we have also used Lemma 3.1 and the fact that

$$\|p^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} \leq C \|u\|_{\hat{H}^{m+1}(\omega_i)}.$$

This last inequality is true because

$$\begin{aligned} \|p^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} &\leq C \|p^+ - v^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} + \|v^+\|_{H^1(\tilde{f}_i^{-1}(\omega_i^+))} \\ &\leq \|v^+\|_{H^{m+1}(\tilde{f}_i^{-1}(\omega_i^+))} \leq C \|u\|_{\hat{H}^{m+1}(\omega_i)}. \end{aligned}$$

On the other side of the interface, we have instead

$$\|u - w\|_{H^1(\omega_i^-)} = \|v - p\|_{H^1(\omega_i^-)} \leq Cd_i^m \|u\|_{\hat{H}^m(\omega_i)},$$

which together with the above calculations gives

$$\|u - w\|_{\hat{H}^1(\omega_i)} \leq Cd_i^m \|u\|_{\hat{H}^m(\omega_i)}.$$

Finally, Poincaré's inequality again gives

$$\|v - p^+ \circ \tilde{f}_i^{-1}\|_{L^2(\omega_i)} \leq Cd_i \|v_1 - p^+ \circ \tilde{f}_i^{-1}\|_{H^1(\omega_i)} \leq Cd_i^{m+1} \|v_1\|_{H^{m+1}(\omega_i)}.$$

Then

$$\begin{aligned} \|u - w\|_{L^2(\omega_i^+)} &\leq C\|(u - v) - p^+ \circ \tilde{f}_i^{-1}\|_{L^2(\omega_i^+)} + \|v - p\|_{L^2(\omega_i^+)} \\ &\leq C(\|v_1 - p^+\|_{L^2(\tilde{f}(\omega_i))} + \|v - p\|_{L^2(\omega_i)}) \leq Cd_i^{m+1}\|u\|_{\hat{H}^{m+1}(\omega_i)}. \end{aligned}$$

Since

$$\|u - w\|_{L^2(\omega_i^-)} = \|v - p\|_{L^2(\omega_i^-)} \leq Cd_i^{m+1}\|u\|_{\hat{H}^{m+1}(\omega_i)},$$

we also have

$$\|u - w\|_{L^2(\omega_i)} \leq Cd_i^{m+1}\|u\|_{\hat{H}^{m+1}(\omega_i)}.$$

The proof is now complete. \square

We are ready to prove that our second assumption, the assumption on approximability is satisfied.

PROPOSITION 3.7. *The sequence of GFEM spaces S_μ satisfies Assumption 1.4.*

Proof. Let $u \in \hat{H}^{m+1}(\Omega) \cap H_0^1(\Omega)$. If $\bar{\omega}_j$ does not intersect $\partial\Omega$ and Γ , we define $w_j \in \Psi_j^\mu = \mathcal{P}_m$ to be the orthogonal projection of u onto \mathcal{P}_m in $H^1(\omega_j)$. Otherwise we define $w_j \in \Psi_j$ using Lemma 3.6. Then we let $w = \sum_j \phi_j w_j$. Lemmas 2.4 and 3.6 together with Condition **B** and **C** then yield

$$\begin{aligned} |u - w|_{\hat{H}^1(\Omega)}^2 &\leq C \sum_j |\phi_j(u - w_j)|_{\hat{H}^1(\omega_j)}^2 \\ &\leq C \sum_j (\|\phi_j\|_{L^\infty}^2 |u - w_j|_{\hat{H}^1(\omega_j)}^2 + \|\nabla \phi_j\|_{L^\infty}^2 \|u - w_j\|_{L^2(\omega_j)}^2) \\ &\leq C \sum_j (d_j^{2m} \|u\|_{\hat{H}^{m+1}(\omega_j)}^2 + d_j^{2m} \|u\|_{\hat{H}^{m+1}(\omega_j)}^2) \leq Ch_\mu^{2m} \|u\|_{\hat{H}^{m+1}(\Omega)}^2, \end{aligned}$$

which gives the desired result. \square

4. Numerical tests. We close the paper with a numerical application of our method. We chose to test the theoretical predictions on a problem that could be implemented with relative ease, but that presented the approximation challenges discussed in the paper. Our work shows, in particular, that transmission problems give rise to approximation issues similar to those of standard Dirichlet problems. We have therefore tested our results for Ω the unit disk in two dimensions and the model problem

$$-\Delta u = 4, \quad u|_{\partial\Omega} = 0, \tag{4.1}$$

which has the exact solution $u(x, y) = 1 - x^2 - y^2$ and allows for more steps in the ‘‘refinement.’’ Our partition of unity is obtained by applying the Shepard procedure to radial functions of the form $\tilde{\phi}_i(x) = \Phi(|x - x_i|/r)$. (We suppress the dependence on μ , for the time being, and consider a single GFE space.) The function $\Phi(x)$ is chosen to be \mathcal{C}^1 , and given as follows. It is equal to 1 for $x \leq 1/8$, equal to 0 for $x \geq 7/8$, and a polynomial of degree 3 for $x \in [1/8, 7/8]$. The resulting partition of unity functions will be denoted ϕ_i . The points x_i are chosen on concentric circles of radii kr , $k \in \mathbb{Z}_+$. The constant r is chosen to be of the form $r = 1/\mu$, with $\mu > 0$ an integer. On each circle, we choose the points to be uniformly distributed at a distance close to r , but $\leq r$. Three basis elements in S_μ : ϕ_i , $x_1\phi_i$, and $x_2\phi_i$ will correspond to each point x_i that is not on the boundary. On the other hand, exactly one basis element will

μ	N	$100e$	N^2e	μ^2e	$100e_a$
2	35	6.6973	2.3441	0.2679	9.8130
3	83	3.1637	2.6258	0.2847	4.3541
4	149	1.9063	2.8403	0.3050	2.4610
6	336	1.0204	3.4286	0.3673	1.1031
8	600	0.5365	3.2189	0.3433	0.6228
12	1354	0.2601	3.5215	0.3745	0.2779
16	2409	0.1353	3.2600	0.3464	0.1567

TABLE 4.1
Convergence rates e of GFE solutions the model problem (4.1).

correspond to each x_i on the boundary (we choose $\lceil 2\pi/r \rceil + 1$ such points, except if $r = 1$, in which case we choose 6 such points), using a linear function that vanishes on the tangent to the boundary at x_i , as in Condition **D**. We show the results of the numerical test in Table 4.1. In that table, N denotes the number of degrees of freedom, e denotes the L^2 error, and e_a denotes the interpolation error, defined using a Taylor type interpolant as in [5].

The results depend greatly on the numerical quadrature. We have also noticed this issue in an independent implementation of this method in one dimension. See also [2]. We have implemented our method with a less precise quadrature formula, in which case we could run more tests, but the condition $e \leq e_a$ was not satisfied any longer.

We conclude that the numerical tests confirm our results, since the scaled errors N^2e and μ^2e have the correct behavior (they are uniformly bounded, in fact nearly constant, in our range). This is consistent with the theoretical error estimate $e_{\text{theoretical}} \leq CN^{-2}$. One-dimensional tests show that values N^2e are nearly constant in a certain range for the number of the degrees of freedom, after which they start growing rather fast. Replacing the quadrature with a more precise quadrature will delay the moment when N^2e starts growing, but does not eliminate this phenomenon (at least in the one-dimensional tests). A similar phenomenon was noticed in two dimensions (but for a less precise quadrature formula than the one used for the results in Table 4.1), while it does seem to appear for N^2e_a , which is remarkably constant in N . The test problem we chose was sufficiently simple that we could implement our method in MATLAB[®] and run the code on a personal computer (the code is available on the authors' web site.) Even in this case, however, issues of memory management arise due to the growth of degrees of freedom with each "refinement". The approach we have described to the numerical implementation of our method can be applied to transmission problems. Indeed, we have shown that from a theoretical standpoint GFEMs for transmission and Dirichlet problems are handled similarly. At the same time, implementing our method for a transmission problem requires significantly more computing power due to a much larger number of degrees of freedom present in the problem. We are planning more numerical tests, also using other norms for the error, to showcase the method in more general situations.

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